universität wien

DISSERTATION / DOCTORAL THESIS

Titel der Dissertation / Title of the Doctoral Thesis

"Cayley–Catalan Combinatorics of Affine Permutation Groups"

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2017 / Vienna 2017

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 405
Dissertationsgebiet lt. Studienblatt / field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Univ.-Prof. Dr. Christian Krattenthaler

Contents

Acknowledgements

First and foremost I want to thank Christian Krattenthaler who always gave me the freedom to pursue my own ideas, but at the same time provided advise and direction whenever it was needed. Thank you also for encouraging and making it possible for me to travel and visit conferences during my studies. I could not have hoped for a better advisor.

I gained a lot from working and discussing with numerous people generously sharing their knowledge with me. Thank you to Cesar Ceballos, Tom Denton, Ilse Fischer, Jim Haglund, Myrto Kallipoliti, Christoph Neumann, Vivien Ripoll, Carsten Schneider, Christian Stump, Marko Thiel, Nathan Williams and Meesue Yoo.

I gratefully acknowledge the financial support from the Austrian Science Fund FWF in the framework of the special research program SFB F50. I would also like to thank the University of Vienna and in particular the Faculty of Mathematics. Throughout my time here I have experienced an environment that was both fascinating mathematically and very enjoyable in general.

I count myself very lucky be part of a large group of awesome young people who worked in Vienna while I was here: Alex, Andrei, Cesar, Chen, Christoph, Daniel, Emma, Florian, Gwendal, Hans, Henri, Judith, Koushik, Lukas, Manjil, Marko, Meesue, Michael, Myrto, Sabine, Stephan, Ting, Tomack, Viviane, Vivien, Yvonne, Zhicong and Zófia. Thank you for drinks had, games played, riddling and travelling the world together!

Finally, thank you to all my Muggle friends (non-mathematical friends) and my family for their genuine interest in what a parking function might be!

Abstract

Cayley–Catalan combinatorics refers to the study of combinatorial objects counted by (generalised) Catalan numbers or Cayley numbers. Examples of classical combinatorial objects treated in this thesis that fall into this category are Dyck paths, parking functions and core partitions. These objects turn out to be closely related to the group of affine permutations and their inversions. Many involved ideas carry over to arbitrary affine Weyl groups. Exploring this connection we review the finite torus, the Shi arrangement and non-nesting parking functions. In particular, we define new combinatorial models for these objects in terms of labelled lattice paths when the crystallographic root system is of classical type.

Several combinatorial statistics on Catalan objects have been introduced to give combinatorial interpretations for polynomials appearing in representation theory or algebraic geometry. For example, Haglund's bounce-statistic, Haiman's dinv-statistic or Armstrong's skew-length of a partition have all been used to define q-analogues of Catalan numbers. We strengthen and expand on previously known symmetry properties of the skew-length statistic. The dinv-statistic is generalised to a statistic on the finite torus, allowing for a new definition of q-Catalan numbers for arbitrary Weyl groups. Furthermore, we extend the notion of Shi tableaux to give a generalisation of the skew-length statistic for affine Weyl groups, thereby enabling us to give a combinatorial definition of rational q-Catalan numbers for Weyl groups.

An important bijection in this field is the so called zeta map. The original zeta map is a bijection on the set of Dyck paths, however, it can be generalised to a uniform bijection attached to any Weyl group. We prove that this bijection transforms the dinv-statistic on elements of the finite torus into the area-statistic on non-nesting parking functions. Furthermore, we develop the lattice path combinatorics of the zeta map for the infinite families of crystallographic root systems in analogy to the connection to Dyck paths when the Weyl group is the symmetric group. This leads to the discovery of two new bijections between ballot paths and lattice paths in a square, both of which are known to be counted by central binomial coefficients.

Zusammenfassung

Die Bezeichnung Cayley–Catalan Kombinatorik bezieht sich auf das Studium von kombinatorischen Objekten, die von (verallgemeinerten) Catalan-Zahlen oder Cayley-Zahlen abgezählt werden. Beispiele für klassische kombinatorische Objekte, die in dieser Dissertation behandelt werden und in diese Kategorie fallen, sind Dyck-Pfade, Parkfunktionen oder Kernpartitionen. Alle diese Objekte sind eng verwandt mit der Gruppe affiner Permutationen und deren Inversionen. Viele der zu beobachtenden Konzepte lassen sich auf allgemeine affine Weyl-Gruppen übertragen. In diesem Zusammenhang besprechen wir den endlichen Torus, das Shi-Gefüge und nichtverschachtelte Parkfunktionen. Insbesondere definieren wir in den Fällen, in denen das zugrundeliegende kristallographische Wurzelsystem von klassischem Typ ist, neue kombinatorische Modelle für diese Objekte anhand von bezeichneten Gitterpunktwegen.

Verschiedene kombinatorische Statistiken wurden auf Catalan-Objekten definiert, um Polynomen, die in der Darstellungstheorie oder der algebraischen Geometrie auftauchen, eine kombinatorische Bedeutung zu verleihen. Zum Beispiel wurden Haglunds bounce-Statistik, Haimans dinv-Statistik und Armstrongs Schieflänge einer Zahlenpartition benutzt, um q-Analoga der Catalan-Zahlen zu definieren. Wir verfeinern und verallgemeinern bisher bekannte Symmetrieeigenschaften der Schieflänge. Die dinv-Statisik wird zu einer Statistik auf den Elementen des endlichen Torus verallgemeinert. Dies erlaubt es uns q-Catalan-Zahlen für beliebige Weyl-Gruppen zu definieren. Zusätzlich erweitern wir den Begriff von Shi-Tafeln, um eine Verallgemeinerung der Schieflänge für affine Weyl-Gruppen zu ermöglichen. Dadurch sind wir in der Lage, eine kombinatorische Definition von rationalen q-Catalan-Zahlen für Weyl-Gruppen anzugeben.

Eine wichtige Bijection in diesem Gebiet ist die sogenannte Zeta-Abbildung. Die ursprüngliche Zeta-Abbildung ist eine Bijektion auf der Menge der Dyck-Pfade, jedoch wurde sie einheitlich zu einer einer beliebigen Weyl-Gruppe zugehörigen Bijektion verallgemeinert. Wir beweisen, dass diese Bijektion die dinv-Statistik auf den Elementen des endlichen Torus auf das Gebiet der entsprechenden nichtverschachtelten Parkfunktion überführt. Weiters entwickeln wir in Analogie zu der Verbindung zu Dyck-Pfaden im Falle der symmetrischen Gruppe die Gitterpunktwegskombinatorik der Zeta-Abbildung für die unendlichen Familien von kristallographischen Wurzelsystemen. Dies führt zur Entdeckung von zwei neuen Bijektionen zwischen Abstimmungspfaden und Gitterpunktwegen, die einem Quadrat eingeschrieben sind, von welchen jeweils bekannt ist, dass sie von Zentralbinomialkoeffizienten abgezählt werden.

CHAPTER 0

Introduction

Three of the most ubiquitous sequences of numbers in combinatorics are the factorials $n!$, the Catalan numbers $\binom{2n}{n}/(n+1)$ and the Cayley numbers $(n+1)^{n-1}$. Factorials count permutations of an n-set which can be represented as bijections between n-sets, as linear orders of an n-set or as collections of labelled cycles. Catalan numbers count a wealth of combinatorial objects such as binary trees, Dyck paths, non-crossing partitions, non-crossing perfect matchings, certain pattern avoiding permutations, rooted plane trees, triangulations of polygons and *many more* [**70**]. Cayley numbers count most obviously maps from an $(n-1)$-set to an $(n+1)$-set, most famously labelled trees, but also parking functions and regions of the Shi arrangement among others.

It is one of the beautiful coincidences in algebraic combinatorics that these numbers are related to each other elegantly by way of a group G acting on a set X such that

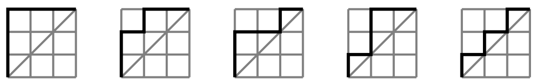
$$\#G = n!\,, \qquad \#X = (n+1)^{n-1} \qquad \text{and} \qquad \#\{Gx : x \in X\} = \frac{1}{n+1}\binom{2n}{n}.$$

As we shall see, a convenient choice is to take the symmetric group $G = \mathfrak{S}_n$ acting on the set $X = \mathrm{PF}_n$ of parking functions. An integer vector with non-negative entries $(f_1, \ldots, f_n) \in \mathbb{N}^n$ is called parking function if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $f_i < \sigma(i)$ for all i. For example, there are sixteen parking functions of length three.

$$\mathrm{PF}_3 = \left\{ \begin{matrix} (0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,0,2),(0,2,0),(2,0,0),(0,1,1), \\ (1,0,1),(1,1,0),(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0) \end{matrix} \right\}$$

Clearly the symmetric group acts on the set of parking functions via permutation of entries. At this point the orbits $\{Gx : x \in X\}$ are easily seen to be indexed by increasing parking functions, which can be identified with Dyck paths. There are five increasing parking functions of length three.

$(0,0,0)$ $(0,0,1)$ $(0,0,2)$ $(0,1,1)$ $(0,1,2)$



Of course there are other equivalent choices for G and X.

The study of the parking function representation has given rise to a whole new branch in algebraic combinatorics. It connects a variety of topics in mathematics – from Macdonald polynomials, and symmetric functions in general [**29, 19**], and the representation theory of Weyl groups and related algebras [**9, 33**], to hyperplane arrangements [**5**], Hilbert schemes [**41, 42**] and knot invariants [**34**]. It is in part responsible for the revival of classical combinatorial objects such as core partitions and abacus diagrams.

Over the years Catalan numbers have been generalised in many different directions, which I prefer to view as dimensions. In this thesis the focus is on the following four dimensions.

The first is the introduction of a rational parameter, which encompasses the introduction of a Fuß-parameter. Many of the objects counted by Catalan numbers listed above can be assigned

Fuß-analogues and rational analogues counted by these two-parameter families of numbers. For example, Dyck paths are lattice paths inside a square that do not cross the main diagonal. The corresponding combinatorial objects counted by the rational Catalan number with parameter n/p, where $n, p \in \mathbb{N}$ are assumed to be relatively prime, are lattice paths inside an n by p rectangle that do not cross the diagonal of slope n/p.

The second generalisation is to pass from Catalan numbers to Cayley numbers. Algebraically this can be viewed as the extension of a family of combinatorial objects by a group. The orbits of our group action are counted by Catalan numbers, whereas the "full" set on which the group acts is counted by Cayley numbers. Combinatorially we pass from unlabelled objects representing the orbits to labelled objects. Typically the group, whose elements are realised as some kind of permutations, acts by rearranging the labels.

Third is the transition to polynomials by introducing a q-statistic and a t-statistic. Here Catalan numbers or Cayley numbers are replaced by the generating function of a family of combinatorial objects counted by these numbers with respect to different statistics. The original sequences of numbers are recovered when all variables are set to one. A great motivation for the study of such statistics is that many of the polynomials that arise also have algebraic interpretations, for example, as the Hilbert series of certain graded algebras, which may carry additional structure such as a group acting on them as a group of linear transformations. Working out the precise connections between polynomials arising in representation theory or algebraic geometry and their combinatorial models is often rather difficult and has led to numerous deep results in the past decades.

The fourth generalisation is to introduce the dependency on an irreducible crystallographic root system. Many of the combinatorial objects in discussion can be viewed as special cases of more general objects attached to a root system. For example, the role of the symmetric group is taken by the corresponding Weyl group. Similarly, Dyck paths are generalised by ideals in the poset of positive roots. The emerging Coxeter–Catalan numbers and Coxeter–Cayley numbers associated to a root system have nice product formulas featuring important invariants of the respective Weyl group.

It is of course desirable, although not always easy, to unify two or more directions of generalisation. This principle serves as the motivation for a lot of the work done in this thesis. Each such (common) generalisation forms a vertex of what I call the *Catalan cube*. Starting at the origin, that is, the original Catalan numbers, we shall explore more and more vertices of this four-dimensional cube, only to discover that the far vertex with all coordinates equal to one remains out of our reach (for now).

This thesis aims at giving a thorough introduction from the point of view of affine Weyl groups, which seem to appear naturally when combinatorial objects corresponding to a root system are generalised to the level of rational Catalan numbers. I hope to make this beautiful topic in algebraic combinatorics accessible to a wide range of readers. The only required prerequisites are some familiarity with root systems of finite Coxeter groups. I include many proofs of known results or sketch how they can be obtained in order to make the presentation more self contained and to collect different techniques together in one place. I have also tried to provide the reader with many conjectures and open problems, some well-known and some of my own.

The thesis is divided into six chapters. In Chapter 1 we recall some needed background and settle conventions, with a focus on affine Weyl groups and their realisations as groups of bijections on the integers. Chapter 2 begins the exploration of several objects related to Dyck paths and parking functions, covering the first three dimensions mentioned above. In Chapter 3 we use core partitions as a link between the combinatorics of Chapter 2 and the affine symmetric group. As well as this, Section 3.4 also contains some original research. The skew-length is a statistic on cores that can be used to define rational q, t-Catalan numbers. We prove some symmetry properties of the skew-length that refine known results and extend to previously untreated cases.

In Chapter 4 Coxeter–Catalan numbers are defined, that is, we take a first step in the dimension of root systems. The main object of study is a quotient of the coroot lattice called the finite torus. New contributions are the definition of a uniform statistic dinv on the finite torus extending a statistic of Haiman, and the introduction of lattice path models for the finite torus in types B_n, C_n and D_n.

Chapter 5 is devoted to the Shi arrangement and non-nesting parking functions, which provide a second possible generalisation of Dyck path combinatorics to the level of Weyl groups. We define new lattice path models for non-nesting parking functions, and extend the notion of Shi tableaux, which encode the regions of the Shi arrangement, to the rational case. These rational Shi tableaux provide us with an analogue of the skew-length statistic for Weyl groups and allows us to define rational q-Coxeter–Catalan numbers.

Finally in Chapter 6 we discuss the zeta map, a bijection that connects the finite torus with non-nesting parking functions. We apply the zeta map to establish a relation between the dinv-statistic on the finite torus and the area-statistic on non-nesting parking function. Moreover we explore the combinatorics of the zeta map types B_n, C_n and D_n using the lattice path interpretations of the objects involved.

CHAPTER 1

Notation

In this Chapter we fix notation. Section 1.1 treats integer partitions, which we encounter in our discussion of core partitions in Chapter 3. Section 1.2 contains notation on lattice paths, which is used in the introduction to Catalan combinatorics based on Dyck paths in Chapter 2 and the treatment of combinatorial zeta maps in Chapter 6. In Sections 1.3 and 1.4 facts on root systems and affine permutation groups are presented, on which we shall rely in Chapters 3–6.

Let $\mathbb{N}$ denote the set of non-negative integers. The cardinality of a finite set S is denoted by $\#S$ or $|S|$. Denote by S^* the set of finite words (of any length) in the alphabet S. For $n \in \mathbb{N}$ define $[n] = \{1, \dots, n\}$ and $[\pm n] = [n] \cup \{-i : i \in [n]\}$. Given a function $f : X \to Y$ the image of a subset $S \subseteq X$ under f is denoted by $f(S) = \{f(s) : s \in S\}$. Denote the power set of a set X by $\mathscr{P}(X) = \{Y \subseteq X\}$.

1.1. Partitions

Let $n \in \mathbb{N}$. An *integer partition* or simply *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers, that is, $\lambda_1 \geq \dots \geq \lambda_r > 0$, such that $\lambda_1 + \dots + \lambda_r = n$. Equivalently, one can regard a partition of n as an infinite sequence $(\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$ and $\sum_i \lambda_i = n$. To move from one description to the other simply append or delete zeroes. The numbers λ_i are called *summands* or *parts* of the partition. The *size* of the partition is the sum of its parts and is denoted by $|\lambda|$. The number of positive parts is called the *length* of the partition and is denoted by $\ell(\lambda)$. Note that there is a unique partition of zero, namely the empty partition. The set of all partitions is denoted by Π.

A pair $z \in \mathbb{Z}^2$ is called a *cell*. Given a cell $z = (i, j) \in \mathbb{Z}^2$ we define its *north*, *east*, *south* and *west neighbours* as

$$\mathbf{n}z = (i-1, j), \qquad \mathbf{e}z = (i, j+1), \qquad \mathbf{s}z = (i+1, j), \qquad \mathbf{w}z = (i, j-1).$$

We identify a partition with a set of cells $\lambda = \{(i, j) \in \mathbb{N}^2 : i \in [\ell(\lambda)], j \in [\lambda_i]\}$ called the *Young diagram* of the partition. Given two partitions λ and μ we say λ *contains* μ if $\mu \subseteq \lambda$ when viewed as Young diagrams. The pair $(\Pi, \subseteq)$ defines a partial order on partitions called *inclusion order*. The *conjugate partition* of a partition λ is defined to be the partition $\lambda' = \{(j, i) : (i, j) \in \lambda\}$ obtained by transposing the Young diagram. The set of partitions of length at most k is denoted by Π^k. The set of *bounded partitions*, that is, partitions with $\lambda_1 \leq k$, is denoted by $\Pi_{\leq k}$. Clearly conjugation is a bijection from Π^k to $\Pi_{\leq k}$.

The *hook-length* of a cell $(i, j) \in \lambda$ is defined as $h_\lambda(i, j) = \lambda_i - i + \lambda'_j - j + 1$. Thus, the hook length of a cell $z \in \lambda$ equals the number of cells in λ that lie in the same row as z and weakly east of z, or in the same column as z and weakly south of z.

A *north-east-path* in λ is a sequence $P = (z_0, z_1, \dots, z_s)$ of cells $z_k \in \lambda$ such that $z_k \in \{\mathbf{n}z_{k-1}, \mathbf{e}z_{k-1}\}$ for all $k \in [s]$. Let $\ell(P) = s$ denote the *length*, $\alpha(P) = z_0$ the *tail* and $\omega(P) = z_s$ the *head* of the path P. A *rim-hook* of λ is a north-east-path h in λ such that $\mathbf{s}\alpha(h) \notin \lambda$, $\mathbf{e}\omega(h) \notin \lambda$ and $\mathbf{es}z \notin \lambda$ for all $z \in h$. For each cell $(i, j) \in \lambda$ there is a (unique) rim-hook h with

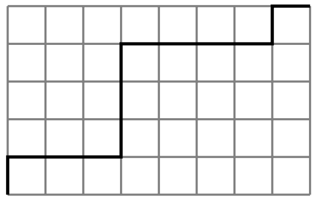


FIGURE 1.1. A lattice path $x = \mathbf{neeennneeeene} \in \mathcal{L}_{8,5}$ with partition $\lambda(x) = (7,3,3,3)$. The rises of x are 2 and 3. Its valleys are $(3,2)$ and $(7,5)$.

$\alpha(h) = (\lambda'_j, j)$ and $\omega(h) = (i, \lambda_i)$. This correspondence is a bijection between λ and the set of rim-hooks of λ.

1.2. Lattice paths

Let $r \in \mathbb{N}$, $a, b \in \mathbb{Z}^2$ and $S \subseteq \mathbb{Z}^2$ be a finite set. A *lattice path* from a to b with *length* r and *steps* in S is a sequence $(z_0, \dots, z_r)$ of points $z_i \in \mathbb{Z}^2$ such that $z_0 = a$, $z_r = b$ and $z_i - z_{i-1} \in S$ for all $i \in [r]$. Instead of the lattice points z_i that the path visits, we can also specify the sequence of steps $z_i - z_{i-1}$. From this point of view a lattice path from a to b with steps in S is a word $s_1 \cdots s_r$ such that $s_i \in S$ for $i \in [r]$ and $\sum_i s_i = b - a$.

Denote by $\mathcal{L}_{m,n}$ the set of lattice paths starting at the origin $(0,0)$ and ending at (m,n) with step set $S = \{\mathbf{e}, \mathbf{n}\}$, where $\mathbf{e} = (1,0)$ is called an *east step* and $\mathbf{n} = (0,1)$ is called a *north step*

There is a third useful way to encode lattice paths if $x \in \mathcal{L}_{m,n}$. For $i \in [n]$ let x_i denote the x-coordinate of the i-th north step, that is, the number of east steps of x preceding the i-th north step of x. Clearly the vector $(x_1, \dots, x_n)$ fully determines x. Setting $\lambda(x)_i = x_{n-i+1}$ for $i \in [n]$ defines a partition $\lambda(x)$. We call $\lambda(x)$ the *partition* of x. This map restricts to a bijection $\lambda : \mathcal{L}_{m,n} \to \Pi^n \cap \Pi_{\leq m}$.

A pattern of the form $\mathbf{nn}$ is called a rise and a pattern of the form $\mathbf{en}$ is called a valley. More precisely let x be a lattice path of length r consisting of east and north steps. A *rise* of x is an index $i \in [r]$ such that the i-th north step of x is immediately followed by another north step. A *valley* of x is a pair $(i,j) \in [r]^2$ such that the i-th east step of x is immediately followed by its j-th north step. See Figure 1.1. Note that in Section 6.1 it is more convenient to work with slightly different conventions when it comes to the valleys of ballot paths.

In order to define certain combinatorial maps on lattice paths we need the following building blocks. Given $j \in \mathbb{Z}$ define a map $\overrightarrow{w}_j^+ : \mathbb{Z}^n \to \{\mathbf{e}, \mathbf{n}\}^*$ where the word $\overrightarrow{w}_j^+(a)$ in the alphabet $\{\mathbf{e}, \mathbf{n}\}$ is obtained as follows: Initialise with the empty word. Read a from left to right. Whenever an entry $a_i = j$ is encountered, append $\mathbf{n}$. Whenever an entry $a_i = j+1$ is encountered, append $\mathbf{e}$. Similarly define $\overleftarrow{w}_j^+(a)$ except now a is read from right to left instead. Moreover define $\overrightarrow{w}_j^-(a)$ as follows: Read a from left to right and append $\mathbf{n}$ for each encountered entry equal to $-j$ and $\mathbf{e}$ for each encountered entry equal to $-j-1$. Define $\overleftarrow{w}_j^-(a)$ analogously.

1.3. Roots, hyperplanes and reflections

In this section we fix notation and recall some facts on root systems and Weyl groups. The reader is referred to [**44**] for further details.

Let Φ be an *irreducible crystallographic root system* with ambient space V, *positive system* Φ^+ and *simple system* $\Delta = \{\alpha_1, \dots, \alpha_r\}$. The positive integer r is called the *rank* of Φ. Any root $\alpha \in \Phi$ can be written as a unique integer linear combination $\alpha = \sum_{i=1}^r c_i \alpha_i$, where all coefficients c_i are non-negative if $\alpha \in \Phi^+$, or all coefficients are non-positive if $\alpha \in -\Phi^+$. Define the *height* of the root α by $\mathrm{ht}(\alpha) = \sum_{i=1}^r c_i$. Thus $\mathrm{ht}(\alpha) > 0$ if and only if $\alpha \in \Phi^+$, and $\mathrm{ht}(\alpha) = 1$ if and only if $\alpha \in \Delta$. There exists a unique *highest root* $\tilde{\alpha}$ such that $\mathrm{ht}(\tilde{\alpha}) \geq \mathrm{ht}(\alpha)$ for all $\alpha \in \Phi$. The *Coxeter number* of Φ can be defined as $h = \mathrm{ht}(\tilde{\alpha}) + 1$.

There are four infinite families of irreducible crystallographic root systems, which are commonly referred to as types A_{n-1}, B_n, C_n and D_n. For each of these types we provide an example below. The chosen conventions are used throughout the thesis. Let $n \in \mathbb{N}$ with $n \geq 2$. The roots, positive roots and simple roots of type A_{n-1} are given by

$$\begin{aligned}\Phi &= \{e_i - e_j : i, j \in [n], i \neq j\},\\ \Phi^+ &= \{e_i - e_j : 1 \leq i < j \leq n\}, \text{and}\\ \Delta &= \{e_i - e_{i+1} : i \in [n-1]\}.\end{aligned}$$

We denote the simple roots by $\alpha_i^A = e_i - e_{i+1}$ for $i \in [n-1]$, and the highest root by $\tilde{\alpha}^A = e_1 - e_n$.

The roots, positive roots and simple roots of type B_n are given by

$$\begin{aligned}\Phi &= \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i : i \in [n]\},\\ \Phi^+ &= \{\pm e_i + e_j : 1 \leq i < j \leq n\} \cup \{e_i : i \in [n]\}, \text{and}\\ \Delta &= \{-e_i + e_{i+1} : i \in [n-1]\} \cup \{e_1\}.\end{aligned}$$

We denote the simple roots by $\alpha_0^B = e_1$ and $\alpha_i^B = e_{i+1} - e_i$ for $i \in [n-1]$, and the highest root by $\tilde{\alpha}^B = e_{n-1} + e_n$.

The roots, positive roots and simple roots of type C_n are given by

$$\begin{aligned}\Phi &= \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : i \in [n]\},\\ \Phi^+ &= \{\pm e_i + e_j : 1 \leq i < j \leq n\} \cup \{2e_i : i \in [n]\}, \text{and}\\ \Delta &= \{-e_i + e_{i+1} : i \in [n-1]\} \cup \{2e_1\}.\end{aligned}$$

We denote the simple roots by $\alpha_0^C = 2e_1$ and $\alpha_i^C = e_{i+1} - e_i$ for $i \in [n-1]$, and the highest root by $\tilde{\alpha}^C = 2e_n$.

Let $n \geq 3$. The roots, positive roots and simple roots of type D_n are given by

$$\begin{aligned}\Phi &= \{\pm e_i \pm e_j : 1 \leq i < j \leq n\},\\ \Phi^+ &= \{\pm e_i + e_j : 1 \leq i < j \leq n\}, \text{and}\\ \Delta &= \{-e_i + e_{i+1} : i \in [n-1]\} \cup \{e_1 + e_2\}.\end{aligned}$$

We denote the simple roots by $\alpha_0^D = e_1 + e_2$ and $\alpha_i^D = e_{i+1} - e_i$ for $i \in [n-1]$, and the highest root by $\tilde{\alpha}^D = e_{n-1} + e_n$.

Let δ be a formal variable. The set of *affine roots* is defined as

$$\widetilde{\Phi} = \{\alpha + k\delta : \alpha \in \Phi, k \in \mathbb{Z}\} \subseteq V \oplus \mathbb{R}\delta.$$

The *height* of an affine root is defined by $\mathrm{ht}(\alpha + k\delta) = \mathrm{ht}(\alpha) + kh$. This yields a linear map $\mathrm{ht} : \widetilde{\Phi} \to \mathbb{Z}$. The sets of *positive* and *simple* affine roots are defined as

$$\widetilde{\Phi}^+ = \Phi^+ \cup \{\alpha + k\delta \in \widetilde{\Phi} : \alpha \in \Phi, k > 0\} \qquad \text{and} \qquad \widetilde{\Delta} = \Delta \cup \{-\tilde{\alpha} + \delta\}.$$

Thus $\alpha + k\delta \in \widetilde{\Phi}^+$ if and only if $\mathrm{ht}(\alpha + k\delta) > 0$, and $\alpha + k\delta \in \widetilde{\Delta}$ if and only if $\mathrm{ht}(\alpha + k\delta) = 1$.

A *hyperplane arrangement* in V is a set of affine hyperplanes $H \subseteq V$. The *regions* of a hyperplane arrangement $\mathscr{A}$ are defined as the connected components of $V - \bigcup_{H \in \mathscr{A}} H$. The *Coxeter arrangement* $\mathrm{Cox}(\Phi)$ consists of all hyperplanes of the form

$$H_\alpha = \{x \in V : \langle x, \alpha \rangle = 0\}$$

for $\alpha \in \Phi$. Its regions are called *chambers*. We define the *dominant chamber* as

$$C_\circ = \big\{x \in V : \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta\big\}.$$

The *Weyl group* W of Φ is the group of linear automorphisms of V generated by all reflections in a hyperplane in $\mathrm{Cox}(\Phi)$. That is, W is generated by the reflections $s_\alpha : V \to V$ defined by

$$s_\alpha(x) = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha,$$

where α ranges over all roots in Φ. The Weyl group acts simply transitively on the chambers. Thus identifying the identity $e \in W$ with the dominant chamber, each chamber corresponds to a unique Weyl group element. Note that by definition of a root system, the Weyl group acts on the root system Φ.

The *affine arrangement* $\mathrm{Aff}(\Phi)$ consists of all hyperplanes of the form

$$H_{\alpha,k} = \big\{x \in V : \langle x, \alpha\rangle = k\big\},$$

where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Its regions are called *alcoves*. We define the *fundamental alcove* as

$$A_\circ = \big\{x \in V : \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta \text{ and } \langle x, \tilde{\alpha}\rangle < 1\big\}.$$

The *affine Weyl group* $\widetilde{W}$ of Φ is the group of affine transformations of V that is generated by all reflections in a hyperplane in $\mathrm{Aff}(\Phi)$. That is, $\widetilde{W}$ is generated by the affine reflections $s_{\alpha,k} : V \to V$ defined by

$$s_{\alpha,k}(x) = x - \frac{2(\langle x, \alpha\rangle - k)}{\langle \alpha, \alpha\rangle}\alpha,$$

where α ranges over Φ and $k \in \mathbb{Z}$. The affine Weyl group acts simply transitively on the set of alcoves. By identifying the identity $e \in \widetilde{W}$ with the fundamental alcove, every alcove corresponds to a unique element of $\widetilde{W}$. An element $\omega \in \widetilde{W}$ is called *dominant* if and only if the alcove $\omega(A_\circ)$ is contained in the dominant chamber $C_\circ$. We denote the set of dominant elements of the affine Weyl group by $\widetilde{W}_+$.

The Weyl group W is generated by the set $S = \{s_{\alpha_1}, \dots, s_{\alpha_r}\}$, and (W, S) is a Coxeter system. The affine Weyl group $\widetilde{W}$ is generated by $\widetilde{S} = S \cup \{s_{\tilde{\alpha},1}\}$, and $(\widetilde{W}, \widetilde{S})$ is also a Coxeter system.

The Weyl group is a parabolic subgroup of the affine Weyl group. Each element $\omega \in \widetilde{W}$ can be assigned a *length* $\ell(\omega)$ indicating the minimal number of generators $t_1, \dots, t_\ell \in \widetilde{S}$ such that ω can be expressed as a product of these generators, that is, $\omega = t_1 \cdots t_\ell$. Each coset $\omega W \in \widetilde{W}/W$ contains a unique representative of minimal length. These representatives are called *Graßmannian*. An element $\omega \in \widetilde{W}$ is Graßmannian if and only if ω^{-1} is dominant.

Given a root $\alpha \in \Phi$ its *coroot* is defined as $\alpha^\vee = 2\alpha/\langle \alpha, \alpha\rangle$. The *coroot lattice* is the integer span of all coroots

$$\check{Q} = \sum_{\alpha \in \Phi} \mathbb{Z}\alpha^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i^\vee \subseteq V.$$

The crystallographic property guarantees that $\check{Q}$ is a discrete free abelian subgroup of rank r in V which is fixed by the action of the Weyl group on V. Thus W acts on $\check{Q}$. For each $q \in \check{Q}$ the translation $t_q : V \to V$ defined by $t_q(x) = x + q$ for all $x \in V$ is an element of the affine Weyl group. Identifying $\check{Q}$ with its translation group we obtain $\widetilde{W} = W \ltimes \check{Q}$. Note that if $\omega \in \widetilde{W}$ is dominant and $\omega = t_q s$, where $q \in \check{Q}$ and $s \in W$, then in particular q lies in the closure of the dominant chamber. Thus $\langle \alpha, q\rangle \geq 0$ for each positive root $\alpha \in \Phi^+$.

To an affine root $\alpha + k\delta$ we associate the half-space

$$\mathcal{H}_{\alpha+k\delta} = \big\{x \in V : \langle x, \alpha\rangle > -k\big\}.$$

The action of the affine Weyl group on half-spaces translates into the following action on $\widetilde{\Phi}$. If $\omega \in \widetilde{W}$ has the unique decomposition $\omega = t_x s$ with $x \in \check{Q}$ and $s \in W$ then

$$\omega \cdot (\alpha + k\delta) = s(\alpha) + (k - \langle x, s(\alpha)\rangle)\delta.$$

Let A be a non-empty intersection of half-spaces of the form $\mathcal{H}_{\alpha+k\delta}$, where $\alpha + k\delta \in \widetilde{\Phi}$. For example, A could be an alcove. A hyperplane $H \in \mathrm{Aff}(\Phi)$ is a *wall* of A if H supports a facet of A. The hyperplane H is said to *separate* A and the fundamental alcove if A and $A_\circ$ lie in different half-spaces bounded by H. In this case we say H is a *separating hyperplane* of A. A wall H of A that separates A and the fundamental alcove $A_\circ$ is called a *floor* of A.

Note that positive affine roots correspond to those half spaces that contain the fundamental alcove. Simple affine roots correspond to half spaces that contain the fundamental alcove and share one of its walls. Floors and separating hyperplanes of an alcove can be expressed in the language of affine roots and elements of the affine Weyl group. Let $\omega \in \widetilde{W}$ and $\alpha + k\delta \in \widetilde{\Phi}^+$. Then the hyperplane $H_{\alpha,-k}$ separates $\omega(A_\circ)$ from $A_\circ$ if and only if $\omega^{-1} \cdot (\alpha + k\delta) \in -\widetilde{\Phi}^+$. The hyperplane $H_{\alpha,-k}$ is a floor of $\omega(A_\circ)$ if and only if $\omega^{-1} \cdot (\alpha + k\delta) \in -\widetilde{\Delta}$.

A positive affine root $\alpha + k\delta \in \widetilde{\Phi}^+$ is an *affine inversion* of $\omega \in \widetilde{W}$ if $\omega \cdot (\alpha + k\delta) \in -\widetilde{\Phi}^+$. It follows from the previous paragraph that the number of affine inversions of an element $\omega \in \widetilde{W}$ equals the number of separating hyperplanes. One can show that this number also equals the length of ω.

1.4. Affine permutations

The Weyl group of type A_{n-1} is easily recognised to be the symmetric group $\mathfrak{S}_n$. We now describe a combinatorial model for the affine Weyl group of type A_{n-1}. A detailed exposition is found in [**17**, Sec. 8.3].

The *affine symmetric group* $\widetilde{\mathfrak{S}}_n$ is the group of bijections $\omega : \mathbb{Z} \to \mathbb{Z}$ such that $\omega(i+n) = \omega(i)+n$ for all $i \in \mathbb{Z}$ and $\omega(1)+\cdots+\omega(n) = n(n+1)/2$. Such a bijection ω is called an *affine permutation*. Each affine permutation is uniquely determined by its *window*

$$[\omega(1), \omega(2), \ldots, \omega(n)].$$

The group $\widetilde{\mathfrak{S}}_n$ has a set of generators called *simple transpositions of type* A_{n-1} given by

$$\begin{aligned} s_i^A &= [1, \ldots, i+1, i, \ldots, n] && \text{for } i \in [n-1] \text{ and} \\ s_n^A &= [0, 2, \ldots, n-1, n+1]. \end{aligned}$$

The affine symmetric group and the set of simple transpositions form a Coxeter system isomorphic to the affine Weyl group of type A_{n-1}. An explicit isomorphism between $\widetilde{\mathfrak{S}}_n$ and the affine Weyl group of type A_{n-1} can be given using the generators by mapping s_i^A to $s_{\alpha_i^A}$ for $i \in [n-1]$ and s_n^A to $s_{\tilde{\alpha}^A,1}$. The *symmetric group* $\mathfrak{S}_n$ can be seen as the subgroup of $\widetilde{\mathfrak{S}}_n$ consisting of all affine permutations whose window is a permutation of $[n]$. This is just the parabolic subgroup of $\widetilde{\mathfrak{S}}_n$ generated by s_i^A for $i \in [n-1]$ and agrees with the image of the Weyl group W under the specified isomorphism. The *length* $\ell(\omega)$ of an affine permutation $\omega \in \widetilde{\mathfrak{S}}_n$ is the minimal number ℓ of simple transpositions in an expression of the form $\omega = s_{i_1}^A s_{i_2}^A \cdots s_{i_\ell}^A$, where $i_j \in [n]$ for all $j \in [\ell]$. The *Graßmannian affine permutations*, that is, the minimal length representatives of the cosets in $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$, are described in [**17**, Prop. 8.3.4].

Proposition 1.4.1. *An affine permutation $\omega \in \widetilde{\mathfrak{S}}_n$ is the minimal length representative of its coset $\omega\mathfrak{S}_n \in \widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$ if and only if its window is increasing, that is,*

$$\omega(1) < \omega(2) < \cdots < \omega(n).$$

We now give a combinatorial description of the decomposition of the affine symmetric group as the semi-direct product of translations and permutations. The coroot lattice of type A_{n-1} is given by

$$\check{Q} = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\}.$$

The affine symmetric group acts on the coroot lattice via the following rules

$$\begin{aligned} s_i^A \cdot x &= (x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) && \text{for } i \in [n-1] \text{ and} \\ s_n^A \cdot x &= (x_n + 1, x_2, \ldots, x_{n-1}, x_1 - 1). \end{aligned}$$

Given $\omega \in \widetilde{\mathfrak{S}}_n$ write $\omega(i) = a_i n + b_i$ for each $i \in \mathbb{Z}$, where $a_i \in \mathbb{Z}$ and $b_i \in [n]$. For $i \in [n]$ set $\sigma(\omega, i) = b_i$, $\mu(\omega, b_i) = a_i$ and $\nu(\omega, i) = -a_i$.

Lemma 1.4.2. [**73**, Lem. 2.2]

(i) The assignment $i \mapsto \sigma(\omega, i)$, where i ranges over $[n]$, defines a permutation $\sigma(\omega) \in \mathfrak{S}_n$.

(ii) The vectors $\mu(\omega) = (\mu(\omega, 1), \ldots, \mu(\omega, n))$ and $\nu(\omega) = (\nu(\omega, 1), \ldots, \nu(\omega, n))$ lie in the coroot lattice $\check{Q}$ and for all $i \in [n]$ we have

$$\mu(s_i^A \omega) = s_i^A \cdot \mu(\omega), \qquad \nu(\omega s_i^A) = s_i^A \cdot \nu(\omega).$$

(iii) We have $\omega \cdot 0 = \mu(\omega)$ and $\omega^{-1} \cdot 0 = \nu(\omega)$.

(iv) We have $\mu(\omega^{-1}) = \nu(\omega)$ and $\sigma(\omega^{-1}) = \sigma(\omega)^{-1}$.

(v) We have $\mu(\omega) = -\sigma(\omega) \cdot \nu(\omega)$.

Proof. Claims (i) and (v) are straight forward. Claims (iii) and (iv) follow immediately from (ii), which can be shown by induction on the length of ω.

Clearly $\mu(e) = \nu(e) = (0, \ldots, 0) \in \check{Q}$. Set $\sigma = \sigma(\omega)$ and suppose that $\mu(\omega) \in \check{Q}$.

Fix $i \in [n-1]$ and choose $j, k \in [n]$ such that $\sigma(j) = i$ and $\sigma(k) = i + 1$. Then $s_i^A \omega(\ell) = \omega(\ell)$ for all $\ell \in [n] - \{j, k\}$ and hence $\mu(s_i^A \omega, \ell) = \mu(\omega, \ell)$ for all $\ell \in [n] - \{i, i+1\}$. Furthermore, $s_i^A \omega(j) = s_i^A(a_j n + i) = a_j n + i + 1$ and $s_i^A \omega(k) = s_i^A(a_k n + i + 1) = a_k n + i$. It follows that $\mu(s_i^A \omega, i) = a_k = \mu(\omega, i+1)$ and $\mu(s_i^A \omega, i+1) = a_j = \mu(\omega, i)$.

Next choose $j, k \in [n]$ such that $\sigma(j) = 1$ and $\sigma(k) = n$. Then $s_n^A \omega(\ell) = \omega(\ell)$ for all $\ell \in [n] - \{j, k\}$. Thus $\mu(s_n^A \omega, \ell) = \mu(\omega, \ell)$ for all $\ell \in [n] - \{1, n\}$. Moreover, $s_n^A \omega(j) = s_n^A(a_j n + 1) = a_j n + 0 = (a_j - 1)n + n$ and $s_n^A \omega(k) = s_n^A(a_k n + n) = a_k n + (n+1) = (a_k + 1)n + 1$. Thus $\mu(s_n^A \omega, 1) = a_k + 1 = \mu(\omega, n) + 1$ and $\mu(s_n^A \omega, n) = a_j - 1 = \mu(\omega, 1) - 1$.

We obtain $\mu(s_i^A \omega) = s_i^A \cdot \mu(\omega)$, and in particular $\mu(s_i^A \omega) \in \check{Q}$, for all $i \in [n]$.

Now suppose $\nu(\omega) \in \check{Q}$ and fix $i \in [n-1]$. Since $\omega s_i^A(j) = \omega(j)$ for all $j \in [n] - \{i, i+1\}$ we have $\nu(\omega s_i^A, j) = \nu(\omega, j)$ for all $j \in [n] - \{i, i+1\}$. Furthermore $\omega s_i^A(i) = \omega(i+1)$ and $\omega s_i^A(i+1) = \omega(i)$, thus $\nu(\omega s_i^A, i) = \nu(\omega, i+1)$ and $\nu(\omega s_i^A, i+1) = \nu(\omega, i)$.

Finally since $\omega s_n^A(j) = \omega(j)$ for all $j \in [n] - \{1, n\}$ we have $\nu(\omega s_n^A, j) = \nu(\omega, j)$ for all $j \in [n] - \{1, n\}$. Moreover $\omega s_n^A(1) = \omega(0) = \omega(n - n) = \omega(n) - n = (a_n - 1)n + b_n$, hence $\nu(\omega s_n^A, 1) = -a_n + 1 = \nu(\omega, n) + 1$. On the other hand $\omega s_n^A(n) = \omega(n+1) = n + \omega(1) = (a_1 + 1)n + b_1$. Thus $\nu(\omega s_n^A, n) = -a_1 - 1 = \nu(\omega, 1) - 1$.

We obtain that $\nu(\omega s_i^A) = s_i^A \cdot \nu(\omega)$, and in particular $\nu(\omega s_i^A) \in \check{Q}$, for all $i \in [n]$. This concludes the proof of claim (ii) and thus the proof of the lemma. ■

For $q \in \check{Q}$ define an affine permutation $t_q \in \widetilde{\mathfrak{S}}_n$ by $t_q(i) = q_i n + i$ for $i \in [n]$. We call an affine permutation $\omega \in \widetilde{\mathfrak{S}}_n$ a *translation* if there exists $q \in \check{Q}$ such that $\omega \cdot x = q + x$ for all $x \in \check{Q}$.

Theorem 1.4.3. [**73**, Thm. 2.3]

(i) Let $\omega \in \widetilde{\mathfrak{S}}_n$ be an affine permutation and set $s = \sigma(\omega)$, $x = \mu(\omega)$ and $y = \nu(\omega)$. Then $\omega = t_x s = s t_{-y}$.

(ii) *Let $x, y \in \check{Q}$. Then $t_x t_y = t_{x+y}$ and $(t_x)^{-1} = t_{-x}$. Hence we may view the coroot lattice $\check{Q}$ as a subgroup of $\widetilde{\mathfrak{S}}_n$.*

(iii) *An affine permutation $\omega \in \widetilde{\mathfrak{S}}_n$ is a translation if and only if $\omega = t_q$ for some $q \in \check{Q}$.*

(iv) *The affine symmetric group is the semi-direct product of the symmetric group and the coroot lattice, that is, $\widetilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \check{Q}$.*

Proof. Claims (i) and (ii) are straightforward calculations. Claim (i) follows from

$$t_x s(i) = \mu(\omega, s(i))n + s(i) = \omega(i) = -\nu(\omega, i)n + s(i) = s(-\nu(\omega, i)n + i) = st_{-y}(i).$$

On the other hand

$$t_x t_y(i) = t_x(y_i n + i) = y_i n + t_x(i) = (x_i + y_i)n + i = t_{x+y}(i)$$

implies (ii). To see (iii) note that $t_q \cdot x = t_q t_x \cdot 0 = t_{q+x} \cdot 0 = q + x$. Conversely, if ω is a translation by $q \in \check{Q}$ then $q + x = \omega \cdot x = q + \sigma(\omega) \cdot x$ for all $x \in \check{Q}$, which implies $\sigma(\omega) = e$. Finally, $\mathfrak{S}_n \check{Q} = \widetilde{\mathfrak{S}}_n$ and $\mathfrak{S}_n$ normalises $\check{Q}$ by (i). Since $\mathfrak{S}_n \cap \check{Q} = \{e\}$, we obtain (iv). ■

Thus the decomposition of an affine permutation into a product of a translation and a permutation can be obtained from its window in a simple and direct fashion. We remark that this combinatorial decomposition has appeared in the literature before, for example in [**16**]. The connection to the algebraic decomposition into a semi-direct product, which exists for any affine Weyl group, is mentioned in [**5**, Sec. 4.2].

Note that the action of the affine symmetric group on left cosets $\widetilde{\mathfrak{S}}_n / \mathfrak{S}_n$ is isomorphic to the action on the coroot lattice since

$$st_q \mathfrak{S}_n = t_{s(q)} \mathfrak{S}_n$$

for all $q \in \check{Q}$ and $s \in \mathfrak{S}_n$. The following lemma describes this action viewed as an action on the set of Graßmannian affine permutations.

Lemma 1.4.4. *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be a Graßmannian affine permutation, $q \in \check{Q}$ and $s \in \mathfrak{S}_n$ such that $\omega = t_q s$.*

(i) *Then for all $i \in [n-1]$ the Graßmannian affine permutation $\omega' \in s_i^A \omega \mathfrak{S}_n$ is given by $\omega' = \omega$ if $s_i^A(q) = q$ and $\omega' = t_{s_i^A(q)} s_i^A s$ otherwise.*

(ii) *The Graßmannian affine permutation $\omega' \in s_n^A \omega \mathfrak{S}_n$ is given by $\omega' = \omega$ if $s_n^A(q) = q$ and $\omega' = t_{s_n^A(q)} us$, where $u = [n, 2, \ldots, n-1, 1]$, otherwise.*

Proof. The claim follows from Proposition 1.4.1 and a simple computation. ■

The affine symmetric group possesses an *involutive automorphism* owing to the symmetry of the Dynkin diagram of type A_{n-1}. Set $(s_i^A)^* = s_{n-i}^A$ for $i \in [n-1]$ and $(s_n^A)^* = s_n^A$. This correspondence extends to an automorphism $\omega \mapsto \omega^*$ on $\widetilde{\mathfrak{S}}_n$, where ω^* is obtained by replacing all instances of s_i^A in any expression of ω in terms of the simple transpositions by $(s_i^A)^*$. The involutive automorphism has a simple explicit description in window notation and fulfils many desirable properties well-known to experts.

Lemma 1.4.5. [**73**, Lem. 2.4]

(i) *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be an affine permutation and $i \in \mathbb{Z}$. Then $w^*(i) = 1 - \omega(1-i)$. In particular the window of ω^* is given by $[n+1-\omega(n), \ldots, n+1-\omega(1)]$.*

(ii) *The involutive automorphism preserves $\mathfrak{S}_n$, translations, Graßmannian affine permutations and dominant affine permutations.*

PROOF. We prove claim (i) by induction on the length of ω. Clearly (i) holds for the identity, as $e^* = e$. Thus assume it is true for ω. For $i \in [n-1]$ the right multiplication of ω by s_i^A corresponds to exchanging the two numbers $\omega(i), \omega(i+1)$ in the window of ω. On the other hand multiplying ω^* by $(s_i^A)^*$ from the right exchanges the numbers $n+1-\omega(i+1), n+1-\omega(i)$. Claim (i) therefore also holds for ωs_i^A.

Claim (ii) follows from (i) and the fact that $(\omega^{-1})^* = (\omega^*)^{-1}$. ■

1.5. Affine signed permutations

The affine Weyl groups of types B_n, C_n and D_n can be realised as groups of certain bijections on integers as well. We still refer to [**17**] for a standard text book on the subject.

Set $N = 2n+1$. A bijection $\omega : \mathbb{Z} \to \mathbb{Z}$ is an *affine signed permutation* if $\omega(i+N) = \omega(i) + N$ and $\omega(-i) = -\omega(i)$ for all $i \in \mathbb{Z}$. The set of all such bijections forms a group under composition which we denote by $\widetilde{\mathfrak{S}}_n^C$. As in type A_{n-1} each affine permutation is fully determined by its *window*

$$[\omega(1), \omega(2), \dots, \omega(n)].$$

The group $\widetilde{\mathfrak{S}}_n^C$ is generated by the following $n+1$ *simple transpositions of type* C_n, defined as

$$\begin{aligned} s_0^C &= [-1, 2, \dots, n], \\ s_i^C &= [1, \dots, i-1, i+1, i, i+2, \dots, n] && \text{for } i \in [n-1] \text{ and} \\ s_n^C &= [1, \dots, n-1, n+1]. \end{aligned}$$

The group $\widetilde{\mathfrak{S}}_n^C$ contains a subgroup $\widetilde{\mathfrak{S}}_n^B$ that consists of all affine permutations ω such that the finite set $\{i \in \mathbb{Z} : i \leq n, \omega(i) > n\}$ has even cardinality. This subgroup is generated by the *simple transpositions of type* B_n, given by

$$\begin{aligned} s_0^B &= [-1, 2, \dots, n], \\ s_i^B &= [1, \dots, i-1, i+1, i, i+2, \dots, n] && \text{for } i \in [n-1] \text{ and} \\ s_n^B &= [1, \dots, n-2, n+1, n+2]. \end{aligned}$$

The group $\widetilde{\mathfrak{S}}_n^B$ contains a subgroup $\widetilde{\mathfrak{S}}_n^D$ that consists of all affine permutations ω such that both finite sets $\{i \in \mathbb{Z} : i \leq n, \omega(i) > n\}$ and $\{i \in \mathbb{Z} : i \geq 0, \omega(i) < 0\}$ have even cardinality. This subgroup is generated by the *simple transpositions of type* D_n, that is,

$$\begin{aligned} s_0^D &= [-1, -2, 3, \dots, n], \\ s_i^D &= [1, \dots, i-1, i+1, i, i+2, \dots, n] && \text{for } i \in [n-1] \text{ and} \\ s_n^D &= [1, \dots, n-2, n+1, n+2]. \end{aligned}$$

Let Φ be a root system of type B_n, C_n or D_n. The group $\widetilde{\mathfrak{S}}_n^\Phi$ is isomorphic to the affine Weyl group $\widetilde{W}$ of Φ. An explicit isomorphism in terms of the generators is obtained by mapping s_i^Φ to $s_{\alpha_i^\Phi}$ for $0 \leq i \leq n-1$, and s_n^Φ to $s_{\tilde{\alpha}^\Phi,1}$. Let $\mathfrak{S}_n^\Phi$ denote the subgroup of $\widetilde{\mathfrak{S}}_n^\Phi$ corresponding to the Weyl group W under this isomorphism. Then an affine permutation $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ lies in $\mathfrak{S}_n^\Phi$ if and only if its window is a subset of $[\pm n]$. More precisely, $\mathfrak{S}_n^B = \mathfrak{S}_n^C$ is the group of *signed permutations* while $\mathfrak{S}_n^D \leq \mathfrak{S}_n^B$ consists of the signed permutations with an even number of sign changes. Furthermore, this isomorphism affords an action of the affine permutations on the coroot lattice $\check{Q}$ of Φ. The coroot lattices of types B_n, C_n and D_n are given by

$$\check{Q}^B = \check{Q}^D = \Big\{ x \in \mathbb{Z}^n : \sum_{i=1}^n x_i \in 2\mathbb{Z} \Big\} \qquad \text{and} \qquad \check{Q}^C = \mathbb{Z}^n.$$

The action of $\widetilde{\mathfrak{S}}_n^\Phi$ on $\check{Q}^\Phi$ is made explicit by the following rules

$$\begin{aligned}
&s_0^C \cdot (x_1 \dots, x_n) = (-x_1, x_2 \dots, x_n),\\
&s_i^C \cdot (x_1 \dots, x_n) = (x_1, \dots, x_{i+1}, x_i, \dots, x_n), \quad \text{for } i \in [n-1],\\
&s_n^C \cdot (x_1 \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n + 1),\\
&s_n^B \cdot (x_1 \dots, x_n) = (x_1, \dots, x_{n-2}, -x_n + 1, -x_{n-1} + 1) \quad \text{and}\\
&s_0^D \cdot (x_1 \dots, x_n) = (-x_1, -x_2, x_3, \dots, x_n).
\end{aligned}$$

A combinatorial description of the Graßmannian affine permutations in $\widetilde{\mathfrak{S}}_n^\Phi$ based on window notation is given by Björner and Brenti.

Proposition 1.5.1. [**17**, Prop. 8.4.4, 8.5.4, and 8.6.4] *Let Φ be a root system of type B_n, C_n or D_n, and $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ an affine permutation. Then ω is Graßmannian if and only if*

$$\begin{cases} 0 < \omega(1) < \omega(2) < \cdots < \omega(n) & \text{if } \Phi \text{ is of type } B_n \text{ or } C_n,\\ 0 < |\omega(1)| < \omega(2) < \cdots < \omega(n) & \text{if } \Phi \text{ is of type } D_n. \end{cases}$$

As in the previous section we next develop a combinatorial description of the decomposition of an affine permutation into a product of a translation by an element of the coroot lattice and an element of the Weyl group, that is, a signed permutation. To this end, for $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ and each $i \in \mathbb{Z}$ write $\omega(i) = a_iN + b_i$ such that $a_i \in \mathbb{Z}$ and $-n \le b_i \le n$. Given $i \in [\pm n]$ define $\sigma(\omega, i) = b_i$, $\mu(\omega, b_i) = -a_i$ and $\nu(\omega, i) = a_i$.

Lemma 1.5.2. [**74**, Lem. 3.2] *Let $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ be an affine permutation.*

(i) The map $i \mapsto \sigma(\omega, i)$, where $i \in [n]$, defines a signed permutation $\sigma(\omega) \in \mathfrak{S}_n^\Phi$.

(ii) The vectors $\mu(\omega) = (\mu(\omega, 1), \dots, \mu(\omega, n))$ and $\nu(\omega) = (\nu(\omega, 1), \dots, \nu(\omega, n))$ lie in the coroot lattice $\check{Q}$ of Φ, and for all i with $0 \le i \le n$ we have

$$\mu(s_i^\Phi \omega) = s_i^\Phi \cdot \mu(\omega), \qquad \nu(\omega s_i^\Phi) = s_i^\Phi \cdot \nu(\omega).$$

(iii) We have $\mu(\omega^{-1}) = \nu(\omega)$ and $\sigma(\omega^{-1}) = \sigma(\omega)^{-1}$.

(iv) We have $\omega \cdot (0, \dots, 0) = \mu(\omega)$ and $\omega^{-1} \cdot (0, \dots, 0) = \nu(\omega)$.

(v) We have $\mu(\omega) = -\sigma \cdot \nu(\omega)$.

Proof. Claims (i) and (v) are immediate from the definitions while (iii) and (iv) follow directly from (ii). Thus it only remains to show (ii) which is done for each type using induction on the length of ω.

First assume that Φ is of type C_n. Clearly $\mu(e) = \nu(e) = (0, \dots, 0) \in \check{Q}$. Suppose that $\mu(\omega) \in \check{Q}$.

Choose $j \in [\pm n]$ such that $\sigma(j) = 1$. Then $s_0^C\omega(k) = \omega(k)$ for all $k \in [n] - \{|j|\}$, hence $\mu(s_0^C\omega, k) = \mu(\omega, k)$ for all $k \in [n] - \{1\}$. Furthermore, $s_0^C\omega(-j) = s_0(-a_jN - 1) = -a_jN + 1$ thus $\mu(s_0^C\omega, 1) = -\mu(\omega, 1)$.

Fix i with $1 \le i \le n-1$ and choose $j, k \in [\pm n]$ such that $\sigma(j) = i$ and $\sigma(k) = i + 1$. Then $s_i^C\omega(\ell) = \omega(\ell)$ for all $\ell \in [n]-\{|j|, |k|\}$. It follows that $\mu(s_i^C\omega, \ell) = \mu(\omega, \ell)$ for all $\ell \in [n]-\{i, i+1\}$. Furthermore, $s_i^C\omega(j) = s_i^C(a_jN + i) = a_jN + i + 1$ and $s_i^C\omega(k) = s_i^C(a_kN + i + 1) = a_kN + i$. Thus $\mu(s_i^C\omega, i) = \mu(\omega, i + 1)$ and $\mu(s_i^C\omega, i + 1) = \mu(\omega, i)$.

Choose $j \in [\pm n]$ such that $\sigma(j) = n$. Then $s_n^C\omega(k) = \omega(k)$ for all $k \in [n] - \{|j|\}$, hence $\mu(s_n^C\omega, k) = \mu(\omega, k)$ for all $k \in [n]-\{n\}$. Moreover, $s_n^C\omega(-j) = s_n^C(-a_jN-n) = -a_jN-n-1 = (-a_j - 1)N + n$ thus $\mu(s_n^C\omega, n) = -\mu(\omega, n) + 1$.

We obtain that $\mu(s_i^C\omega) = s_i^C \cdot \mu(\omega)$, and in particular that $\mu(s_i^C\omega) \in \check{Q}$ for all i with $0 \le i \le n$.

Now assume that $\nu(\omega) \in \check{Q}$. Clearly $\omega s_0^C(j) = \omega(j)$ for all $j \in [n] - \{1\}$, hence $\nu(\omega s_0^C, j) = \nu(\omega, j)$ for all $j \in [n] - \{1\}$. Furthermore, $\omega s_0^C(1) = -\omega(1)$ thus $\nu(\omega s_0^C, 1) = -\nu(\omega, 1)$.

Fix i with $1 \le i \le n-1$ then $\omega s_i^C(j) = \omega(j)$ for all $j \in [n] - \{i, i+1\}$. Hence $\nu(\omega s_i^C, j) = \nu(\omega, j)$ for all $j \in [n] - \{i, i+1\}$ Moreover $\omega s_i^C(i) = \omega(i+1)$ and $\omega s_i^C(i+1) = \omega(i)$. It follows that $\nu(\omega s_i^C, i) = \nu(\omega, i+1)$ and $\nu(\omega s_i^C, i+1) = \nu(\omega, i)$.

Finally $\omega s_n^C(j) = \omega(j)$ for all $j \in [n-1]$ and therefore $\nu(\omega s_n^C, j) = \nu(\omega, j)$ for all $j \in [n-1]$. On the other hand $\omega s_n^C(n) = \omega(N-n) = N - \omega(n) = (-a_n+1)N - b_n$ thus $\nu(\omega s_n^C, n) = -\nu(\omega, n) + 1$.

We conclude that $\nu(\omega s_i^C) = s_i^C \cdot \nu(\omega)$ and in particular $\nu(\omega s_i^C) \in \check{Q}$ for all i with $0 \le i \le n$.

In type B_n there is only one new case to treat, namely the simple transposition s_n^B.

Suppose that $\mu(\omega) \in \check{Q}$. Choose $j, k \in [\pm n]$ such that $\sigma(j) = n-1$ and $\sigma(k) = n$. Then $s_n^B\omega(\ell) = \omega(\ell)$ for all $\ell \in [n] - \{|j|, |k|\}$, hence $\mu(s_n^B\omega, \ell) = \mu(\omega, \ell)$ for all $\ell \in [n-2]$. Moreover, $s_n^B\omega(-j) = s_n^B(-a_jN - n + 1) = -a_jN - n - 1 = (-a_j - 1)N + n$ and $s_n^B\omega(-k) = s_n^B(-a_kN - n) = -a_kN - n - 2 = (-a_k - 1)N + n - 1$ thus $\mu(s_n^B\omega, n-1) = a_k + 1 = -\mu(\omega, n) + 1$ and $\mu(s_n^B\omega, n) = a_j + 1 = -\mu(\omega, n-1) + 1$.

Now assume that $\nu(\omega) \in \check{Q}$. Clearly $\omega s_n^B(j) = \omega(j)$ for all $j \in [n-2]$ and therefore $\nu(\omega s_n^B, j) = \nu(\omega, j)$ for all $j \in [n-2]$. On the other hand $\omega s_n^B(n-1) = \omega(n+1) = \omega(N-n) = N - \omega(n) = (-a_n+1)N - b_n$ and $\omega s_n^B(n) = \omega(n+2) = \omega(N-n+1) = N - \omega(n-1) = (-a_{n-1}+1)N - b_{n-1}$. Hence $\nu(\omega s_n^B, n-1) = -a_n + 1 = -\nu(\omega, n) + 1$ and $\nu(\omega s_n^B, n) = -a_{n-1} + 1 = -\nu(\omega, n-1) + 1$.

We conclude that $\mu(s_n^B\omega) = s_n^B \cdot \mu(\omega) \in \check{Q}$ and $\nu(\omega s_n^B) = s_n^B \cdot \nu(\omega) \in \check{Q}$ as needed.

Type D_n is similar to B_n, except for the treatment of the simple transposition s_0^D.

Suppose that $\mu(\omega) \in \check{Q}$. Choose $j, k \in [\pm n]$ such that $\sigma(j) = 1$ and $\sigma(k) = 2$. Then $s_0^D\omega(k) = \omega(\ell)$ for all $\ell \in [n] - \{|j|, |k|\}$, hence $\mu(s_0^D\omega, \ell) = \mu(\omega, \ell)$ for all $\ell \in [n] - \{1, 2\}$. Furthermore, $s_0^D\omega(-j) = s_0^D(-a_jN - 1) = -a_jN + 2$ and $s_0^D\omega(-k) = s_0^D(-a_kN - 2) = -a_kN + 1$, thus $\mu(s_0^D\omega, 1) = a_k = -\mu(\omega, 2)$ and $\mu(s_0^D\omega, 2) = a_j = -\mu(\omega, 1)$.

Now assume that $\nu(\omega) \in \check{Q}$. Clearly $\omega s_0^D(j) = \omega(j)$ for all $j \in [n] - \{1, 2\}$, hence $\nu(\omega s_0^D, j) = \nu(\omega, j)$ for all $j \in [n] - \{1, 2\}$. Furthermore, $\omega s_0^D(1) = -\omega(2)$ and $\omega s_0^D(2) = -\omega(1)$. Hence $\nu(\omega s_0^D, 1) = -\nu(\omega, 2)$ and $\nu(\omega s_0^D, 2) = -\nu(\omega, 1)$.

We conclude that $\mu(s_0^D\omega) = s_0^D \cdot \mu(\omega) \in \check{Q}$ and $\nu(\omega s_0^D) = s_0^D \cdot \nu(\omega) \in \check{Q}$ as needed. ■

For $q \in \check{Q}^\Phi$ define an affine permutation $t_q \in \widetilde{\mathfrak{S}}_n^\Phi$ by setting $t_q(i) = -q_iN + i$ for $i \in [n]$. We call an affine permutation $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ a *translation* by $q \in \check{Q}^\Phi$ if $\omega \cdot x = x + q$ for all $x \in \check{Q}^\Phi$. Thus by definition the translations in $\widetilde{\mathfrak{S}}_n^\Phi$ correspond to translations in $\widetilde{W}$.

Theorem 1.5.3. [**74**, Prop. 3.3]

(i) Let $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ be an affine permutation and set $s = \sigma(\omega)$, $x = \mu(\omega)$ and $y = \nu(\omega)$. Then $\omega = t_x s = s t_{-y}$.

(ii) Let $x, y \in \check{Q}^\Phi$. Then $t_x t_y = t_{x+y}$ and $(t_x)^{-1} = t_{-x}$. Hence we may view the coroot lattice $\check{Q}^\Phi$ as a subgroup of $\widetilde{\mathfrak{S}}_n^\Phi$.

(iii) An affine permutation $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ is a translation if and only if $\omega = t_q$ for some $q \in \check{Q}^\Phi$.

(iv) The affine symmetric group is the semi-direct product of the symmetric group and the coroot lattice, that is, $\widetilde{\mathfrak{S}}_n^\Phi = \mathfrak{S}_n^\Phi \ltimes \check{Q}^\Phi$.

Proof. To prove claim (i) let $i \in [n]$ and $\omega(i) = a_iN + b_i$ such that $a_i \in \mathbb{Z}$ and $b_i \in [\pm n]$. Then

$$t_x s(i) = t_x(\sigma(\omega, i)) = t_x(b_i) = -\mu(\omega, b_i)N + b_i = a_iN + b_i,$$
$$s t_{-y}(i) = s(\nu(\omega, i)N + i) = s(a_iN + i) = a_iN + s(i) = a_iN + b_i.$$

To see claim (ii), note that for each $i \in [n]$ we have

$$t_x t_y(i) = t_x(-y_iN + i) = -y_iN + t_x(i) = -(y_i + x_i)N + i = t_{x+y}(i).$$

To see (iii) note that $t_q \cdot x = t_q t_x \cdot 0 = t_{q+x} \cdot 0 = q + x$, where we use (i), (ii) and Lemma 1.5.2 (iv). Thus t_q is indeed a translation. Conversely, if $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ is a translation by $q \in \check{Q}^\Phi$ then $q + x =$

$\omega \cdot x = q + \sigma(\omega) \cdot x$ for all $x \in \check{Q}^\Phi$, which implies $\sigma(\omega) = e$. Finally, $\mathfrak{S}_n^\Phi \check{Q}^\Phi = \widetilde{\mathfrak{S}}_n^\Phi$ and $\mathfrak{S}_n^\Phi$ normalises $\check{Q}^\Phi$ by (i). Since $\mathfrak{S}_n^\Phi \cap \check{Q}^\Phi = \{e\}$, we obtain (iv). ■

The following lemma is an easy consequence of the above that we will refer to in later sections.

LEMMA 1.5.4. [**74**, Lem. 3.4] *Let $\omega \in \widetilde{\mathfrak{S}}_n^\Phi$ be the minimal length coset representative of $\omega\mathfrak{S}_n^\Phi$, set $x = \mu(\omega) \in \check{Q}$ and $s = \sigma(\omega) \in \mathfrak{S}_n^\Phi$. Then for each $i \in [n]$*

$$\left|s^{-1}(i)\right| = \#\big\{k \in [n] : |x_k N - k| \leq |x_i N - i|\,\big\}.$$

If Φ is of type B_n or C_n, or if Φ is of type D_n and $\left|s^{-1}(i)\right| \neq 1$, then $s^{-1}(i) > 0$ if and only if $x_i \leq 0$. However, if Φ is of type D_n and $\left|s^{-1}(i)\right| = 1$, then $s^{-1}(i) > 0$ if and only if either $x_i \leq 0$ and the number of positive entries of x is even, or $x_i > 0$ and the number of positive entries of x is odd.

PROOF. Since ω is a minimal length coset representative, the absolute values of the entries of the window of ω must be increasing, that is, $0 < |\omega(1)| < |\omega(2)| < \cdots < |\omega(n)|$. On the other hand, $|s(j)| = i$ if and only if $\omega(j) = |x_i N - i|$. Hence $|s(j)| = i$ is equivalent to

$$j = \#\big\{k \in [n] : |x_k N - k| \leq |x_i N - i|\,\big\}.$$

Furthermore, if Φ is of type B_n or C_n, or if Φ is of type D_n and $\left|s^{-1}(i)\right| \neq 1$, then $\omega(i) > 0$. Hence $s(i) > 0$ if and only if $-x_i N + i > 0$, which is the case if and only if $x_i \leq 0$.

If Φ is of type D_n then the sign of $\omega(1)$ possibly has to be changed such that there is an even number of integers $j \in \mathbb{Z}$ with $j \geq 0$ and $\omega(j) < 0$. ■

CHAPTER 2

Exploring the Catalan-cube

The present chapter provides an overview of several combinatorial objects related to Dyck paths and parking functions to illustrate different generalisations of the Catalan numbers. In Section 2.1 Catalan numbers and Dyck paths are introduced. In Section 2.2 we generalise to Catalan numbers depending on a rational parameter and Dyck paths that stay above a rational slope. Section 2.3 introduces Cayley numbers and parking functions, which are shown to have an interpretation as labelled Dyck paths. In Section 2.4 rational Dyck paths and parking functions are combined to define rational Cayley numbers. In Section 2.5 several statistics on Dyck paths are introduced, which we use to define q,t-Catalan numbers. Section 2.6 combines the ideas from Sections 2.3 and 2.5 to define polynomials versions of the Cayley numbers. Finally, in Section 2.7 we take q,t-Catalan numbers to the rational level.

None of the contents of this chapter are new, and ample references are provided in the text. To make this introduction as self-contained as possible, I have included proofs whenever this seemed reasonable. The generalisation of the objects discussed in the present chapter to the Weyl group setting will be a recurring theme and an important motivation in the remainder of the thesis.

2.1. Dyck paths and Catalan numbers

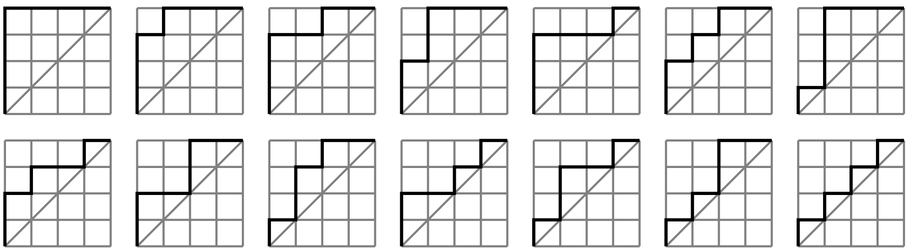


FIGURE 2.1. The fourteen Dyck paths of length four.

The *Catalan numbers* [**64**, A000108] are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n+1}.$$

A plethora of combinatorial problems involving these numbers can be found in a book of Stanley [**70**], including an appendix on the history of the Catalan numbers written by Pak [**54**]. In these initial sections we shall focus our attention on just one kind of Catalan objects, namely, Dyck paths.

A *Dyck path* of *length* n is a lattice path $x \in \mathcal{L}_{n,n}$ visiting no lattice point $(i,j) \in \mathbb{Z}^2$ with $i > j$. Equivalently, a Dyck path is a word $s_1 \cdots s_{2n}$ consisting of n east steps and n north steps such that every initial subword $s_1 \cdots s_r$ contains at least as many north steps as east steps. Denote the set of Dyck paths in $\mathcal{L}_{n,n}$ by $\mathfrak{D}_n$. For example the paths in $\mathfrak{D}_4$ are shown in Figure 2.1.

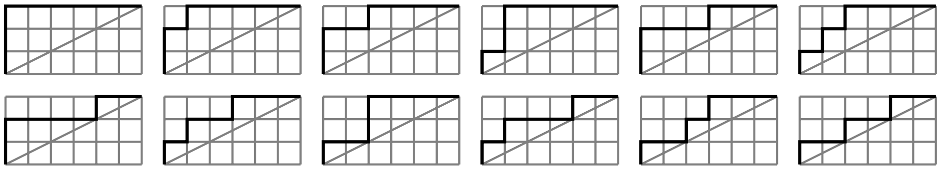


FIGURE 2.2. The twelve 2-Dyck paths of length three.

PROPOSITION 2.1.1. *The number of Dyck paths of length n is given by the n-th Catalan number. That is,* $\#\mathfrak{D}_n = C_n$.

PROOF. A classical proof of this result uses the so called *reflection principle*. The number of all lattice paths from $(0,0)$ to (n,n) using east and north steps is $\#\mathcal{L}_{n,n} = \binom{2n}{n}$. Thus it suffices to show that the number of paths in $\mathcal{L}_{n,n}$ that go below the diagonal are counted by $\binom{2n}{n+1}$. If x is such a path then there exists a unique Dyck path y and lattice path z such that $x = y\mathbf{e}z$. If we now *reflect* z, that is, replace each east step by a north step and vice versa, then we obtain a lattice path $\phi(x) = y\mathbf{e}z' \in \mathcal{L}_{n+1,n-1}$. Conversely, for every lattice path $x \in \mathcal{L}_{n+1,n-1}$ there exists a unique Dyck path y and lattice path z such that $x = y\mathbf{e}z$. Reflecting once more yields a lattice path $\phi(x) = y\mathbf{e}z'$ in $\mathcal{L}_{n,n}$ that goes below the diagonal. Hence the map $x \mapsto \phi(x)$ is a bijection and the claim follows from $\#\mathcal{L}_{n+1,n-1} = \binom{2n}{n+1}$. ■

For more details on the reflection principle see [**83, 31**].

PROPOSITION 2.1.2. *The Catalan numbers satisfy the recursion* $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ *for all* $n \geq 0$*, where* $C_0 = 1$.

PROOF. A convenient way of proving the result is to use a simple combinatorial decomposition of Dyck paths. For each non-empty Dyck path x there exist unique (possibly empty) Dyck paths y and z such that $x = \mathbf{n}y\mathbf{e}z$. Conversely, given two Dyck paths y and z, the concatenation $x = \mathbf{n}y\mathbf{e}z$ is again a Dyck path. ■

Using the recursion from Proposition 2.1.2 the generating function for Catalan numbers can be obtained by a routine computation.

COROLLARY 2.1.3. *The generating function of the Catalan numbers satisfies the equation* $\mathcal{F}(z) = 1 + z\mathcal{F}(z)^2$ *and has the closed formula*

$$\sum_{n\geq 0} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

■

2.2. Rational Catalan numbers

A first generalisation of Catalan numbers are the *Fuß–Catalan numbers* that depend on two parameters $m, n \in \mathbb{N}$ and are defined as

$$C_{n,nm+1} = \frac{1}{mn+1}\binom{mn+n}{n}.$$

Evidently, the Fuß–Catalan numbers reduce to the Catalan numbers when $m = 1$.

Many of the objects encountered in the study of Catalan numbers have been given generalisations counted by Fuß–Catalan numbers. For example, there is a family of lattice paths counted by Fuß–Catalan numbers that generalises Dyck paths. An *m-Dyck path* of *length n* is a lattice

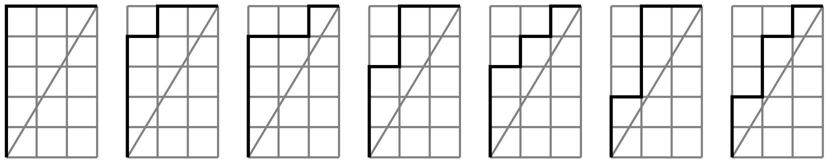


FIGURE 2.3. The seven Dyck paths of rational slope 5/3.

path $x \in \mathcal{L}_{mn,n}$ that visits no lattice point (i,j) with $i > mj$. Denote the set of m-Dyck paths of length n by $\mathfrak{D}_n^{(m)}$. The paths in $\mathfrak{D}_3^{(m)}$ are shown in Figure 2.2.

Instead of spending much time on Fuß–Catalan objects we move one step ahead. The *rational Catalan numbers* depend on a rational parameter n/p where $n, p \in \mathbb{N}$ are assumed to be relatively prime. They are defined by

$$C_{n,p} = \frac{1}{n+p}\binom{n+p}{n}.$$

It is easily verified that rational Catalan numbers reduce to Fuß–Catalan numbers when $p = mn + 1$.

There are yet more variations of Dyck paths that are counted by rational Catalan numbers. A lattice path $x \in \mathcal{L}_{p,n}$ is called a *rational Dyck path* of *slope* n/p if it does not go below the diagonal with slope n/p. That is, if $x = (z_0, \dots, z_{n+p})$ then x is a rational Dyck path if and only if $nz_{i1} \leq pz_{i2}$ for all $i \in [n+p]$. Denote the set of rational Dyck paths with slope n/p by $\mathfrak{D}_{n,p}$. For example, the paths in $\mathfrak{D}_{5,3}$ are shown in Figure 2.3.

More generally, define a function $w_n^p : \mathbb{Z}^2 \to \mathbb{Z}$ where the *weight* of a lattice point (i,j) is given by $w_n^p(i,j) = jp - in$. The statistic $\mathrm{neg} : \mathcal{L}_{p,n} \to \mathbb{N}$ counts the number of points visited by a lattice path that have a negative weight. That is, for $x \in \mathcal{L}_{p,n}$ we set

$$\mathrm{neg}(x) = \#\big\{i \in [n+p] : w_n^p(z_i) < 0\big\}.$$

Clearly $\mathfrak{D}_{n,p} = \{x \in \mathcal{L}_{p,n} : \mathrm{neg}(x) = 0\}$. To count the fibres of neg we use the following result due to Spitzer, which is often referred to as the Cycle Lemma.

THEOREM 2.2.1. [**66**, Thm. 2.1] *Let $a_1, \dots, a_n \in \mathbb{R}$ be such that $\sum_i a_i = 0$, but $a_r + \cdots + a_{r+s-1} \neq 0$ for all $r \in [n]$ and $s \in [n-1]$, where we set $a_{k+n} = a_k$ for all $k \in [n]$. That is, the numbers $a_1, \dots, a_n$ sum up to zero, but no non-trivial cyclically connected sub-sum vanishes. Then for each $k \in \{0, \dots, n-1\}$ there exists a unique $r \in [n]$ such that exactly k sub-sums of the form $a_r + \cdots + a_{r+s-1}$ with $s \in [n]$ are negative.*

PROOF. By assumption the sub-sums $b_s = a_1 + \cdots + a_s$ with $s \in [n]$ are pairwise distinct. Hence there exists a unique permutation $\sigma \in \mathfrak{S}_n$ such that $b_{\sigma(1)} < \cdots < b_{\sigma(n)}$. The number of negative sub-sums of the form $a_r + \cdots + a_{r+s-1}$ where $s \in [n]$ is given by $\sigma^{-1}(r) - 1$ for all $r \in [n]$. ■

We derive a result due to Bizley [**15**], namely that rational Dyck paths are counted by rational Catalan numbers.

THEOREM 2.2.2. *Let $n, p \in \mathbb{N}$ be relatively prime, and $k \in \{0, \dots, n+p-1\}$. Then $\#\{x \in \mathcal{L}_{p,n} : \mathrm{neg}(x) = k\} = C_{n,p}$. In particular, $\#\mathfrak{D}_{n,p} = C_{n,p}$.*

PROOF. Assign weights $w_n^p(\mathbf{n}) = p$ and $w_n^p(\mathbf{e}) = -n$. Given a path $x \in \mathcal{L}_{p,n}$ set $a_i = w_n^p(s_i)$. Clearly $a_1 + \cdots + a_{n+p} = 0$, and since n and p are relatively prime, this is the only vanishing cyclically connected sub-sum. Moreover, the sub-sums $a_1 + \cdots + a_s$ are precisely the weights $w_n^p(z_s)$ of the points visited by x. The cyclic group of order $n+p$ acts on $\mathcal{L}_{p,n}$ by cyclic

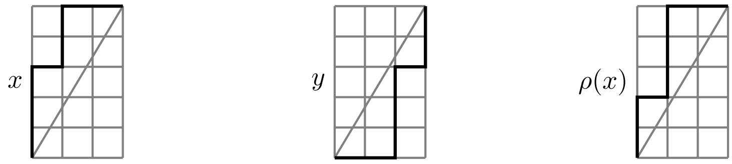


FIGURE 2.4. The paths $x \in \mathfrak{D}_{5,3}$ and $y \in \mathcal{L}_{3,5}$ lie in the same orbit under cyclic permutation of the steps. The path $\rho(x) \in \mathfrak{D}_{5,3}$ is the reverse path of y.

permutation of the steps. By Theorem 2.2.1 each orbit under this action contains exactly one path from each fibre of neg. ∎

An immediate corollary is the following.

COROLLARY 2.2.3. *Let $n, p \in \mathbb{N}$ be relatively prime, then the rational Catalan number $C_{n,p}$ is an integer.* ∎

Furthermore Theorem 2.2.2 can be used to count m-Dyck paths.

COROLLARY 2.2.4. *For all $n, m \in \mathbb{N}$ the number of m-Dyck paths of length n is given by the Fuß–Catalan number $C_{n,mn+1}$.*

PROOF. We claim that the map $\phi : \mathfrak{D}_n^{(m)} \to \mathfrak{D}_{n,mn+1}$ defined by $x \mapsto x\mathbf{e}$ is a bijection. To see this it suffices to verify that $j(mn) < in$ implies $j(mn+1) \leq in$ for all $j \in [n]$. ∎

Next, note that $\{x \in \mathcal{L}_{p,n} : \mathrm{neg}(x) = n+p-1\}$ consists of all paths in $\mathcal{L}_{p,n}$ that do not go above the diagonal of slope n/p. These paths turn into rational Dyck paths when the sequence of steps is reversed. By the proof of Theorem 2.2.2 each orbit of $\mathcal{L}_{p,n}$ under cyclic permutation of the steps contains exactly one path from each fibre of neg. Let $\rho : \mathfrak{D}_{n,p} \to \mathfrak{D}_{n,p}$ denote the map that sends x to the reverse path of y where $y \in \mathcal{L}_{p,n}$ is the unique lattice path with $\mathrm{neg}(y) = n+p-1$ that is a cyclic permutation of x. See Figure 2.4 for an example.

COROLLARY 2.2.5. *The map $\rho : \mathfrak{D}_{n,p} \to \mathfrak{D}_{n,p}$ is an involution.* ∎

The map ρ appears in different places in the literature. For example Xin [**80**] calls it the *rank complement*, while Ceballos, Denton and Hanusa [**21**] call it *conjugation* on Dyck paths.

Recently Ceballos and González D'León [**22**] have generalised many Catalan objects beyond the rational level. These objects are counted by so called s-Catalan numbers, which depend on a signature $s \in \mathbb{N}^k$.

2.3. Cayley numbers and parking functions

The *Cayley numbers* [**64**, A000272] are defined as

$$\mathscr{C}_n = (n+1)^{n-1}.$$

Cayley [**20**] famously proved that they count labelled trees. This can be seen via different beautiful combinatorial arguments such as Prüfer sequences [**56**] or the proof of Joyal [**46**]. See also [**1**, Chap. 30].

The objects counted by Cayley numbers we are most interested in are parking functions. A vector $f \in \mathbb{N}^n$ is called a *parking function* of *length* n if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $f_{\sigma(i)} < i$ for all $i \in [n]$. Equivalently, f is a parking function if $\#\{j \in [n] : f_j < i\} \geq i$ for all $i \in [n]$. Let PF_n denote the set of all parking functions of length n. Note that the symmetric group $\mathfrak{S}_n$ acts on PF_n by permuting entries.

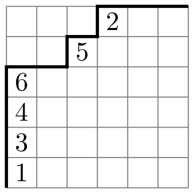


FIGURE 2.5. A vertically labelled Dyck path of length six.

Parking functions were first considered and enumerated by Konheim and Weiss [**47**] in connection with hash functions [1]. Before we derive the number of parking functions ourselves, we need an important connection between parking functions and Dyck paths.

Let $x \in \mathcal{L}_{m,n}$ be a lattice path. Recall that a rise of x is an index $i \in [n]$ such that the i-th north step of x is immediately followed by another north step. A valley of x is a pair $(i, j) \in [m] \times [n]$ such that the i-th east step of x is immediately followed by the j-th north step of x. Let $\sigma \in \mathfrak{S}_n$ be a permutation. A *descent* of σ is a number $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. Otherwise i is called an *ascent*.

A *vertically labelled Dyck path* of length n is a pair (σ, x) of a Dyck path $x \in \mathfrak{D}_n$ and a permutation $\sigma \in \mathfrak{S}_n$ such that every rise of x is an ascent of σ. If we picture a vertically labelled Dyck path by placing the label $\sigma(i)$ next to the i-th north step of x then this condition translates to labels increasing along columns. We denote the set of all vertically labelled Dyck paths of length n by $\mathrm{Vert}(A_{n-1})$. For an example see Figure 2.5.

Alternatively define an equivalence relation on $\mathfrak{S}_n \times \mathfrak{D}_n$ by $(\sigma, x) \sim (\tau, y)$ if and only if $x = y$ and $\sigma H = \tau H$, where $H \leq \mathfrak{S}_n$ is the subgroup generated by the simple transpositions

$$\left\{s_i^A : i \text{ is a rise of } x\right\}.$$

We may then view vertically labelled Dyck paths as equivalence classes

$$\mathrm{Vert}(A_{n-1}) = \left\{[\sigma, x]_\sim : x \in \mathfrak{D}_n, \sigma \in \mathfrak{S}_n\right\}.$$

More precisely each equivalence class $[\tau, x]_\sim$ contains a unique vertically labelled Dyck path (σ, x), where σ is the representative of the coset τH of minimal length. The set $\mathrm{Vert}(A_{n-1})$ thus inherits a natural $\mathfrak{S}_n$-action given by $\tau \cdot [\sigma, x] = [\tau\sigma, x]$. The following well-known result establishes that this action is isomorphic to the action on PF_n by permuting entries.

PROPOSITION 2.3.1. *Parking functions and vertically labelled Dyck paths are isomorphic $\mathfrak{S}_n$-sets.*

PROOF. An $\mathfrak{S}_n$-equivariant bijection $\phi_A : \mathrm{Vert}(A_{n-1}) \to \mathrm{PF}_n$ is given by $\phi_A(\sigma, x) = f$, where $f_{\sigma(i)} = j$ if the i-th north step of x is preceded by j east steps of x. ■

For example, the vertically labelled Dyck path from Figure 2.5 is mapped to the parking function $f = (0, 3, 0, 0, 2, 0)$.

COROLLARY 2.3.2. *The orbits of PF_n under the $\mathfrak{S}_n$-action defined by permutation of entries, which are indexed by increasing parking functions, are counted by the Catalan numbers, that is, $\#\{\mathfrak{S}_n \cdot f : f \in \mathrm{PF}_n\} = C_n$.* ■

Next we give a bijection between parking functions and labelled trees. The first such bijection is due to Schützenberger [**58**].

[1] Konheim and Weiss also included an explanation for the name "parking function", although it was later pointed out to be "politically incorrect" by Stanley [**68**].

PROPOSITION 2.3.3. *Parking functions are counted by the Cayley numbers, that is,* $\# \mathrm{PF}_n = \mathscr{C}_n$.

PROOF. We give a bijection from $\mathrm{Vert}(A_{n-1})$ to the set of rooted labelled forests on n vertices, which in turn are in bijection with labelled trees on $n+1$ vertices. Together with Proposition 2.3.1 this yields the claim.
Define a map ϕ recursively. Given $(\sigma, x) \in \mathrm{Vert}(A_n)$, let y and z be Dyck paths determined by $x = \mathbf{n}y\mathbf{e}z$. Furthermore let $k = \sigma(1)$, $S \subseteq [n]$ be the set of labels assigned to the north steps of y and $\overline{S} = [n] - (S \cup \{k\})$ be the set of labels assigned to the north steps of z. Then ϕ maps the y-portion of (σ, x) to an S-labelled rooted forest F_1 and the z-portion to an $\overline{S}$-labelled rooted forest F_2. The image F of (σ, x) under ϕ is defined as the union of F_1 and the tree obtained by joining the roots of all trees in F_2 to a new root vertex labelled k.
Conversely, let F be a labelled rooted forest on n vertices. Then the forests F_1 and F_2 are obtained as $F_1 = F - T$ and $F_2 = T - \{r\}$, where T is the tree in F whose root r has the minimal label among all roots of F. Hence ϕ is a bijection. ∎

From the recursive definition of the bijection in the proof of Proposition 2.3.3 it is particularly easy to see that it is surjective and invertible. One can also give a more explicit description of the same bijection. See [**37**, Sec. 4] or [**36**, Chap. 5].
Furthermore note that for $n > 2$ the $\mathfrak{S}_n$-action on labelled trees on $n+1$ vertices defined via permuting labels is not isomorphic to PF_n.

2.4. Rational parking functions

The *rational Cayley numbers* depend on a rational parameter n/p, where $n, p \in \mathbb{N}$ are assumed to be relatively prime, and are defined as

$$\mathscr{C}_{n,p} = p^{n-1}.$$

These numbers count rational parking functions, which are in bijection with vertically labelled rational Dyck paths.
A vector $f \in \mathbb{N}^n$ is a *rational parking function* of *slope* n/p if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that $f_{\sigma(i)} \leq (i-1)p/n$ for all $i \in [n]$. Equivalently f is a rational parking function if and only if

$$\#\{j \in [n] : f_j \leq (i-1)p/n\} \geq i$$

for all $i \in [n]$, which is the case if and only if

$$\#\{j \in [n] : f_j < i\} \geq \frac{in}{p}$$

for all $i \in [p]$. We denote the set of rational parking functions with slope n/p by $\mathrm{PF}_{n,p}$. Clearly the symmetric group $\mathfrak{S}_n$ acts on $\mathrm{PF}_{n,p}$ via permutation of entries. Moreover note that $\mathrm{PF}_n = \mathrm{PF}_{n,n+1}$.
Compared to classical parking functions, the interest in rational parking functions is quite recent. See for example [**8, 33**].
Our proof of the fact that rational parking functions are counted by rational Cayley numbers is taken from [**75**] and relates parking functions to residue classes modulo p. Given a vector $f \in (\mathbb{Z}/p\mathbb{Z})^n$ choose a representative $f = (f_1, \ldots, f_n)$ with $0 \leq f_i < p$ for all $i \in [n]$ and set $\mathrm{sum}(f) = f_1 + \cdots + f_n$. This defines a statistic $\mathrm{sum} : (\mathbb{Z}/p\mathbb{Z})^n \to \mathbb{N}$.

THEOREM 2.4.1. *Let* $n, p \in \mathbb{N}$ *be relatively prime, and let* $A \leq (\mathbb{Z}/p\mathbb{Z})^n$ *denote the subgroup generated by the element* $(1, \ldots, 1)$. *Then each coset in* $(\mathbb{Z}/p\mathbb{Z})^n/A$ *contains a unique rational parking function. The bijection mapping a rational parking function to its coset is an isomorphism of* $\mathfrak{S}_n$*-sets. In particular* $\# \mathrm{PF}_{n,p} = \mathscr{C}_{n,p}$.

PROOF. Since n and p are relatively prime, for each coset $H \in (\mathbb{Z}/p\mathbb{Z})^n/A$ the sums $\sum_i x_i$, where x ranges over H, are distinct modulo p. Hence there is a unique element $f \in H$ such that $\operatorname{sum}(f)$ is minimal. It suffices to show that a vector $f \in \mathbb{N}^n$ lies in $\mathrm{PF}_{n,p}$ if and only if $\operatorname{sum}(f)$ equals the minimum of $\{\operatorname{sum}(x) : x \in f + A\}$. To see this note that for all $i \in [p]$ the inequality

$$\operatorname{sum}(f) \leq \operatorname{sum}\big(f - (i, \dots, i)\big) = \operatorname{sum}(f) - ni + p \cdot \#\big\{j \in [n] : f_j - i < 0\big\}$$

is equivalent to $\#\{j \in [n] : f_j < i\} \geq in/p$. ∎

Next consider the set $\operatorname{Vert}(A_{n-1}, p)$ of vertically labelled rational Dyck paths. The definition is completely analogous to the definition in Section 2.3. A *vertically labelled rational Dyck path* of *slope* n/p is a pair (σ, x) of a permutation $\sigma \in \mathfrak{S}_n$ and a Dyck path $x \in \mathfrak{D}_{n,p}$ such that rises of x are ascents of σ. Equivalently we may view $\operatorname{Vert}(A_{n-1}, p)$ as a set of equivalence classes $\{[\sigma, x] : \sigma \in \mathfrak{S}_n, x \in \mathfrak{D}_{n,p}\}$, where $(\sigma, x) \sim (\tau, y)$ if and only if $x = y$ and $\sigma H = \tau H$ where $H \leq \mathfrak{S}_n$ is generated by the simple transpositions corresponding to the rises of x. Thus $\operatorname{Vert}(A_{n-1}, p)$ inherits a natural $\mathfrak{S}_n$-action, which turns out to be isomorphic to the action on parking functions.

The proof of Proposition 2.4.2 below is the same as the proof of Proposition 2.3.1.

PROPOSITION 2.4.2. *Let $n, p \in \mathbb{N}$ be relatively prime. Then $\mathrm{PF}_{n,p}$ and $\operatorname{Vert}(A_{n-1}, p)$ are isomorphic $\mathfrak{S}_n$-sets.* ∎

As a corollary of Proposition 2.4.2 and Theorem 2.2.2 we obtain the number of increasing parking functions of slope n/p.

COROLLARY 2.4.3. *Let $n, p \in \mathbb{N}$ be relatively prime. Then the number of $\mathfrak{S}_n$-orbits of $\mathrm{PF}_{n,p}$ equals $C_{n,p}$.* ∎

2.5. The q, t-Catalan numbers

There are many ways to introduce q-analogues of the Catalan numbers [**28**]. We are particularly interested in generating polynomials of Dyck paths with respect to certain statistics. Our starting point will be a q-analogue of the recursion from Proposition 2.1.2. The *Carlitz–Riordan q-Catalan numbers* [**18**] are defined by way of the recursion

$$C_{n+1}(q) = \sum_{k=0}^{n} q^k C_k(q) C_{n-k}(q)$$

for all $n \in \mathbb{N}$, with initial condition $C_0(q) = 1$. Clearly $C_n(1) = C_n$ due to Proposition 2.1.2. We will give three interpretations of $C_n(q)$ as the generating function of Dyck paths with respect to different statistics later on. Here we start out with the most natural one.

First define the *inversion* statistic $\operatorname{inv} : \mathcal{L}_{n,m} \to \mathbb{N}$. Given a lattice path $x \in \mathcal{L}_{n,m}$ corresponding to the word $s_1 \cdots s_{m+n}$ set

$$\operatorname{inv}(x) = \#\big\{(i, j) \in [m+n]^2 : i < j, s_i = \mathbf{n} \text{ and } s_j = \mathbf{e}\big\}.$$

Thereby $\operatorname{inv}(x)$ measures the area below the path x.

Define the *area statistic* $\operatorname{area} : \mathfrak{D}_n \to \mathbb{N}$ as the number of lattice points $(i, j) \in \mathbb{Z}^2$ with positive weight $w_n^{n+1}(i, j) = n(j - i) + j$ that lie below a Dyck path. Equivalently, for $x \in \mathfrak{D}_n$ let

$$\operatorname{area}(x) = \operatorname{inv}(x) - \binom{n+1}{2}.$$

In some sense the area statistic measures the area between a Dyck path and the main diagonal. Turn $P = \{(i, j) \in [n]^2 : i < j\}$ into a poset by equipping it with the order defined by $(i, j) \leq (k, \ell)$

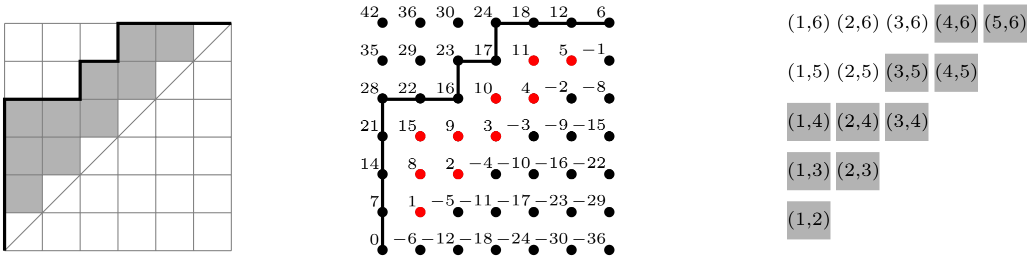


FIGURE 2.6. Different interpretations of the area statistic of a Dyck path $x \in \mathfrak{D}_6$, where $\operatorname{area}(x) = 10$.

if and only if $k \leq i < j \leq \ell$. The set of valleys of a Dyck path forms an anti-chain in P. Given $x \in \mathfrak{D}_n$ define an order ideal

$$I_x = P - \bigcup_{(i,j) \text{ is a valley of } x} \big\{(k,\ell) \in P : (i,j) \leq (k,\ell)\big\}. \tag{2.1}$$

Then $\operatorname{area}(x) = |I_x|$ and the pairs $(i,j) \in I_x$ correspond to unit squares between the path x and the main diagonal. The poset P is just the poset of positive roots in type A_{n-1}, thus we get a first glimpse at possible generalisations to other root systems.

We now demonstrate that the statistic area naturally leads to the q-Catalan numbers.

PROPOSITION 2.5.1. *For all $n \in \mathbb{N}$ we have* $C_n(q) = \sum_{x \in \mathfrak{D}_n} q^{\operatorname{area}(x)}$.

PROOF. The proof is a straightforward generalisation of the proof of Proposition 2.1.2. Let $y \in \mathfrak{D}_k$ and $z \in \mathfrak{D}_{n-k}$ be Dyck paths and set $x = \mathbf{n}y\mathbf{e}z$. Then $x \in \mathfrak{D}_{n+1}$ and $\operatorname{area}(x) = k + \operatorname{area}(y) + \operatorname{area}(z)$. Hence the area generating function and the q-Catalan numbers satisfy the same recursion and initial condition. ∎

Garsia and Haiman defined q,t*-Catalan numbers* algebraically as the Hilbert series of the sign component $\mathcal{DH}_n^{\epsilon}$ of a bigraded representation $\mathcal{DH}_n$ of the symmetric group $\mathfrak{S}_n$ called *diagonal harmonics*.

$$C_n(q,t) = \operatorname{Hilb}(\mathcal{DH}_n^{\epsilon}; q, t) = \sum_{i,j} \dim(\mathcal{DH}_n^{\epsilon})_{i,j} q^i t^j.$$

For more information on this representation see for example [**40, 30**]. Haglund conjectured [**35**] and later proved together with Garsia [**29**] that this series has a combinatorial interpretation as the bivariate generating polynomial of Dyck paths with respect to certain statistics. Namely, they proved that

$$C_n(q,t) = \sum_{x \in \mathfrak{D}_n} q^{\operatorname{dinv}(x)} t^{\operatorname{area}(x)} = \sum_{x \in \mathfrak{D}_n} q^{\operatorname{area}(x)} t^{\operatorname{bounce}(x)}.$$

Two new statistics make an appearance in addition to the area statistic. The bounce statistic was defined by Haglund in [**35**]. The dinv statistic was found by Haiman [**36**, Chap. 3][2]. The two combinatorial models were shown to be equivalent by Haglund using the so called zeta map. Two constructions used by Haglund, namely the bounce path and bounce points of a Dyck path,

[2]Allegedly Garsia informed Haiman in an email of Haglund's discovery of a suitable statistic without indicating what that statistic was. Thus Haiman sat down and worked out such a statistic of his own. Of course the two statistics turned out not to be the same. Garsia later expressed his regrets for not writing this email two years earlier.

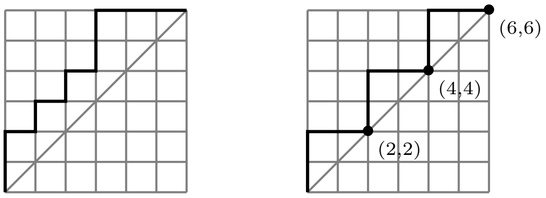


FIGURE 2.7. A Dyck path $x \in \mathfrak{D}_6$ (left) and its bounce path with bounce points (right), where $\mathrm{bounce}(x) = 6$.

as well as the inverse of the zeta map actually date back even further to a paper of Andrews, Krattenthaler, Orsina and Papi [**3**].

The *area vector* $(a_1, \dots, a_n)$ of a Dyck path $x \in \mathfrak{D}_n$ is defined by

$$a_i = i - j - 1$$

where j is the number of east steps of x preceding the i-th north step of x. Equivalently a_i equals the number of lattice points $(j,k) \in \mathbb{Z}^2$ below the path x such that the weight $w_n^{n+1}(j,k)$ is positive and $w_n^{n+1}(j,k) \equiv i-1$ modulo n. The area vector determines the Dyck path uniquely. It is easy to show that $(a_1, \dots, a_n) \in \mathbb{N}^n$ is the area vector of a Dyck path if and only if $a_1 = 0$ and $a_{i+1} \leq a_i + 1$ for all $i \in [n-1]$. Moreover the sum of the entries of the area vector equals the area statistic. In particular, the entry a_i encodes the area between the i-th north step of the Dyck path and the diagonal.

The second statistic involved is the somewhat mysterious *dinv statistic* $\mathrm{dinv} : \mathfrak{D}_n \to \mathbb{N}$, which is defined as

$$\mathrm{dinv}(x) = \#\big\{(i,j) \in [n]^2 : i < j \text{ and either } a_i = a_j \text{ or } a_i = a_j + 1\big\},$$

where $(a_1, \dots, a_n)$ is the area vector of $x \in \mathfrak{D}_n$. The word dinv stands for “diagonal inversion”.

Finally, there is the *bounce statistic* $\mathrm{bounce} : \mathfrak{D}_n \to \mathbb{N}$. Given $x \in \mathfrak{D}_n$ we first construct another Dyck path $y \in \mathfrak{D}_n$ called the *bounce path* of x as follows. Starting at $(0,0)$ draw north steps until the starting point of an east step of x is reached. Then draw east steps until the path touches the diagonal. Now draw north steps again until the starting point of an east step of x is reached, and repeat this process until the path ends at (n,n). Suppose the bounce path of x has the form $\mathbf{n}^{i_1}\mathbf{e}^{i_1} \cdots \mathbf{n}^{i_r}\mathbf{e}^{i_r}$. That is, the bounce path returns to the diagonal at the points (b_j, b_j) where $b_j = i_1 + \dots + i_j$ for $j \in [r]$. Then the bounce statistic is given by

$$\mathrm{bounce}(x) = rn - \sum_{j=1}^{r} i_j(r-j+1) = \sum_{j=1}^{r} (n - b_j).$$

See Figure 2.7 for an example.

The three statistics are related by the following map on Dyck paths called the *zeta map*. Let $x \in \mathfrak{D}_n$ be a Dyck path with area vector $a = (a_1, \dots, a_n)$. Set

$$\zeta_A(x) = \overrightarrow{w}_0^-(a)\overrightarrow{w}_{-1}^-(a)\cdots\overrightarrow{w}_{-n}^-(a).$$

A priori $\zeta_A(x)$ is a word in the alphabet $\{\mathbf{e}, \mathbf{n}\}$. Since every entry of a contributes exactly one copy of the letter $\mathbf{n}$ and one copy of the letter $\mathbf{e}$, where the north step occurs before the east step, we see that $\zeta_A(x)$ gives rise to a Dyck path. Thus we view $\zeta_A : \mathfrak{D}_n \to \mathfrak{D}_n$ as a map to Dyck paths. For example, the words used to construct $\zeta_A(x)$ in Figure 2.8 are $\overrightarrow{w}_0^-(a) = \mathbf{nn}$, $\overrightarrow{w}_{-1}^-(a) = \mathbf{enen}$, $\overrightarrow{w}_{-2}^-(a) = \mathbf{enne}$ and $\overrightarrow{w}_{-3}^-(a) = \mathbf{ee}$.

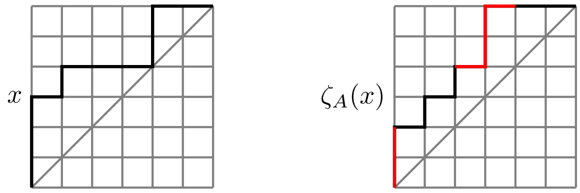


FIGURE 2.8. The dyck path $x \in \mathfrak{D}_6$ with area vector $a = (0,1,2,2,0,1)$ (left) and the path $\zeta_A(x) \in \mathfrak{D}_6$ (right).

THEOREM 2.5.2. [**36**, Thm. 3.15] *The map $\zeta_A : \mathfrak{D}_n \to \mathfrak{D}_n$ is a bijection and satisfies*

$$\operatorname{dinv}(x) = \operatorname{area}(\zeta_A(x)), \qquad \operatorname{area}(x) = \operatorname{bounce}(\zeta_A(x))$$

for all $x \in \mathfrak{D}_n$.

PROOF. For each Dyck path $y \in \mathfrak{D}_n$ there exists a unique decomposition $y = y_0 y_1 \cdots y_{m+1}$ into words y_j in the alphabet $\{\mathbf{e}, \mathbf{n}\}$, where $m \in \mathbb{N}$, such that the following conditions are fulfilled: (i) y_0 consists of north steps, (ii) y_j begins with an east step for all $j \in [m+1]$, and (iii) the number of east steps in y_j equals the number of north steps in y_{j-1} for all $j \in [m+1]$.

Let $x \in \mathfrak{D}_n$ be a Dyck path with area vector a. Set $b_i = \#\{j \in [n] : a_j = i\}$ and let m be the maximal entry of a. Then $\overrightarrow{w}^-_{-j}(a)$ is empty for $j > m+1$ and the decomposition

$$\zeta_A(x) = \overrightarrow{w}^-_0(a)\overrightarrow{w}^-_{-1}(a) \cdots \overrightarrow{w}^-_{-m-1}(a)$$

satisfies the properties (i)–(iii). It follows that ζ_A is injective and hence bijective.

It is not hard to see that the bounce path of $\zeta_A(x)$ has precisely $m+1$ bounce points, namely $(\sum_{i=0}^{j-1} b_i, \sum_{i=0}^{j-1} b_i)$ for $j \in [m]$. Therefore

$$\operatorname{bounce}(\zeta_A(x)) = \sum_{j=1}^{m+1} \Big(n - \sum_{i=0}^{j-1} b_i\Big) = \sum_{j=1}^{m} \sum_{i=j}^{n} b_i = \sum_{j=1}^{m} j b_j = \operatorname{area}(x).$$

Finally, dividing the area of $\zeta_A(x)$ into the squares between $\zeta_A(x)$ and its bounce path and the squares below the bounce path, we obtain

$$\operatorname{area}(\zeta_A(x)) = \sum_{j=1}^{m} \left(\binom{b_i}{2} + \operatorname{inv}\left(\overrightarrow{w}^-_{-j}(a)\right)\right) = \operatorname{dinv}(x),$$

where $\overrightarrow{w}^-_{-j}(a)$ is viewed as a lattice path in $\mathcal{L}_{b_{j-1}, b_j}$. ■

Note that the decomposition of a Dyck path described in the above proof can be obtained by dissecting x at the peaks of its bounce path. See Figures 2.7 and 2.8. This yields a simple combinatorial description of the inverse zeta map.

As a consequence of Theorem 2.5.2 the three introduced statistics all have the same distribution.

COROLLARY 2.5.3. *For all $n \in \mathbb{N}$ we have*

$$C_n(q) = \sum_{x \in \mathfrak{D}_n} q^{\operatorname{dinv}(x)} = \sum_{x \in \mathfrak{D}_n} q^{\operatorname{area}(x)} = \sum_{x \in \mathfrak{D}_n} q^{\operatorname{bounce}(x)}.$$

■

Note that it is a trivial consequence of the definition of $\mathcal{DH}_n$ that the q,t-Catalan polynomials are symmetric in q and t, that is,

$$C_n(q,t) = C_n(t,q).$$

Corollary 2.5.3 yields the much weaker result $C_n(q,1) = C_n(1,q)$. It would be desirable to have a combinatorial explanation for the symmetric joint distribution of the statistics giving rise to the q,t-Catalan numbers. An example of such a result for two entirely different statistics is given in the next paragraph.

Let $x \in \mathfrak{D}_n$. The *initial rise* $\mathrm{inr} : \mathfrak{D}_n \to \mathbb{N}$ is defined by $\mathrm{inr}(x) = r$ where $x = \mathbf{n}^r\mathbf{e}y$ begins with r north steps. Moreover $\mathrm{ret} : \mathfrak{D}_n \to \mathbb{N}$ counts the number of *returns* of a Dyck path to the diagonal, that is, $\mathrm{ret}(x)$ denotes the number of indices $i \in [n]$ such that x visits the lattice point (i,i). Note that $\mathrm{ret}(x) = \mathrm{inr} \circ \zeta(x)$ for all $x \in \mathfrak{D}_n$.

Proposition 2.5.4. *The statistics* inr *and* ret *are jointly symmetrically distributed.*

Proof. Define a map $\phi : \mathfrak{D}_n \to \mathfrak{D}_n$ recursively as follows. Map the empty path to itself. Given $x \in \mathfrak{D}_n$ for $n \geq 1$ let y and z be Dyck paths such that $x = \mathbf{n}y\mathbf{e}z$. Set $\phi(x) = \mathbf{n}\phi(z)\mathbf{e}\phi(y)$. The map ϕ is clearly an involution and exchanges the two statistics inr and ret. ■

Unfortunately such a simple combinatorial explanation for the symmetry of the q,t-Catalan numbers seems to be completely out of reach so far.

Open Problem 2.5.5. [**36**, Prob. 3.11] *Find a bijection (maybe even an involution) $\phi : \mathfrak{D}_n \to \mathfrak{D}_n$ exchanging the statistics* area *and* bounce.

2.6. The q,t-Cayley numbers

Instead of considering only the sign component of the diagonal harmonics $\mathcal{DH}_n$ it is natural to ask for the full Hilbert series

$$\mathscr{C}_n(q,t) = \mathrm{Hilb}(\mathcal{DH}_n; q,t) = \sum_{i,j} \dim(\mathcal{DH}_n)_{i,j} q^i t^j.$$

As it turns out combinatorial models for these polynomials have been found, extending those of the previous section to labelled Dyck paths.

Define the statistic $\mathrm{dinv}' : \mathrm{Vert}(A_{n-1}) \to \mathbb{N}$ as follows. Given a vertically labelled Dyck path $(\sigma,x) \in \mathrm{Vert}(A_{n-1})$ set

$$\begin{aligned}\mathrm{dinv}'(\sigma,x) = \#\big\{(i,j) \in [n]^2 : i<j \text{ and } a_i = a_j \text{ and } \sigma(i) < \sigma(j)\big\}\\ + \#\big\{(i,j) \in [n]^2 : i<j \text{ and } a_i = a_j + 1 \text{ and } \sigma(j) < \sigma(i)\big\}.\end{aligned}$$

Haglund and Loehr [**37**] conjectured the following combinatorial interpretation of the bivariate Hilbert series of diagonal harmonics, which serves as our definition of *q,t-Cayley numbers*.

$$\mathscr{C}_n(q,t) = \sum_{(\sigma,x)\in\mathrm{Vert}(A_{n-1})} q^{\mathrm{dinv}'(\sigma,x)} t^{\mathrm{area}(x)}$$

In the same paper they also provided an alternative model that relies on the bounce statistic, and a generalised zeta map connecting the two approaches.

A *diagonally labelled Dyck path* of *length* n is a pair (σ,x) of a permutation $\sigma \in \mathfrak{S}_n$ and a Dyck path $x \in \mathfrak{D}_n$ such that $\sigma(i) < \sigma(j)$ for all valleys (i,j) of x. Let $\mathrm{Diag}(A_{n-1})$ denote the set of all diagonally labelled Dyck paths of length n. The set $\mathrm{Diag}(A_{n-1})$ is endowed with a modified area statistic. Recall that $\mathrm{area}(x)$ equals the number of pairs (i,j) in the order ideal I_x defined in (2.1). Define the statistic $\mathrm{area}' : \mathrm{Diag}(A_{n-1}) \to \mathbb{N}$ by letting

$$\mathrm{area}'(\sigma,x) = \big\{(i,j) \in I_x : \sigma(i) < \sigma(j)\big\}.$$

The *diagonal reading word* of a pair $(\sigma,x) \in \mathfrak{S}_n \times \mathfrak{D}_n$ is the permutation $\mathrm{drw}_A(\sigma,x) \in \mathfrak{S}_n$ given by

$$\mathrm{drw}_A(\sigma,x)(i) = \#\big\{r \in [n] : n\mu_r + r \leq n\mu_i + i\big\},$$

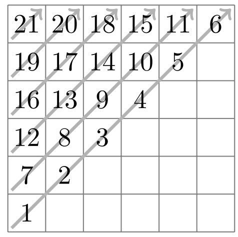


FIGURE 2.9. The diagonal reading order.

where μ is the area vector of x. Alternatively the diagonal reading word can be read off quickly by scanning all unit squares that may contain a label in the *diagonal reading order* depicted in Figure 2.9, and writing down the labels in the order in which they are encountered.

Define a *zeta map* on labelled Dyck paths $\zeta_A : \mathfrak{S}_n \times \mathfrak{D}_n \to \mathfrak{S}_n \times \mathfrak{D}_n$ via

$$\zeta_A(\sigma, x) = (\mathrm{drw}_A(\sigma, x), \zeta_A(x)).$$

The properties of the map ζ_A are closely tied to valleys and rises of labelled Dyck paths. Let $(\sigma, x) \in \mathfrak{S}_n \times \mathfrak{D}_n$. Given a rise i of x, we say that i is a *rise* of (σ, x) *labelled* by $(\sigma(i), \sigma(i+1))$. Similarly, let (i, j) be a valley of x. We say (i, j) is a *valley* of (σ, x) *labelled* by $(\sigma(i), \sigma(j))$.

The next result demonstrates the significance of the diagonal reading word.

LEMMA 2.6.1. *Let $(\sigma, x) \in \mathfrak{S}_n \times \mathfrak{D}_n$ and $a, b \in [n]$. Then (σ, x) has a rise labelled (a, b) if and only if $\zeta_A(\sigma, x)$ has a valley labelled (a, b).*

PROOF. To see this, note that the valleys of $\zeta_A(x)$ only occur within a sequence $\overrightarrow{w}^{-}_{-j}(\mu)$, where μ denotes the area vector of x, and correspond precisely to the rises of x. It is left to the reader to verify that the labels work out. See also Sections 6.3, 6.4 and 6.5 for similar computations. ∎

Note that Lemma 2.6.1 furnishes a second description of the map ζ_A. The valleys of $\zeta_A(x)$ can be filled in using the diagonal reading word $\mathrm{drw}_A(e, x)$. Since each Dyck path is determined uniquely by its valleys, we obtain $\zeta_A(x)$.

Lemma 2.6.1 establishes the main result of this section.

THEOREM 2.6.2. [**37**, pp. 17–20] *The map ζ_A restricts to a bijection $\zeta_A : \mathrm{Vert}(A_{n-1}) \to \mathrm{Diag}(A_{n-1})$. Moreover*

$$\mathrm{dinv}'(\sigma, x) = \mathrm{area}'(\zeta_A(\sigma, x)).$$

for all $(\sigma, x) \in \mathrm{Vert}(A_{n-1})$.

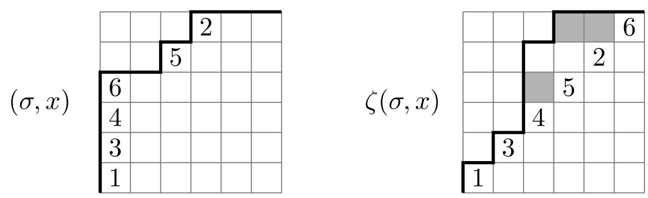


FIGURE 2.10. A vertically labelled Dyck path $(\sigma, x) \in \mathrm{Vert}(A_5)$ with area vector $(0, 1, 2, 3, 2, 2)$, diagonal reading word $\mathrm{drw}_A(\sigma, x) = [1, 3, 4, 5, 2, 6]$ and $\mathrm{dinv}'(\sigma, x) = 3$. Furthermore its image under the zeta map $\zeta(\sigma, x) \in \mathrm{Diag}(A_5)$ with $\mathrm{area}' \circ \zeta(\sigma, x) = 3$.

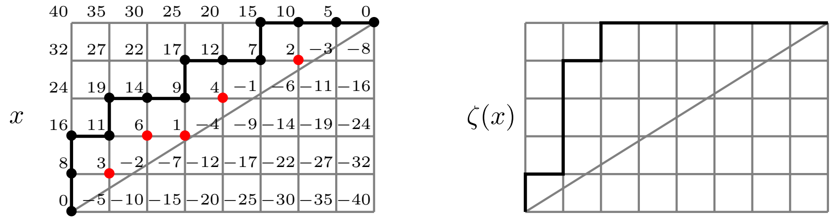


FIGURE 2.11. A Dyck path $x \in \mathfrak{D}_{5,8}$ and its image under the zeta map $\zeta(x) \in \mathfrak{D}_{5,8}$. The permutation used to sort the steps is $u = [1, 13, 10, 2, 7, 12, 4, 9, 6, 11, 3, 8, 5] \in \mathfrak{S}_{13}$. We have $\operatorname{area}(x) = 5$.

PROOF. It is easy to see that $\zeta_A : \mathfrak{S}_n \times \mathfrak{D}_n \to \mathfrak{S}_n \times \mathfrak{D}_n$ is bijective since $\zeta_A : \mathfrak{D}_n \to \mathfrak{D}_n$ is bijective. Lemma 2.6.1 guarantees that $(\sigma, x) \in \operatorname{Vert}(A_{n-1})$ if and only if $\zeta_A(\sigma, x) \in \operatorname{Diag}(A_{n-1})$. The proof of the statement on the statistics is omitted here. It can be verified in the spirit of Theorem 2.5.2 and Lemma 2.6.1. ■

As a consequence we obtain the second combinatorial interpretation of the Hilbert series of diagonal harmonics as

$$\mathscr{C}_n(q,t) = \sum_{(\sigma,x)\in\operatorname{Diag}(A_{n-1})} q^{\operatorname{area}'(\sigma,x)} t^{\operatorname{bounce}(\sigma,x)}.$$

The equivalence of the combinatorial and the algebraic definitions of $\mathscr{C}_n(q,t)$ was established only recently when Carlsson and Mellit proved the compositional shuffle conjecture [**19**]. For more information on related conjectures (open and proved) see for example [**36, 38, 39, 82**].

Note that, as in previous section, there remains the problem of finding a combinatorial explanation for the symmetry $\mathscr{C}_n(q,t) = \mathscr{C}_n(t,q)$.

2.7. Rational q,t-Catalan numbers

The area statistic generalises naturally to the rational case. Define the statistic $\operatorname{area} : \mathfrak{D}_{n,p} \to \mathbb{N}$ by letting $\operatorname{area}(x)$ equal the number of points $(i,j) \in \mathbb{Z}^2$ below the path x such that the weight $w_n^p(i,j)$ is positive.

Furthermore given a lattice path $x \in \mathcal{L}_{p,n}$, recall that the weights $w_n^p(z_{i-1})$ of the lattice points visited by x, where $i \in [n+p]$, are pairwise distinct as in the proof of Theorem 2.2.2. Therefore there exists a unique permutation $u \in \mathfrak{S}_n$ such that $w_n^p(z_{u(i)-1}) < w_n^p(z_{u(i+1)-1})$ for all $i \in [n+p]$. Define the *rational zeta map*, which is also called *sweep map*, $\zeta : \mathcal{L}_{p,n} \to \mathcal{L}_{p,n}$ by mapping x to the path

$$\zeta(x) = s_{u(1)} s_{u(2)} \cdots s_{u(n+p)},$$

where s_i denotes the i-th step of x. This definition of the rational zeta map is due to Armstrong, Loehr and Warrington [**7**]. They also discuss multiple equivalent descriptions that have appeared in the literature, some of which we will treat in later chapters.

Note that a cyclic shift of the steps does not change the image of a path under the zeta map. That is, ζ is constant on the orbits of $\mathcal{L}_{p,n}$ under the action of the cyclic group of order $n+p$.

THEOREM 2.7.1. *The zeta map restricts to a bijection $\zeta : \mathfrak{D}_{n,p} \to \mathfrak{D}_{n,p}$.*

Armstrong, Loehr and Warrington [**7**, Prop. 3.2, Conj 3.3] showed that the sweep map sends Dyck paths to Dyck paths, and conjectured its bijectivity. Thomas and Williams [**79**] recently proved that the sweep map is a bijection.

It is not too difficult to show that the rational zeta map generalises the map ζ_A on Dyck paths defined in Section 2.5.

PROPOSITION 2.7.2. *The zeta map and the rational zeta map agree under the identification of* $\mathfrak{D}_n$ *and* $\mathfrak{D}_{n,n+1}$. *That is,* $\zeta_A(x)\mathbf{e} = \zeta(x\mathbf{e})$ *for all* $x \in \mathfrak{D}_n$.

The proof of Proposition 2.7.2 is omitted. The interested reader may wish to compare with the proof of Theorem 6.3.7, which is very similar in style.

In analogy to Section 2.5 we define the *rational* q,t*-Catalan numbers* as the generating polynomials of rational Dyck paths with respect to the area statistic and its push forward under the sweep map.

$$C_{n,p}(q,t) = \sum_{x \in \mathfrak{D}_{n,p}} q^{\mathrm{area}(x)} t^{\mathrm{area}(\zeta(x))}$$

Rational q,t-Catalan numbers and related conjectures were introduced for example in [**32, 6, 7, 33**]. Often their definition serves as a motivation for the introduction of a map related to the sweep map in some form.

Also the rational q,t-Catalan polynomials exhibit a symmetry in the variables q and t. The only known proof of this fact relies on the proof of the rational shuffle conjecture due to Mellit [**52**], which provides an equivalent algebraic definition of the polynomials $C_{n,p}(q,t)$.

We shall encounter rational q,t-Cayley numbers later in this thesis. For now our investigation of the Catalan-cube using Dyck paths comes to an end as we turn to combinatorial objects of a different flavour and prepare to bring Weyl groups into the picture.

CHAPTER 3

Core partitions

Core partitions are a special set of integer partitions and were originally introduced in the modular representation theory of the symmetric group. See for example [**45**, Chap. 2]. Recently cores and in particular simultaneous cores have experienced a resurgence in combinatorics.

Lascoux [**51**] made the connection between cores and affine permutations. Cores naturally appeared in the attempt to attack the positivity conjecture for Macdonald polynomials by introducing k-Schur functions. See for example the exposition in [**48**]. Armstrong, Hanusa and Jones [**6**] popularised the study of the finite set of simultaneous cores. For example, the average size and the poset structure with respect to inclusion have been of interest. We refer to [**77, 25**] and the references therein.

In Section 3.1 cores are defined and equipped with an action of the affine symmetric group following the ideas of Lascoux. In Section 3.2 we expand on the results of Section 3.1 by showing how the length function of the affine symmetric group fits into the picture. Section 3.3 is devoted to the relation between simultaneous cores, rational Dyck paths and their counterparts inside the affine symmetric group. In Section 3.4 we treat a statistic on core partitions called skew-length, which was introduced by Armstrong, Hanusa and Jones in order to define q,t-Catalan numbers via cores. Here the exposition follows [**73**].

Most results collected in this chapter are already known, except for Section 3.4 where a number of new results and conjectures are presented.

3.1. Cores and the affine symmetric group

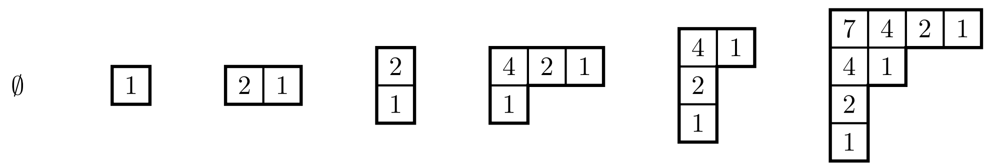


FIGURE 3.1. All seven partitions that are both 3-cores and 5-cores with their hook-lengths filled in.

Let $n \in \mathbb{N}$. Recall that the *hook-length* $h_\lambda(x)$ of a cell x in the Young diagram of a partition λ is defined as the number of cells in the same row and east of x, plus the number of cells in the same column and south of x, plus one (for the cell itself). A partition λ is called an *n-core* if no cell of λ has hook-length equal to n. Let $\mathfrak{C}_n$ denote the set of all n-cores. For example, Figure 3.1 shows all the partitions that are both a 3-core and a 5-core.

LEMMA 3.1.1. *Let κ be a partition. Then the following are equivalent: (i) The partition κ is an n-core. (ii) No rim-hook of κ has length divisible by n. (iii) No cell in κ has hook-length divisible by n.*

PROOF. Dividing a rim-hook of length divisible by n into connected subsets of size n yields at least one rim hook of length n. See Figure 3.2. A rim hook of length m corresponds to a cell

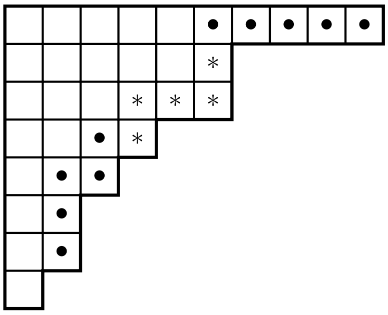


FIGURE 3.2. The existence of a rim-hook of length fifteen implies the existence of a rim-hook of length five.

with hook-length m, so (i) implies (ii) and (ii) implies (iii). Finally, (i) is a trivial consequence of (iii). ■

Define functions $c : \mathbb{Z}^2 \to \mathbb{Z}$ and $r_n : \mathbb{Z}^2 \to \mathbb{Z}/n\mathbb{Z}$ as follows. The *content* of a cell $x = (i, j)$ in the i-th row and the j-th column is defined as $c(x) = j - i$. The *n-residue* $r_n(x)$ of a cell is defined as its content modulo n. See Figure 3.3. A cell x is called *addable* for λ if $x \notin \lambda$ and $\lambda \cup \{x\}$ is again the Young diagram of a partition. A cell $x \in \lambda$ is called *removable* for λ if $\lambda - \{x\}$ is again the Young diagram of a partition.

LEMMA 3.1.2. *An n-core cannot have an addable cell and a removable cell with the same n-residue.*

PROOF. Let x, y be two cells with n-residue i, and assume that x is addable and y removable for a partition λ. Any rim hook h with head $\alpha(h) = x$ and tail $\omega(h) = y$ has length $mn + 1$ for some $m \geq 1$. Thus λ has a rim hook of length mn and cannot be an n-core by Lemma 3.1.1. ■

We now state the two fundamental results of this section relating cores and the affine symmetric group, both of which are due to Lascoux [**51**].

THEOREM 3.1.3. *The affine symmetric group $\tilde{\mathfrak{S}}_n$ acts on the set $\mathfrak{C}_n$ via the following rules: Let κ be an n-core and s_i^A a simple transposition. (i) If there are any addable cells for κ of n-residue i, then $s_i^A \cdot \kappa$ is obtained from κ by adding all of them. (ii) If instead there are any removable cells of n-residue i, then $s_i^A \cdot \kappa$ is obtained from κ by removing all of them. (iii) Otherwise κ is left invariant.*

An example for the action defined in Theorem 3.1.3 is found in Figure 3.4.

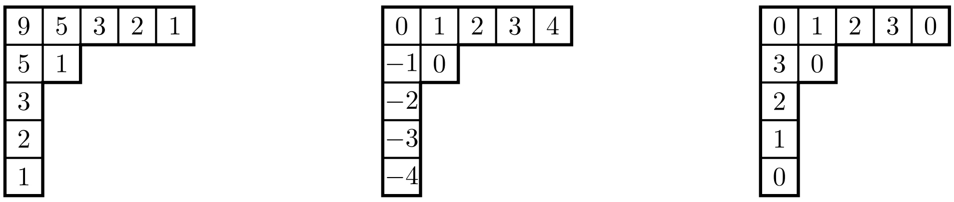


FIGURE 3.3. The 4-core $\kappa = (5, 2, 1^3)$ filled with its hook-lengths (left), its contents (middle), and its 4-residues (right).

$$\kappa = s_2^A \cdot \kappa \quad \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 0 \\ \hline 3 & 0 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \qquad s_1^A \cdot \kappa \quad \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 0 & 1 \\ \hline 3 & 0 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \qquad s_4^A \cdot \kappa \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

FIGURE 3.4. A 4-core κ and its images under the action by three simple transpositions.

THEOREM 3.1.4. *The action of the affine symmetric group $\widetilde{\mathfrak{S}}_n$ on $\mathfrak{C}_n$ is isomorphic to the canonical action on the left cosets $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$. In particular, the symmetric group $\mathfrak{S}_n$ is the stabiliser subgroup of the empty partition, and the induced map $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n \to \mathfrak{C}_n$ sending each affine Graßmannian permutation ω to $\omega \cdot \emptyset$ is a bijection.*

We first attempt a direct proof of Theorem 3.1.3 using the understanding of n-cores we have developed so far. However, this turns out to be not fully satisfactory. Below a different approach involving other combinatorial objects is demonstrated that proves to be much more insightful and extends naturally to a proof of Theorem 3.1.4.

PROOF OF THEOREM 3.1.3 (SKETCH). By Lemma 3.1.2 $s_i^A \cdot \kappa$ is a well-defined partition. We show that $s_i^A \cdot \kappa$ is an n-core. Suppose we are in case (i). If $s_i^A \cdot \kappa$ is not an n-core then it contains a rim hook h of length n. Since κ is an n-core, h contains a newly added cell x of residue i, and x is the head or the tail of h. Without loss of generality assume that x is the head of h. Then the tail of h is adjacent to an addable cell of residue i which is a contradiction.
Similarly, in case (ii) if $s_i^A \cdot \kappa$ contains a rim hook h of length n then the head (or tail) of h must be adjacent to a cell x which was removed from κ. But then the tail (respectively the head) of h is removable and of the same residue as x.
It remains to show that the rules (i)–(iii) define a group action. The obvious idea is to check that the relations between the generators s_i^A are compatible with these rules. Clearly, $s_i^A \cdot (s_i^A \cdot \kappa) = \kappa$ and $s_i^A \cdot (s_j^A \cdot \kappa) = s_j^A \cdot (s_i^A \cdot \kappa)$ when $|j - i| > 2$ modulo n. To show that the braid relations

$$s_i^A \cdot (s_{i+1}^A \cdot (s_i^A \cdot \kappa)) = s_{i+1}^A \cdot (s_i^A \cdot (s_{i+1}^A \cdot \kappa))$$

are fulfilled one needs to distinguish a lot of cases. While this can be done, it is cumbersome and we do not attempt it here. ∎

Instead we want to develop a different description of n-cores in terms of abaci.
A subset $A \subseteq \mathbb{Z}$ is called *abacus* if there exist integers $a, b \in \mathbb{Z}$ such that $z \in A$ for all z with $z < a$, and $z \notin A$ for all z with $z > b$. The elements of A are called *beads* and the elements of $\mathbb{Z} - A$ are called *gaps*. We visualise an abacus as an arrangement of k columns called *runners* labelled $1, \ldots, k$. The runner i contains the integers $z_{ij} = jk + i$ for $j \in \mathbb{Z}$. The *level* of z_{ij} is defined to be $j + 1$, such that the number 0 appears in level zero. We draw the runners in increasing order from left to right and arrange their elements in increasing order from top to bottom. That is, imagine $-\infty$ at the top and ∞ at the bottom, such that numbers with the same level are horizontally aligned. Finally the beads of A are circled.
An abacus is *normalised* if zero is a gap and there are no negative gaps. An abacus is *balanced* if the number of positive beads equals the number of non-positive gaps, that is,

$$\#(A \cap \mathbb{N}^+) = \#(\mathbb{Z} - (A \cup \mathbb{N}^+)).$$

An abacus A is *n-flush* if $z - n \in A$ for all $z \in A$.

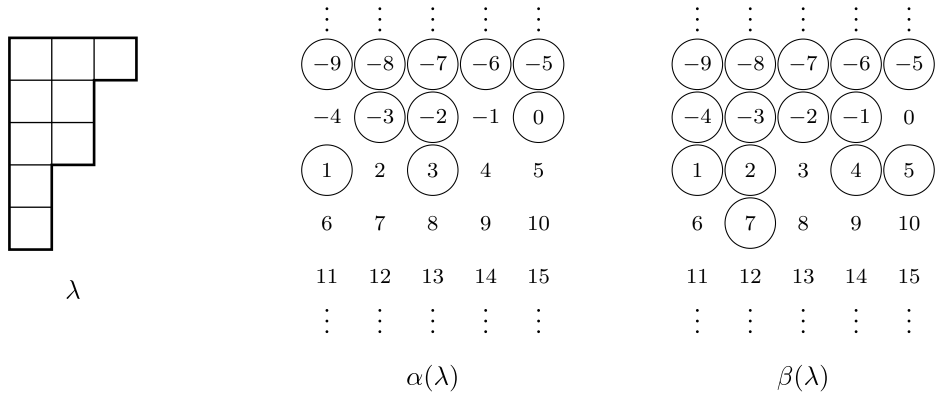


FIGURE 3.5. The partition $\lambda = (3, 2, 2, 1, 1)$ and the corresponding balanced abacus $\alpha(\lambda)$ and normalised abacus $\beta(\lambda)$. Note that $\alpha(\lambda)$ is not 5-flush, however, it is, for example, 6-flush.

Note that the formal definition of an abacus does not a priori determine the number of runners. If we want to emphasise a fixed number of runners we shall call A an abacus on k runners or simply a k-abacus.

The theorem below is a version of the classical result that n-cores correspond to abacus diagrams that are n-flush (see for example [**45**, 2.7.13]). Let λ be a partition and H denote the set of hook-lengths appearing in the first column of λ. Define two maps $\alpha : \Pi \to \mathscr{P}(\mathbb{Z})$, $\beta : \Pi \to \mathscr{P}(\mathbb{Z})$ via

$$\alpha(\lambda) = \{\lambda_i - i + 1 : i \geq 1\} \qquad \text{and} \qquad \beta(\lambda) = H \cup \{z \in \mathbb{Z} : z < 0\},$$

where by convention $\lambda_i = 0$ when $i > \ell(\lambda)$. Note that $\beta(\lambda) = \{z + \ell(\lambda) - 1 : z \in \alpha(\lambda)\}$. See Figure 3.5 for an example.

THEOREM 3.1.5. *The map α is a bijection between partitions and balanced abaci. The map β is a bijection between partitions and normalised abaci. Furthermore, the following are equivalent: (i) λ is an n-core. (ii) $\alpha(\lambda)$ is n-flush. (iii) $\beta(\lambda)$ is n-flush.*

PROOF. Clearly $A = \alpha(\lambda)$ and $B = \beta(\lambda)$ are abaci. The abacus B is normalised by definition, contains $\ell(\lambda)$ positive beads, and β is a bijection. Since A is obtained from B by subtracting $\ell(\lambda) - 1$ from each bead it must be balanced, and α is bijective as well.

The equivalence of (ii) and (iii) is obvious, thus it suffices to show that (i) and (ii) are equivalent. The key idea is pictured in Figure 3.6. Assume that λ contains a rim-hook h of length n with head in row i and tail in row $i + d$. Let μ be the partition obtained from λ by removing h. Then

$$\mu_i = \lambda_{i+1} - 1, \quad \mu_{i+1} = \lambda_{i+2} - 1, \quad \ldots, \quad \mu_{i+d-1} = \lambda_{i+d} - 1$$

and $\mu_{i+d} = \lambda_i - n + d$. Thus, $\alpha(\mu)$ is obtained from A by replacing $\lambda_i - i + 1$ by

$$(\lambda_i - n + d) - (i + d) + 1 = (\lambda_i - i + 1) - n.$$

This corresponds to moving the bead $\lambda_i - i + 1$ upward by one level. Conversely, if a bead $b \in A$ *can* be moved upward because $b - n \notin A$, then let

$$b = \lambda_i - i + 1, \quad \ldots, \quad \lambda_{i+d} - (i + d) + 1$$

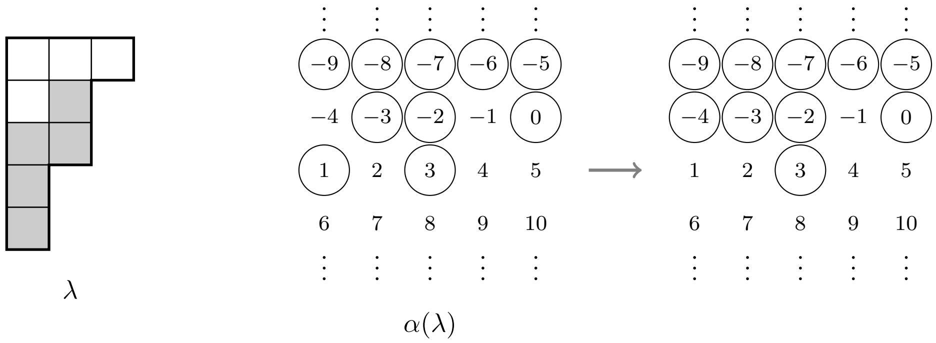


FIGURE 3.6. Removing a rim-hook of length five from the partition $\lambda = (3,2,2,1,1)$ corresponds to moving the bead $b = 1$ in the balanced abacus $\alpha(\lambda)$ up by one level. The resulting partition is a 5-core, and the resulting abacus is 5-flush.

be all beads between b and $b-n$. To show that there is a rim-hook of length n with head in row i and tail in row $i+d$ it suffices to show that $\lambda_{i+d+1} \leq \lambda_i - n + d$. But this is equivalent to

$$\lambda_{i+d+1} - (i+d+1) + 1 < (\lambda_i - n + d + 1) - i - d = (\lambda_i - i + 1) - n$$

which is true by assumption. ∎

From the above proof we immediately get another surprising result.

COROLLARY 3.1.6. [**45**, 2.7.16] *For each partition λ there exists a unique n-core which can be obtained by successively removing rim-hooks of length n.* ∎

We now carry the action of $\widetilde{\mathfrak{S}}_n$ on the coroot lattice $\check{Q}$ over to balanced n-flush abaci. See Figure 3.7.

THEOREM 3.1.7. *The canonical action of $\widetilde{\mathfrak{S}}_n$ on left cosets $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$ is isomorphic to the following action on balanced n-flush abaci.*
Let A be such an abacus on n runners. Then $s_i^A \cdot A$ is obtained from A in the following way: For all integers $z = i + jn$ compare z and $z+1$. If one is a gap and the other is a bead then exchange the roles of the two.

PROOF. Let A be an n-flush abacus on n runners and let $x_1, \dots, x_n$ denote the levels of the lowest bead in each runner. Since A is balanced if and only if $\sum x_i = 0$ the map ϕ defined by $A \mapsto (x_1, \dots, x_k)$ is a bijection from the set of balanced n-flush abaci to the coroot lattice $\check{Q}$. It suffices to verify that $s_i^A \cdot \phi(A) = \phi(s_i^A \cdot A)$.
Clearly, if $i \in [n-1]$ then applying s_i^A to A exchanges the levels x_i and $x_i + 1$ (since z and $z+1$ are in the same level). On the other hand, applying s_n^A to A exchanges x_1 and x_n and then removes one bead in runner n and adds one bead in runner 1 (since here z appears in the level above $z+1$). This corresponds exactly to the action of $\widetilde{\mathfrak{S}}_n$ on $\check{Q}$, which is isomorphic to the action on left cosets. ∎

The obtained correspondence between Graßmannian affine permutations and n-flush abaci can easily be made explicit by mapping ω to the abacus

$$\omega \cdot \{z \in \mathbb{Z} : z \leq 0\} = \{\omega(z) : -z \in \mathbb{N}\}.$$

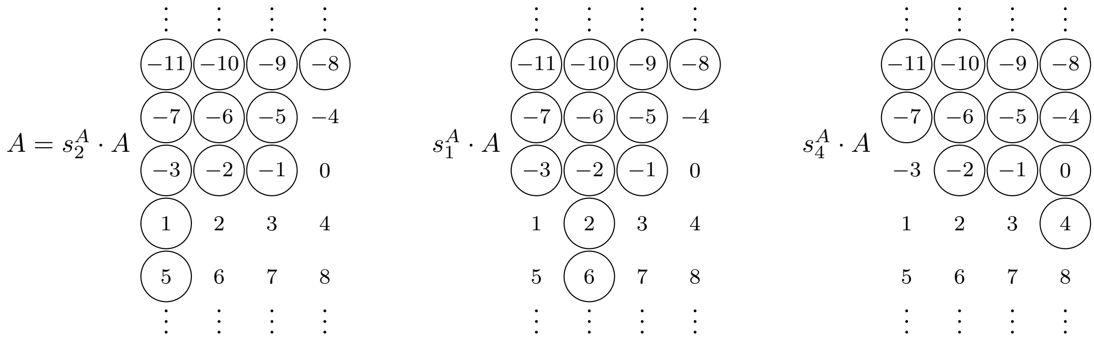


FIGURE 3.7. The balanced 4-flush abacus $A = \alpha(\kappa)$ corresponding to the 4-core κ from Figure 3.4 and its images under the action of three simple transpositions.

We also formulate this bijection in terms of dominant affine permutations. Define $\gamma : \widetilde{\mathfrak{S}}_n \to \mathscr{P}(\mathbb{Z})$ by

$$\gamma(\omega) = \{z \in \mathbb{Z} : \omega(z) \leq 0\}.$$

PROPOSITION 3.1.8. *The map γ restricts to a bijection from dominant affine permutations to balanced n-flush abaci. If $\gamma(\omega)$ is viewed as an n-abacus then the set of minimal gaps in each runner of $\gamma(\omega)$ equals $\{\omega^{-1}(1), \ldots, \omega^{-1}(n)\}$, that is, the minimal gaps of $\gamma(\omega)$ constitute the window of the Graßmannian affine permutation ω^{-1}. The levels $x_1, \ldots, x_n$ of the maximal beads in each runner of $\gamma(\omega)$ are given by $(x_1, \ldots, x_n) = \omega^{-1} \cdot 0$.* ■

See Figure 3.8 for an example.

By now all the necessary tools are at our disposal.

PROOF OF THEOREMS 3.1.3 AND 3.1.4. Due to Theorem 3.1.5 and Theorem 3.1.7 it suffices to verify that $\alpha(s_i^A \cdot \kappa) = s_i^A \cdot \alpha(\kappa)$ for all n-cores $\kappa \in \mathfrak{C}_n$.

Let $b_j = \kappa_j - j + 1$. There is an addable cell u of residue i in the j-th row of κ if and only if $b_j \equiv i \mod n$ and $b_j + 1 \notin \alpha(\kappa)$ (either because $j = 1$ in which case b_j is the largest bead of $\alpha(\kappa)$, or because $\kappa_{j-1} > \kappa_j$ when $b_{j-1} \geq b_j + 2$). Adding the cell u increases the length of row j by one, which corresponds to moving the bead b_j from runner i to runner $i+1$.

There is a removable cell u of residue i in the j-the row of κ if and only if $b_j \equiv i + 1 \mod n$ and $b_j - 1 \notin \alpha(\kappa)$ (as $\kappa_{j+1} < \kappa_j$ implies $b_{j+1} \leq b_j - 2$). Removing the cell u decreases the length of row j by one, which corresponds to moving the bead b_j from runner $i+1$ to runner i.

We conclude that indeed $\alpha(s_i^A \cdot \kappa) = s_i^A \cdot \beta(\kappa)$ and the claims follow. ■

We end this section with a result connecting conjugation on cores to the involutive automorphism on affine permutations. Clearly $\kappa \in \mathfrak{C}_n$ if and only if $\kappa' \in \mathfrak{C}_n$.

PROPOSITION 3.1.9. [**73**, Prop. 2.17] *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be a dominant affine permutation. Then $\alpha^{-1} \circ \gamma(\omega^*)$ is the conjugate partition of $\alpha^{-1} \circ \gamma(\omega)$.*

PROOF. This is best understood using a different description of the map $\alpha^{-1} \circ \gamma$. Following [**48**, Sec. 1.2] we read the one-line notation of the dominant affine permutation ω from left to right, drawing a north step for each encountered non-positive number and an east step for each encountered positive number. The resulting path P outlines the south-west boundary of the partition $\alpha^{-1} \circ \gamma(\omega)$.

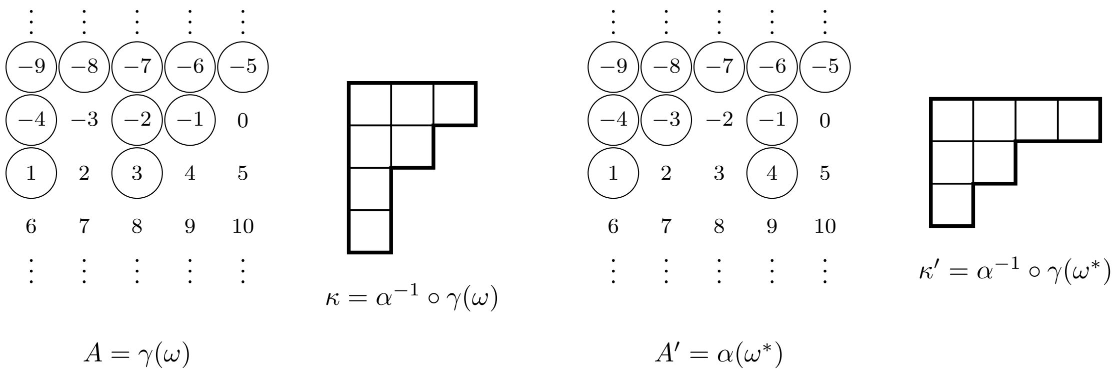


FIGURE 3.8. The balanced 5-flush abaci A, A' and the 5-cores κ, κ'corresponding to the dominant affine permutations $\omega = [-1, 6, 0, 3, 7]$ and $\omega^* = [-1, 3, 6, 0, 7]$. Note that the window of $\omega^{-1} = [-3, 0, 4, 6, 8]$ consists of the minimal gaps in each runner of A, and $\omega^{-1}(0) = (1, -1, 1, 0, -1)$ agrees with the levels of the maximal beads in each runner of A.

By Lemma 1.4.5 (i) $(\omega^{-1})^*(i) = 1 - \omega^{-1}(1-i)$ Hence $(\omega^{-1})^*(i) \leq 0$ if and only if $\omega^{-1}(1-i)$ is positive. Reading the one-line notation of $(\omega^{-1})^*$ from left to right and drawing a path as prescribed therefore yields the reverse path of P with north and east step exchanged. ■

See Figure 3.8.

3.2. Affine inversions

In this section we strengthen Theorem 3.1.4 by taking the length function of $\widetilde{\mathfrak{S}}_n$ into account, thereby deepening our understanding of the inversion statistic on the affine symmetric group.

Let $\omega \in \widetilde{\mathfrak{S}}_n$ be an affine permutation. An *affine inversion* of ω is a pair $(i, j) \in [n] \times \mathbb{N}$ such that $i < j$ and $\omega(i) > \omega(j)$. Note that this definition coincides with the usual definition of inversions if $\omega \in \mathfrak{S}_n$. Clearly each affine permutation has only finitely many affine inversions. We may thus define the statistic $\mathrm{inv} : \widetilde{\mathfrak{S}}_n \to \mathbb{N}$ by letting $\mathrm{inv}(\omega)$ denote the number of affine inversions of ω. Moreover define a refinement of the inversion statistic $\mathbf{inv} : \widetilde{\mathfrak{S}}_n \to \mathbb{N}^n$ by letting $\mathbf{inv}(\omega, i)$ denote the number of inversions of ω of the form $(j, kn + i)$ with $j, k \in \mathbb{N}$. If ω is a Graßmannian affine permutation then $\mathbf{inv}(\omega)$ is decreasing and $\mathbf{inv}(\omega, n) = 0$. This yields a map $\mathbf{inv} : \widetilde{\mathfrak{S}}_n / \mathfrak{S}_n \to \Pi^{n-1}$ from Graßmannian affine permutations to partitions with at most $n - 1$ positive parts.

LEMMA 3.2.1. *Let* $\omega \in \widetilde{\mathfrak{S}}_n$. *Then*

$$\mathrm{inv}(\omega) = \sum_{i,j} \left| \left\lfloor \frac{\omega(j) - \omega(i)}{n} \right\rfloor \right|,$$

where the sum is taken over all $i, j \in [n]$ *such that* $i < j$.

PROOF. Let $i, j \in [n]$ with $i < j$. If $\omega(i) < \omega(j)$ then $(j, kn + i)$ is an affine inversion of ω if and only if $k \geq 1$ and

$$\omega(i + kn) < \omega(j) \Leftrightarrow kn < \omega(j) - \omega(i) \Leftrightarrow k \leq \left\lfloor \frac{\omega(j) - \omega(i)}{n} \right\rfloor.$$

On the other hand if $\omega(i) > \omega(j)$ then $(i, kn+j)$ is an affine inversion of ω if and only if $k \geq 0$ and

$$k \leq \left\lfloor \frac{\omega(i)-\omega(j)}{n} \right\rfloor = \left| \left\lfloor \frac{\omega(j)-\omega(i)}{n} \right\rfloor \right| - 1.$$

■

LEMMA 3.2.2. *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be a Graßmannian affine permutation and choose $x \in \check{Q}$ and $s \in \mathfrak{S}_n$ such that $\omega = t_x s$. Then*

$$\mathbf{inv}(\omega, i) = \sum_{j \in [n], i<j} \left\lfloor \frac{\omega(j)-\omega(i)}{n} \right\rfloor = \sum_{j\in[n], x_k < x_j} \left(x_j - x_k + \left\lfloor \frac{j-k}{n} \right\rfloor \right),$$

where $k = s(i)$.

PROOF. If ω is Graßmannian then we are always in the first case in the proof of Lemma 3.2.1. The claim follows from a simple computation. ■

The numbers $\left|\left\lfloor \frac{\omega(j)-\omega(i)}{n} \right\rfloor\right|$ were first considered by Shi who connected the inversion statistic to the length function by proving Theorem 3.2.3 below. This theorem was rediscovered by Björner and Brenti who also studied the affine symmetric group via affine inversions. Lemma 3.2.1 should be compared to the discussion preceding [**17**, Prop. 8.3.1].

The next theorem extends a well-known fact about permutations, namely that their length equals their number of inversions, to the affine symmetric group.

THEOREM 3.2.3. [**60**, Lem. 4.2.2] *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be an affine permutation. Then* $\mathrm{inv}(\omega) = \ell(\omega)$.

Lemma 3.2.2 together with Lemma 1.4.4 reveal the effect that acting from the left by a simple transposition has on the affine inversions of Graßmannian affine permutations.

LEMMA 3.2.4. *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be a Graßmannian affine permutation, $x \in \check{Q}$ and $s \in \mathfrak{S}_n$ such that $\omega = t_x s$.*

(i) For all $i \in [n-1]$

$$\mathbf{inv}(\omega', j) = \begin{cases} \mathbf{inv}(\omega, j) + 1 & \text{if } s(j) = i+1 \text{ and } x_{i+1} < x_i, \\ \mathbf{inv}(\omega, j) - 1 & \text{if } s(j) = i \text{ and } x_i < x_{i+1}, \\ \mathbf{inv}(\omega, j) & \text{otherwise,} \end{cases}$$

where ω' is the Graßmannian representative of the coset $s_i^A \omega \mathfrak{S}_n$.

(ii) Furthermore,

$$\mathbf{inv}(\omega', j) = \begin{cases} \mathbf{inv}(\omega, j) + 1 & \text{if } s(j) = 1 \text{ and } x_1 < x_n + 1, \\ \mathbf{inv}(\omega, j) - 1 & \text{if } s(j) = n \text{ and } x_n < x_1 - 1, \\ \mathbf{inv}(\omega, j) & \text{otherwise,} \end{cases}$$

where ω' is the Graßmannian representative of the coset $s_n^A \omega \mathfrak{S}_n$.

PROOF. First consider (i). Set $y = s_i^A \cdot x$. The claim is trivial if $y = x$ thus assume the contrary. Lemma 1.4.4 yields $\omega' = t_y s_i^A s$. Set $J = s(j)$ and $k = s_i^A(J)$ and compute

$$\begin{aligned} \mathbf{inv}(\omega', j) &= \sum_{\ell \in [n], y_k < y_\ell} \left(y_\ell - y_k + \left\lfloor \frac{\ell - k}{n} \right\rfloor \right) = \sum_{\ell\in[n], x_J < x_\ell} \left(x_\ell - x_J + \left\lfloor \frac{s_i^A(\ell) - s_i^A(J)}{n} \right\rfloor \right) \\ &= \mathbf{inv}(\omega, j) + \sum_{\ell \in [n], x_J < x_\ell} \left(\left\lfloor \frac{s_i^A(\ell) - s_i^A(J)}{n} \right\rfloor - \left\lfloor \frac{\ell - J}{n} \right\rfloor \right). \end{aligned}$$

κ $\delta(\kappa)$

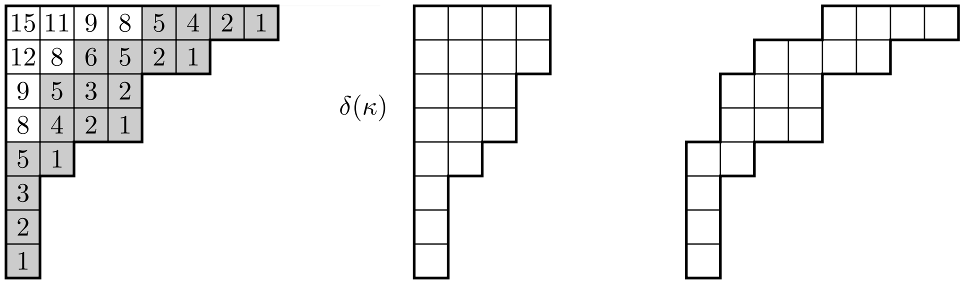


FIGURE 3.9. A 7-core κ (left), its bounded partition $\delta(\kappa)$ (middle), and the reconstruction of the core (right). We have $|\kappa|_7 = 19$.

The first claim follows since

$$\left\lfloor \frac{s_i^A(\ell) - s_i^A(J)}{n} \right\rfloor - \left\lfloor \frac{\ell - J}{n} \right\rfloor = 0$$

for all $\ell, J \in [n]$ with $\ell \neq J$ unless $\{\ell, J\} = \{i, i+1\}$.

The proof of (ii) is completely analogous but the computations are more tedious. ∎

The inversion statistic on affine permutations has a nice counterpart in the world of cores. For any partition λ define the *n-size* $|\lambda|_n$ as the number of cells with hook-length at most n.

Define a map $\delta : \mathfrak{C}_n \to \Pi_{\leq n-1}$ from core partitions to bounded partitions as follows. For any n-core κ let $\delta(\kappa) = \lambda$ where λ_i is defined as the number of cells in the i-th row of κ that have a hook-length less then n. See Figure 3.9.

PROPOSITION 3.2.5. [**50**, Thm. 7] *The map $\delta : \mathfrak{C}_n \to \Pi_{\leq n-1}$ is a bijection between n-cores and partitions with parts at most $n-1$, such that $|\kappa|_n = |\delta(\kappa)|$.*

PROOF. The only non-trivial claim is that δ is bijective. We will describe the inverse map. Given $\lambda \in \Pi_{\leq n-1}$, the core κ is obtained as follows. Take λ and leave its last row unchanged. Then shift all other rows east until the cells in the second to last row have hook-lengths less than n in the resulting skew diagram. Repeat this shifting procedure for each row of λ. If μ/ν is the obtained skew diagram then $\kappa = \mu$ is the desired n-core. ∎

The following result keeps track of the effect that acting by an affine permutation has on the bounded partition $\delta(\kappa)$.

LEMMA 3.2.6. *Let κ be an n-core such that $s_i^A \cdot \kappa$ is obtained from κ by adding cells of n-residue i. Let $x = (r, s)$ be the cell added to κ such that s is minimal. Then $\delta(s_i \cdot \kappa)$ is obtained from $\delta(\kappa)$ by adding a single cell in row r.*

PROOF. Let $x_1, \dots, x_d$ be the addable cells for κ of n-residue i ordered from west to east, that is, $x = x_1$. Then the rim-hook connecting two consecutive cells x_j and x_{j+1} has length $(n-1) + mn$. Since $s_i^A \cdot \kappa$ contains no rim-hook of length n and no addable cells of n-residue i, we must have $m = 0$. Thus x_j and x_{j+1} appear in consecutive diagonals of n-residue i.

The rim-hook connecting them corresponds to a cell y_{j+1} with hook-lengths $h_\kappa(y_{j+1}) = n - 1$ and $h_{s_i^A \cdot \kappa}(y_{j+1}) = n + 1$. Moreover it is easy to see that these cells are the *only* cells which have small hook-length in κ but large hook-length in $s_i^A \cdot \kappa$. (Such a cell y must align horizontally and vertically with a newly added cell. In addition, these two cells must lie in consecutive diagonals of n-residue i as otherwise the original hook-length $h_\kappa(y)$ is too large.)

Since y_{j+1} and x_{j+1} appear in the same row, their effects on $\delta(s_i^A \cdot \kappa)$ cancel each other. Hence $\delta(s_i^A \cdot \kappa)$ is obtained from $\delta(\kappa)$ by adding a single cell in row r corresponding to x. ∎

Lemma 3.2.7. *Let κ be an n-core with corresponding abacus $\alpha(\kappa)$ and bounded partition $\lambda = \delta(\kappa)$, and set $b = \kappa_r - r + 1$. Then λ_r equals the number of gaps of $\alpha(\kappa)$ between b and $b - n$.*

Proof. A cell $(r, \kappa_r - n + i + 1)$ with $i \in [n-1]$ has hook-length less than n if and only if $\kappa_{r+i} \leq \kappa_r - n + i$. Hence λ_r equals the number of indices $i \in [n-1]$ such that the bead $b' = \kappa_{r+i} - r - i + 1$ satisfies $b' \leq b - n$. But this is just the number of gaps g of $\alpha(\kappa)$ with $b < g < b - n$. ∎

These preparations lead to the main result of the present section. Theorem 3.2.8 was mentioned in [**50**]. A very compact proof was provided in [**49**, Prop. 8.15].

Theorem 3.2.8. *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be a Graßmannian affine permutation and $\kappa = \omega \cdot \emptyset$ the corresponding n-core. Then the partition $\mathbf{inv}(\omega) \in \Pi^{n-1}$ is the conjugate of $\delta(\kappa) \in \Pi_{\leq n-1}$. In particular, $\ell(\omega) = |\kappa|_n$.*

Proof. Apply induction on the length of ω. Clearly, $\mathbf{inv}(e) = (0, \ldots, 0) = \delta(\emptyset)'$. Thus it suffices to show that $\mathbf{inv}(\omega) = \delta(\kappa)'$ implies $\mathbf{inv}(s_i^A \omega) = \delta(s_i^A \cdot \kappa)'$ whenever $\ell(s_i^A \omega) = \ell(\omega) + 1$.
Let $i \in [n-1]$ and set $x = \omega(0)$. By Lemma 3.2.4 (i) $\mathbf{inv}(s_i^A \omega)$ is obtained from $\mathbf{inv}(\omega)$ by adding a single cell to row j, where $\omega(j) \equiv i + 1$ modulo n, and $x_{i+1} < x_i$. In terms of abaci (see Theorem 3.1.7) this means that beads from runner i are moved to runner $i+1$. By Theorem 3.1.3 $s_i^A \cdot \kappa$ is obtained from κ by adding cells of n-residue i.
If $i = n$ then by Lemma 3.2.4 (ii) the partition $\mathbf{inv}(s_n^A \omega)$ is obtained from $\mathbf{inv}(\omega)$ by adding a single cell to row j, where $\omega(j) \equiv 1$ modulo n, and $x_1 < x_n + 1$. Therefore beads are moved from runner n to runner 1 in the abacus setting, and cells of n-residue 0 are added in the core setting.
Let $u = (r, s)$ be the cell of n-residue i added to κ when s_i^A is applied that minimises s. Let $b = \kappa_r - r + 2$ be the bead of $\alpha(s_i^A \cdot \kappa)$ corresponding to the row of $s_i^A \cdot \kappa$ containing u. Due to Lemmas 3.2.6 and 3.2.7 the partition $\delta(s_i^A \cdot \kappa)$ is obtained from $\delta(\kappa)$ by adding a cell to column m where m is the number of gaps of $\alpha(s_i^A \cdot \kappa)$ between b and $b - n$. But b is just the minimal gap in runner $i + 1$ of $\alpha(\kappa)$. Hence m equals the number of gaps g of $\alpha(\kappa)$ such that $b \leq g < b - n$. This implies $j = m$ where we use Proposition 3.1.8. ∎

We remark that on a similar note cores and their bounded partitions have been used to obtain canonical reduced expressions of Graßmannian affine permutations in terms of simple transpositions [**50**]. Furthermore, similar theories relating permutations to abaci and cores have also been developed for the other infinite families of root systems [**43**].

An immediate consequence of Theorem 3.2.8 is the following.

Corollary 3.2.9. *The induced map*

$$\mathbf{inv} : \widetilde{\mathfrak{S}}_n / \mathfrak{S}_n \to \Pi^{n-1}$$

is a bijection between Graßmannian affine permutations and partitions of length at most $n - 1$. ∎

In the literature sometimes a different convention for encoding the inversions of affine permutations in a vector is preferred. Define a second refinement of the inversion statistic in $\mathbf{inv}^* : \widetilde{\mathfrak{S}}_n \to \mathbb{N}^n$ by letting $\mathbf{inv}^*(\omega, i)$ denote the number of affine inversions of ω of the form (i, j) with $j \in \mathbb{N}$. If ω is Graßmannian then $\mathbf{inv}^*$ is increasing and $\mathbf{inv}^*(\omega, 1) = 0$. Thus we obtain another map $\mathbf{inv}^* : \widetilde{\mathfrak{S}}_n / \mathfrak{S}_n \to \Pi^{n-1}$. It is not difficult to see that $\mathbf{inv}^*(\omega) = \mathbf{inv}(\omega^*)$ and therefore $\mathbf{inv}^*$ induces a bijection between Graßmannian affine permutations and bounded partitions. This result is due to Björner and Brenti [**16**, Thm. 4.4].

The domain of the bijection in Corollary 3.2.9 can be extended to encompass the whole affine symmetric group. This nice consequence can be seen as a generalisation of the fact that each permutation is uniquely determined by its Lehmer code to the affine case.

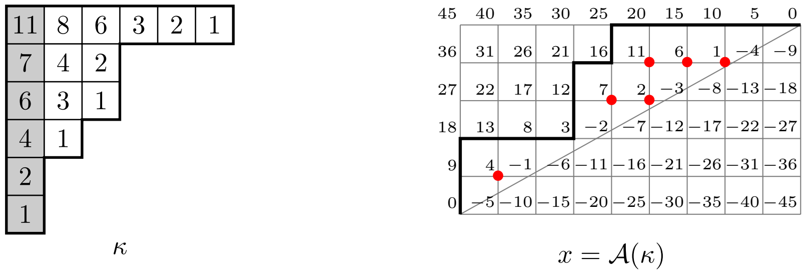


FIGURE 3.10. A simultaneous $5, 9$-core $\kappa \in \mathfrak{C}_{5,9}$ and its image $x \in \mathfrak{D}_{5,9}$ under Anderson's bijection.

THEOREM 3.2.10. [**16**, Thm. 4.6] *The map*

$$\mathbf{inv}^* : \widetilde{\mathfrak{S}}_n \to \big\{(z_1, \ldots, z_n) \in \mathbb{N}^n : z_i = 0 \textit{ for some } i \in [n]\big\}$$

is a bijection between affine permutations and non-positive vectors with non-negative integer entries.

Moreover we obtain an involution on the set of bounded partitions Π^{n-1} by sending $\mathbf{inv}(\omega)$ to $\mathbf{inv}^*(\omega)$. By Proposition 3.1.9 this is equivalent to sending $\delta(\kappa)$ to $\delta(\kappa')$. This involution is called *n-conjugation* in [**48**].

Having different perspectives on this involution can be very useful. For example, every partition in Π^{n-1} can be viewed as a partition in Π^n. It is then trivial from the core setting that n-conjugation "converges" to the ordinary conjugation of partitions as n tends to infinity.

3.3. Simultaneous core partitions

In this section we make the connection between cores and Dyck paths. Given $n, p \in \mathbb{N}$ a partition κ is called *simultaneous n, p-core* or simply *n, p-core* if it is both an n-core and a p-core. Let $\mathfrak{C}_{n,p} = \mathfrak{C}_n \cap \mathfrak{C}_p$ denote the set of simultaneous n, p-cores. By Theorem 3.1.5 simultaneous cores correspond to abaci that are both n-flush and p-flush.

The following construction due to Anderson [**2**] relates n, p-cores to rational slope Dyck paths. Assume that n and p are relatively prime. For any finite set $H = \{h_1, \ldots, h_k\}$ of positive integers there exists a unique partition λ such that H is the set of hook-lengths of the cells in the first column of λ. For $x \in \mathfrak{D}_{n,p}$ let $\varphi(x)$ be the partition such that the set of hook-lengths of the cells in its first column equals the set $H(x)$ of positive weights $w_n^p(i, j) = jp - in$ of lattice points $(i, j) \in \mathbb{Z}^2$ below the path x.

THEOREM 3.3.1. [**2**, Prop. 1] *Let n and p be relatively prime. Then the map $\varphi : \mathfrak{D}_{n,p} \to \mathfrak{C}_{n,p}$ is a bijection. Furthermore* $\operatorname{area}(x) = \ell(\varphi(x))$ *for all $x \in \mathfrak{D}_{n,p}$.*

PROOF. The claim follows from Theorem 3.1.5 since the map sending x to the set H of positive weights of lattice points below x is clearly a bijection from $\mathfrak{D}_{n,p}$ to the set of normalised abaci that are both n-flush and p-flush, and since $\varphi(x) = \beta^{-1}(H(x) \cup \{z \in \mathbb{Z} : z < 0\})$ by definition. ∎

Let $\mathcal{A} : \mathfrak{C}_{n,p} \to \mathfrak{D}_{n,p}$ denote the inverse of the map φ above. An example is found in Figure 3.10. As a consequence it is now easy to enumerate simultaneous cores.

COROLLARY 3.3.2. *The set $\mathfrak{C}_{n,p}$ is finite if and only if n and p are relatively prime. If this is the case then $|\mathfrak{C}_{n,p}| = C_{n,p}$.* ∎

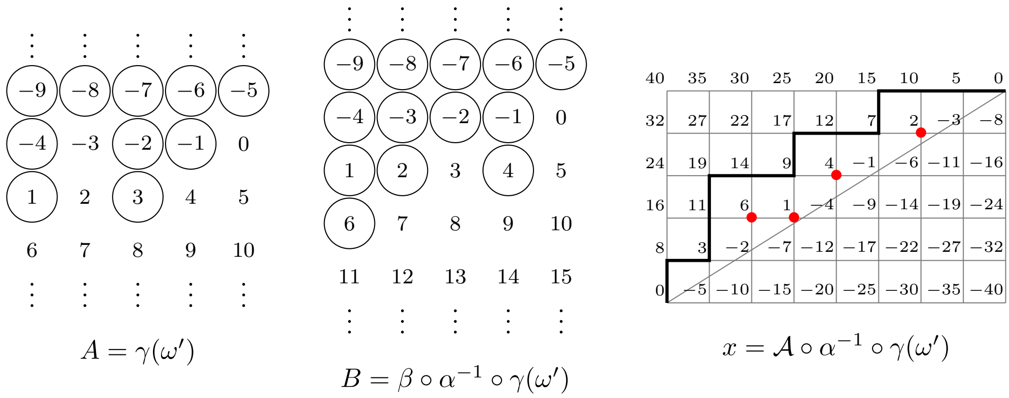


FIGURE 3.11. The balanced abacus A (left), the normalised abacus B (middle) and the rational Dyck path x (right) corresponding to the affine permutation $\omega' = [0, 8, -1, 2, 6] \in \widetilde{\mathfrak{S}}_5^8$. Note that $\omega' = [3, 1, 2, 5, 4] \cdot \omega$, where $\omega = [-1, 6, 0, 3, 7] \in \widetilde{\mathfrak{S}}_5^8$ is dominant, thus A equals $\alpha(\omega)$ from Figure 3.8. We have $M_{\omega'} = -3$, $(g_1, g_2, g_3, g_4, g_5) = (0, 3, 11, 9, 7)$ and $\sigma = [3, 1, 4, 5, 2]$. Hence $(\sigma, x) = \mathcal{A}_A(\omega') \in \mathrm{Vert}(A_4, 8)$.

It is natural to ask for the counterpart of n, p-cores in the setting of the affine symmetric group. The appropriate set of affine permutations was studied by Gorsky, Mazin and Vazirani [**33**]. An affine permutation $\omega \in \widetilde{\mathfrak{S}}_n$ is called *p-stable* if $\omega(i) < \omega(i + p)$ for all $i \in \mathbb{Z}$. Let $\widetilde{\mathfrak{S}}_n^p$ denote the set of all p-stable elements of $\widetilde{\mathfrak{S}}_n$.

PROPOSITION 3.3.3. *Let n and p be relatively prime and $\omega \in \widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$ be a Graßmannian affine permutation. Then $\omega \cdot \emptyset \in \mathfrak{C}_{n,p}$ if and only if $\omega^{-1} \in \widetilde{\mathfrak{S}}_n^p$. That is, simultaneous n, p-cores are in bijection with dominant p-stable elements of $\widetilde{\mathfrak{S}}_n$.*

PROOF. This is an immediate consequence of the results of Section 3.1. ∎

Conjugation of the core also has a nice counterpart on rational Dyck paths under the Anderson map.

PROPOSITION 3.3.4. [**21**, Thm. 4.1] *Let n and p be relatively prime and $\kappa \in \mathfrak{C}_{n,p}$. Then $\mathcal{A}(\kappa') = \rho(\mathcal{A}(\kappa))$, where $\rho : \mathfrak{D}_{n,p} \to \mathfrak{D}_{n,p}$ is the involution of Corollary 2.2.5.*

The proof is simple, but we omit it.

Gorsky, Mazin and Vazirani [**33**, Sec. 3.1] defined a function called *Anderson map* $\mathcal{A}_A : \widetilde{\mathfrak{S}}_n^p \to \mathrm{Vert}(A_{n-1}, p)$ that yields a bijective correspondence between the set of p-stable affine permutations and the set of rational parking functions. Their map should be seen as the Cayley analogue of the bijection of Anderson. Instead of Dyck paths it involves labelled paths. On the other hand, dominant p-stable affine permutations can be seen as orbits of $\widetilde{\mathfrak{S}}_n^p$ under an action of $\mathfrak{S}_n$. See Section 5.4.

Given $\omega \in \widetilde{\mathfrak{S}}_n^p$ define

$$\mathcal{A}_A(\omega) = (\sigma, \mathcal{A} \circ \alpha^{-1} \circ \gamma(\omega)),$$

where $\sigma \in \mathfrak{S}_n$ is defined by $\sigma(i) = \omega(g_i + M_\omega)$, M_ω is the minimal gap of $\gamma(\omega)$ and g_i is the minimal gap of $\beta \circ \alpha^{-1} \circ \gamma(\omega)$ congruent to $(i-1)p$ modulo n. An example is found in Figure 3.11. Note that $\gamma(\omega) = \gamma(\omega')$, where $\omega' \in \mathfrak{S}_n\omega$ is dominant.

THEOREM 3.3.5. [**33**, Thm. 3.4] *Let n and p be relatively prime. Then the map $\mathcal{A}_A : \widetilde{\mathfrak{S}}_n^p \to \mathrm{Vert}(A_{n-1}, p)$ is a bijection.*

Proof (Sketch). Set $\kappa = \alpha^{-1} \circ \gamma(\omega)$, $x = \mathcal{A}(\kappa)$ and $B = \beta(\kappa)$. It is not difficult to see that $\mathcal{A}_A(\omega) = (\sigma, x)$ really is a vertically labelled Dyck path.

First note that

$$B = \{z - M_\omega : z \in \gamma(\omega)\}.$$

Hence if g_i is a minimal gap of B in its residue class modulo n then $g_i + M_\omega$ is a minimal gap of $\gamma(\omega)$. Thus $\omega(g_i + M_\omega) > 0$ but $\omega(g_i + M_\omega - n) \le 0$. It follows that $\sigma(i) \in [n]$ for all $i \in [n]$ and σ is indeed a permutation.

Secondly, if $i \in [n-1]$ is a rise of x then $g_{i+1} = g_i + p$ and therefore $\sigma(i) < \sigma(i+1)$ because ω is assumed to be p-stable.

To see that $\mathcal{A}_A$ is injective let $\sigma \in \mathfrak{S}_n$ and $\omega \in \mathfrak{S}_n^p$ be dominant such that $\sigma\omega \in \mathfrak{S}_n^p$. Then $\mathcal{A}_A(\sigma\omega) = (\sigma\tau, x)$, where $(\tau, x) = \mathcal{A}_A(\omega)$, and the injectivity of $\mathcal{A}_A$ follows from Theorem 3.3.1.

To see that $\mathcal{A}_A$ is surjective it suffices to show that $\sigma\omega \in \mathfrak{S}_n^p$ for all dominant $\omega \in \mathfrak{S}_n^p$ and all $\sigma \in \mathfrak{S}_n$ such that $(\sigma, x) \in \mathrm{Vert}(A_{n-1}, p)$. This requires some work and we omit the details. ∎

Note that $\mathcal{A}_A$ does not map dominant p-stable affine permutations to increasing parking functions. However, the image of the set of dominant stable affine permutations has a simple characterisation using the diagonal reading word in the case $p = n + 1$.

Proposition 3.3.6. *Let $\omega \in \widetilde{\mathfrak{S}}_n^{n+1}$ be dominant. Then $\mathcal{A}_A(\omega)$ is of the form* $(\mathrm{drw}_A(e, x)^{-1}, x)$.

Proof. Let $(\sigma, x) = \mathcal{A}_A(\omega)$ and regard $B = \beta \circ \alpha^{-1} \circ \gamma(\omega)$ as an abacus on n runners. Define $\tau \in \mathfrak{S}_n$ by $\tau(i) \equiv i - 1$ modulo n.

The diagonal reading word $\mathrm{drw}(e, x)$ sorts the minimal gaps of B increasingly. If $\omega = t_q u$ with $q \in \check{Q}$ and $u \in \mathfrak{S}_n$ then u^{-1} sorts the minimal gaps of $\gamma(\omega)$ increasingly. It follows that $\mathrm{drw}(e, x) = \tau^{-M_\omega} u^{-1}$.

On the other hand $\sigma(i) = \omega(g_i + M_\omega)$ where g_i is the minimal gap of B in runner $\tau(i)$. Equivalently $\sigma(i) = \omega(G_i)$ where G_i is the minimal gap of $\gamma(\omega)$ in runner $\tau^{M_\omega}(i)$. Now $G_i = \omega^{-1}(j)$ for some $j \in [n]$, and $\sigma(i) = j$ if and only if $u\tau^{M_\omega}(i) = j$ It follows that $\sigma = u\tau^{M_\omega}$. ∎

As a consequence of Theorem 3.3.5 we obtain that p-stable affine permutations are counted by the rational Cayley numbers.

Corollary 3.3.7. *Let n and p be relatively prime. Then $|\widetilde{\mathfrak{S}}_n^p| = \mathscr{C}_{n,p}$.*

Alternatively we could derive this enumerative result in a different way (see Theorem 4.4.6) and use Corollary 3.3.7 to complete the proof of Theorem 3.3.5.

3.4. The skew-length statistic

Armstrong, Hanusa and Jones [**6**] devised the skew-length statistic to give an alternative definition of the polynomials $C_n(q, t)$ using core partitions.

Let λ be a partition. Given the hook-length h of a cell x in the top row of λ we denote by $H^c(h)$ the set of hook-lengths of cells in the same column as x. Given the hook-length h of a cell x in the first column of λ we denote by $H^r(h)$ the set of hook-lengths of cells in the same row as x. If h is not the hook-length of a suitable cell, set $H^r(h) = \emptyset$ respectively $H^c(h) = \emptyset$. For example, in Figure 3.12 we have $H^c(12) = \{12, 5\}$ and $H^r(12) = \{12, 10, 5, 3, 1\}$ and $H^r(11) = \emptyset$.

For our purposes in this section we need the following simple consequence of Theorem 3.1.5.

Lemma 3.4.1. *Let $\kappa \in \mathfrak{C}_n$ be an n-core and $z \ge 0$. Then $z + n \in H^r(h)$ implies $z \in H^r(h)$, and $z + n \in H^c(h)$ implies $z \in H^c(h)$.*

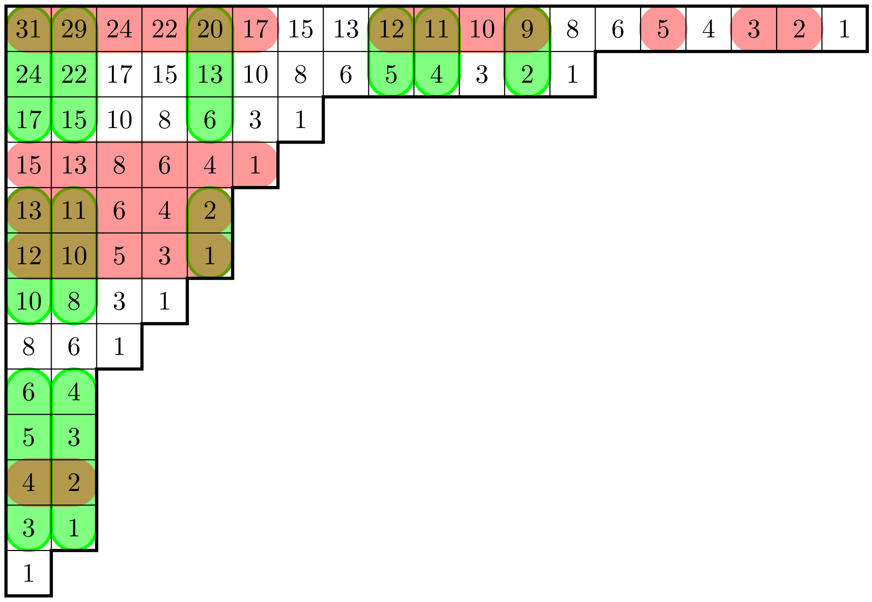


FIGURE 3.12. A simultaneous 7, 16-core $\kappa \in \mathfrak{C}_{7,16}$ with the multisets $H_{7,16}(\kappa)$ in red and $H_{16,7}(\kappa)$ in green.

PROOF. By Theorem 3.1.5 the set of n-cores is characterised by the property that $z + n \in H^c(m)$ implies $z \in H^c(m)$ for all $z \geq 0$, where m is the maximal hook-length of κ. The claim follows since conjugation and deletion of the first column map n-cores to n-cores. ■

Let $\kappa \in \mathfrak{C}_{n,p}$ be a simultaneous core with maximal hook-length m. Moreover choose an element $h \in H^c(m)$. We call the row of κ with maximal hook-length h an *n-row* (respectively a *p-row*) if $h+n \notin H^c(m)$ (respectively $h+p \notin H^c(m)$). Similarly, given $h \in H^r(m)$ we call the column with maximal hook-length h an *n-column* (respectively a *p-column*) if $h + n \notin H^r(m)$ (respectively $h + p \notin H^r(m)$). In Figure 3.12 the maximal hook-lengths of the 7-rows are $31, 15, 13, 12$ and 4, and the maximal hook-lengths of the 7-columns are $31, 29, 20, 12, 11$ and 9.

Define the *skew-length statistic* $\operatorname{skl} : \mathfrak{C}_{n,p} \to \mathbb{N}$ by letting $\operatorname{skl}(\kappa)$ equal the number of cells that are contained in an n-row of κ and have hook-length less than p.

Guoce Xin [**80**] and independently Ceballos, Denton and Hanusa [**21**] recently proved that this statistic fulfils certain symmetry properties in the case where n and p are relatively prime. See Corollaries 3.4.4 and 3.4.5 below. While neither of these results are obvious at first glance, it will become clear in Section 5.4 that these symmetries should be viewed as shadows of the involutive automorphism of $\widetilde{\mathfrak{S}}_n$ and follow from Proposition 3.1.9. However, they are also shadows of a much stronger combinatorial symmetry which only becomes apparent using a different definition of skew-length.

Given $\kappa \in \mathfrak{C}_{n,p}$ denote by $H_{n,p}(\kappa)$ the multiset of hook-lengths of cells that are contained both in an n-row and in a p-column of κ. See Figure 3.12 for an example. The multi-set $H_{n,p}(\kappa)$ allows for a new equivalent definition of $\operatorname{skl}(\kappa)$.

PROPOSITION 3.4.2. [**73**, Prop. 1.4] *Let $\kappa \in \mathfrak{C}_{n,p}$ be an n, p-core. Then* $\operatorname{skl}(\kappa) = \#H_{n,p}(\kappa)$.

PROOF. Fix an n-row with largest hook-length h. On the one hand by Lemma 3.4.1 a cell x in this row has hook-length less than p if and only if $h_\kappa(x)$ is the minimal representative of its residue class modulo p in $H^r(h)$. On the other hand x is contained in a p-column if and only if $h_\kappa(x)$ is the maximal representative of its residue class modulo p in $H^r(h)$. Thus both $\mathrm{skl}(\kappa)$ and $\#H_{n,p}(\kappa)$ count the number of residue classes modulo p with a representative in $H^r(h)$. ■

This section's first main result is a surprising symmetry property of the multiset $H_{n,p}(\kappa)$.

THEOREM 3.4.3. [**73**, Thm. 1.3] *Let $\kappa \in \mathfrak{C}_{n,p}$ be an n,p-core. Then $H_{n,p}(\kappa) = H_{p,n}(\kappa)$.*

PROOF. We prove the claim by induction on the size of κ. Denote by $\tilde{\kappa}$ the partition obtained from κ by deleting the first column. Clearly $\tilde{\kappa} \in \mathfrak{C}_{n,p}$ and we may assume that $H_{n,p}(\tilde{\kappa}) = H_{p,n}(\tilde{\kappa})$.
Let m denote the maximal hook-length in κ. Note that each n-row of $\tilde{\kappa}$ is an n-row of κ. The only p-column of $\tilde{\kappa}$ that is not a p-column of κ has maximal hook-length $m-p$. Thus there exist sets $A \subseteq H^c(m)$ and $B \subseteq H^c(m-p)$ with $H_{n,p}(\kappa) = (H_{n,p}(\tilde{\kappa}) \cup A) - B$. Similarly $H_{p,n}(\kappa) = (H_{p,n}(\tilde{\kappa}) \cup C) - D$ for some sets $C \subseteq H^c(m)$ and $D \subseteq H^c(m-n)$. It suffices to show that $A - B = C - D$ and $B - A = D - C$.
Suppose $z \in A$ but $z \notin B$. Then $z \in H^c(m)$ and $z + n \notin H^c(m)$. On the one hand we obtain $z \notin H^c(m-n)$ and $z \notin D$. It follows that $A \cap D = \emptyset$. On the other hand we obtain $z+n+p \notin H^c(m)$ and therefore $z+n \notin H^c(m-p)$. Since $z \notin B$ this implies $z \notin H^c(m-p)$ and consequently $z+p \notin H^c(m)$. We obtain $z \in C$. Therefore $A - B \subseteq C - D$, and $A - B = C - D$ by symmetry.
Conversely suppose $z \in B$ but $z \notin A$. By symmetry we have $B \cap C = \emptyset$ and $z \notin C$. On the other hand $z + n \notin H^c(m-p)$ implies $z+n+p \notin H^c(m)$ and thus $z + p \notin H^c(m-n)$. Moreover $z \in H^c(m-p)$ implies $z+p \in H^c(m)$ and therefore $z \in H^c(m)$. Since $z \notin A$ we obtain $z + n \in H^c(m)$ and $z \in H^c(m-n)$. We conclude that $z \in D$ and the proof is complete. ■

From Theorem 3.4.3 we immediately recover the two mentioned results due to Guoce Xin and Ceballos, Denton and Hanusa. Note that we do not require n and p to be relatively prime in Theorem 3.4.3. Thus our results extend to previously untreated territory.

COROLLARY 3.4.4. *The skew-length of an n,p-core is independent of the order of n and p.* ■

COROLLARY 3.4.5. *Let $\kappa \in \mathfrak{C}_{n,p}$ be an n,p-core with conjugate κ'. Then* $\mathrm{skl}(\kappa) = \mathrm{skl}(\kappa')$. ■

Indeed, with our alternative definition of skew-length given in Proposition 3.4.2 the statements of Corollaries 3.4.4 and 3.4.5 are identical.

The multiset $H_{n,p}(\kappa)$ promises to have further interesting property that need to be investigated and might shed some light on the nature of skew-length statistic. We present two equivalent conjectural properties of $H_{n,p}(\kappa)$.

CONJECTURE 3.4.6. [**73**, Conj. 1.7] *Let n and p be relatively prime, and $\kappa \in \mathfrak{C}_{n,p}$. Then the hook-length of each cell of κ appears in $H_{n,p}(\kappa)$ with multiplicity at least one.*

CONJECTURE 3.4.7. [**73**, Conj. 1.8] *Let n and p by relatively prime, $z \in \mathbb{N}$ and $\kappa \in \mathfrak{C}_{n,p}$. Then $z + n \in H_{n,p}(\kappa)$ implies $z \in H_{n,p}(\kappa)$.*

Next we present a way for computing the skew-length of a core $\kappa \in \mathfrak{C}_{n,p}$ using the corresponding rational Dyck path $\mathcal{A}^{-1}(\kappa)$.

Given a rational Dyck path $x \in \mathfrak{D}_{n,p}$, we assign to each of its steps the label $w_n^p(i,j)$, where (i,j) is the starting point of the step. Let $(g_1, g_2, \ldots, g_n)$ be the vector consisting of the labels of the north steps of x ordered increasingly. As before denote by $H(x)$ the set of positive labels of lattice points below x. See Figure 3.13.

A *codinv pair* of x is a pair of integers (g,b) such that g is the label of a north step of x and $b \in H(x)$ and $g < b < g + p$. The *codinv tableau* of x is the collection of numbers

$$d_{i,j}(x) = \#\big\{(g_i, b) : b \in H(x), g_i < b < g_i + p \text{ and } b \equiv g_j \mod n\big\} \tag{3.1}$$

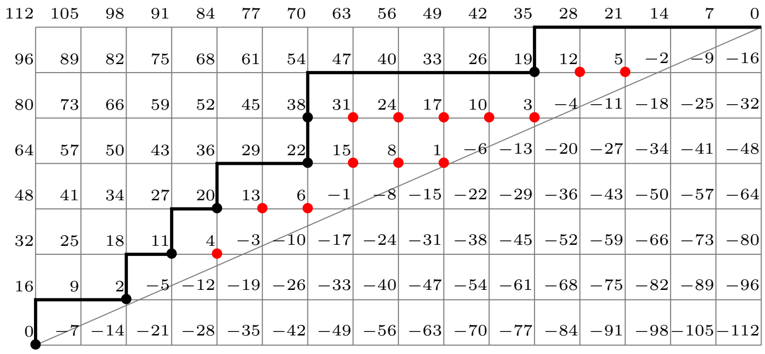


FIGURE 3.13. A Dyck path $x \in \mathfrak{D}_{7,16}$ with $H(x) = \{1,3,4,5,6,8,10,12,13,15,17,24,31\}$, and north steps labelled by $(g_1, g_2, g_3, g_4, g_5, g_6, g_7) = (0,2,11,19,20,22,38)$.

where $i, j \in [n]$ with $i < j$. For example, the Dyck path in Figure 3.13 has codinv pairs

$$\begin{gathered}(0,4),(0,6),(0,13),(0,1),(0,8),(0,15),(0,3),(0,10),(0,5),(0,12),\\(2,4),(2,6),(2,13),(2,8),(2,15),(2,3),(2,10),(2,17),(2,5),(2,12),\\(11,13),(11,15),(11,17),(11,24),(11,12),\\(19,24),(19,31),(20,24),(20,31),(22,24),(22,31).\end{gathered}$$

Its codinv tableau is found in Figure 3.14.

Similar constructions have appeared in the literature before. First we remark that the codinv tableau is related to, albeit not the same as, the laser fillings of Ceballos, Denton and Hanusa [**21**, Def. 5.13]. The row-sums and column-sums of the codinv tableau and the laser filling of a Dyck path agree. However, the codinv tableau is always of staircase shape while the laser filling sits inside the boxes below the rational Dyck path. Secondly we note that codinv pairs have been considered by Gorsky and Mazin [**32**] using slightly different notation. However, they only considered the column-sums of the codinv tableau. See also the remarks following Theorem 6.1.1.

The skew-length of $\mathcal{A}(x)$ can be computed from the codinv tableau by taking the sum of all entries.

PROPOSITION 3.4.8. *Let n and p be relatively prime and $x \in \mathfrak{D}_{n,p}$ a rational Dyck path. Then*

$$\operatorname{skl} \circ \mathcal{A}^{-1}(x) = \sum_{i,j} d_{i,j}(x).$$

$$\begin{matrix} d_{1,7} & d_{2,7} & d_{3,7} & d_{4,7} & d_{5,7} & d_{6,7} \\ d_{1,6} & d_{2,6} & d_{3,6} & d_{4,6} & d_{5,6} & \\ d_{1,5} & d_{2,5} & d_{3,5} & d_{4,5} & & \\ d_{1,4} & d_{2,4} & d_{3,4} & & & \\ d_{1,3} & d_{2,3} & & & & \\ d_{1,2} & & & & & \end{matrix} \quad = \quad \begin{matrix} 2 & 3 & 2 & 2 & 2 & 2 \\ 3 & 2 & 1 & 0 & 0 & \\ 2 & 2 & 1 & 0 & & \\ 2 & 2 & 1 & & & \\ 1 & 1 & & & & \\ 0 & & & & & \end{matrix}$$

FIGURE 3.14. The codinv tableau of the Dyck path in Figure 3.13.

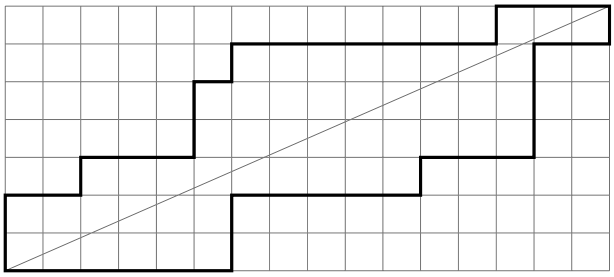


FIGURE 3.15. The image $\zeta(x)$ of the rational Dyck path in Figure 3.13 under the zeta map, and the (rotated) path $\eta(x)$ below the diagonal.

PROOF. Consider the normalised abacus $A = H(x) \cup \{z \in \mathbb{Z} : z < 0\}$ as an abacus on n runners. The numbers $g_1, \ldots, g_n$ are the minimal gaps of A in each runner. The sum $\Sigma_j = \sum_i d_{i,j}(x)$ counts pairs of the form (g, b) where g is a minimal gap of A and b is a bead of A in runner j such that $g < b < g + p$. Hence Σ_j also counts pairs of the form (g, b) where b is the maximal bead in runner j of A and g is a gap such that $b - p < g < b$. By Lemma 3.2.7 Σ_j equals the number of cells with hook-length less then p in the n-row of residue j. ■

A third interpretation of the skew-length coming from the universe of p-stable affine permutations using rational Shi tableaux is developed in Section 5.4.

Define the n by p *staircase* $\Delta_{n,p}$ as the maximal partition that fits inside the n by p rectangle and above the diagonal of slope n/p. That is,

$$(\Delta_{n,p})_i = \left\lfloor \frac{(n-i)p}{n} \right\rfloor.$$

Each rational Dyck path $x \in \mathfrak{D}_{n,p}$ is the south-east boundary of a partition $\lambda(x) \subseteq \Delta_{n,p}$. Recall that $\lambda(x)$ is called the partition of x. In fact, we obtain a bijection

$$\lambda : \mathfrak{D}_{n,p} \to \left\{\mu \in \Pi : \mu \subseteq \Delta_{n,p}\right\}.$$

For example, the partition of the Dyck path x in Figure 3.13 is the partition $\lambda = (11, 6, 6, 4, 3, 2, 0)$.

The zeta map on rational Dyck paths defined in Section 2.7 comes with a dual map $\eta : \mathfrak{D}_{n,p} \to \mathfrak{D}_{n,p}$, called *eta map* by Ceballos, Denton and Hanusa, which is defined by $\eta(x) = \zeta(\rho(x))$.

The following result relating the skew-length and the zeta map has appeared in the literature in many guises.

THEOREM 3.4.9. [**73**, Thm. 3.4] *Let n, p be positive coprime integers and $x \in \mathfrak{D}_{n,p}$. Then the partition of $\zeta(x)$ equals the row-sums of the codinv tableau of x. Moreover, the partition of $\eta(x)$ equals the column-sums of the codinv tableau of x.*

PROOF. Consider the j-th north step of $\zeta(x)$. An east step of x precedes this north step in $\zeta(x)$ if and only if its label is less than g_j. Let L be the set of lines of slope n/p through a lattice point labelled by $g_j - kn$ for some $k > 0$. The label of an east step of x is less than g_j if and only if the east step is intersected by a line in L.

Moreover each east step of x is intersected by at most one such line. Thus the number of east steps preceding the j-th north step of $\zeta(x)$ is counted by the number of intersections of an east step of x and a line in L.

For each line in L the number of intersected east steps equals the number of intersected north steps. Thus the number of east steps preceding the j-th north step of $\zeta(x)$ is counted by the number of intersections of a north step of x and a line in L.

It takes a moment of thought to verify that this number is given by the sum $\sum_{i=1}^{j-1} d_{i,j}(x)$ of a row of the codinv tableau. Indeed the number $d_{i,j}(x)$ counts the number of intersections of the north step of x labelled g_i and a line in L. equals $f_{n-j+1}(\omega)$.

The dual statement concerning the eta map could be deduced analogously. However, it follows immediately using the involutive automorphism (see Proposition 5.4.13 and Proposition 3.1.9). ■

Our proof is inspired by the resemblance of codinv tableaux and laser fillings. Theorem 3.4.9 should therefore be compared with [**21**, Thm. 5.15]. Moreover we remark that taking the column-sums of the codinv tableau actually coincides with the definition of a map of Gorsky and Mazin [**32**, Def. 3.3], which is the eta map in our notation. Thus it is well worth comparing Theorem 3.4.9 to [**7**, Thm. 4.12]. Finally, Theorem 3.4.9 together with the proof of Proposition 3.4.8 provides a different description of the zeta map using simultaneous cores. This coincides with Armstrong's definition of the zeta map given in [**7**, Sec. 4.2].

Having verified that $\mathrm{skl}(\mathcal{A}^{-1}(x)) = (n-1)(p-1)/2 - \mathrm{area}(\zeta(x))$ we obtain another combinatorial interpretation of the rational q,t-Catalan numbers.

Corollary 3.4.10. *Let n and p be relatively prime. Then*

$$C_{n,p}(q,t) = \sum_{\kappa \in \mathfrak{C}_{n,p}} q^{\ell(\kappa)} t^{(p-1)(n-1)/2 - \mathrm{skl}(\kappa)}.$$

■

As we have seen rational Dyck paths are in bijection with two different sets of partitions: simultaneous n,p-cores on the one hand and partitions that fit inside the n by p staircase on the other hand. We close this section with a conjecture that relates the poset structures of these sets of partitions with respect to inclusion via the zeta map. To this end view ζ as a map from $\mathfrak{C}_{n,p}$ to $\{\lambda \in \Pi : \lambda \subseteq \Delta_{n,p}\}$ by defining $\zeta(\kappa)$ to be the partition of $\zeta(\mathcal{A}(\kappa))$.

Conjecture 3.4.11. *Let n and p be relatively prime and $\kappa, \lambda \in \mathfrak{C}_{n,p}$. Then $\zeta(\lambda) \subseteq \zeta(\kappa)$ implies $\lambda \subseteq \kappa$.*

Conjecture 3.4.11 is open beyond the Catalan case $p = n + 1$ where a proof was found by Aigner and the author. To see that Conjecture 3.4.11 is quite strong note that it readily implies Theorem 2.7.1. Suppose $\zeta(\lambda) = \zeta(\kappa)$ then $\lambda \subseteq \kappa$ and $\kappa \subseteq \lambda$. Hence ζ is injective. Furthermore Conjecture 3.4.11 implies another non-trivial result on the poset structure of cores, namely the existence of a maximum with respect to inclusion [**25**, Thm. 5.1].

Secondly note that ζ^{-1} is merely an order preserving bijection, but not an isomorphism of posets. In fact the two posets under consideration are not isomorphic.

Finally it should be mentioned that also other posets of partitions have been considered in connection with simultaneous cores and parking functions. See for example [**78**]. Additional relations between these different types of partitions likely wait to be discovered.

CHAPTER 4

The finite torus

Having seen first connections between Catalan objects such as rational Dyck paths and core partitions and the affine Weyl group of type A_{n-1}, we now fully turn our attention to root systems.

In Section 4.1 we review some background from invariant theory and define Coxeter–Catalan numbers. Our main object of study in this chapter, the finite torus, is introduced in Section 4.2. The enumerative results stated in this section are due to Haiman. In Section 4.3 we explore the relation between the finite torus of type A_{n-1} and vertically labelled Dyck paths, and define similar lattice path models for the other three infinite families of root systems. The contents of these sections were anticipated in [**74**] but are now presented in a more general form. In Section 4.4 we discuss a generalisation of p-stable affine permutations and the Anderson map due to Thiel. Finally in Section 4.5 we define q-analogues of Coxeter–Catalan and Coxeter–Cayley numbers using a uniform dinv statistic on the finite torus that generalises Haiman's statistic from Section 2.5 and a type C_n analogue found by Thiel and the author.

4.1. Coxeter–Catalan numbers

Before we are able to generalise Catalan numbers to the level of crystallographic root systems we need to define certain invariants associated to the Weyl group of such a root system. Coincidentally these invariants come up in invariant theory.

In fact, an important motivation for the study of reflection groups comes from invariant theory. Let G be a finite group with a representation $\rho : G \to \mathrm{GL}(V)$ for some real vector space V. Furthermore let $\mathbb{R}[V]$ be the algebra of polynomial functions on V, that is, the $\mathbb{R}$-algebra generated by the linear functionals V^* with pointwise multiplication. Then there is a natural way to turn $\mathbb{R}[V]$ into a representation of G. The eponymous problem of invariant theory is to describe the ring $\mathbb{R}[V]^G$ of polynomials left invariant under this action.

As an example consider the standard representation of the symmetric group $\mathfrak{S}_n$ on $V = \mathbb{R}^n$ that is defined by permutation of the standard basis. Then $\mathbb{R}[V] = \mathbb{R}[x_1, \dots, x_n]$ is the polynomial ring generated by the coordinate functions. The ring of invariant polynomials $\Lambda = \mathbb{R}[V]^{\mathfrak{S}_n}$ is the ring of symmetric polynomials that was already studied by Newton. It turns out that Λ is again a polynomial ring, that is, it is generated as an algebra over the reals by algebraically independent polynomials. For example, as sets of algebraically independent generators we could choose the elementary symmetric polynomials

$$e_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

or the power sum symmetric polynomials

$$p_k(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^k$$

where $k \in [n]$ in each case.

The celebrated Theorem of Chevalley–Shephard–Todd asserts that this desirable behaviour characterises finite reflection groups. The original proof is due to Shephard and Todd and relies on the classification of finite reflection groups. Chevalley gave the first uniform proof.

THEOREM 4.1.1. [**59**, Thm. 5.1] *Let V be an r-dimensional Euclidean vector space and $G \leq \mathrm{GL}(V)$ be a finite group. Then the following three statements are equivalent. (i) The group G is generated by reflections. (ii) There exist r algebraically independent invariant polynomials of degrees $d_1, \dots, d_r$ such that $\#G = \prod_{i=1}^{r} d_i$. (iii) The ring of invariants $\mathbb{R}[V]^G$ is generated by r algebraically independent polynomials.*

Let W be the reflection group of an irreducible root system Φ. The numbers $d_1, \dots, d_r$ are invariants of the group W and are called the *degrees* of Φ. For example, the degrees of the reflection representation of the symmetric group $\mathfrak{S}_n$ are $2, \dots, n$. Interestingly there is a second interpretation for these numbers.

A *Coxeter element* of W is a product $c = s_1 \cdots s_r$ of all simple transpositions. The order h of a Coxeter element is called *Coxeter number* of Φ. A priori c depends on a choice of simple roots and on the chosen order on the simple roots, however, it is not difficult to show that any two Coxeter elements are conjugate to each other in W. Thus the Coxeter number is an invariant of the group W. An explanation why this definition is equivalent to the definition given in Section 1.3 is found in [**44**, Chap. 3.20]. For example, consider the symmetric group $\mathfrak{S}_n$. The Coxeter elements are the long cycles and the Coxeter number is n.

Coxeter [**24**] showed the following.

THEOREM 4.1.2. *Let W be a finite reflection group of rank r with Coxeter element c and Coxeter number h. Then the eigenvalues of c are given by $\omega^{e_1}, \dots, \omega^{e_r}$, where $\omega = e^{2i\pi/h}$ and*

$$\{e_1 + 1, \dots, e_r + 1\} = \{d_1, \dots, d_r\}$$

are the degrees of W.

The numbers $e_1, \dots, e_r$ are also called the *exponents* of Φ.

With these definitions at our disposal we are set to define the *Coxeter–Catalan numbers* as

$$C_\Phi = \prod_{i=1}^{r} \frac{e_i + h + 1}{e_i + 1} = \frac{1}{|W|} \prod_{i=1}^{r} (d_i + h).$$

Although there is no need to be so restrictive, Φ will always denote an irreducible crystallographic root system.

Coxeter–Catalan numbers have appeared in many different contexts. For example, they count positive sign types studied by Shi [**62, 63**] and ideals in the root poset Φ^+. Athanasiadis [**11**] proved that the dominant regions of the generalised Shi arrangement, to which we return in Section 5.1, are counted by *Coxeter–Fuß–Catalan numbers*

$$C_{\Phi, mn+1} = \frac{1}{|W|} \prod_{i=1}^{n} (d_i + mh).$$

Haiman [**40**] studying the action of the Weyl group on a quotient of the root lattice, which is the focus of Section 4.2, encountered the *rational Coxeter–Catalan numbers*

$$C_{\Phi, p} = \frac{1}{|W|} \prod_{i=1}^{n} (e_i + p),$$

where p and the Coxeter number h of Φ are assumed to be relatively prime. We also refer to [**4**] for more background and an excellent exposition of the non-crossing side of the story, which is not treated in this thesis.

For now our knowledge only just suffices to verify that the Coxeter–Catalan numbers specialise to the numbers defined in the previous chapters when Φ is of type A_{n-1}. However, we shall soon see how the uniformly defined objects mentioned above, which are of a more geometric flavour, relate to combinatorial objects such as lattice paths.

4.2. The finite torus

The Cayley numbers associated to an irreducible crystallographic root system Φ are defined as

$$\mathscr{C}_\Phi = (h+1)^r$$

where h denotes the Coxeter number and r denotes the rank of Φ. If Φ is of type A_{n-1} then we recover the Cayley numbers $\mathscr{C}_n$. Similarly, we define *rational Coxeter–Cayley numbers* as

$$\mathscr{C}_{\Phi,p} = p^r$$

where the parameter $p \in \mathbb{N}$ is assumed to be relatively prime to the Coxeter number h of Φ. The (rational) Coxeter–Cayley numbers count the elements in the *finite torus* defined as the quotient

$$\check{Q}/p\check{Q}$$

of the coroot lattice $\check{Q}$ of Φ. If we restrict ourselves to the Fuß–Catalan case then the Coxeter–Cayley numbers also count the regions of the Shi arrangement of Φ as we shall see in Section 5.1.

The Weyl group W acts on the coroot lattice $\check{Q}$ and fixes the sub-lattice $p\check{Q}$. Thus also the finite torus is equipped with a natural action of the Weyl group. Moreover, the affine Weyl group $\widetilde{W}$ contains the subgroup $\widetilde{W}_p = W \ltimes p\check{Q}$ that acts on the coroot lattice. Haiman has determined the orbits and the stabiliser subgroups of these actions.

Theorem 4.2.1. [**40**, Lem. 7.4.1] *Let Φ be an irreducible crystallographic root system with Weyl group W, coroot lattice $\check{Q}$ and Coxeter number h, and let $p \in \mathbb{N}$ be relatively prime to h. Then the following objects are in bijection: (i) W-orbits of $\check{Q}/p\check{Q}$, (ii) $\widetilde{W}_p$-orbits of $\check{Q}$, and (iii) coroot lattice points in $\check{Q} \cap p\overline{A_\circ}$. Furthermore, let $X \in \check{Q}/p\check{Q}$ be an element of the orbit corresponding to $q \in \check{Q} \cap p\overline{A_\circ}$. Then the stabiliser subgroup $H \leq W$ of X is generated by the reflections*

$$\big\{s_\alpha : q \text{ is contained in the wall of } p\overline{A_\circ} \text{ perpendicular to } \alpha \in \Phi^+\big\}.$$

Proof. In fact, the $\widetilde{W}_p$-orbit of an element $q \in \check{Q}$ is precisely the W-orbit of $q+p\check{Q} \in \check{Q}/p\check{Q}$. Moreover the dilated fundamental alcove $p\overline{A_\circ}$ is a fundamental domain for $\widetilde{W}_p$. Thus $\check{Q} \cap p\overline{A_\circ}$ is a system of representatives for the orbits of $\check{Q}$ under the action of $\widetilde{W}_p$. The last claim follows from the general fact about reflection groups that the stabiliser subgroup of any element $x \in V$ is generated by the reflections it contains. See [**44**, Chap. 1.12 and 4.8] for the omitted details. ∎

Theorem 4.2.1 allows for a nice description of the elements of the finite torus in terms of a *canonical representative*.

Lemma 4.2.2. [**74**, Lem. 2.5] *Let Φ be an irreducible crystallographic root system with Weyl group W, coroot lattice $\check{Q}$ and Coxeter number h, and let $p \in \mathbb{N}$ be relatively prime to h. Then every element $X \in \check{Q}/p\check{Q}$ can be written uniquely as $w(q) + p\check{Q}$, where $q \in \check{Q} \cap p\overline{A_\circ}$ and $w \in W$ satisfies $w(\alpha) \in \Phi^+$ for all $\alpha \in \Delta$ with $\langle q, \alpha\rangle = 0$, and $w(-\tilde{\alpha}) \in \Phi^+$ if $\langle q, \tilde{\alpha}\rangle = p$.*

Proof. The claim follows from the fact that the stabiliser subgroup $H \leq W$ of an element X is a parabolic subgroup of W, which is shown in the proof of [**65**, Thm. 4.6]. ∎

Haiman proceeds to count the W-orbits of $\check{Q}/p\check{Q}$, which now amounts to counting $\check{Q}$-lattice points in the dilated simplex $p\overline{A_\circ}$.

THEOREM 4.2.3. [**40**, Thm. 7.4.4] *Let Φ be an irreducible crystallographic root system with Weyl group W, coroot lattice $\check{Q}$ and Coxeter number h, and let $p \in \mathbb{N}$ be relatively prime to h. Then the number of W-orbits of $\check{Q}/p\check{Q}$ equals $C_{\Phi,p}$.*

Haiman's proof of Theorem 4.2.3 is very instructive as it combines ideas from rather different areas of combinatorics. By Theorem 4.2.1 the proof is reduced to counting lattice points in a dilated simplex which is achieved by Ehrhart theory. Another ingredient is the following formula due to Shephard–Todd [**59**, Thm. 5.3],

$$\sum_{w \in W} t^{n-\dim \mathrm{Fix}(w)} = \prod_{i=1}^{n}(1+e_i t),$$

which relates the exponents of W to the dimensions of the fixed spaces of the elements of W. The connection between the number of orbits and the Shephard–Todd-formula is made via Pólya theory. The proof also uses explicitly the fact that h and p are relatively prime.

4.3. Lattice path models for the finite torus

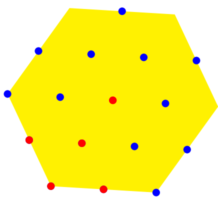


FIGURE 4.1. The finite torus $\check{Q}/4\check{Q}$ of type A_2 forms a hexagon.

Let Φ by a root system of type A_{n-1} and $p \in \mathbb{N}$ be relatively prime to the Coxeter number n. Clearly the Coxeter–Cayley numbers $\mathscr{C}_{\Phi,p}$ agree with the rational Cayley numbers $\mathscr{C}_{n,p}$ in this case. Let us explain why the finite torus generalises parking functions. The coroot lattice of type A_{n-1} is given by

$$\check{Q} = \Big\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \Big\}.$$

Hence the finite torus can be identified with

$$\check{Q}/p\check{Q} = \Big\{ x \in (\mathbb{Z}/p\mathbb{Z})^n : \sum_{i=1}^{n} x_i \equiv 0 \mod p \Big\}.$$

If n and p are relatively prime then each coset in

$$(\mathbb{Z}/p\mathbb{Z})^n/A,$$

where $A \leq (\mathbb{Z}/p\mathbb{Z})^n$ denotes the subgroup generated by the element $(1,\ldots,1)$, contains exactly one representative $x \in (\mathbb{Z}/p\mathbb{Z})^n$ such that $x_i \in \{0,\ldots,p\}$ for all $i \in [n]$ and $\sum_i x_i \equiv 0$ modulo p. Thus the finite torus is in bijection with $(\mathbb{Z}/p\mathbb{Z})^n/A$. Using Theorem 2.4.1 we obtain the following result.

PROPOSITION 4.3.1. *Let Φ be a root system of type A_{n-1} with coroot lattice $\check{Q}$ and $p \in \mathbb{N}$ relatively prime to n. Let $\psi_A(x + p\check{Q})$ denote the parking function representative of the coset $-x + p\mathbb{Z}^n + A$. Then $\psi_A : \check{Q}/p\check{Q} \to \mathrm{PF}_{n,p}$ is an $\mathfrak{S}_n$-equivariant bijection from the finite torus to the set of rational parking functions.* ■

It follows that the rational Coxeter–Catalan numbers $C_{\Phi,n}$ of type A_{n-1} equal the rational Catalan numbers $C_{n,p}$. However, the correspondence described above does not restrict to a bijection between increasing parking functions and the elements of $\check{Q} \cap p\overline{A_\circ}$. Together with the image of the set of dominant p-stable affine permutations under the Anderson map this yields three different distinguished subsets of $\mathrm{PF}_{n,p}$ of Catalan cardinality, all of which form a system of representatives for the $\mathfrak{S}_n$-orbits.

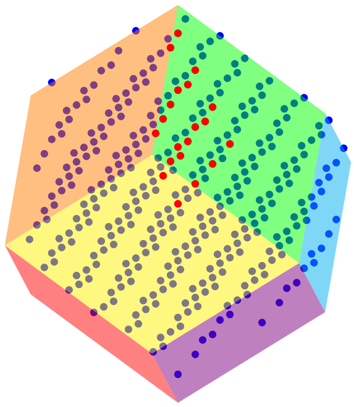


FIGURE 4.2. The finite torus $\check{Q}/7\check{Q}$ of type B_3 forms a rhombic dodecahedron[1].

4.3.1. Type B_n. Let Φ be a root system of type B_n. The Coxeter number of Φ is $h = 2n$. Let $p \in \mathbb{N}$ be relatively prime to h. The coroot lattice of Φ is given by

$$\check{Q} = \Big\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \in 2\mathbb{Z} \Big\}.$$

Moreover a system of representatives for the orbits of the finite torus under the action of the Weyl group $\mathfrak{S}_n^B$ is given by

$$\check{Q} \cap p\overline{A_\circ} = \Big\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \in 2\mathbb{Z}, 0 \leq x_1 \leq \cdots \leq x_n \text{ and } x_{n-1} + x_n \leq p \Big\}.$$

The stabiliser subgroup $H \leq \mathfrak{S}_n^B$ of $x + p\check{Q}$ for $x \in \check{Q} \cap p\overline{A_\circ}$ is generated by S, where $s_i^B \in S$ with $i \in [n-1]$ whenever $x_i = x_{i+1}$, $s_0^B \in S$ if $x_1 = 0$, and $s_{\tilde{\alpha}^B} \in S$ if $x_{n-1} + x_n = p$. Note that Athanasiadis [**12**, Sec. 5.3] already considered the Fuß–Catalan case $p = mh + 1$. An example of the finite torus is found in Figure 4.2.

Given a lattice path $x \in \mathcal{L}_{(p-1)/2,n}$ let $H_x \leq \mathfrak{S}_n^B$ denote the subgroup generated by S, where $s_i^B \in S$ if and only if $i \in [n-1]$ is a rise of x and $s_0^B \in S$ if and only if S begins with a north step. Define an equivalence relation $\sim$ on the set $\mathfrak{S}_n^B \times \mathcal{L}_{(p-1)/2,n}$ via $(s,x) \sim (t,y)$ if and only if $x = y$ and $sH_x = tH_x$. The set of equivalence classes is denoted by

$$\mathrm{Vert}(B_n, p) = \big\{ [s,x]_\sim : s \in \mathfrak{S}_n^B, x \in \mathcal{L}_{(p-1)/2,n} \big\}$$

and carries a natural action of the group of signed permutations given by $s \cdot [t,x] = [st,x]$ for all $s,t \in \mathfrak{S}_n^B$ and $x \in \mathcal{L}_{(p-1)/2,n}$. We call $\mathrm{Vert}(B_n,p)$ the set of *vertically labelled lattice paths*. To see why note that cosets in $\mathfrak{S}_n^B/H_x$ are in bijection with signed permutations $s \in \mathfrak{S}_n^B$ satisfying $s(i) < s(i+1)$ for all rises i of x and $s(1) > 0$ if x begins with a north step. These signed

[1]Coincidentally the rhombic dodecahedron belongs to a family called Catalan solids.

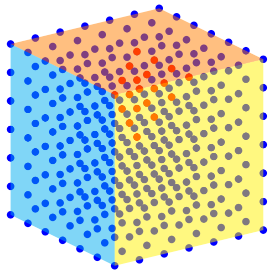


FIGURE 4.3. The finite torus $\check{Q}/7\check{Q}$ of type C_3 forms a cube.

permutations are precisely those of minimal length in their respective cosets. Hence $\operatorname{Vert}(B_n, p)$ has a nice combinatorial interpretation similar to the vertically labelled Dyck paths of type A_{n-1}. By writing $(s, x) \in \operatorname{Vert}(B_n, p)$ we mean (s, x) to be the canonical representative of $[s, x]_\sim$. A lattice path $x \in \mathcal{L}_{(p-1)/2,n}$ with labelled north steps lies in $\operatorname{Vert}(B_n, p)$ if and only if labels are increasing along columns and labels in the "zeroth" column are positive.

The following lemma was known to Athanasiadis in the Fuß–Catalan case and asserts that $\mathfrak{S}_n^B$-orbits of the finite torus $\check{Q}/p\check{Q}$ can be identified with lattice paths in $\mathcal{L}_{(p-1)/2,n}$. To this end recall that $x \in \mathcal{L}_{(p-1)/2,n}$ is encoded by the numbers x_i counting the number of east steps preceding the i-th north step of x. Define a vector $q =\in \mathbb{N}^n$ by $q_i = x_i$ for $i \in [n-1]$ and

$$q_n = \begin{cases} 2x_n - x_{n-1} & \text{if } x_1 + \cdots + x_{n-2} \text{ is even,} \\ p - 2x_n + x_{n-1} & \text{if } x_1 + \cdots + x_{n-2} \text{ is odd.} \end{cases}$$

We denote $q = \psi_B(x)$.

LEMMA 4.3.2. [**74**, Prop. 6.1] *Let Φ be a root system of type B_n with coroot lattice $\check{Q}$ and let p be relatively prime to the Coxeter number. The map $\psi_B : \mathcal{L}_{(p-1)/2,n} \to \check{Q} \cap p\overline{A_\circ}$ is a bijection.*

PROOF. By definition $q \in \check{Q} \cap p\overline{A_\circ}$. Moreover ψ_B is injective. Finally ψ_B is surjective since $q_{n-1} + q_n < p$ and $q_{n-1} \leq q_n$ imply $x_n \leq p/2$. ∎

The next theorem extends the above bijection to an $\mathfrak{S}_n^B$-equivariant bijection from $\operatorname{Vert}(B_n, p)$ to the finite torus. For $x \in \mathcal{L}_{(p-1)/2,n}$ define a signed permutation $u_x \in \mathfrak{S}_n^B$ by

$$u_x = [1, \ldots, n-1, (-1)^{x_{n-1}+x_n} n],$$

where x_i is defined as above. Moreover given $[s, x] \in \operatorname{Vert}(B_n, p)$ set $\psi_B([s, x]) = su_x(q) + p\check{Q}$.

THEOREM 4.3.3. [**74**, Prop. 6.3] *Let Φ be a root system of type B_n with coroot lattice $\check{Q}$, and let p be relatively prime to the Coxeter number. Then the map $\psi_B : \operatorname{Vert}(B_n, p) \to \check{Q}/p\check{Q}$ given by $\psi_B([s, x]) = su_x(q) + p\check{Q}$ is a well defined $\mathfrak{S}_n^B$-equivariant bijection.*

PROOF. Let $x \in \mathcal{L}_{(p-1)/2,n}$ and $q = \psi_B(x) \in \check{Q} \cap p\overline{A_\circ}$. We claim that the stabiliser subgroup $H \leq \mathfrak{S}_n^B$ of $q + p\check{Q}$ equals $u_x H_x u_x^{-1}$. That is, H is a conjugate of the subgroup generated by the rises (and an initial north step) of x. To see this note that $u_x s_n^B u_x^{-1} = s_n^B$ if $q_n = 2x_n - x_{n-1}$ and $u_x s_n^B u_x^{-1} = s_{\tilde{\alpha}^B}$ if $q_n = p - 2x_n + x_{n-1}$. The claim follows. ∎

4.3.2. Type C_n. Next consider a root system Φ of type C_n and let $p \in \mathbb{N}$ be relatively prime to the Coxeter number $h = 2n$ of Φ. The coroot lattice is simply given by $\check{Q} = \mathbb{Z}^n$. Hence the finite torus equals

$$\check{Q}/p\check{Q} = (\mathbb{Z}/p\mathbb{Z})^n.$$

A system of representatives for the orbits of the finite torus under the action of the Weyl group $\mathfrak{S}_n^B$ is given by

$$\check{Q} \cap p\overline{A_\circ} = \{q \in \mathbb{Z}^n : 0 \leq q_1 \leq \cdots \leq q_n \leq (p-1)/2\},$$

and the stabiliser subgroup $H \leq \mathfrak{S}_n^B$ of such a q is generated by the simple transpositions s_i^B for each $i \in [n-1]$ with $q_i = q_{i+1}$, and the simple transposition s_0^B if $q_1 = 0$. This was pointed out by Athanasiadis [**12**, Sec. 5.2] in the case where $p = mh + 1$.

The connection between the finite torus of type C_n and vertically labelled lattice paths is even smoother than in type B_n.

Theorem 4.3.4. [**74**, Prop. 4.2] *Let Φ be a root system of type C_n with coroot lattice $\check{Q}$, and let p be relatively prime to the Coxeter number. Then the map $\psi_C : \mathrm{Vert}(B_n, p) \to \check{Q}/p\check{Q}$ given by $\psi_C([s, x]) = s(q) + p\check{Q}$, where q_i denotes the number of east steps preceding the i-th north step of the lattice path $x \in \mathcal{L}_{(p-1)/2,n}$, is a well-defined $\mathfrak{S}_n^B$-equivariant bijection.*

Proof. Given a path $x \in \mathcal{L}_{(p-1)/2,n}$ let q_i be the number of east steps preceding the i-th north step of x. The resulting vector q satisfies $0 \leq q_1 \leq \cdots \leq q_n \leq (p-1)/2$ and this correspondence is a bijection from $\mathcal{L}_{(p-1)/2,n}$ to $\check{Q} \cap p\overline{A_\circ}$. Moreover the stabiliser subgroup $H \leq \mathfrak{S}_n^B$ of $q + p\check{Q}$ is equal to the subgroup of H_x generated by the rises (an an initial north step) of x. To see this note that no element $q \in \check{Q}$ satisfies $\langle q, \tilde{\alpha}^C \rangle = p$ since p is odd by assumption. ∎

We also write $\mathrm{Vert}(C_n, p) = \mathrm{Vert}(B_n, p)$. As is remarked in [**74**] the Theorems 4.3.3 and 4.3.4 provide an explicit $\mathfrak{S}_n^B$-equivariant bijection between the finite tori of types B_n and C_n. While the existence of such a bijection is not surprising (the quotients $\check{Q}/p\check{Q}$ and Q/pQ, where Q denotes the root lattice, are isomorphic as W-sets), the author is unaware of a previous appearance of such an isomorphism in the literature.

4.3.3. Type D_n. Finally we turn to type D_n. Let Φ be a root system of type D_n and let $p \in \mathbb{N}$ be relatively prime to the Coxeter number $h = 2n - 2$. The coroot lattice and finite torus equal those of type B_n and a system of representatives for the $\mathfrak{S}_n^D$-orbits of $\check{Q}/p\check{Q}$ is given by

$$\check{Q} \cap p\overline{A_\circ} = \left\{q \in \mathbb{Z} : \sum_{i=1}^{n} q_i \in 2\mathbb{Z}, 0 \leq |q_1| \leq q_2 \leq \cdots \leq q_n \text{ and } q_{n-1} + q_n \leq p\right\}.$$

The stabiliser subgroup $H \leq \mathfrak{S}_n^D$ of $q + p\check{Q}$ is generated by S, where $s_i^D \in S$ for each $i \in [n-1]$ with $q_i = q_{i+1}$, $s_{\tilde{\alpha}^D} \in S$ if $q_{n-1} + q_n = p$, and $s_0^D \in S$ if $q_1 = -q_2$. Recall that $s_{\tilde{\alpha}^D}$ exchanges the entries q_{n-1} and q_n and changes the signs of these entries, and s_0^D exchanges q_1 and q_2 and changes the signs of these entries. Again this was pointed out by Athanasiadis [**12**, Sec. 5.4] in the Fuß–Catalan case.

A *signed lattice path* is a lattice path in $\mathcal{L}_{m,n}$ except that if it begins with an east step then this east step is replaced by a signed east step from the set $\{\mathbf{e}^+, \mathbf{e}^-\}$. Denote the set of all such paths by $\mathcal{L}^\bullet_{m,n}$. We also define a *sign* function $\epsilon : \mathcal{L}^\bullet_{m,n} \to \{\pm 1\}$ on signed lattice paths by setting $\epsilon(x) = -1$ if x contains $\mathbf{e}^-$ and $\epsilon(x) = 1$ otherwise.

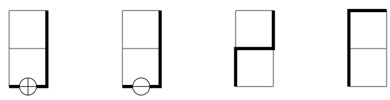


Figure 4.4. The signed lattice paths in $\mathcal{L}^\bullet_{1,2}$.

For example the set $\mathcal{L}^\bullet_{1,2} = \{\mathbf{e}^+\mathbf{nn}, \mathbf{e}^-\mathbf{nn}, \mathbf{nen}, \mathbf{nne}\}$ is displayed in Figure 4.4.

Given a signed lattice path $x \in \mathcal{L}^{\bullet}_{(p-1)/2,n}$ let x_i denote the number of east steps (with or without sign) of x that precede the i-th north step of x. Define a vector $q \in \mathbb{Z}^n$ by setting $q_1 = \epsilon(x)x_1$, $q_i = x_i$ for $1 < i < n$ and

$$q_n = \begin{cases} 2x_n - x_{n-1} & \text{if } x_1 + \cdots + x_{n-2} \text{ is even,} \\ p - 2x_n + x_{n-1} & \text{if } x_1 + \cdots + x_{n-2} \text{ is odd.} \end{cases}$$

Denote $q = \psi_D(x)$.

It is easy to see that signed lattice paths represent the orbits of the finite torus under the action of the Weyl group.

Proposition 4.3.5. [**74**, Prop. 5.3] *Let Φ be a root system of type D_n with coroot lattice $\check{Q}$, and let p be relatively prime to the Coxeter number. Then the map $\psi_D : \mathcal{L}^{\bullet}_{(p-1)/2,n} \to \check{Q} \cap p\overline{A_\circ}$ defined above is a bijection.*

Proof. Suppose $x \in \mathcal{L}^{\bullet}_{n-1,n}$ is a signed lattice path. Clearly $|q_1| \leq q_2 \leq \cdots \leq q_{n-1}$. Moreover, $x_{n-1} \leq 2x_n - x_{n-1} < p - x_{n-1}$ and $x_{n-1} < p - 2x_n + x_{n-1} \leq p - x_{n-1}$ hence $q_{n-1} \leq q_n \leq p - q_{n-1}$. Since $q_1 + \cdots + q_n$ is even by definition, we conclude that $q \in \check{Q} \cap p\overline{A_\circ}$. Moreover, ψ_D is clearly injective and surjective. ∎

By adding suitable labels to the signed lattice paths, we obtain a combinatorial model for the finite torus of type D_n. This definition is very much in the spirit of the vertically labelled Dyck paths in type A_{n-1} and the vertically labelled lattice paths in types B_n and C_n, although things are a bit less obvious in type D_n.

Given a signed lattice path $x \in \mathcal{L}^{\bullet}_{(p-1)/2,n}$ define $H_x \leq \mathfrak{S}_n^D$ as the subgroup generated by S, where $s_i^D \in S$ whenever i is a rise of x, and $s_0^D \in S$ if x begins with two north steps. Moreover let

$$u_x = [\epsilon(x), 2, \ldots, n-1, (-1)^{q_{n-1}+q_n} n] \in \mathfrak{S}_n^B, \tag{4.1}$$

where $q = \psi_D(x)$ is defined as in Proposition 4.3.5. Define an equivalence relation on $\mathfrak{S}_n^D u_x \times \mathcal{L}^{\bullet}_{(p-1)/2,n}$ via $(s,x) \sim (t,y)$ if and only if $x = y$ and $sH_x = tH_x$. The set of *vertically labelled signed lattice paths* is defined as the set of equivalence classes

$$\mathrm{Vert}(D_n, p) = \left\{ [s,x]_\sim : s \in \mathfrak{S}_n^D u_x, x \in \mathcal{L}^{\bullet}_{(p-1)/2,n} \right\}$$

and carries a natural $\mathfrak{S}_n^D$ action.

The picture to keep in mind is as follows. Each vertically labelled signed lattice path $[t,x] \in \mathrm{Vert}(D_n,p)$ contains a unique pair $(s,x) \sim (t,x)$ of a signed permutation $s \in \mathfrak{S}_n^B$ and a signed lattice path $x \in \mathcal{L}^{\bullet}_{(p-1)/2,n}$ such that $s(i) < s(i+1)$ for each rise i of x, $|s(1)| < s(2)$ if x begins with two north steps, and

$$\prod_{i=1}^{n} \mathrm{sgn}(s(i)) = \epsilon(x)(-1)^{q_{n-1}+q_n}.$$

By writing $(s,x) \in \mathrm{Vert}(D_n,p)$ we mean (s,x) to be the canonical representative of $[s,x]$. See Figure 4.5.

The fact that s is not necessarily an even signed permutation is unintuitive at first sight. The reason why it is natural to work with the "odd" signed permutation s is that it satisfies $s(\alpha) \in \Phi^+$ for all $\alpha \in \Delta$ with $s_\alpha \in S$. The even signed permutation su_x thus satisfies $su_x(\alpha) \in \Phi^+$ for all $\alpha \in \Delta \cup \{-\tilde{\alpha}\}$ with $s_\alpha \in u_x S u_x^{-1}$. The significance of the conjugate subgroup $u_x H_x u_x^{-1}$ becomes clear from the next theorem.

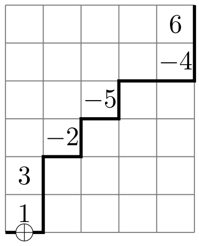


FIGURE 4.5. A vertically labelled signed lattice paths $(s, x) \in \mathrm{Vert}(D_6, 11)$. We have $\psi_D(x) = (1, 1, 2, 3, 5, 6)$, $\epsilon(x) = 1$, $u_x = [1, 2, 3, 4, 5, -6]$, $s = [1, 3, -2, -5, -4, 6] \in \mathfrak{S}_6^D u_x$, and $\psi_D(s, x) = (1, -2, 1, -5, -3, -6) + 11\check{Q}$.

THEOREM 4.3.6. [**74**, Prop. 5.6] *Let Φ be a root system of type D_n with coroot lattice $\check{Q}$, and let $p \in \mathbb{N}$ be relatively prime to the Coxeter number. Then the map $\psi_D : \mathrm{Vert}(D_n, p) \to \check{Q}/p\check{Q}$ given by $\psi_D([s, x]) = su_x(\psi_D(x)) + p\check{Q}$ is a well-defined $\mathfrak{S}_n^D$-equivariant bijection.*

PROOF. By Proposition 4.3.5 the orbits of both sets are in bijection. Suppose $q = \psi_D(x) \in \check{Q} \cap p\overline{A_\circ}$ corresponds to $x \in \mathcal{L}^\bullet_{(p-1)/2,n}$. Then the stabiliser subgroup $H \leq \mathfrak{S}_n^D$ of $q + p\check{Q}$ equals $u_x H_x u_x^{-1}$ where $H_x \leq \mathfrak{S}_n^D$ is generated by the rises (and two initial north steps) of x and $u_x \in \mathfrak{S}_n^B$ is defined in (4.1). To see this note that $u_x s_1^D u_x^{-1} = s_0^D$ if $\epsilon(x) = -1$ and $u_x s_{n-1}^D u_x^{-1} = s_{\tilde{\alpha}^D}$ if $q_{n+1} + q_n$ is odd. The claim follows. ■

4.4. p-stable elements of the affine Weyl group

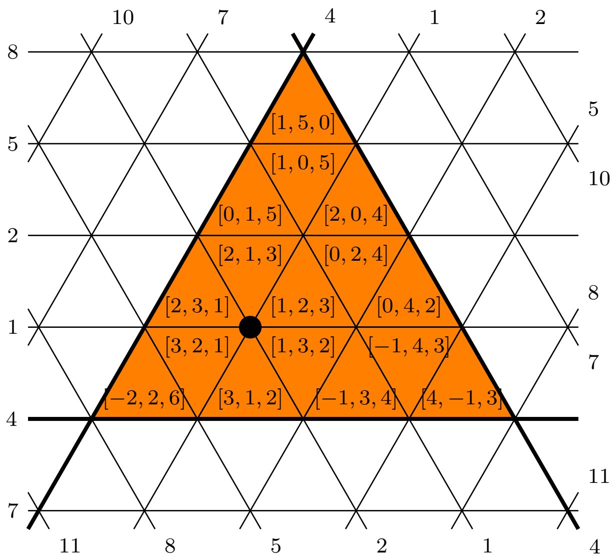


FIGURE 4.6. The Sommers region of type A_2.

We have seen in Section 3.3 that the set of vertically labelled Dyck paths $\mathrm{Vert}(A_{n-1}, p)$ is in bijection with p-stable elements of the affine Weyl group $\widetilde{\mathfrak{S}}_n$ of type A_{n-1}. As part of his thesis [**75**] Thiel generalised this correspondence to a bijection between the elements of the finite torus and a subset of the affine Weyl group of the same type.

Let Φ be an irreducible crystallographic root system and p be a positive integer relatively prime to the Coxeter number h. An element $\omega \in \widetilde{W}$ is called *p-stable* if it has no inversions of height p. That is,

$$\omega(\widetilde{\Phi}_p) \subseteq \widetilde{\Phi}^+,$$

where $\widetilde{\Phi}_p = \{\alpha + k\delta \in \widetilde{\Phi} : \mathrm{ht}(\alpha + k\delta) = p\}$. Let $\widetilde{W}^p$ denote the set of p-stable elements in $\widetilde{W}$. It is easy to see that the definition of p-stable elements of the affine Weyl group of type A_{n-1} coincides with the set of p-stable affine permutations considered by Gorsky, Mazin and Vazirani.

Note that $\omega \in \widetilde{W}^p$ if and only if $\omega^{-1}(A_\circ)$ lies in the region bounded by the hyperplanes of height p. This region is called *Sommers region*. One can show that the Sommers region is not only bounded but isometric to the p-fold dilation of the r-dimensional simplex $A_\circ$. In fact an even stronger statement holds.

THEOREM 4.4.1. [**75**, Thms. 3.6.3 and 3.6.4] *Let Φ be an irreducible crystallographic root system and p relatively prime to the Coxeter number. Then there exists a unique element $\omega_p \in \widetilde{W}$ of the affine Weyl group that maps the Sommers region bijectively to the dilated fundamental alcove $pA_\circ$.*

Instead of discussing the proof of Theorem 4.4.1 we provide an explicit description of the element ω_p in those cases, where we need it.

LEMMA 4.4.2. *(i) If Φ is a root system of type B_n then the element $\omega^B_{h+1} = t_x s \in \widetilde{W}$, where*

$$x = \begin{cases} (1,2,\dots,n-1,n) \\ (1,2,\dots,n-1,n+1) \end{cases} \quad \textit{and} \quad s = \begin{cases} [1,2,\dots,n-1,n] & \textit{if } n \equiv 0,3 \mod 4, \\ [1,2,\dots,n-1,-n] & \textit{if } n \equiv 1,2 \mod 4, \end{cases}$$

maps the Sommers region to the dilated fundamental alcove $(h+1)\overline{A_\circ}$.

(ii) If Φ is a root system of type C_n then the element $\omega^C_{h+1} = t_x s \in \widetilde{W}$, where

$$x = (1,2,\dots,n-1,n) \quad \textit{and} \quad s = [-n,-n+1,\dots,-2,-1],$$

maps the Sommers region to the dilated fundamental alcove $(h+1)\overline{A_\circ}$.

(iii) If Φ is a root system of type D_n then the element $\omega^D_{h+1} = t_x s \in \widetilde{W}$, where

$$x = \begin{cases} (0,1,2,\dots,n-2,n-1) \\ (0,1,2,\dots,n-2,n) \end{cases} \quad \textit{and} \quad s = \begin{cases} [1,2,\dots,n-1,n] & \textit{if } n \equiv 0,3 \mod 4, \\ [-1,2,\dots,n-1,-n] & \textit{if } n \equiv 1,2 \mod 4, \end{cases}$$

maps the Sommers region to the dilated fundamental alcove $(h+1)\overline{A_\circ}$.

PROOF. It suffices to show that

$$\omega_{h+1}\big((\Delta+\delta) \cup \{-\tilde{\alpha}+2\delta\}\big) = \Delta \cup \{-\tilde{\alpha}+(h+1)\delta\},$$

which can be verified by a routine computation.

To see claim (i) first suppose $n \equiv 0,3$ modulo 4. Then $\omega^B_{h+1}(\alpha+\delta) = \alpha$ for all $\alpha \in \Delta$ and $\omega^B_{h+1}(-\tilde{\alpha}^B + 2\delta) = -\tilde{\alpha}^B + (2n+1)\delta$. If instead $n \equiv 1,2$ modulo 4 then the only difference is that $\omega^B_{h+1}(e_n - e_{n-1}) + \delta = -\tilde{\alpha}^B + (h+1)\delta$ and $\omega^B_{h+1}(-\tilde{\alpha}^B + 2\delta) = e_n - e_{n-1}$.

Claim (ii) follows from $\omega^C_{h+1}(e_{i+1} - e_i + \delta) = e_{n-i+1} - e_{n-i}$ for all $i \in [n-1]$, $\omega^C_{h+1}(2e_1 + \delta) = -\tilde{\alpha}^C + (h+1)\delta$ and $\omega^C_{h+1}(\tilde{\alpha}^C + 2\delta) = 2e_1$.

To see claim (iii) first suppose $n \equiv 0,3$ modulo 4. Then $\omega^D_{h+1}(\alpha+\delta) = \alpha$ for all $\alpha \in \Delta$ and $\omega^D_{h+1}(-\tilde{\alpha}^D + 2\delta) = -\tilde{\alpha}^D + (2n+1)\delta$. If $n \equiv 1,2$ modulo 4 then $\omega^D_{h+1}(e_n - e_{n-1}) + \delta = -\tilde{\alpha}^D + (h+1)\delta$ and $\omega^D_{h+1}(-\tilde{\alpha}^D + 2\delta) = e_n - e_{n-1}$ as in type B_n above, and now $\omega^D_{h+1}(e_2 - e_1) + \delta = e_1 + e_2$ and $\omega^D_{h+1}(e_1 + e_2 + \delta) = e_2 - e_1$. ∎

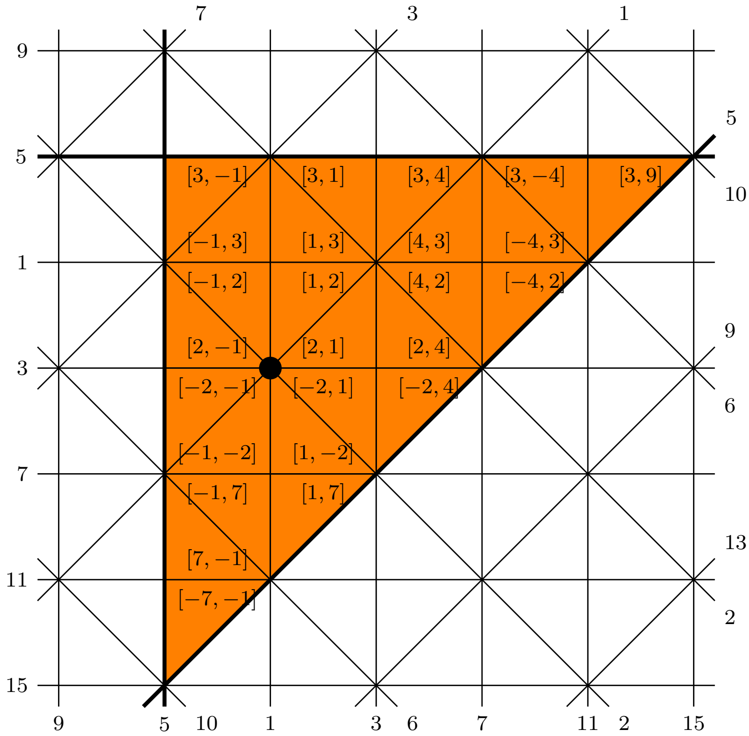


FIGURE 4.7. The Sommers region of type C_2.

We remark that the element ω_p is related to a problem we have already touched on at the end of Section 3.4, namely the existence of a maximal n,p-core with respect to inclusion. If Φ is of type A_{n-1} then the Anderson map sends ω_p to the maximum of $\mathfrak{C}_{n,p}$. Lascoux showed that this is equivalent to ω_p being a maximum of $\widetilde{W}^p_+$, the set of dominant p-stable elements of the affine Weyl group, with respect to the Bruhat order [**51**, Prop. 1]. Fayers [**25**, Sec. 5] pointed out that even more is true. Indeed the element ω_p is a maximum of $\widetilde{W}^p_+$ with respect to the weak order. Thiel and Williams conjecture that this should hold for arbitrary root systems.

CONJECTURE 4.4.3. [**77**, Conj. 6.14] *Let Φ be an irreducible root system and p relatively prime to the Coxeter number. Then ω_p is a maximum of $\widetilde{W}^p_+$ with respect to the weak order.*

As a consequence of Theorem 4.4.1 we obtain the desired bijection $\mathcal{A}:\widetilde{W}^p\to\check{Q}/p\check{Q}$ between p-stable elements and the finite torus by letting

$$\mathcal{A}(\omega)=-\omega\omega_p^{-1}(0)+p\check{Q}.$$

THEOREM 4.4.4. [**75**, Thm. 3.6.6] *Let Φ be an irreducible crystallographic root system and p a positive integer relatively prime to the Coxeter number. Then the map $\mathcal{A}:\widetilde{W}^p\to\check{Q}/p\check{Q}$ is a bijection.*

PROOF. By Theorem 4.4.1 the map $\omega\mapsto\omega_p\omega^{-1}$ is a bijection from $\widetilde{W}^p$ to the set of alcoves contained in $pA_\circ$. Since $A_\circ$ is a fundamental domain for the action of $\widetilde{W}$ on V, $pA_\circ$ is a fundamental domain for the action of $\widetilde{W}_p$ on V and the alcoves contained in $pA_\circ$ form a system of representatives for the right cosets in $\widetilde{W}_p\setminus\widetilde{W}$. Another system of representatives is formed by the translations t_x where x ranges over a system of representatives for $\check{Q}/p\check{Q}$. Hence the map from $\{\omega \in \widetilde{W} : \omega(A_\circ) \subseteq pA_\circ\}$ to $\check{Q}/p\check{Q}$ sending ω to a translation representative of its coset is a bijection. Explicitly, if $\omega = t_x s$ with $x \in \check{Q}$ and $s \in W$ then the claim follows from

$$\widetilde{W}_p t_x s = \widetilde{W}_p t_{s^{-1}(x)} = \widetilde{W}_p t_{-\omega^{-1}(0)}.$$

∎

The map $\mathcal{A}$ is called the *uniform Anderson map*. It generalises the Anderson map of Gorsky, Mazin and Vazirani in the following sense.

THEOREM 4.4.5. [**75**, Thm. 3.7.1] *Let Φ be a root system of type A_{n-1}, let p be relatively prime to n and $\omega \in \widetilde{\mathfrak{S}}_n^p$. Then the parking function corresponding to $\mathcal{A}_A(\omega)$ matches the parking function corresponding to $\mathcal{A}(\omega)$. That is,*

$$\psi_A \circ \mathcal{A} = \phi \circ \mathcal{A}_A,$$

where $\psi_A : \check{Q}/(n+1)\check{Q} \to \mathrm{PF}_n$ is defined as in Proposition 4.3.1 and $\phi_A : \mathrm{Vert}(A_{n-1}) \to \mathrm{PF}_n$ is defined as in Proposition 2.3.1.

The proof of Theorem 4.4.5 is surprisingly involved so we leave it at a reference.

Using Theorem 4.4.1 it becomes clear that the number of p-stable elements is in fact given by a power p^r. Moreover applying the uniform Anderson it can be shown that the dominant elements of $\widetilde{W}^p$ are counted by rational Catalan numbers.

THEOREM 4.4.6. [**75**, Cors. 3.6.5 and 3.9.4] *Let Φ be an irreducible crystallographic root system and p a positive integer relatively prime to the Coxeter number. The number of p-stable elements in the affine Weyl group $\widetilde{W}$ equals $\mathscr{C}_{\Phi,p}$. The number of dominant p-stable elements in $\widetilde{W}$ is given by $C_{\Phi,p}$.*

4.5. The dinv-statistic

We conclude our treatment of the finite torus by defining first q-analogues of the Coxeter–Catalan and Coxeter–Cayley numbers.

Define the *uniform dinv statistic* $\mathrm{dinv} : \check{Q} \cap (h+1)\overline{A_\circ} \to \mathbb{N}$ by

$$\mathrm{dinv}(x) = \#\big\{\alpha \in \Phi^+ : \langle x, \alpha\rangle \in \{\mathrm{ht}(\alpha), \mathrm{ht}(\alpha)+1\}\big\}.$$

Furthermore define the statistic $\mathrm{dinv}' : \check{Q}/(h+1)\check{Q} \to \mathbb{N}$ by

$$\begin{aligned}\mathrm{dinv}'(X) = {} & \#\big\{\alpha \in \Phi^+ : \langle x, \alpha\rangle = \mathrm{ht}(\alpha) \text{ and } w(\alpha) \in \Phi^+\big\} \\ & + \#\big\{\alpha \in \Phi^+ : \langle x, \alpha\rangle = \mathrm{ht}(\alpha)+1 \text{ and } w(\alpha) \in -\Phi^+\big\},\end{aligned}$$

where $w(x) \in X$ is assumed to be the canonical representative as in Lemma 4.2.2. Define the *q-Coxeter–Catalan numbers* and the *q-Coxeter–Cayley numbers* respectively as

$$C_\Phi(q) = \sum_{x \in \check{Q} \cap (h+1)\overline{A_\circ}} q^{\mathrm{dinv}(x)} \qquad \text{and} \qquad \mathscr{C}_\Phi(q) = \sum_{X \in \check{Q}/(h+1)\check{Q}} q^{\mathrm{dinv}'(X)}.$$

The following theorem asserts that these q-analogues specialise to the polynomials $C_n(q)$ and $\mathscr{C}_n(q)$ when Φ is of type A_{n-1}.

THEOREM 4.5.1. *Let Φ be a root system of type A_{n-1} with coroot lattice $\check{Q}$, $(s,\pi) \in \mathrm{Vert}(A_{n-1})$ a vertically labelled Dyck path and $w(x) + (h+1)\check{Q} = \psi_A^{-1} \circ \phi_A(s,\pi)$ the corresponding element of the finite torus in the sense of Propositions 2.3.1 and 4.3.1. Then*

$$\mathrm{dinv}(x) = \mathrm{dinv}(\pi) \qquad \textit{and} \qquad \mathrm{dinv}'\big(w(x) + (h+1)\check{Q}\big) = \mathrm{dinv}'(s,\pi).$$

We avoid a direct proof of Theorem 4.5.1 and postpone it until Section 6.2.

A dinv statistic was also defined in type C_n in [**74**]. Compared to Theorem 4.5.1 it is quite straightforward to check that the uniform dinv statistic specialises to dinv_C when Φ is of type C_n.

CHAPTER 5

The Shi arrangement

The second important geometric object in non-nesting Catalan combinatorics is the Shi arrangement. The definition of the Shi arrangement and the so called non-nesting parking functions together with some background are presented in Section 5.1. Section 5.2 offers an explanation how these objects are related to Dyck paths and parking functions, and provides lattice path models for the non-nesting parking functions of types B_n, C_n and D_n, which are also found in [**74**]. In Section 5.3 we define a uniform area statistic on non-nesting parking functions to give a second interpretation of q-Coxeter–Catalan and q-Coxeter–Cayley numbers that uses the Shi arrangement instead of the finite torus. In Section 5.4 we turn our attention to a construction of Fishel, Tzanaki and Vazirani who encoded dominant regions of the Shi arrangement as tableaux on the root poset. Following [**73**] we generalise Shi tableaux to the rational Catalan level. Rational Shi tableaux are then used to define rational q-Coxeter–Catalan numbers. Furthermore a conjecture on the invertibility of the construction of rational Shi tableaux is presented, which we finally prove in type A_{n-1} by exploiting some beautiful connections to the combinatorics of cores and the contents of Chapter 3.

5.1. Non-nesting parking functions and the Shi arrangement

Let Φ be an irreducible crystallographic root system with Weyl group W and affine Weyl group $\widetilde{W}$. Shi [**60, 61**] indexed the regions of the affine arrangement $\mathrm{Aff}(\Phi)$ as follows. Given $\alpha \in \Phi^+$ and $\omega \in \widetilde{W}$ define $k(\alpha, \omega)$ as the integer satisfying

$$k(\alpha,\omega) < \langle x, \alpha \rangle < k(\alpha,\omega) + 1$$

for all $x \in \omega(A_\circ)$. The induced map $\mathbf{k}(\omega) : \Phi^+ \to \mathbb{N}$ defined by $\alpha \mapsto k(\alpha,\omega)$ is called the *address* or the *Shi coordinates* of ω. Clearly no two elements of $\widetilde{W}$ have the same Shi coordinates, that is, the map $\mathbf{k} : \widetilde{W} \to \{k : \Phi^+ \to \mathbb{N}\}$ is a bijection onto its image. Shi proceeds to characterise the set $\mathbf{k}(\widetilde{W})$ of functions that appear as the Shi coordinates of some element $\omega \in \widetilde{W}$.

Regarding only the signs of the integers $k(\alpha,\omega)$ one obtains the *sign type* of ω. The set of sign types is in bijection with the regions of the hyperplane arrangement

$$\mathrm{Shi}(\Phi) = \big\{H_{\alpha,k} : \alpha \in \Phi^+, k \in \{0,1\}\big\},$$

which we call the *Shi arrangement* of Φ. The regions of $\mathrm{Shi}(\Phi)$ are commonly called *Shi regions*. Indeed, two elements $\omega, \omega' \in \widetilde{W}$ have the same sign type if and only if $\omega(A_\circ)$ and $\omega'(A_\circ)$ lie in the same region of $\mathrm{Shi}(\Phi)$. Shi [**62**] proved that the sign types of Φ are counted by the Cayley numbers $(h+1)^r$.

The regions of the Shi arrangement that lie inside the dominant chamber correspond precisely to the sign types that have no negative entries. In [**63**] Shi counted the number of dominant Shi regions by relating them to ideals in the root poset Φ^+. We collect the results of Shi in the following theorem.

THEOREM 5.1.1. *Let Φ be an irreducible crystallographic root system. The number of sign types of Φ equals the number of regions of* $\mathrm{Shi}(\Phi)$ *and is given by the Cayley number $\mathscr{C}_\Phi$. The map* *sending each dominant region R of* $\mathrm{Shi}(\Phi)$ *to the set*

$$\big\{\alpha \in \Phi^+ : H_{\alpha,1} \text{ is a floor of } R\big\}$$

is a bijection between the dominant regions of the Shi arrangement and the anti-chains in the root poset Φ^+*. The number of dominant regions of* $\mathrm{Shi}(\Phi)$ *is given by the Catalan number* C_Φ*.*

A few years later deformations of the Coxeter arrangement rose in popularity [**10, 55**] and people started to study the *generalised Shi arrangement* or *m-Shi arrangement* that depends on a positive integer m and is defined as

$$\mathrm{Shi}^m(\Phi) = \big\{H_{\alpha,k} : \alpha \in \Phi^+, -m < k \leq m\big\}.$$

The m-Shi arrangement retains many of the nice enumerative properties of the Shi arrangement. Yoshinaga [**81**] used the theory of free arrangements to show that the regions of the m-Shi arrangement are counted by generalised Cayley numbers. Athanasiadis [**11, 12**] proved that the dominant regions of the m-Shi arrangement are counted by Fuß–Catalan numbers by setting up a bijection with certain chains of ideals in the root poset. We collect these results in the following theorem.

THEOREM 5.1.2. *Let Φ be an irreducible crystallographic root system and m be a positive integer. Then the number of regions of the m-Shi arrangement* $\mathrm{Shi}^m(\Phi)$ *is given by* $\mathscr{C}_{\Phi,mh+1}$*. The number of dominant regions of* $\mathrm{Shi}^m(\Phi)$ *is given by* $C_{\Phi,mh+1}$*.*

In order to work with the regions of the Shi arrangement it is very useful to associate them with certain elements of the affine Weyl group. The following theorem was proven by Shi in the case $m = 1$, by Athanasiadis for dominant regions and by Thiel [**75**, Sec. 4] in its general form.

THEOREM 5.1.3. *Let Φ be an irreducible crystallographic root system and m be a positive integer. Then every region of* $\mathrm{Shi}^m(\Phi)$ *contains a unique alcove $\omega(A_\circ)$, where $\omega \in \widetilde{W}$, that satisfies the following equivalent conditions.*

- *(i) All floors of $\omega(A_\circ)$ are elements of* $\mathrm{Shi}^m(\Phi)$*.*
- *(ii) The floors of $\omega(A_\circ)$ are precisely the floors of the region R of the m-Shi arrangement that contains $\omega(A_\circ)$.*
- *(iii) We have $k(\alpha,\omega) \leq k(\alpha,\omega')$ for all $\alpha \in \Phi^+$ and all $\omega' \in \widetilde{W}$ such that $\omega(A_\circ)$ and $\omega'(A_\circ)$ are contained in the same region of the m-Shi arrangement.*

The alcoves described by Theorem 5.1.3 are called *Shi alcoves* or *minimal alcoves* of a Shi region.

Armstrong, Reiner and Rhoades [**9**] found a description of the Shi arrangement that carries a natural action of the Weyl group. Denote the set of anti-chains in the root poset by $\mathrm{AC}(\Phi^+)$. Define an equivalence relation on $W \times \mathrm{AC}(\Phi^+)$ via $(s,A) \sim (t,B)$ if and only if $A = B$ and $sH = tH$, where $H \leq W$ is the subgroup generated by the reflections $\{s_\alpha : \alpha \in A\}$. The *non-nesting parking functions* of Φ are defined as equivalence classes

$$\mathrm{Park}(\Phi) = \big\{[s,A]_\sim : s \in W, A \subseteq \Phi^+ \text{ is an anti-chain}\big\}.$$

Clearly, the Weyl group acts on the non-nesting parking functions via $s \cdot [t,A] = [st,A]$ for all $s,t \in W$ and all anti-chains $A \subseteq \Phi^+$.

The following lemma allows the selection of a *canonic representative* in each non-nesting parking function.

LEMMA 5.1.4. *Let $t \in W$ and $A \subseteq \Phi^+$ be an anti-chain. Then the non-nesting parking function $[t,A]$ contains a unique element (s,A) such that $s(A) \subseteq \Phi^+$.*

PROOF. This follows from the result of Sommers [**65**, Thm. 6.4] that the subgroup $H \leq W$ generated by the reflections $\{s_\alpha : \alpha \in A\}$ is parabolic. ∎

Non-nesting parking functions can be related to the regions of the Shi arrangement by extending the correspondence between dominant regions and anti-chains from Theorem 5.1.1.

THEOREM 5.1.5. [**76**, Thm. 15.2] *Let Φ be an irreducible crystallographic root system. Then the map $\Theta : \{R$ is a region of $\mathrm{Shi}(\Phi)\} \to \mathrm{Park}(\Phi)$ that sends each region R of $\mathrm{Shi}(\Phi)$ to the non-nesting parking function $[s, A]$, where $R \subseteq sC_\circ$ and $\alpha \in A$ if and only if $\alpha \in \Phi^+$ and $H_{\alpha,1}$ is a floor of $s^{-1}R$, is a bijection.*

Note that a similar bijection using ceilings instead of floors was given in [**9**, Prop. 10.3].
The concept of non-nesting parking functions was extended to the Fuß–Catalan level to encompass the m-Shi arrangement by Rhoades [**57**].

5.2. Lattice path models for non-nesting parking functions

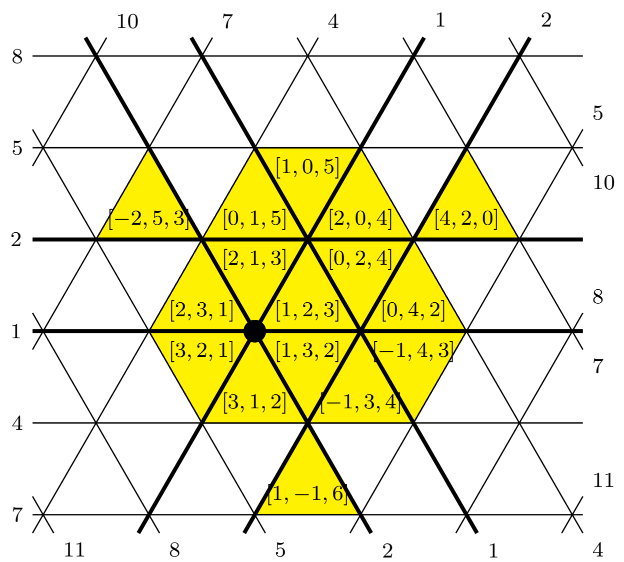


FIGURE 5.1. The Shi arrangement of type A_2.

The regions of the Shi arrangement of type A_{n-1} have early on been related to classical parking functions. Bijections between the two sets are due to Pak and Stanley [**67, 69**] and Athanasiadis and Linusson [**13**]. Later Armstrong [**5**] gave an interpretation of the q,t-Catalan numbers in terms of the dominant regions of the Shi arrangement. Gorsky, Mazin and Vazirani [**33**] extended the above ideas to the rational case.
The foundation of this correspondence is the well-known fact that ideals (respectively, order filters or anti-chains) in the root poset Φ^+ of type A_{n-1} are in bijection with Dyck paths. Given $\pi \in \mathfrak{D}_n$ set

$$A_\pi^A = \{e_i - e_j : (i,j) \text{ is a valley of } \pi\}.$$

LEMMA 5.2.1. *Let Φ be a root system of type A_{n-1}. Then the map $\varphi_A : \mathfrak{D}_n \to \mathrm{AC}(\Phi^+)$ given by $\pi \mapsto A_\pi^A$ is a bijection between Dyck paths and anti-chains in the root poset.* ■

Extending Lemma 5.2.1 we construct a bijection between diagonally labelled Dyck paths and non-nesting parking functions of type A_{n-1}. The regions of the Shi arrangement are therefore in bijection with diagonally labelled Dyck paths.

PROPOSITION 5.2.2. *Let Φ be a root system of type A_{n-1}. Then the map $\varphi_A : \mathrm{Diag}(A_{n-1}) \to \mathrm{Park}(\Phi)$ defined by $(s,\pi) \mapsto [s, A^A_\pi]$ is a bijection between diagonally labelled Dyck paths and non-nesting parking functions.*

PROOF. The claim follows from Lemmas 5.1.4 and 5.2.1, since clearly $s(e_i - e_j) \in \Phi^+$ if and only if $s(i) < s(j)$ for all $i, j \in [n]$ with $i < j$. ■

We give similar combinatorial interpretations for the non-nesting parking functions of types B_n, C_n and D_n. Note that m-non-nesting parking functions and the regions of the m-Shi arrangement are connected to m-Dyck paths. In the discussion below we limit ourselves to the Catalan case, although the more general setting should be explored in the future.

5.2.1. Type B_n. The correct lattice paths to consider in type B_n are ballot paths. A *ballot path* is a lattice path starting at the origin $(0,0)$ using only north and east steps such that every initial sub-word of steps $s_1 \cdots s_k$ contains at least as many north steps as east steps. Thus the path never goes below the main diagonal $x = y$. Denote the set of ballot paths with n steps by $\mathcal{B}_n$. Moreover we fix the following convention concerning the valleys of ballot paths. Let $\beta \in \mathcal{B}_n$. As usual a pair (i,j) is a *valley* of β if the i-th east step of β is immediately followed by its j-th north step. Moreover we call $(i, n-i+1)$ a *valley* of β if the last step of β is its i-th east step.
Let Φ be a root system of type B_n and $\beta \in \mathcal{B}_{2n}$ be a ballot path with valley (i,j). Define the root $\alpha_{i,j} \in \Phi^+$ as

$$\alpha_{i,j} = \begin{cases} e_{n+1-i} - e_{n+1-j} & \text{if } j < n+1, \\ e_{n+1-i} & \text{if } j = n+1, \\ e_{n+1-i} + e_{j-n-1} & \text{if } j > n+1. \end{cases}$$

Furthermore set

$$A^B_\beta = \{\alpha_{i,j} : (i,j) \text{ is a valley of } \beta\}.$$

The following folklore lemma asserts that this correspondence identifies ballot paths with anti-chains in the root poset.

LEMMA 5.2.3. [**74**, Lem. 6.6] *Let Φ be a root system of type B_n. The map $\varphi_B : \mathcal{B}_{2n} \to \mathrm{AC}(\Phi^+)$ given by $\beta \mapsto A^B_\beta$ is a bijection between ballot paths of length $2n$ and the anti-chains in the root poset.* ■

A *diagonally labelled ballot path* of type B_n is a pair (s,β) of a signed permutation $s \in \mathfrak{S}^B_n$ and a ballot path $\beta \in \mathcal{B}_{2n}$ such that for each valley (i,j) of β we have

$$s(n+1-i) > s(n+1-j).$$

Let $\mathrm{Diag}(B_n)$ denote the set of all diagonally labelled ballot paths of type B_n.
We visualise such a labelled path by writing the entries $s(i)$ for $i = n, n-1 \ldots, 1, 0, -1, \ldots, -n$, in the main diagonal next to the ballot path. Thereby a pair $(s,\beta) \in \mathfrak{S}^B_n \times \mathcal{B}_{2n}$ lies in $\mathrm{Diag}(B_n)$ if and only if for each valley the number below is greater than the number to its right. See Figure 5.2.

PROPOSITION 5.2.4. [**74**, Prop. 6.9] *The map $\varphi_B : \mathrm{Diag}(B_n) \to \mathrm{Park}(B_n)$ given by $(s,\beta) \mapsto [s, A^B_\beta]$ is a bijection.*

PROOF. Let $\beta \in \mathcal{B}_{2n}$ be a ballot path and $s \in \mathfrak{S}^B_n$ be a signed permutation. By Lemmas 5.2.3 and 5.1.4 it suffices to show that $(s,\beta) \in \mathrm{Diag}(B_n)$ if and only if $s(A^B_\beta) \subseteq \Phi^+$. Suppose

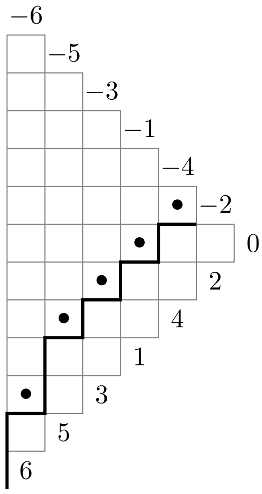


FIGURE 5.2. A diagonally labelled ballot path $(s, \beta) \in \mathrm{Diag}(B_6)$, where $s = [6, 5, 3, 1, 4, 2]$.

$s(A^B_\beta) \subseteq \Phi^+$. Then an easy case-by-case check reveals

$$\begin{aligned}(i, j) \text{ is a valley of } \beta \Leftrightarrow \alpha_{i,j} \in A^B_\beta \\ \Rightarrow s(\alpha_{i,j}) \in \Phi^+ \Leftrightarrow s(n+1-i) > s(n+1-j).\end{aligned}$$

Conversely if $(s, \beta) \in \mathrm{Diag}(B_n)$ then

$$\begin{aligned}\alpha_{i,j} \in A^B_\beta \Leftrightarrow (i, j) \text{ is a valley of } \beta \\ \Rightarrow s(n+1-i) > s(n+1-j) \Leftrightarrow s(\alpha_{i,j}) \in \Phi^+.\end{aligned}$$

■

5.2.2. Type C_n. Since the root posets of types B_n and C_n are isomorphic it is no surprise that ballot paths are the correct paths to use in type C_n as well. Let $\beta \in \mathcal{B}_{2n}$ be a ballot path and (i, j) a valley of β. Define the positive root

$$\alpha_{i,j} = \begin{cases} e_{n+1-i} - e_{n+1-j} & \text{if } j \leq n, \\ e_{n+1-i} + e_{j-n} & \text{if } j > n. \end{cases}$$

Furthermore set

$$A^C_\beta = \{\alpha_{i,j} : (i, j) \text{ is a valley of } \beta\}.$$

The following result is well-known and identifies ballot paths with the anti-chains in the root poset of type C_n.

LEMMA 5.2.5. [**74**, Lem. 4.5] *Let Φ be a root system of type C_n. The map $\varphi_C : \mathcal{B}_{2n} \to \mathrm{AC}(\Phi^+)$ given by $\beta \mapsto A^C_\beta$ is a bijection between ballot paths of length $2n$ and the set of anti-chains in the root poset Φ^+.* ■

A *diagonally labelled ballot path* of type C_n is a pair (s, β) of a signed permutation $s \in \mathfrak{S}^B_n$ and a ballot path $\beta \in \mathcal{B}_{2n}$ such that

$$s(n+1-i) > \begin{cases} s(n+1-j) & \text{if } j \leq n, \\ s(j-n) & \text{if } j > n. \end{cases}$$

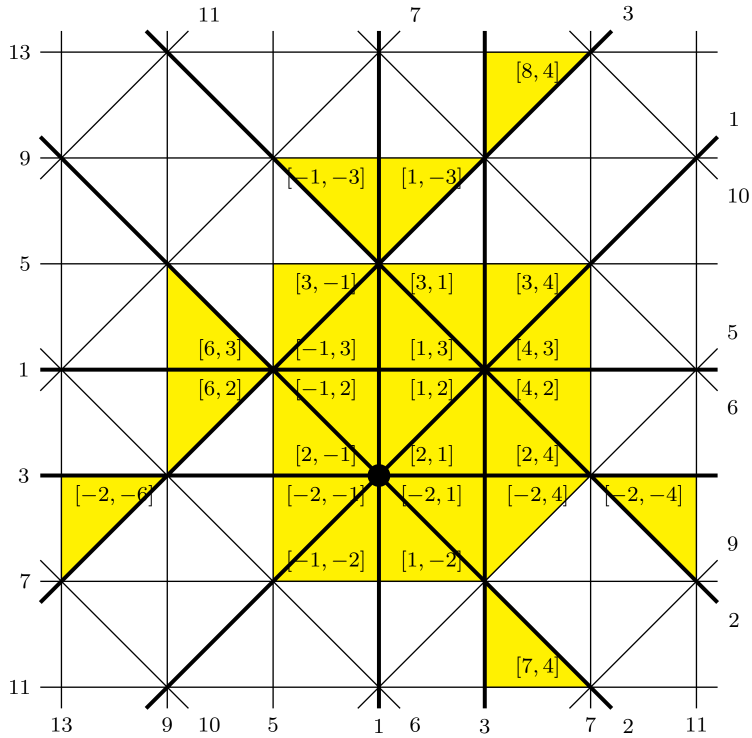


FIGURE 5.3. The Shi arrangement of type C_2.

for each valley (i,j) of β. We denote the set of all diagonally labelled ballot paths of type C_n by $\mathrm{Diag}(C_n)$.

We picture diagonally labelled ballot paths of type C_n as follows. Given $s \in \mathfrak{S}_n^B$ and $\beta \in \mathcal{B}_{2n}$ place the labels $s(i)$, where $i = n, n-1, \dots, 1, -1, \dots, -n$, in the diagonal as in Figure 5.4. Then $(s,\beta) \in \mathrm{Diag}(C_n)$ if and only if for each valley of β the label to its right is smaller than the label below it. In particular, if the path ends with an east step then the label below must be positive.

Taking labels into account, we extend Lemma 5.2.5 to a bijection between diagonally labelled ballot paths and non-nesting parking functions.

PROPOSITION 5.2.6. [**74**, Prop. 4.7] *The map* $\varphi_C : \mathrm{Diag}(C_n) \to \mathrm{Park}(C_n)$ *given by* $(s,\beta) \mapsto [s, A_\beta^C]$ *is a bijection.*

PROOF. Let $s \in \mathfrak{S}_n^B$ a signed permutation and $\beta \in \mathcal{B}_{2n}$ be a ballot path. Using Lemmas 5.2.5 and 5.1.4 it suffices to show that $(s,\beta) \in \mathrm{Diag}(C_n)$ if and only if $s(A_\beta^C) \subseteq \Phi^+$. Assume $j \le n$. If $(s,\beta) \in \mathrm{Diag}(C_n)$ then

$$\begin{aligned}\alpha_{i,j} \in A_\beta^C &\Leftrightarrow (i,j) \text{ is a valley of } \beta \\ &\Rightarrow s(n+1-i) > s(n+1-j) \Leftrightarrow s(\alpha_{i,j}) \in \Phi^+.\end{aligned}$$

Conversely, if $s(A_\beta^C) \subseteq \Phi^+$ then

$$\begin{aligned}(i,j) \text{ is a valley of } \beta &\Leftrightarrow \alpha_{i,j} \in A_\beta^C \\ &\Rightarrow s(\alpha_{i,j}) \in \Phi^+ \Leftrightarrow s(n+1-i) > s(n+1-j).\end{aligned}$$

The case $j > n$ is treated similarly. ∎

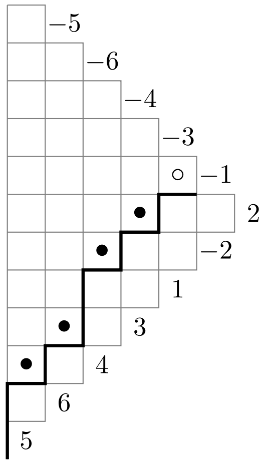


FIGURE 5.4. The diagonally labelled ballot path $([-2,1,3,4,6,5], \mathbf{nnenennenene}) \in \mathrm{Diag}(C_6)$.

We remark that a different combinatorial model for the regions of the type C_n Shi arrangement has been considered by Mészáros [**53**]. Her approach is related to the bijection of Athanasiadis and Linusson for type A_{n-1}.

5.2.3. Type D_n. Finally we present a lattice path interpretation of the non-nesting parking functions of type D_n in terms of labelled ballot paths of odd length. One aspect in which the root system of type D_n differs from the other infinite families is that the root poset is not planar. For this reason its anti-chains are seldom associated with lattice paths in the literature. Still there is a natural way to identify anti-chains with ballot paths of odd length by adding a sign to a certain east step.

A *signed ballot path* $\beta \in \mathcal{B}^{\bullet}_{2n-1}$ is a ballot path with $2n-1$ steps except that if its n-th north step is followed by an east step, then this east step is replaced by a *signed east step* from the set $\{\mathbf{e}^+, \mathbf{e}^-\}$. Define a *sign* function $\epsilon : \mathcal{B}^{\bullet}_{2n-1} \to \{\pm 1\}$ in the same way as for signed lattice paths, that is, $\epsilon(\beta) = -1$ if β contains the step $\mathbf{e}^-$ and $\epsilon(\beta) = 1$ otherwise. For example, the paths in $\mathcal{B}^{\bullet}_3 = \{\mathbf{nen}, \mathbf{nne}^+, \mathbf{nne}^-, \mathbf{nnn}\}$ are drawn in Figure 5.5.

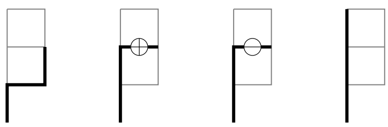


FIGURE 5.5. The set $\mathcal{B}^{\bullet}_3$ of signed ballot paths of length three.

Valleys of signed ballot paths obey the same conventions as valleys of ballot paths, with one exception. If β has a valley of the form (i,n) and the n-th north step of β is not followed by an east step, then $(i, n+1)$ is also counted as a valley of β. Let $\beta \in \mathcal{B}^{\bullet}_{2n-1}$ be a signed ballot path and (i,j) a valley of β. Define the corresponding positive root as

$$\alpha_{i,j} = \begin{cases} e_{n+1-i} - e_{n+1-j} & \text{if } j \leq n-1, \\ e_{n+1-i} - \epsilon(\beta) e_1 & \text{if } j = n, \\ e_{n+1-i} + \epsilon(\beta) e_1 & \text{if } j = n+1, \\ e_{n+1-i} + e_{j-n} & \text{if } j \geq n+2. \end{cases}$$

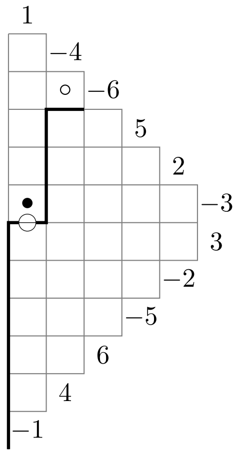


FIGURE 5.6. A diagonally labelled signed ballot path $(s, \beta) \in \mathrm{Diag}(D_n)$. We have $\epsilon(\beta) = -1$, $s = [-3, -2, -5, 6, 4, -1]$, and $A_\beta^D = \{e_6 - e_1, e_5 + e_4\}$.

Furthermore set

$$A_\beta^D = \big\{\alpha_{i,j} : (i,j) \text{ is a valley of } \beta\big\}.$$

Note that if β has a valley of the form (i, n) and the n-th north step of β is not followed by an east step, then both $\alpha_{i,n}$ and $\alpha_{i,n+1}$ are added to A_β. It is easy to check that this correspondence identifies signed ballot paths with anti-chains in the root poset.

LEMMA 5.2.7. [**74**, Prop. 5.8] *Let Φ be a root system of type D_n. The map $\varphi_D : \mathcal{B}^\bullet_{2n-1} \to \mathrm{AC}(\Phi^+)$ given by $\beta \mapsto A_\beta^D$ is a bijection between signed ballot paths and anti-chains in the root poset.* ■

A *diagonally labelled signed ballot path* (s, β) is a pair of an even signed permutation $s \in \mathfrak{S}_n^D$ and a signed ballot path $\beta \in \mathcal{B}^\bullet_{2n-1}$ such that for each valley (i, j) of β we have

$$s(n+1-i) > \begin{cases} s(n+1-j) & \text{if } j \leq n-1, \\ \epsilon(\beta)s(1) & \text{if } j = n, \\ -\epsilon(\beta)s(1) & \text{if } j = n+1, \\ s(n-j) & \text{if } j \geq n+2. \end{cases}$$

Denote the set of all diagonally labelled signed ballot paths by $\mathrm{Diag}(D_n)$.

Diagonally labelled signed lattice paths can be visualised as follows. Given an even signed permutation $s \in \mathfrak{S}_n^D$ and a signed ballot path $\beta \in \mathcal{B}^\bullet_{2n-1}$, place the labels $s(i)$, where $i = n, n-1, \ldots, 2, \epsilon(\beta), -\epsilon(\beta), -2, \ldots, -n$, on the diagonal as in Figure 5.6. Thereby, $(s, \beta) \in \mathrm{Diag}(D_n)$ if and only if for each valley the label to its right is smaller than the label below it.

We conclude this section by extending Lemma 5.2.7 to a bijection between diagonally labelled signed ballot paths an non-nesting parking functions of type D_n.

PROPOSITION 5.2.8. [**74**, Prop. 5.12] *The map $\varphi_D : \mathrm{Diag}(D_n) \to \mathrm{Park}(D_n)$ given by $(s, \beta) \mapsto [s, A_\beta^D]$ is a bijection.*

PROOF. Let $s \in \mathfrak{S}_n^D$ be an even signed permutation and $\beta \in \mathcal{B}^\bullet_{2n-1}$ a signed ballot path. Using Lemmas 5.2.7 and 5.1.4 it suffices to show that $(s, \beta) \in \mathrm{Diag}(D_n)$ if and only if $s(A_\beta^D) \subseteq$

Φ^+. This can be accomplished easily distinguishing a few cases. For example, assume that $(s,\beta) \in \mathrm{Diag}(D_n)$. Then

$$\begin{aligned} \alpha_{i,n} = e_{n+1-i} - \epsilon(\beta)e_1 \in A^D_\beta &\Leftrightarrow (i,n) \text{ is a valley of } \beta \\ &\Rightarrow s(n+1-i) > \epsilon(\beta)s(1) \Leftrightarrow s(\alpha_{i,n}) \in \Phi^+. \end{aligned}$$

Conversely, suppose that $s(A^D_\beta) \subseteq \Phi^+$. Then

$$\begin{aligned} (i,n) \text{ is a valley of } \beta &\Leftrightarrow \alpha_{i,n} \in A^D_\beta \\ &\Rightarrow s(\alpha_{i,n}) \in \Phi^+ \Leftrightarrow s(n+1-i) > \epsilon(\beta)s(1). \end{aligned}$$

Other roots are treated similarly. ■

5.3. The area-statistic

In this section the non-nesting parking functions are used to define q-analogues of the Coxeter–Catalan and Coxeter–Cayley numbers.

Let Φ be an irreducible crystallographic root system. To each anti-chain $A \subseteq \Phi^+$ associate an order ideal

$$I(A) = \Phi^+ - \bigcup_{\alpha \in A} \{\beta \in \Phi^+ : \alpha \leq \beta\}.$$

Note that $I(A)$ is the complement of the minimal order filter containing A, hence it is the maximal order ideal not containing any elements of A. Define the *uniform area statistic* $\mathrm{area} : \mathrm{AC}(\Phi^+) \to \mathbb{N}$ as the cardinality

$$\mathrm{area}(A) = |I(A)|\,.$$

Moreover, define the statistic $\mathrm{area}' : \mathrm{Park}(\Phi) \to \mathbb{N}$ by

$$\mathrm{area}'(X) = \#\big\{\alpha \in I(A) : s(\alpha) \in \Phi^+\big\},$$

where $(s,A) \in X$ is assumed to be the canonical representative as in Lemma 5.1.4.

Equivalent definitions of these statistics make use of the Shi arrangement and are due to Armstrong [**5**] and Stump [**72**].

The following proposition establishes that the statistics area and area$'$ can be used to generalise the q-Catalan numbers of Section 2.5 and their Cayley analogues from Section 2.6 to the level of Weyl groups. In fact we shall see in Theorem 6.2.2 that the polynomials

$$C_\Phi(q) = \sum_{A \in \mathrm{AC}(\Phi^+)} q^{\mathrm{area}(A)} \qquad \text{and} \qquad \mathscr{C}_\Phi(q) = \sum_{[s,A] \in \mathrm{Park}(\Phi)} q^{\mathrm{area}'([s,A])}$$

agree with the polynomials $C_\Phi(q)$ and $\mathscr{C}_\Phi(q)$ defined in Section 4.5.

PROPOSITION 5.3.1. *Let Φ be the root system of type A_{n-1} and $(s,\pi) \in \mathrm{Diag}(A_{n-1})$. Then* $\mathrm{area}(\pi) = \mathrm{area}(A_\pi)$ *and* $\mathrm{area}'(s,\pi) = \mathrm{area}'([s,A_\pi])$*. In particular,*

$$C_n(q) = \sum_{A \in \mathrm{AC}(\Phi^+)} q^{\mathrm{area}(A)} \qquad \textit{and} \qquad \mathscr{C}_n(q) = \sum_{[s,A] \in \mathrm{Park}(\Phi)} q^{\mathrm{area}'([s,A])}.$$

PROOF. The claim follows directly from Lemma 5.2.1 and Proposition 5.2.2 and the definitions of the bijections used therein. ■

Combinatorial area statistics were also defined in type C_n [**74**]. Let $(s,\beta) \in \mathrm{Diag}(C_n)$ be a diagonally labelled ballot path. Then $\mathrm{area}_C(\beta)$ denotes the number of unit squares between the ballot path and the diagonal. Moreover, $\mathrm{area}'_C(s,\beta)$ denotes the number of unit squares below the path such that the label to its right is less than the label below it. See Figure 5.7

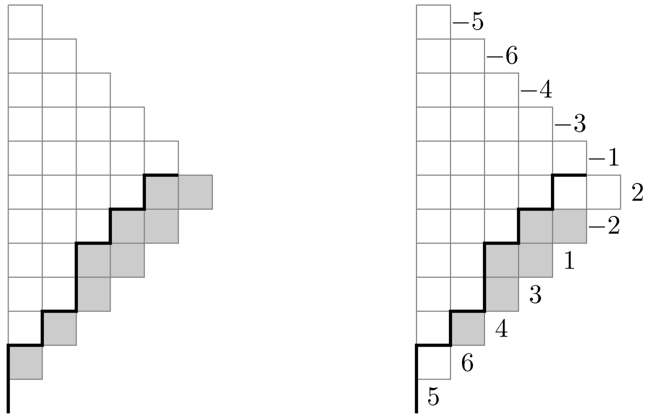


FIGURE 5.7. A ballot path $\beta \in \mathcal{B}_{12}$ with $\operatorname{area}_C(\beta) = 9$ (left), and a diagonally labelled ballot path $(s, \beta) \in \operatorname{Diag}(C_6)$ with $\operatorname{area}'_C(s, \beta) = 6$ (right). The contributing squares below the paths are shaded grey.

It is an easy consequence of Lemma 5.2.5 and Proposition 5.2.6 that the uniform statistics area and area′ generalise their combinatorial counterparts in type C_n.

PROPOSITION 5.3.2. *Let Φ be a root system of type C_n and $(s, \beta) \in \operatorname{Diag}(C_n)$. Then* $\operatorname{area}_C(\beta) = \operatorname{area}(A_\beta)$ *and* $\operatorname{area}'_C(s, \beta) = \operatorname{area}'([s, A_\beta])$. ■

5.4. Rational Shi tableaux

Having found such a nice Fuß-analogue of the Shi arrangement, namely the m-Shi arrangement, it is only natural to ask for a rational analogue. Since the m-Shi arrangement consists precisely of the hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$ such that $|\operatorname{ht}(-\alpha + k\delta)| < mh+1$, a first natural candidate for a rational analogue is the arrangement

$$\{H_{\alpha,k} : \alpha \in \Phi^+, k \in \mathbb{Z}, |\operatorname{ht}(-\alpha + k\delta)| < p\}.$$

In terms of Shi alcoves this corresponds to the set of elements $\omega \in \widetilde{W}$ such that all floors of $\omega(A_\circ)$ have height less than p. Unfortunately these sets do not seem to have similarly appealing enumerative properties as the Shi arrangement itself. Instead a different interpretation of Shi alcoves comes to the rescue.

THEOREM 5.4.1. [**75**, Thm. 4.3.6] *Let Φ be an irreducible crystallographic root system, $m \in \mathbb{N}$ and $\omega \in \widetilde{W}$ an element of the affine Weyl group. Then $\omega(A_\circ)$ is the minimal alcove of a region of* $\operatorname{Shi}^m(\Phi)$ *if and only if ω has no inversions of height $mh + 1$, that is, $\omega \in \widetilde{W}^{mh+1}$.*

Hence the p-stable elements of the affine Weyl group known to us from Section 4.4 are a perfectly suitable rational analogue of Shi alcoves. In the remainder of this section we aim to strengthen this claim by investigating whether additional properties of Shi alcoves are inherited by the elements of $\widetilde{W}^p$. We first demonstrate that the set of p-stable elements of the affine Weyl group can be equipped with a W-action similar to the action on non-nesting parking functions.

Define an equivalence relation on the set of pairs $W \times \widetilde{W}^p_+$ by letting $(s, \omega) \sim (t, \omega')$ if and only if $\omega = \omega'$ and $sH = tH$, where $H \leq W$ is the subgroup generated by the set

$$\{s_\alpha : \alpha \in \Phi^+ \cap \omega \cdot (\widetilde{\Phi}_p)\}.$$

The set of equivalence classes

$$\widetilde{W}^{(p)} = \big\{[s,\omega]_\sim : s \in W, \omega \in \widetilde{W}^p_+\big\}$$

carries a natural W-action, given by $s \cdot [t,\omega] = [st,\omega]$ for all $s,t \in W$ and $\omega \in \widetilde{W}^p_+$.

PROPOSITION 5.4.2. *Let Φ be an irreducible crystallographic root system and p be relatively prime to the Coxeter number. Then each class $[t,\omega] \in \widetilde{W}^{(p)}$ contains a unique element (s,ω) such that $s\omega \in \widetilde{W}^p$. The induced map $\phi : \widetilde{W}^{(p)} \to \widetilde{W}^p$ defined by $[t,\omega] \mapsto s\omega$ is a bijection.*

PROOF. First note that $\Phi^+ \cap \omega \cdot (\widetilde{\Phi}_p)$ is an anti-chain in the root poset Φ^+. To see this suppose that $\alpha, \beta, \alpha+\beta \in \Phi^+$. Then

$$\mathrm{ht}(\omega^{-1}(\alpha+\beta)) = \mathrm{ht}(\omega^{-1}(\alpha)) + \mathrm{ht}(\omega^{-1}(\beta)) \neq \mathrm{ht}(\omega^{-1}(\alpha)).$$

For $s \in W$ and $\omega \in \widetilde{W}^p$ we have $s\omega \in \widetilde{W}^p$ if and only if $s(\alpha) \in \Phi^+$ for all $\alpha \in \Phi^+ \cap \omega \cdot (\widetilde{\Phi}_p)$. Thus the existence of a unique representative $(s,\omega) \in [t,\omega]$ with $s\omega \in \widetilde{W}^p$ follows from [**65**, Thm. 6.4].
The map ϕ is clearly injective. Moreover [**75**, Lem. 3.9.2] asserts that for any $\omega \in \widetilde{W}^p$ and $s \in W$ with $\omega(A_\circ) \subseteq sC_\circ$ also $s^{-1}\omega \in \widetilde{W}^p$. Hence the ϕ is also surjective. ■

Furthermore, we would like the fact that each dominant Shi alcove is uniquely determined by the Shi hyperplanes that separate it from the fundamental alcove to hold in the rational setting. See Conjecture 5.4.10 for a precise formulation. In passing we use the set $\widetilde{W}^p$ to define a common generalisation of the polynomials $C_{n,p}(q)$ and $C_\Phi(q)$. Thus we reach the rational q-Coxeter–Catalan numbers in the Catalan cube.
However, we first need to return to the study affine inversions initiated in Section 3.2.
For each $\alpha \in \Phi^+$ and $\omega \in \widetilde{W}$ the number $|k(\alpha,\omega)|$ counts how many hyperplanes of the form $H_{\alpha,k}$ with $k \in \mathbb{Z}$ separate $\omega(A_\circ)$ from the fundamental alcove. Equivalently, $|k(\alpha,\omega)|$ counts the inversions of ω^{-1} of the form $\alpha + k\delta$. That is,

$$|k(\alpha,\omega)| = \#\big\{\pm\alpha + k\delta \in \widetilde{\Phi}^+ \cap \omega \cdot (-\widetilde{\Phi}^+)\big\}.$$

The sum

$$\sum_{\alpha\in\Phi^+} |k(\alpha,\omega)|$$

therefore gives the number of inversions of ω and also equals the length of ω. Furthermore the map $\mathbf{t}(\omega) : \Phi^+ \to \mathbb{N}$ given by $\alpha \mapsto |k(\alpha,\omega)|$ equals the address of ω if and only if ω is dominant. We call $\mathbf{t}(\omega)$ the *inversion table* of ω. Note that two elements of $\widetilde{W}$ can have the same inversion table. However, if we restrict the domain to dominant elements of the affine Weyl group, then the map $\mathbf{t} : \widetilde{W}_+ \to \{t : \Phi^+ \to \mathbb{N}\}$ is injective. All of this was first observed by Shi [**60, 62**] and later rediscovered by Björner and Brenti [**16**] in type A_{n-1}.
The inversion table can be computed directly using the decomposition of the affine Weyl group into a semi-direct product.

LEMMA 5.4.3. [**73**, Lem. 2.7] *Let $\alpha \in \Phi^+$ be a positive root and $\omega \in \widetilde{W}$ an element of the affine Weyl group. If $\omega = t_q s$, where $q \in \check{Q}$ and $s \in W$, then*

$$|k(\alpha,\omega)| = \begin{cases} |\langle q,\alpha\rangle| & \text{if } s^{-1}\cdot\alpha \in \Phi^+, \\ |\langle q,\alpha\rangle - 1| & \text{if } s^{-1}\cdot\alpha \in -\Phi^+. \end{cases}$$

PROOF. Set $\beta = s^{-1}(\alpha)$. For $k \geq 0$

$$\omega^{-1}\cdot(\alpha + k\delta) = \beta + (k + \langle q,\alpha\rangle)\delta \in -\widetilde{\Phi}^+$$

$$\begin{array}{llllll} k_{1,7} & k_{2,7} & k_{3,7} & k_{4,7} & k_{5,7} & k_{6,7} \\ k_{1,6} & k_{2,6} & k_{3,6} & k_{4,6} & k_{5,6} & \\ k_{1,5} & k_{2,5} & k_{3,5} & k_{4,5} & & \\ k_{1,4} & k_{2,4} & k_{3,4} & & & \\ k_{1,3} & k_{2,3} & & & & \\ k_{1,2} & & & & & \end{array} \quad = \quad \begin{array}{cccccc} 5 & 5 & 3 & 2 & 2 & 2 \\ 3 & 2 & 1 & 0 & 0 & \\ 2 & 2 & 1 & 0 & & \\ 2 & 2 & 1 & & & \\ 1 & 1 & & & & \\ 0 & & & & & \end{array}$$

FIGURE 5.8. The inversion table of the affine permutation $\omega = [-2, 15, -1, 16, -14, 10, 4] \in \widetilde{\mathfrak{S}}_7$ with inverse $\omega^{-1} = [-12, -10, -1, 7, 8, 10, 26]$.

if and only if either $k < -\langle q, \alpha\rangle$ or both $k = -\langle q, \alpha\rangle$ and $\beta \in -\Phi^+$. On the other hand for $k \geq 1$

$$\omega^{-1} \cdot (-\alpha + k\delta) = -\beta + (k - \langle q, \alpha\rangle)\delta \in -\widetilde{\Phi}^+$$

if and only if $k < \langle q, \alpha\rangle$ or both $k = \langle q, \alpha\rangle$ and $\beta \in \Phi^+$. Combined this implies the claim. ∎

The type A_{n-1} inversion table is closely related to the contents of Section 3.2. For $i, j \in [n]$ with $i < j$ and $\omega \in \widetilde{\mathfrak{S}}_n$ set $k_{i,j}(\omega) = |k(e_i - e_j, \omega)|$. We arrange the numbers $k_{i,j}(\omega)$ in a staircase tableaux as in Figure 5.8.

PROPOSITION 5.4.4. [**73**, Prop. 2.8] *Let $i, j \in [n]$ with $i < j$ and $\omega \in \widetilde{\mathfrak{S}}_n$. Then*

$$k_{i,j}(\omega) = \left| \left\lfloor \frac{\omega^{-1}(j) - \omega^{-1}(i)}{n} \right\rfloor \right|.$$

PROOF. Write $\omega = t_q s$, where $q \in \check{Q}$ and $s \in \mathfrak{S}_n$. The claim follows from Lemma 5.4.3 and

$$\begin{aligned} \left| \left\lfloor \frac{\omega^{-1}(j) - \omega^{-1}(i)}{n} \right\rfloor \right| &= \left| \left\lfloor \frac{-q_j n + s^{-1}(j) - (-q_i n + s^{-1}(i))}{n} \right\rfloor \right| \\ &= \begin{cases} |q_i - q_j| & \text{if } s^{-1}(i) < s^{-1}(j), \\ |q_i - q_j - 1| & \text{if } s^{-1}(i) > s^{-1}(j). \end{cases} \end{aligned}$$ ∎

Recall that the numbers $k_{i,j}(\omega)$ already appeared in Lemma 3.2.1 with the only difference that now we consider them from the perspective of dominant affine permutations instead of Graßmannian affine permutations. If ω is dominant, then by Lemma 3.2.2 the partition given by the row-sums of the inversion table of ω equals $\mathbf{inv}(\omega^{-1})$. That is,

$$\mathbf{inv}(\omega^{-1}, j) = \sum_{i=1}^{n} k_{i,j}(\omega)$$

for $j \in [n]$.

A simple computation reveals the effect of the involutive automorphism on the inversion table.

LEMMA 5.4.5. [**73**, Prop. 2.6] *Let $\omega \in \widetilde{\mathfrak{S}}_n$ be an affine permutation. Then the inversion table of ω^* is the transpose of the inversion table of ω. That is, $k_{i,j}(\omega^*) = k_{n+1-j,n+1-i}(\omega)$ for all $i, j \in [n]$ with $i < j$.*

PROOF. We compute

$$\begin{aligned} k_{i,j}(\omega^*) &= \left| \left\lfloor \frac{n+1-(\omega^*)^{-1}(i)-(n+1)+(\omega^*)^{-1}(j)}{n} \right\rfloor \right| \\ &= \left| \left\lfloor \frac{(\omega^{-1})^*(n+1-i)-(\omega^{-1})^*(n+1-j)}{n} \right\rfloor \right| \\ &= \left| \left\lfloor \frac{\omega^{-1}(n+1-i)-\omega^{-1}(n+1-j)}{n} \right\rfloor \right| = k_{n+1-j,n+1-i}(\omega). \end{aligned}$$ ∎

As a consequence of Lemma 5.4.5 the partition formed by the column-sums $\sum_j k_{i,j}(\omega)$of the inversion table of ω equals $\mathbf{inv}^*(\omega^{-1})$. We obtain an interesting alternative description of the n-conjugation on bounded partitions by partitioning the summands of the bounded partition in a clever way and recombining them afterwards.

Fishel, Tzanaki and Vazirani [**27**] encoded the dominant regions of the m-Shi arrangement using Shi tableaux, which are closely related to inversion tables. Although they only considered type A_{n-1} at the time, their definition applies to the uniform case without changes.

Given $\omega \in \widetilde{W}_+^{mh+1}$ and $\alpha \in \Phi^+$ set

$$t^{mh+1}(\alpha, \omega) = \min(k(\alpha, \omega), m).$$

The map $\mathbf{t}^{mh+1}(\omega) : \Phi^+ \to \mathbb{N}$ given by $\alpha \mapsto t^{mh+1}(\alpha, \omega)$ is called the *m-Shi tableau* of ω. The idea behind the definition of the Shi tableau of $\omega \in \widetilde{W}_+^{mh+1}$ is to consider only those inversions that stem from a hyperplane in $\mathrm{Shi}^m(\Phi)$. The set of Shi tableaux was characterised by the work of Athanasiadis [**11**] on geometric chains of ideals in the root poset.

THEOREM 5.4.6. [**27**, Lem. 2.5] *The map*

$$\mathbf{t}^{mh+1} : \widetilde{W}_+^{mh+1} \to \{t : \Phi^+ \to \mathbb{N}\}$$

is a bijection onto its image. Furthermore $t : \Phi^+ \to \mathbb{N}$ lies in the image of $\mathbf{t}^{mh+1}$ if and only if

$$\min(t(\alpha) + t(\beta), m) \leq t(\alpha + \beta) \leq \min(t(\alpha) + t(\beta) + 1, m)$$

whenever $\alpha, \beta, \alpha + \beta \in \Phi^+$.

We generalise Shi tableaux to the rational level of Catalan combinatorics as follows. Given $\omega \in \widetilde{W}_+^p$ and $\alpha \in \Phi^+$ define

$$t^p(\alpha, \omega) = \#\big\{ -\alpha + k\delta \in \widetilde{\Phi}^+ \cap \omega \cdot (-\widetilde{\Phi}^+_{<p}) \big\}.$$

The map $\mathbf{t}^p(\omega) : \Phi^+ \to \mathbb{N}$ defined by $\alpha \mapsto t^p(\alpha, \omega)$ is called *rational Shi tableau* of ω. Note that $t^p(\alpha, \omega)$ counts certain affine inversions of ω^{-1}, which correspond to separating hyperplanes of ω. Proposition 5.4.9 below shows that $\mathbf{t}^p(\omega)$ is well-defined when $p = mh + 1$.

The following is a useful lemma on dominant elements of the affine Weyl group.

LEMMA 5.4.7. *Let $\alpha \in \Phi^+$ be a positive root and $\omega \in \widetilde{W}_+$ be dominant with $\omega = t_q s$, where $q \in \check{Q}$ and $s \in W$. If $\langle q, \alpha \rangle = 0$ then $s^{-1}(\alpha) \in \Phi^+$.*

PROOF. Suppose $\langle q, \alpha \rangle = 0$. The height function $\mathrm{ht} : \Phi \to \mathbb{R}$ extends to a linear functional on V. Thus we may choose $v \in V$ with $\langle v, \beta \rangle = \mathrm{ht}(\beta)/h$ for all $\beta \in \Phi$, where h is the Coxeter number of Φ. Note that $v \in A_\circ$ by definition. Thus $\langle \omega(v), \alpha \rangle > 0$ since ω is dominant. We compute

$$\frac{\mathrm{ht}(s^{-1}(\alpha))}{h} = \langle v, s^{-1}(\alpha) \rangle = \langle s(v), \alpha \rangle = \langle q + s(v), \alpha \rangle = \langle \omega(v), \alpha \rangle > 0.$$ ∎

The rational Shi tableau of a dominant p-stable element of the affine Weyl group can be computed directly using the decomposition of the affine Weyl group into a semi-direct product.

Proposition 5.4.8. [**73**, Prop. 2.11] *Let $m \in \mathbb{N}$ and $r \in [h-1]$, where h is the Coxeter number of Φ, and $\alpha \in \Phi^+$ be a positive root. Set $p = mh + r$ and let $\omega \in \widetilde{W}^p_+$ be dominant and p-stable. If $\omega = t_q s$, where $q \in \check{Q}$ and $s \in W$, then*

$$t^p(\alpha,\omega) = \begin{cases} \min(k(\alpha,\omega), m) & \textit{if } r-h < \mathrm{ht}(s^{-1}(\alpha)) < 0 \textit{ or } r < \mathrm{ht}(s^{-1}(\alpha)), \\ \min(k(\alpha,\omega), m+1) & \textit{otherwise.} \end{cases}$$

Proof. Set $\beta = s^{-1}(\alpha)$. Recall that $\langle q, \alpha\rangle \geq 0$ because ω is dominant. If $\mathrm{ht}(\beta) = r$ then $\beta + m\delta \in \widetilde{\Phi}_p$ and therefore

$$\alpha + (m - \langle q,\alpha\rangle)\delta = \omega \cdot (\beta + m\delta) \in \widetilde{\Phi}^+$$

since $\omega \in \widetilde{W}^p$. If instead $\mathrm{ht}(\beta) = r - h$ then $\beta + (m+1)\delta \in \widetilde{\Phi}_p$ and

$$\alpha + (m + 1 - \langle q,\alpha\rangle)\delta = \omega \cdot (\beta + (m+1)\delta) \in \widetilde{\Phi}^+.$$

Consequently $\mathrm{ht}(\beta) = r$ implies $\langle q,\alpha\rangle \leq m$, and $\mathrm{ht}(\beta) = r-h$ implies $\langle q,\alpha\rangle \leq m+1$. Now let $k \geq 1$. Then

$$0 < \mathrm{ht}\left(-\omega^{-1} \cdot (-\alpha + k\delta)\right) = \mathrm{ht}(\beta) + (\langle q,\alpha\rangle - k)h < p = mh + r$$

if and only if one of the following (mutually exclusive) cases occurs

$$\begin{gathered} m > 0 \text{ and } k = \langle q,\alpha\rangle \text{ and } 0 < \mathrm{ht}(\beta), \\ m = 0 \text{ and } k = \langle q,\alpha\rangle \text{ and } 0 < \mathrm{ht}(\beta) < r, \\ -m < k - \langle q,\alpha\rangle < 0, \\ m > 0 \text{ and } -m = k - \langle q,\alpha\rangle \text{ and } \mathrm{ht}(\beta) < r, \\ -m-1 = k - \langle q,\alpha\rangle \text{ and } \mathrm{ht}(\beta) < -h + r. \end{gathered}$$

Equivalently $-\alpha + k\delta$ contributes to $t^p(\alpha,\omega)$ if and only if

$$k \in \{1,2,\dots\} \cap \begin{cases} \{\langle q,\alpha\rangle - m + 1, \dots, \langle q,\alpha\rangle\} & \text{if } r \leq \mathrm{ht}(\beta), \\ \{\langle q,\alpha\rangle - m, \dots, \langle q,\alpha\rangle\} & \text{if } 0 < \mathrm{ht}(\beta) \leq r, \\ \{\langle q,\alpha\rangle - m, \dots, \langle q,\alpha\rangle - 1\} & \text{if } r - h \leq \mathrm{ht}(\beta) < 0, \\ \{\langle q,\alpha\rangle - m - 1, \dots, \langle q,\alpha\rangle - 1\} & \text{if } \mathrm{ht}(\beta) \leq r - h. \end{cases}$$

The claim now follows from Lemma 5.4.3. Note that in the last two cases $\langle q,\alpha\rangle - 1 \geq 0$ is ensured by Lemma 5.4.7. ∎

Given $\omega \in \widetilde{W}^p_+$ the map $\chi^p(\omega) : \Phi^+ \to \mathbb{N}$ defined by

$$\chi^p(\alpha,\omega) = \begin{cases} m & \text{if } r - h < \mathrm{ht}(s^{-1}\cdot\alpha) < 0 \text{ or } r < \mathrm{ht}(s^{-1}\cdot\alpha), \\ m+1 & \text{otherwise,} \end{cases}$$

where $p = mh + r$ with $m \in \mathbb{N}$ and $r \in [h-1]$ and $\omega = t_q s$ with $q \in \check{Q}$ and $s \in W$, is called *min-tableau* of ω.

Armed with Proposition 5.4.8 we can affirm that the rational Shi tableau generalises the m-Shi tableau of Fishel, Tzanaki and Vazirani. That is, the two definitions of $t^{mh+1}(\alpha,\omega)$ agree.

Proposition 5.4.9. *Let $\alpha \in \Phi^+$ be a positive root and $\omega \in \widetilde{W}^{mh+1}_+$. Then*

$$\min(k(\alpha,\omega), m) = \#\left\{ -\alpha + k\delta \in \widetilde{\Phi}^+ \cap \omega \cdot (-\widetilde{\Phi}^+_{<mh+1})\right\}.$$

Proof. By Proposition 5.4.8 it suffices to show that $k(\alpha,\omega) \leq m$ whenever $\chi^{mh+1}(\alpha,\omega) = m+1$, which is the case if and only if $s^{-1}(\alpha) \in \Delta \cup \{-\tilde{\alpha}\}$. If $s^{-1}(\alpha) = -\tilde{\alpha}$ then

$$\alpha + (m+1-\langle q,\alpha\rangle)\delta = \omega(-\tilde{\alpha} + (m+1)\delta) \in \omega(\widetilde{\Phi}^{mh+1}) \subseteq \widetilde{\Phi}^+$$

since $\omega \in \widetilde{W}_+^{mh+1}$ and thus $\langle q,\alpha\rangle - 1 \leq m$. If $s^{-1}(\alpha) \in \Delta$ then

$$\alpha + (m-\langle q,\alpha\rangle)\delta = \omega(s^{-1}(\alpha) + m\delta) \in \omega(\widetilde{\Phi}^{mh+1}) \subseteq \widetilde{\Phi}^+$$

and therefore $\langle q,\alpha\rangle \leq m$. The claim follows from Lemma 5.4.3. ■

Note that the min-tableau of an element $\omega \in \widetilde{W}_+^{mh+1}$ is not in general constant.

The following conjecture and open problem are in the spirit of Theorem 5.4.6.

Conjecture 5.4.10. [**73**, Conj. 2.13] *Let Φ be an irreducible crystallographic root system and p be a positive integer relatively prime to the Coxeter number. Then the map $\mathbf{t}^p : \widetilde{W}_+^p \to \{t : \Phi^+ \to \mathbb{N}\}$ is injective.*

Open Problem 5.4.11. *Characterise the set $\mathbf{t}^p(\widetilde{W}_+^p) \subseteq \{t : \Phi^+ \to \mathbb{N}\}$.*

Conjecture 5.4.10 is known to be true if $p = mh \pm 1$. The case $p = mh - 1$ is related to bounded regions of the m-Shi arrangement and was studied by Athanasiadis and Tzanaki [**14**]. We show that the conjecture holds in type A_{n-1}.

Theorem 5.4.12. [**73**, Thm. 2.14] *Let Φ be a root system of type A_{n-1} and p be relatively prime to n. Then the map $\mathbf{t}^p : \widetilde{W}_+^p \to \{t : \Phi^+ \to \mathbb{N}\}$ is injective.*

The proof of Theorem 5.4.12 turns out to be very combinatorial as it uses many of the bijections encountered in our study of Dyck paths and cores. First we observe that the rational Shi tableau behaves similarly to the inversion table under the involutive automorphism. To simplify notation set $t^p_{i,j}(\omega) = t^p(e_i - e_j, \omega)$ and $\chi^p_{i,j}(\omega) = \chi^p(e_i - e_j, \omega)$ for $i, j \in [n]$ with $i < j$ when Φ is of type A_{n-1}.

Proposition 5.4.13. [**73**, Prop. 2.9] *Let $\omega \in \widetilde{\mathfrak{S}}_n^p$ be a dominant p-stable affine permutation. Then the rational Shi tableau of ω^* is the transpose of the rational Shi tableau of ω. That is, $t^p_{i,j}(\omega^*) = t^p_{n+1-j,n+1-i}(\omega)$ for all $i, j \in [n]$ with $i < j$.*

Proof. The claim follows from Lemma 5.4.5, Proposition 5.4.8 and the fact that the involutive automorphism also transposes the min-tableau. To see this note that $\mathrm{ht}(e_i - e_j) = j - i$ and

$$k \leq s(j) - s(i) \Leftrightarrow k \leq s^*(n+1-i) - s^*(n+1-j),$$

for all $k \in \mathbb{Z}$, $i, j \in [n]$ and $s \in \mathfrak{S}_n$ due to Lemma 1.4.5 (i). ■

The following theorem is a key step towards the proof of Theorem 5.4.12.

Theorem 5.4.14. [**73**, Thm. 3.2] *Let n, p be positive coprime integers and $\omega \in \widetilde{\mathfrak{S}}_n^p$ be a dominant p-stable affine permutation. Then the rational Shi tableau of ω equals the codinv tableau of $\mathcal{A} \circ \alpha^{-1} \circ \gamma(\omega)$. That is, $t^p_{i,j}(\omega) = d_{i,j}(\mathcal{A} \circ \alpha^{-1} \circ \gamma(\omega))$ for all $i, j \in [n]$ with $i < j$.*

Proof. Let $\omega^{-1} = t_q s$ where $q \in \check{Q}$ and $s \in \mathfrak{S}_n$, and fix $i, j \in [n]$ such that $i < j$. By Lemma 3.2.2 $t^p_{i,j}(\omega)$ equals the number of inversions $(j, kn+i)$ of ω^{-1} such that

$$\omega^{-1}(j) - \omega^{-1}(kn+i) < p.$$

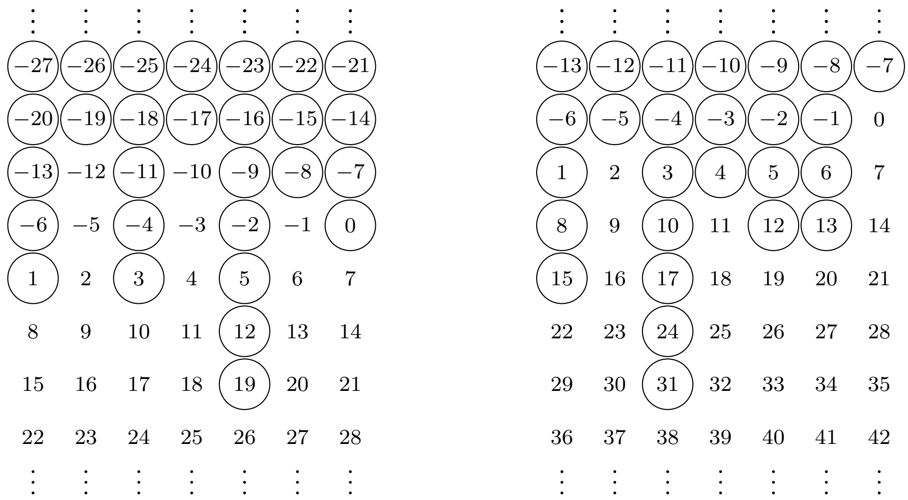


FIGURE 5.9. The balanced abacus $A = \gamma(\omega)$, where $\omega = [-2, 15, -1, 16, -14, 10, 4]$, (left) and the normalised abacus $B = \beta \circ \alpha^{-1}(A)$ (right), both depicted on 7 runners. With the notation of the proof of Theorem 5.4.14 we have $\omega^{-1} = [-12, -10, -1, 7, 8, 10, 26]$ and $s = [2, 4, 6, 7, 1, 3, 5]$ and $\sigma = [7, 2, 4, 5, 6, 1, 3]$. Moreover $\mathcal{A} \circ \alpha^{-1}(A) = \mathcal{A} \circ \beta^{-1}(B)$ is the rational Dyck path of Figure 3.13.

Let $A = \gamma(\omega)$ be an abacus on n runners (see Figure 5.9). Then $\omega^{-1}(j)$ is the minimal gap of A in the runner $s(j)$. Moreover, $(j, kn+i)$ is an inversion of ω^{-1} contributing to $t^p_{i,j}(\omega)$ if and only if $\omega^{-1}(kn+i)$ is a non-minimal gap of A in the runner $s(i)$ and

$$\omega^{-1}(j) - p < \omega^{-1}(nk+i) < \omega^{-1}(j).$$

Hence $t^p_{i,j}(\omega)$ counts the number of non-minimal gaps g in runner $s(i)$ such that $m - p < g < m$ where m is the minimal gap in runner $s(j)$.

Equivalently $t^p_{i,j}(\omega)$ counts the number of beads b in runner $s(j)$ such that $m < b < m+p$ where m is the minimal gap in runner $s(i)$. Define a normalised abacus on n runners, namely

$$B = \beta \circ \alpha^{-1} \circ \gamma = \{z + \ell - 1 : z \in A\},$$

where ℓ is the length of the partition $\alpha^{-1} \circ \gamma(\omega)$. Moreover define $\sigma \in \mathfrak{S}_n$ by $\sigma(i) \equiv s(i) + \ell - 1$ modulo n. Then $t^p_{i,j}(\omega)$ counts the number of beads b in the runner $\sigma(j)$ of B such that $m < b < m + p$ where m is the minimal gap in the runner $\sigma(i)$ of B.

Set $x = \mathcal{A} \circ \alpha^{-1} \circ \gamma(\omega) \in \mathfrak{D}_{n,p}$. Since B is normalised, the minimal gap of each runner of B is non-negative. Thus it is the same to consider only positive beads of B. But the positive beads of B are just the hook-lengths of the cells in the first column of $\alpha^{-1} \circ \gamma$ and therefore make up the set $H(x)$. Moreover the minimal gaps of the runners of B are just the labels of the north steps of x. (See the definition of a codinv pair.)

The theorem now follows from the observation that σ sorts the minimal gaps of B increasingly. This is implied by the fact that s sorts the minimal gaps of A increasingly, which form the window of the affine Graßmannian permutation ω^{-1}. ■

From Theorem 5.4.14 we derive many interesting consequences. Together with Proposition 5.4.13 and Proposition 3.1.9 we obtain an interesting result for free.

COROLLARY 5.4.15. [**73**, Cor. 3.3] *Let n, p be positive coprime integers and $x \in \mathfrak{D}_{n,p}$ be a rational Dyck path. Then the codinv tableau of $\rho(x)$ is the transpose of the codinv tableau of x.*

Corollary 5.4.15 is another strengthening of Corollary 3.4.5.

Pak and Stanley [**67, 69**] found a bijection between the regions of the (m-extended) Shi arrangement of type A_{n-1} and the set of parking functions $\mathrm{PF}_{n,mn+1}$. Gorsky, Mazin and Vazirani [**33**, Def. 3.8] generalised this bijection to a map $\mathbf{f} : \widetilde{\mathfrak{S}}_n^p \to \mathrm{PF}_{n,p}$. The *rational Pak–Stanley labelling* $\mathbf{f}(\omega)$ is defined by

$$f(\omega, i) = \#\big\{(a, b) \in [n] \times \mathbb{N} : a < b < a + p, \omega(a) > \omega(b) \text{ and } \omega(b) \equiv i \text{ modulo } n\big\},$$

for $i \in [n]$. If ω is a dominant p-stable affine permutation then the Pak–Stanley labelling is obtained by taking the row-sums of the Shi tableau of ω. That is,

$$f(\omega, j) = \sum_{i=1}^{j-1} t_{i,j}^p(\omega).$$

Consequently the *dual Pak–Stanley labelling*, which we define by $f^*(\omega, i) = f(\omega^*, i)$, is obtained by taking the column-sums of the Shi tableau of ω. That is,

$$f^*(\omega, i) = \sum_{j=n-i+2}^{n} t_{n-i+1,j}^p(\omega).$$

As consequence of Theorem 5.4.14 together with Theorem 3.4.9 we obtain a connection between the Anderson map, the zeta map and the Pak–Stanley labelling that was already observed in [**33**, Thm. 5.3].

COROLLARY 5.4.16. [**73**, Thm. 3.4] *Let n, p be positive coprime integers and $\omega \in \widetilde{\mathfrak{S}}_n^p$ be a dominant p-stable affine permutation. Then the partition of $\zeta(\mathcal{A}_A(\omega))$ equals the (reversed[1]) Pak–Stanley labelling $f(\omega)$. Moreover, the partition of $\eta(\mathcal{A}_A(\omega))$ equals the (reversed) dual Pak–Stanley labelling $f^*(\omega)$.*

We are now in a position to use a result of Ceballos, Denton and Hanusa to prove that each rational Dyck path is determined uniquely by its codinv tableau, and equivalently, that each dominant p-stable affine permutation is determined uniquely by its rational Shi tableau.

PROOF OF THEOREM 5.4.12. We deduce the claim from [**21**, Thm. 6.3], which asserts that any rational Dyck path x can be reconstructed from the pair $(\zeta(x), \eta(x))$. Let $\omega \in \widetilde{W}_+^p$ and $x = \mathcal{A} \circ \alpha^{-1}\gamma(\omega)$. By Corollary 5.4.16 the rational Shi tableau of ω encodes both $\zeta(x)$ and $\eta(x)$ in terms of column-sums and row-sums. Therefore it contains enough information to determine the path x uniquely. Using the Anderson map again we we recover ω. ∎

The bijectivity of the zeta map is equivalent to the fact that we obtain a well-defined area-preserving involution on the set $\{\lambda \in \Pi : \lambda \subseteq \Delta_{n,p}\}$ by mapping $\mathbf{f}(\omega)$ to $\mathbf{f}^*(\omega)$. We call this involution *n, p-conjugation.* In some sense n, p-conjugation is to n-conjugation what n-conjugation is to ordinary conjugation of partitions. For example, n, p-conjugation "converges" to n-conjugation as p tends to infinity.
Ceballos, Denton and Hanusa posed the following problem in [**21**].

OPEN PROBLEM 5.4.17. *Find a combinatorial description of n, p-conjugation that does not use the inverse zeta map.*

[1] In our convention partitions are weakly decreasing while $f(\omega)$ is weakly increasing.

Rational Shi tableaux offer a new perspective on this involution and show that $\mathbf{f}^*(\omega)$ can be computed from $\mathbf{f}(\omega)$ by partitioning each summand in a clever way and recombining the parts afterwards. However, this does not solve Problem 5.4.17 because we still need the inverse zeta map to compute ω and its rational Shi tableau from $\mathbf{f}(\omega)$.

If $p = mh + 1$ then Fishel, Kallipoliti and Tzanaki [**26**, Thm. 4.2] proved an explicit (recursive) formula for the entries of the rational Shi tableau which they attribute to Conflitti. Ceballos and the author have found a conjectural generalisation of this formula, however, we require that the min-tableau is already known.

Conjecture 5.4.18. *Let n and p be relatively prime and $\omega \in \widetilde{\mathfrak{S}}_n^p$ be dominant. Then*

$$t_{i,j}^p(\omega) = \min\left\{ \chi_{i,j}^p(\omega), \left\lceil \frac{\left(f(\omega,j) - \sum_{k=1}^{i-1} t_{k,j}^p(\omega)\right)\chi_{i,j}^p(\omega) + \sum_{k=i+1}^{j-1} t_{i,k}^p(\omega)\chi_{i,k}^p(\omega)}{1 + \sum_{k=i+1}^{j} \chi_{i,k}^p(\omega)} \right\rceil \right\}.$$

Given an irreducible crystallographic root system Φ with rank r and a positive integer p relatively prime to the Coxeter number h, define the *rational q-Coxeter–Catalan numbers* as

$$C_{\Phi,p}(q) = q^{(p-1)r/2} \sum_{\omega \in \widetilde{W}_+^p} \prod_{\alpha\in\Phi^+} q^{-t^p(\alpha,\omega)}.$$

The sum of the entries of the Shi tableau generalises the height statistic used by Stump in [**72**, Conj. 3.14]. Thus the polynomials $C_{\Phi,p}(q)$ generalise the q-Fuß–Catalan numbers proposed therein. The conjecture of Stump concernes an algebraic interpretation of these q-Fuß–Catalan numbers and is still open beyond type A_{n-1} even in the Catalan case.

Proposition 5.4.19. *The polynomials $C_{\Phi,p}(q)$ are a common generalisation of the rational q-Catalan numbers defined in Section 2.7 when Φ is of type A_{n-1}, and the q-Coxeter–Catalan numbers defined in Section 5.3 when $p = n + 1$.*

Proof. The first claim is a consequence of Corollary 5.4.16 and the fact that the zeta map is a bijection. The second claim follows directly from the definitions. ∎

Note that the apparently complicated skew-length statistic can be given a uniform definition for different Weyl groups. At the same time the natural length statistic has turned out to be much more resistant in this regard.

Gorsky, Mazin and Vazirani use the Pak–Stanley labelling and the Anderson map to define *rational q,t-Cayley numbers* as

$$\mathscr{C}_{n,p}(q,t) = \sum_{\omega\in\mathfrak{S}_n^p} q^{(p-1)(n-1)/2 - \sum \mathcal{A}_A(\omega,i)} t^{(p-1)(n-1)/2 - \sum f(\omega,i)}.$$

Two natural questions that remain open are whether the polynomials $C_{\Phi,p}(q)$ lead to an interesting Cayley analogue $\mathscr{C}_{\Phi,p}(q)$, and if there is an algebraic interpretation perhaps coming from rational Cherednik algebras.

Using the stronger Theorem 2.7.1 instead of [**21**, Thm. 6.3] in the proof of Theorem 5.4.12 allows for a proof that each dominant p-stable affine permutation is determined uniquely by its Pak–Stanley labelling. The equivalent statement for the set of all p-stable affine permutations is still open.

Conjecture 5.4.20. [**33**, Conj. 1.4] *Let n and p be relatively prime. Then the rational Pak–Stanley labelling $\mathbf{f} : \widetilde{\mathfrak{S}}_n^p \to \mathrm{PF}_{n,p}$ is a bijection.*

In some sense Conjecture 5.4.20 is to Theorem 2.7.1 what Theorem 3.2.10 is to Corollary 3.2.9.

We conclude this section by stating a somewhat wild conjecture that should be seen as attempt to generalise the rational Pak–Stanley labelling on dominant elements of the affine symmetric group to arbitrary crystallographic root systems.

CONJECTURE 5.4.21. *Let Φ be a root system of type A_{n-1}, B_n, C_n or D_n. Then the map assigning to each dominant p-stable element of the affine Weyl group the vector obtained by taking the row-sums of its rational Shi tableau is injective.*

Note that the root posets of these types are commonly pictured by arranging the positive roots in a "staircase" (see Sections 5.2.1–5.2.3). Row-sums in the above conjecture are defined by arranging the numbers $t^p(\alpha, \omega)$, where α ranges over Φ^+, in the same way.

Conjecture 5.4.21 is ambitious in the sense that I currently do not know what the inherent meaning of row-sums should be for different types of root systems. Quite possible there exists a different formulation of the same phenomenon that turns out to be more natural.

CHAPTER 6

The zeta map

In this section we connect the worlds of the finite torus and its orbits under the action of the Weyl group, and of non-nesting parking functions and (chains of) ideals in the root poset. In Section 6.1 we introduce and discuss a uniform zeta map that was defined by Thiel. In Section 6.2 we use the zeta map to relate the area statistic of Section 5.3 to the dinv statistic of Section 4.5. In type A_{n-1} the uniform zeta map is equivalent to the maps on (labelled) Dyck paths studied by Haglund and Loehr that were introduced in Sections 2.5 and 2.6. In Sections 6.3–6.5 we describe the lattice path combinatorics for the other three infinite families of root systems C_n, D_n and B_n. The presentation closely follows [**74**]. Throughout this section we restrict ourselves to the Coxeter–Catalan case, that is, $p = h + 1$. Since the finite torus, non-nesting parking functions and the uniform zeta map have all been defined at least at the Fuß–Catalan level, it is certainly a project for the future to generalise the combinatorics developed here to the more general case $p = mh + 1$. Note that the usual order of first generalising type A_{n-1} combinatorics to type B_n or C_n and then finding a uniform description of the witnessed phenomena is reversed here. The uniform zeta map predates the combinatorics in types different from A_{n-1}. However, it is still worth investigating what the combinatorial picture in other types looks like. First, this leads to new combinatorial results that are interesting by themselves. For example, we obtain two new bijections between square lattice paths $\mathcal{L}_{n,n}$ and ballot paths $\mathcal{B}_{2n}$, both of which are known to be counted by central binomial coefficients. Secondly, we have seen in Section 5.4 that a profound understanding of the combinatorics behind provides a better intuition and allows for proofs that are not feasible otherwise. In particular it would be interesting if the combinatorial maps described in Sections 6.3–6.5 led to a better understanding of the intricacies of the rational combinatorics of their respective root systems.

6.1. The uniform zeta map

Recall that the zeta map from Section 2.6 is a bijection from vertically labelled Dyck paths onto diagonally labelled Dyck paths. In the past chapters these objects were shown to be the type A_{n-1} instances of the finite torus and non-nesting parking functions respectively. Thiel [**76**] found a Weyl group analogue of the zeta map that maps the elements of the finite torus bijectively to the set of non-nesting parking functions.

The uniform zeta map originates in a paper by Cellini and Papi [**23**] who gave a bijection between the W orbits of the finite torus $\check{Q}/(h+1)\check{Q}$ and the set of anti-chains in the root poset, which are in turn in bijection with the dominant regions of the Shi arrangement. Athanasiadis [**12**] extended this map to the Fuß–Catalan case, that is, he showed that the W orbits of the finite torus $\check{Q}/(mh+1)\check{Q}$ are in bijection with the dominant regions of the m-extended Shi arrangement. Finally, Thiel lifted these results to a bijection from $\check{Q}/(mh+1)\check{Q}$ to the set of m-non-nesting parking functions.

Here, we restrict ourselves to the case $m = 1$.

Let Φ be an irreducible crystallographic root system. Define the *uniform zeta map*

$$\zeta : \check{Q}/(h+1)\check{Q} \to \operatorname{Park}(\Phi)$$

via $\zeta = \Theta^{-1} \circ \mathcal{A}^{-1}$, where Θ is the bijection from Theorem 5.1.5 and $\mathcal{A}$ is the uniform Anderson map from Theorem 4.4.4.

The following theorem provides a more explicit description of the zeta map. Moreover it shows that the zeta map is W-equivariant.

THEOREM 6.1.1. [**74**, Props. 2.10 and 2.11] *Let Φ be an irreducible crystallographic root system with Weyl group W, coroot lattice $\check{Q}$ and Coxeter number h. Let $\omega \in \widetilde{W}_{+}^{h+1}$ and $v \in W$ such that $v\omega \in \widetilde{W}^{h+1}$. Choose $x, y \in \check{Q}$ and $s, u \in W$ such that $\omega_{h+1} = t_x s$ and $\omega = t_y u$. Let $q \in \check{Q} \cap (h+1)\overline{A_\circ}$ and $w \in W$ be such that $w(A) \subseteq \Phi^+$, where*

$$A = \big\{\alpha \in \Delta \cup \{-\tilde{\alpha}\} : q \text{ lies in the wall of } (h+1)\overline{A_\circ} \text{ perpendicular to } \alpha\big\},$$

and such that

$$w(q) + (h+1)\check{Q} = -v\omega\omega_{h+1}^{-1}(0) + (h+1)\check{Q}.$$

Note that $w(q)$ is a canonical representative in the sense of Lemma 4.2.2. Then

$$\zeta\big(w(q) + (h+1)\check{Q}\big) = [wsu^{-1}, us^{-1}(A)].$$

PROOF. First note that by assumption

$$w(q) + (h+1)\check{Q} = -vt_y us^{-1} t_{-x}(0) + (h+1)\check{Q} = vus^{-1}(x - su^{-1}(y)) + (h+1)\check{Q}$$

Since q and

$$x - su^{-1}(y) = t_x su^{-1} t_{-y}(0) = \omega_{h+1}\omega^{-1}(0)$$

lie in the same W-orbit and are both elements of $(h+1)\overline{A_\circ}$ Theorem 4.2.1 asserts that

$$q = x - su^{-1}(y).$$

Let $\beta \in \Phi^+$. Then

$$\begin{aligned}
H_{\beta,1} \text{ is a floor of } \omega(A_\circ) &\Leftrightarrow \omega^{-1}(-\beta + \delta) \in -\widetilde{\Delta} \\
&\Leftrightarrow \omega_{h+1}\omega^{-1}(\beta - \delta) \in \omega_{h+1}(\widetilde{\Delta}) = (\Delta - \delta) \cup \{-\tilde{\alpha} + h\delta\} \\
&\Leftrightarrow t_x su^{-1} t_{-y}(\beta) = t_q su^{-1}(\beta) \in \Delta \cup \{-\tilde{\alpha} + (h+1)\delta\} \\
&\Leftrightarrow \begin{cases} su^{-1}(\beta) \in \Delta \text{ and } \langle q, su^{-1}(\beta)\rangle = 0 \text{ or} \\ su^{-1}(\beta) = -\tilde{\alpha} \text{ and } \langle q, su^{-1}(\beta)\rangle = h+1 \end{cases} \\
&\Leftrightarrow su^{-1}(\beta) \in A.
\end{aligned}$$

Thus we conclude that $\zeta(w(q) + (h+1)\check{Q}) = [v, us^{-1}(A)]$.

It remains to show that $v = wsu^{-1}$. By the above computation $su^{-1}v^{-1}w$ lies in the stabiliser subgroup of $q+(h+1)\check{Q}$ in W, which is generated by the reflections s_α for $\alpha \in A$ by Theorem 4.2.1. Therefore

$$v^{-1}wsu^{-1} \in us^{-1}\langle s_\alpha : \alpha \in A\rangle su^{-1} = \langle s_{us^{-1}(\alpha)} : \alpha \in A\rangle = \langle s_\beta : \beta \in su^{-1}(A)\rangle.$$

In other words $vH = wsu^{-1}H$ where $H = \langle s_\beta : \beta \in su^{-1}(A)\rangle$. On the one hand $vsu^{-1}(A) \subseteq \Phi^+$ due to the fact that $v\omega(A_\circ)$ is a Shi alcove contained in the chamber $v(C_\circ)$ [**76**, Sec. 12.3]. On the other hand $wsu^{-1}(su^{-1}(A)) = w(A) \subseteq \Phi^+$ by choice of w. Since there is only one element in the coset vH mapping $su^{-1}(A)$ to a subset of Φ^+, we must have $v = wsu^{-1}$ as claimed. ∎

Since the zeta map is W-equivariant it induces a bijection $\zeta : \check{Q} \cap (h+1)\overline{A_\circ} \to \mathrm{AC}(\Phi^+)$, which is equivalent to the map of Cellini and Papi mentioned above. If Φ is of type A_{n-1} the uniform zeta map on orbits specialises to the zeta map on Dyck paths. Moreover, the uniform zeta map with domain equal to the finite torus corresponds to the zeta map from vertically labelled Dyck paths to diagonally labelled Dyck paths.

THEOREM 6.1.2. [**75**, Thm. 5.6.3] *Let Φ be a root system of type A_{n-1}. Then $\zeta_A = \varphi_A^{-1} \circ \zeta \circ \psi_A^{-1} \circ \phi_A$.*

PROOF. Let $(s,x) \in \mathrm{Vert}(A_{n-1})$ such that $s = \mathrm{drw}(e,x)^{-1}$. Then Theorem 4.4.5 and Proposition 3.3.6 imply that $\mathcal{A} \circ \psi_A^{-1} \circ \phi_A(s,x) = \mathcal{A}_A(s,x)$ is dominant. Hence $\zeta \circ \psi_A^{-1} \circ \phi(s,x) = [e, A]$ for some anti-chain $A \in \mathrm{AC}(\Phi^+)$. Set $(e,y) = \varphi_A^{-1}([e,A])$. We apply a useful trick due to Thiel. The stabiliser of (s,x) is generated by

$$\big\{ s_\alpha : \alpha = e_{s(i)} - e_{s(i+1)} \text{ where } i \text{ is a rise of } x \big\}.$$

The stabiliser of $[e,A]$ equals

$$\big\{ s_\alpha : \alpha = e_i - e_j \text{ where } (i,j) \text{ is a valley of } y \big\}.$$

Since $\zeta \circ \psi_A^{-1} \circ \phi_A$ is $\mathfrak{S}_n$-equivariant it follows that the two stabilisers coincide. It follows that the valleys of y correspond precisely to the rises of x which proves $\zeta_A(s,x) = (e,y)$ because of Lemma 2.6.1.

The claim for arbitrary s follows by using again the fact that the involved functions are $\mathfrak{S}_n$-equivariant. ∎

6.2. From area to dinv

In this section the dinv-statistic on the finite torus is connected to the area-statistic on non-nesting parking functions via the zeta map.

LEMMA 6.2.1. *Let $\alpha \in \Phi^+$ be a positive root and $x \in \check{Q}$, $s \in W$ such that $\omega_{h+1} = t_x s$. Then*

$$\langle \omega_{h+1}(0), \alpha \rangle = \begin{cases} \mathrm{ht}(\alpha) & \textit{if } s^{-1}(\alpha) \in \Phi^+, \\ \mathrm{ht}(\alpha) + 1 & \textit{if } s^{-1}(\alpha) \in -\Phi^+. \end{cases}$$

PROOF. Choose $v \in V$ such that $\langle v, \alpha \rangle = \mathrm{ht}(\alpha)/h$ for all $\alpha \in \Phi^+$. By [**75**, Thm. 3.6.2] $\omega_{h+1}(v) = (h+1)v$. It follows that

$$\langle \omega_{h+1}(0), \alpha \rangle = \langle (h+1)v - s(v), \alpha \rangle = \mathrm{ht}(\alpha) + \frac{1}{h}\big(\mathrm{ht}(\alpha) - \mathrm{ht}(s^{-1}(\alpha))\big).$$

The claim follows since $\langle x, \alpha \rangle$ is an integer. ∎

THEOREM 6.2.2. *Let Φ be an irreducible crystallographic root system with Weyl group W, coroot lattice $\check{Q}$, Coxeter number h and zeta map ζ, and let $X \in \check{Q}/(h+1)\check{Q}$. Let $q \in \check{Q} \cap (h+1)\overline{A_\circ}$ and $w \in W$ such that $w(q) \in X$ is the canonical representative as in Lemma 4.2.2. Then*

$$\mathrm{dinv}(q) = \mathrm{area}(\zeta(q)) \qquad \textit{and} \qquad \mathrm{dinv}'(X) = \mathrm{area}'(\zeta(X)).$$

PROOF. Choose $\omega \in \widetilde{W}_+^{h+1}$, $x, y \in \check{Q}$ and $s, u, v \in W$ such that $\omega_{h+1} = t_x s$, $\omega = t_y u$, $v\omega \in \widetilde{W}^{h+1}$ and

$$w(q) = -v\omega\omega_{h+1}^{-1}(0) = -v(y) + vus^{-1}(x).$$

Theorem 6.1.1 yields

$$\zeta(w(q) + (h+1)\check{Q}) = [wsu^{-1}, us^{-1}A].$$

We prove the more precise statement that the bijection

$$I(us^{-1}A) \to \big\{ \pm\, su^{-1}(\beta) : \beta \in I(us^{-1}A)\big\} \cap \Phi^+$$

maps the positive roots contributing to $\operatorname{area}(\zeta(q))$ to the positive roots contributing to $\operatorname{dinv}(q)$. Furthermore this map restricts to a bijection between the positive roots contributing to area' and dinv'.

On the one hand a computation shows that

$$\begin{aligned}\langle q,\alpha\rangle &= \langle x - su^{-1}(y),\alpha\rangle\\ &= \operatorname{ht}(\alpha) + \frac{1}{h}\big(\operatorname{ht}(\alpha)-\operatorname{ht}(s^{-1}(\alpha))\big) - \langle y, us^{-1}(\alpha)\rangle\\ &= \operatorname{ht}(\alpha) - \langle y, us^{-1}(\alpha)\rangle + \begin{cases}0 & \text{if } s^{-1}(\alpha)\in\Phi^+,\\ 1 & \text{if } s^{-1}(\alpha)\in-\Phi^+,\end{cases}\end{aligned}$$

for all $\alpha\in\Phi^+$ due to Lemma 6.2.1.

On the other hand for each $\beta\in\Phi^+$ we have $\beta\in us^{-1}A$ if and only if $H_{\beta,1}$ is a floor of $\omega(A_\circ)$ by the definition of the zeta map. Thus

$$\begin{aligned}\beta\in I(us^{-1}A) &\Leftrightarrow H_{\beta,1} \text{ does not separate } \omega(A_\circ) \text{ and } A_\circ\\ &\Leftrightarrow \omega^{-1}(-\beta+\delta) = -u^{-1}(\beta) + (1-\langle y,\beta\rangle)\delta \in \widetilde{\Phi}^+\\ &\Leftrightarrow \langle y,\beta\rangle = 0 \text{ or } \langle y,\beta\rangle = 1, -u^{-1}(\beta)\in\Phi^+.\end{aligned}$$

The proof is completed by distinguishing a few cases.

Let $\beta\in\Phi^+$ and choose $\alpha\in\{\pm su^{-1}(\beta)\}\cap\Phi^+$. First assume $\langle y,\beta\rangle = 0$ then $u^{-1}(\beta)\in\Phi^+$ by Lemma 5.4.7. Also $0=\langle y,\beta\rangle = \langle y, us^{-1}(\alpha)\rangle$. If $su^{-1}(\beta)\in\Phi^+$ then $s^{-1}(\alpha) = u^{-1}(\beta)\in\Phi^+$ and $\langle q,\alpha\rangle = \operatorname{ht}(\alpha)$. If $su^{-1}(\beta)\in-\Phi^+$ then $s^{-1}(\alpha) = -u^{-1}(\beta)\in-\Phi^+$. Hence $\langle q,\alpha\rangle = \operatorname{ht}(\alpha)+1$. Moreover $wsu^{-1}(\beta)\in\Phi^+$ implies that $w(\alpha)\in\Phi^+$ if and only if $su^{-1}(\beta)\in\Phi^+$.

Secondly suppose that $\langle y,\beta\rangle = 1$ and $u^{-1}(\beta)\in-\Phi^+$. If $su^{-1}(\beta)\in\Phi^+$ then $s^{-1}(\alpha)\in-\Phi^+$ and $\langle y, us^{-1}(\alpha)\rangle = \langle y,\beta\rangle = 1$. It follows that $\langle q,\alpha\rangle = \operatorname{ht}(\alpha)-1+1 = \operatorname{ht}(\alpha)$. If $su^{-1}(\beta)\in-\Phi^+$ then $s^{-1}(\alpha)\in\Phi^+$ and $\langle y, us^{-1}(\alpha)\rangle = -1$. Thus $\langle q,\alpha\rangle = \operatorname{ht}(\alpha)+1$. Moreover, $wsu^{-1}(\beta)\in\Phi^+$ implies that $w(\alpha)\in\Phi^+$ if and only if $su^{-1}(\beta)\in\Phi^+$.

Conversely let $\alpha\in\Phi^+$ and choose $\beta\in\{\pm us^{-1}(\alpha)\}\cap\Phi^+$. First assume $\langle q,\alpha\rangle = \operatorname{ht}(\alpha)$. If $s^{-1}(\alpha)\in\Phi^+$ then $0=\langle y, us^{-1}(\alpha)\rangle = \langle y,\beta\rangle$. By Lemma 5.4.7 $u^{-1}(\beta)\in\Phi^+$ and thus $\beta = us^{-1}(\alpha)$. Consequently $w(\alpha)\in\Phi^+$ implies $wsu^{-1}(\beta) = w(\alpha)\in\Phi^+$. If $s^{-1}(\alpha)\in-\Phi^+$ then $\langle y, us^{-1}(\alpha)\rangle = 1$ and therefore $us^{-1}(\alpha)\in\Phi^+$ since $y\in\overline{C_\circ}$. We obtain $\langle y,\beta\rangle = 1$ and $u^{-1}(\beta) = s^{-1}(\alpha)\in-\Phi^+$. Moreover $w(\alpha)\in\Phi^+$ implies $wsu^{-1}(\beta)\in\Phi^+$ as above.

Finally assume that $\langle q,\alpha\rangle = \operatorname{ht}(\alpha)+1$. If $s^{-1}\in\Phi^+$ then $\langle y, us^{-1}(\alpha)\rangle = -1$ and therefore $us^{-1}(\alpha)\in-\Phi^+$ since $y\in\overline{C_\circ}$. Hence $\langle y,\beta\rangle = 1$ and $-u^{-1}(\beta) = s^{-1}(\alpha)\in\Phi^+$. Moreover, $w(\alpha)\in-\Phi^+$ implies $wsu^{-1}(\beta) = -w(\alpha)\in\Phi^+$. If $s^{-1}\in-\Phi^+$ then $0=\langle y, us^{-1}(\alpha)\rangle = \langle y,\beta\rangle$ and thus $u^{-1}(\beta)\in\Phi^+$. It follows that $\beta = -us^{-1}(\alpha)$ and hence $w(\alpha)\in-\Phi^+$ implies $wsu^{-1}(\beta) = -w(\alpha)\in\Phi^+$. ■

Finally we give a proof that the dinv statistic of type A_{n-1} is a special case of the uniform dinv statistic.

Proof of Theorem 4.5.1. Let $(s,x) \in \mathrm{Vert}(A_{n-1})$ be a vertically labelled Dyck path. Then

$$\begin{aligned}\mathrm{dinv}'(s,x) &= \mathrm{area}' \circ \zeta_A(s,x)\\ &= \mathrm{area}' \circ \varphi_A \circ \zeta_A(s,x)\\ &= \mathrm{dinv}' \circ \zeta^{-1} \circ \varphi_A \circ \zeta_A(s,x)\\ &= \mathrm{dinv}' \circ \psi_A^{-1} \circ \phi_A(s,x),\end{aligned}$$

where we use Theorem 2.6.2, Proposition 5.3.1, Theorem 6.2.2 and Theorem 6.1.2 in that order. ∎

There exist generalisations of the algebraic background motivating the definition of q,t-Catalan numbers that replace the symmetric group by a Weyl group. Consequently q,t-Coxeter–Catalan numbers $C_\Phi(q,t)$ can be defined as the bivariate Hilbert series of certain W-modules. See for example [**71**, Appendix A] for a list of the first few polynomials obtained in this way. It is an open problem to find a combinatorial interpretation of these polynomials.

Open Problem 6.2.3. *Find a statistic* tstat *on the anti-chains in the root poset* Φ^+ *such that*

$$C_\Phi(q,t) = \sum_{A \in \mathrm{AC}(\Phi^+)} q^{\mathrm{area}(A)} t^{\mathrm{tstat}(A)}.$$

Alternatively, find a statistic tstat *on the W-orbits of the finite torus $\check{Q}/(h+1)\check{Q}$ such that*

$$C_\Phi(q,t) = \sum_{q \in \check{Q} \cap (h+1)\overline{A_\circ}} q^{\mathrm{dinv}(q)} t^{\mathrm{tstat}(q)}.$$

Problem 6.2.3 was suggested by Stump and remains unsolved beyond type A_{n-1}. Note that even a partial solution in the form of a conjectured statistic say in type B_n or C_n would be of interest.

One might assume at first glance that knowing the dinv statistic as well as the area statistic should be sufficient to obtain the q,t-Catalan numbers, as is the case in type A_{n-1}. However, let me emphasise that the dinv statistic is only known for W-orbits of the finite torus, while the area statistic is only known for anti-chains of the root poset. It is a spectacular coincidence that in type A_{n-1} these two objects both correspond naturally to Dyck paths in such a way that the area statistic, which is natural in the world of anti-chains, serves as the mysterious statistic tstat in the world of W-orbits.

6.3. Combinatorics in type C_n

Let $\pi \in \mathcal{L}_{n,n}$ and for $i \in [n]$ let q_i denote the number of east steps of π preceding its i-th north step. Moreover let $x \in \check{Q}$ and $s \in \mathfrak{S}_n^B$ be as in Lemma 4.4.2 (ii) such that $\omega_{h+1}^C = t_x s$. Define the type C_n *area vector* of π as

$$\mu = s(q-x) = s(q_1-1, q_2-2\ldots, q_n-n) = (n-q_n, \ldots, 2-q_2, 1-q_1).$$

Indeed observe that μ_{n-i+1} counts the number of unit squares in the i-th row between the path π and the path $(\mathbf{en})^n \in \mathcal{L}_{n,n}$ consisting of alternating north and east steps. In this regard μ is quite similar to the type A_{n-1} area vector of a Dyck path. The entries of the type C_n area vector are negative whenever π is east of $(\mathbf{en})^n$ in the respective row. See Figure 6.1.

The following is an auxiliary result on area vectors for later use.

Lemma 6.3.1. [**74**, Lem. 4.11] *Let $\pi \in \mathcal{L}_{n,n}$ be a lattice path with type C_n area vector μ.*

(i) Let $i,j \in [n]$ with $i<j$ such that $\mu_j = \mu_i - 1$ and $\mu_\ell \notin \{\mu_i - 1, \mu_i\}$ for all ℓ with $i<\ell<j$. Then $j = i+1$.

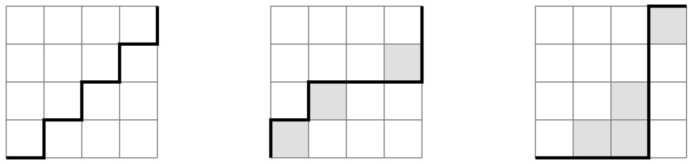


FIGURE 6.1. The lattice paths with type C_n area vectors $(0,0,0,0), (0,-1,1,1)$ and $(1,0,-1,-2)$.

(ii) *For all* $i \in [n-1]$ *we have* $\mu_i \leq \mu_{i+1} + 1$.
(iii) *Let* $j \in [n]$ *such that* $\mu_j < 0$. *Then there exist* $i \in [j-1]$ *with* $\mu_i = \mu_j + 1$.
(iv) *Let* $i \in [n]$ *such that* $\mu_i > 1$. *Then there exists* $j \in [n]$ *with* $i < j$ *such that* $\mu_j = \mu_i - 1$.
(v) *Let* $i \in [n]$ *such that* $\mu_i = 1$ *and* $\mu_\ell \notin \{0,1\}$ *for all* $\ell \in [n]$ *with* $i < \ell$. *Then* $i = n$.

PROOF. We start by proving claim (i). From $\mu = s(q-x)$ we obtain $q_{n+1-\ell} = n+1-\ell-\mu_\ell \notin \{n+1-\ell-\mu_i, n+2-\ell-\mu_i\}$ for all ℓ with $i < \ell < j$. Since $q_{n+1-j} = n+2-j-\mu_i$ and $q_{n-\ell} \leq q_{n+1-\ell}$ it follows inductively that $n+2-\ell-\mu_i < q_{n+1-\ell}$ for all ℓ with $i < \ell < j$. But this yields a contradiction for $\ell = i+1$, namely $q_{n-i} > n+1-i-\mu_i = q_{n+1-i}$. Thus $j = i+1$. Claim (ii) is an immediate consequence of $q_{n-i} \leq q_{n+1-i}$ for all $i \in [n-1]$. Claim (iii) follows from (ii) and $\mu_1 \geq 0$. Similarly claims (iv) and (v) follow from (ii) and $\mu_n \leq 1$. ■

Let $(w, \pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ be a vertically labelled lattice path and μ the type C_n area vector of π. Define the type C_n *diagonal reading word* $\mathrm{drw}_C(w, \pi)$ as follows: For each $i = 0, 1, \ldots, n$ first write down the negative labels $-w(j)$ of the rows with $\mu_{n+1-j} = -i$ from top to bottom, then write down the labels $w(j)$ of rows with $\mu_{n+1-j} = i+1$ from bottom to top.

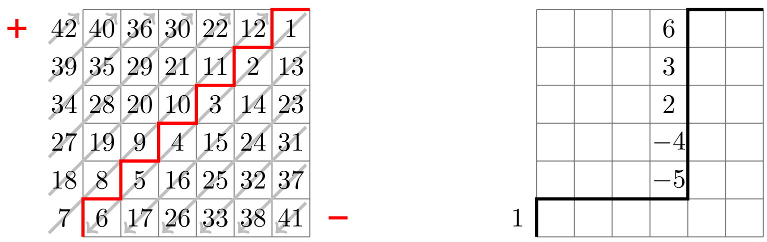


FIGURE 6.2. The diagonal reading order of type C_6 indicated left, and a vertically labelled lattice path with $\mathrm{drw}_C(w, \pi) = [-2, 1, 3, 4, 6, 5]$ on the right.

The diagonal reading word of type C_n can also be read off quickly by scanning all unit squares that may contain labels according to the *diagonal reading order*, which is indicated in Figure 6.2 (left).

The next result confirms that the diagonal reading word is the correct signed permutation. We abbreviate $\mathrm{Vert}(C_n) = \mathrm{Vert}(C_n, h+1)$.

PROPOSITION 6.3.2. [**74**, Prop. 4.14] *Let* $(w, \pi) \in \mathrm{Vert}(C_n)$ *be a vertically labelled lattice path with area vector* μ, *let* s *be defined as in Lemma 4.4.2* (ii), *and define* $u \in \mathfrak{S}_n^B$ *such that* $t_\mu u^{-1}$ *is Graßmannian. Then* $\mathrm{drw}_C(w, \pi) = wsu^{-1}$.

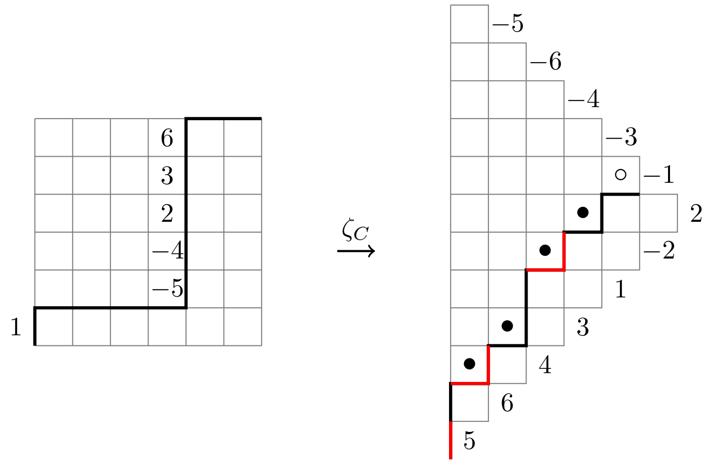


FIGURE 6.3. A vertically labelled lattice path and its image under the zeta map.

PROOF. Let $i, j \in [n]$ and recall that $N = 2n+1$. By definition we have $|\mathrm{drw}_C(w,\pi)(i)| = |w(j)|$ if and only if

$$\begin{aligned} i &= \#\big\{r \in [n], |\mu_r| < |\mu_{n+1-j}|\,\big\} + \#\big\{r \in [n+1-j] : \mu_r = \mu_{n+1-j} > 0\big\} \\ &\quad + \#\big\{r \in [n] : r \geq n+1-j, \mu_r = \mu_{n+1-j} \leq 0\big\} + \#\big\{r \in [n], \mu_r = -\mu_{n+1-j} > 0\big\} \\ &= \big\{r \in [n] : |\mu_r N - r| \leq |\mu_{n+1-j}N - (n+1-j)|\,\big\}. \end{aligned}$$

By Lemma 1.5.4 we obtain $\left|u^{-1}(i)\right| = n+1-j$, hence

$$|\mathrm{drw}_C(w,\pi)(i)| = |w(j)| = \left|w(n+1-\left|u^{-1}(i)\right|)\right| = \left|wsu^{-1}(i)\right|.$$

Moreover $\mathrm{drw}_C(w,\pi)(i) = w(j)$ if and only if $\mu_{n+1-j} > 0$. On the other hand $su^{-1}(i) = j$ is equivalent to $u^{-1}(i) = -(n+1-j) < 0$, which is the case if and only if $\mu_{n+1-j} > 0$ by Lemma 1.5.4. Thus we may drop the absolute values in the identity above and the proof is complete. ■

We are now in a position to define the combinatorial zeta map of type C_n, which is made up of the building blocks defined in Section 1.2.

Define the type C_n *zeta map* on lattice paths $\zeta_C : \mathcal{L}_{n,n} \to \mathcal{B}_{2n}$ by

$$\zeta_C(\pi) = \overleftarrow{w}_n^-(\mu)\overrightarrow{w}_n^+(\mu)\overleftarrow{w}_{n-1}^-(\mu)\overrightarrow{w}_{n-1}^+(\mu)\cdots\overleftarrow{w}_1^-(\mu)\overrightarrow{w}_1^+(\mu)\overleftarrow{w}_0^-(\mu)\overrightarrow{w}_0^+(\mu),$$

where μ is the type C_n area vector of $\pi \in \mathcal{L}_{n,n}$. Define the type C_n *zeta map* on labelled paths $\zeta_C : \mathfrak{S}_n^B \times \mathcal{L}_{n,n} \to \mathfrak{S}_n^B \times \mathcal{B}_{2n}$ by

$$\zeta_C(w,\pi) = \big(\mathrm{drw}_C(w,\pi), \zeta_C(\pi)\big).$$

It is easy to see that $\zeta_C(\pi)$ really is a ballot path for all $\pi \in \mathcal{L}_{n,n}$.

Our first main result of this section is the fact that the zeta map can be inverted using a construction reminiscent of the bounce path of a Dyck path in type A_{n-1}.

THEOREM 6.3.3. [**74**, Thm. 4.17] *The map $\zeta_C : \mathcal{L}_{n,n} \to \mathcal{B}_{2n}$ is a bijection.*

PROOF. Let $\pi \in \mathcal{L}_{n,n}$ be a lattice path with type C_n area vector μ. For each k with $0 \leq k \leq n$ let α_k denote the number of indices $i \in [n]$ such that $|\mu_i| = k$.

Define the *bounce path* of a ballot path $\beta \in \mathcal{B}_{2n}$ as follows: start at the end point of β and go south until you hit the diagonal. Bounce off it and travel to the west until you reach the upper end of a north step of β. Bounce off the path β to the south until you hit the diagonal again, and repeat until you arrive at $(0,0)$.

Now suppose that $\beta = \zeta_C(\pi)$. By definition of the zeta map the end point of β is $(n-\alpha_0, n+\alpha_0)$. The bounce path of β meets the diagonal for the first time in the point $(n-\alpha_0, n-\alpha_0)$, and then travels west to the point $(n-\alpha_0-\alpha_1, n-\alpha_0)$, which is the starting point of the segment $\overleftarrow{w}_0^-(\mu)\overrightarrow{w}_0^+(\mu)$. We claim that this point is a peak of the bounce path.

To see this note that Lemma 6.3.1 (iii) and (iv) imply that each non-empty segment $\overleftarrow{w}_j^-(\mu)$ or $\overrightarrow{w}_j^+(\mu)$ ends with a north step, except possibly $\overrightarrow{w}_0^+(\mu)$. In particular, the starting point of any segment $\overleftarrow{w}_j^-(\mu)\overrightarrow{w}_j(\mu)$ is either $(0,0)$ or the endpoint of a north step of β.

Inductively the peaks of the bounce path therefore encode the numbers $\alpha_0, \alpha_1, \dots, \alpha_n$.

Knowing α_0 and α_1 we can recover the number and relative order of zeroes, ones and minus ones in μ from the segment $\overleftarrow{w}_0^-(\mu)\overrightarrow{w}_0^+(\mu)$. Since $\overleftarrow{w}_0^-(\mu)$ ends with a north step, we first obtain the sequences $\overleftarrow{w}_0^-(\mu)$ and $\overrightarrow{w}_0^+(\mu)$ and thus the number occurrences of ones and minus ones in μ. Moreover these paths encode the relative order of zeroes and minus ones, respectively the relative order of zeroes and ones. The relative order of ones and minus ones is implied by the following observation: If $\mu_i = 1$ and $\mu_j = -1$ for some $i, j \in [n]$ with $i < j$ then there exists ℓ with $i < \ell < j$ and $\mu_\ell = 0$ due to Lemma 6.3.1 (ii).

Similarly one now reconstructs the numbers of twos and minus twos, as well as the relative order of zeroes, ones, minus ones, twos and minus twos, using the segment $\overleftarrow{w}_1^-(\mu)\overrightarrow{w}_1^+(\mu)$. Continuing in this fashion one recovers the entire area vector μ. Thus ζ_C is injective. Since $\mathcal{L}_{n,n}$ and $\mathcal{B}_{2n}$ both have cardinality $\binom{2n}{n}$ it is also bijective. ∎

As an easy consequence of Theorem 6.3.3 it follows that the zeta map on labelled paths is a bijection as well. More interesting is the result that this map restricts to a bijection from vertically labelled lattice paths to diagonally labelled ballot paths.

THEOREM 6.3.4. [**74**, Thm. 4.18] *The zeta map $\zeta_C : \mathrm{Vert}(C_n) \to \mathrm{Diag}(C_n)$ is a bijection.*

Theorem 6.3.4 follows from Theorem 6.3.5 below that relates the rises of a vertically labelled path to the valleys of the corresponding diagonally labelled path.

Let $(w, \pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ and $i \in [n]$ be a rise of π. We say i is a *rise* of (w, π) *labelled* by $(w(i), w(i+1))$. Similarly, let $(v, \beta) \in \mathfrak{S}_n^B \times \mathcal{B}_{2n}$ and (i, j) be a valley of β. Then we say

$$(i,j) \text{ is a } \textit{valley} \text{ of } (v,\beta) \textit{ labelled} \text{ by } \begin{cases} \big(v(n+1-i), v(n+1-j)\big) & \text{if } j \leq n, \\ \big(v(n+1-i), v(n-j)\big) & \text{if } j > n. \end{cases}$$

Note that with our usual way of picturing diagonally labelled ballot paths, each valley is labelled by the number below it and the number to its right.

THEOREM 6.3.5. [**74**, Thm. 4.20] *Let $(w, \pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ and $a, b \in w([n])$. Then (w, π) has a rise labelled by (a, b) if and only if $\zeta_C(w, \pi)$ has a valley labelled by (b, a) or $(-a, -b)$. Moreover π begins with a north step and $w(1) = a$ if and only if $\zeta_C(w, \pi)$ has a valley labelled by $(a, -a)$.*

PROOF. **(Part 1)** We first assume that we are given a valley of $\zeta_C(w, \pi)$ and show that there exists a fitting rise in (w, π). Let μ be the area vector of π. Choose $u \in \mathfrak{S}_n^B$ such that $t_\mu u^{-1}$ is a Graßmannian affine permutation, and define $s \in \mathfrak{S}_n^B$ as in Lemma 4.4.2 (ii). A valley of $\zeta_C(\pi)$ can either occur within a sequence $\overleftarrow{w}_k^-(\mu)$ or $\overrightarrow{w}_k^+(\mu)$, or if $\zeta_C(\pi)$ ends with an east step. We treat these three cases independently. No valley may arise at the join of two such sequences because of Lemma 6.3.1 (iii) and (iv).

(1.1) There is a valley within the sequence $\overleftarrow{w}_k^-(\mu)$. Then there exist indices $i, j \in [n]$ with $i < j$ such that $\mu_i = -k$, $\mu_j = -k-1$ and $\mu_\ell \notin \{-k-1, -k\}$ for all ℓ with $i < \ell < j$. By Lemma 6.3.1 (i) we have $j = i+1$. Hence $q_{n-i} = n - i - \mu_{i+1} = n + 1 - i - \mu_i = q_{n+1-i}$

and $n-i$ is a rise of π. We claim that the labels of our valley are compatible with the labels $(w(n-1), w(n-i+1))$ of this rise.

Suppose (x,y) is our valley. Then x equals the number of east steps in the sequence

$$\overleftarrow{w}_n^-(\mu)\overrightarrow{w}_n^+(\mu)\cdots\overleftarrow{w}_{k+1}^-(\mu)\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu_{i+1},\ldots,\mu_n).$$

In other words

$$\begin{aligned}x &= \#\big\{r\in[n]:|\mu_r|>k+1\big\}+\#\big\{r\in[n]:r\geq i+1,\mu_r=-k-1\big\}\\ &= \#\big\{r\in[n]:|\mu_rN-r|\geq|(-k-1)N-(i+1)|\,\big\}.\end{aligned}$$

By Lemma 1.5.4 we obtain

$$n+1-x=\#\big\{r\in[n]:|\mu_rN-r|\leq|\mu_{i+1}N-(i+1)|\,\big\}=|u(i+1)|\,.$$

Since $\mu_{i+1}\leq 0$ we have $u(i+1)>0$ and thus

$$\mathrm{drw}_C(w,\pi)(n+1-x)=wsu^{-1}(n+1-x)=ws(i+1)=-w(n-i).$$

Similarly y equals the number of north steps in the sequence

$$\overleftarrow{w}_n^-(\mu)\overrightarrow{w}_n^+(\mu)\cdots\overleftarrow{w}_{k+1}^-(\mu)\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu_i,\ldots,\mu_n).$$

We may rewrite this as

$$\begin{aligned}y &= \#\big\{r\in[n]:|\mu_r|>k\big\}+\#\big\{r\in[n]:r\geq i,\mu_r=-k\big\}\\ &= \#\big\{r\in[n]:|\mu_rN-r|\geq|-kN-i|\,\big\}.\end{aligned}$$

As before Lemma 1.5.4 provides

$$n+1-y=\#\big\{r\in[n]:|\mu_rN-r|\leq|\mu_iN-i|\,\big\}=|u(i)|=u(i)$$

and we compute

$$\mathrm{drw}_C(w,\pi)(n+1-y)=wsu^{-1}(n+1-y)=ws(i)=-w(n+1-i).$$

(1.2) The valley appears within the sequence $\overrightarrow{w}_k^+(\mu)$. Then there exist indices $i,j\in[n]$ with $i<j$ such that $\mu_i=k+1$, $\mu_j=k$ and $\mu_\ell\notin\{k,k+1\}$ for all ℓ with $i<\ell<j$. We obtain $j=i+1$ and $n-i$ is a rise of π just as in (1.1).

Let (x,y) be our valley. Then x equals the number of east steps in

$$\overleftarrow{w}_n^-(\mu)\overrightarrow{w}_n^+(\mu)\cdots\overleftarrow{w}_k^-(\mu)\overrightarrow{w}_k^+(\mu_1,\ldots,\mu_i).$$

Equivalently

$$\begin{aligned}x &= \#\big\{r\in[n]:|\mu_r|>k+1\big\}+\#\big\{r\in[n]:\mu_r=-k-1\big\}+\#\big\{r\in[i]:\mu_r=k+1\big\}\\ &= \#\big\{r\in[n]:|\mu_rN-r|\geq|(k+1)N-i|\,\big\}.\end{aligned}$$

Since $\mu_i>0$ Lemma 1.5.4 implies

$$n+1-x=\#\big\{r\in[n]:|\mu_rN-r|\leq|\mu_iN-i|\,\big\}=|u(i)|=-u(i)$$

and we compute

$$\mathrm{drw}_C(w,\pi)(n+1-x)=wsu^{-1}(n+1-x)=-ws(i)=w(n+1-i).$$

On the other hand y equals the number of north steps in the sequence

$$\overleftarrow{w}_n^-(\mu)\overrightarrow{w}_n^+(\mu)\cdots\overleftarrow{w}_k^-(\mu)\overrightarrow{w}_k^+(\mu_1,\ldots,\mu_{i+1}).$$

If $k>0$ then

$$\begin{aligned}y &= \#\big\{r\in[n]:|\mu_r|>k\big\}+\#\big\{r\in[n]:\mu_r=-k\big\}+\#\big\{r\in[i+1]:\mu_r=k\big\}\\ &= \#\big\{r\in[n]:|\mu_rN-r|\geq|kN-(i+1)|\,\big\},\end{aligned}$$

and Lemma 1.5.4 provides

$$n+1-y = \#\big\{r \in [n] : |\mu_r N - r| \leq |\mu_{i+1} N - (i+1)|\,\big\} = |u(i+1)| = -u(i+1).$$

In particular $y \leq n$ and we compute

$$\mathrm{drw}_C(w,\pi)(n+1-y) = wsu^{-1}(n+1-y) = -ws(i+1) = w(n-i).$$

Otherwise $k = 0$ and

$$y = n + \#\big\{r \in [i+1] : \mu_r = 0\big\} = n + \#\big\{r \in [n] : |\mu_r N - r| \leq i+1\big\}.$$

From Lemma 1.5.4 we obtain

$$y - n = \#\big\{r \in [n] : |\mu_r N - r| \leq |\mu_{i+1} N - (i+1)|\,\big\} = |u(i+1)| = u(i+1).$$

Since $y > n$ the second label of the valley is given by

$$\mathrm{drw}_C(w,\pi)(n-y) = -wsu^{-1}(y-n) = -ws(i+1) = w(n-i).$$

(1.3) The path $\zeta_C(\pi)$ ends with an east step. Then there exists $i \in [n]$ such that $\mu_i = 1$ and $\mu_\ell \notin \{0,1\}$ for all ℓ with $i < \ell$. From Lemma 6.3.1 (v) we know that $i = n$. Consequently $q_1 = 1 - \mu_n = 0$ and π begins with a north step.

Let (x,y) be the valley above the final east step. Then x equals the number of east steps in $\zeta_C(\pi)$, that is,

$$x = \#\big\{r \in [n] : |\mu_r| > 0\big\} = \#\big\{r \in [n] : |\mu_r N - r| \geq n+1\big\}$$
$$n+1-x = \#\big\{r \in [n] : |\mu_r N - r| \leq |\mu_n N - n|\,\big\} = |u(n)| = -u(n).$$

The valley's first label is

$$\mathrm{drw}_C(w,\pi)(n+1-x) = wsu^{-1}(n+1-x) = -ws(n) = w(1).$$

On the other hand, y equals the number of north steps of $\zeta_C(\pi)$ plus one. We have

$$y = n+1+\#\big\{r \in [n] : |\mu_r N - r| \leq n\big\}$$
$$y - n = \#\big\{r \in [n] : |\mu_r N - r| \geq n+1\big\} = -u(n).$$

Hence the valley's second label is

$$\mathrm{drw}_C(w,\pi)(n-y) = -wsu^{-1}(y-n) = ws(n) = -w(1).$$

(Part 2) To complete the proof we need to demonstrate the reverse implication. Thus assume that $i \in [n-1]$ is a rise of π, that is, $q_i = q_{i+1}$. Then $\mu_{n+1-i} = i - q_i$ and $\mu_{n-i} = i+1-q_{i+1} = \mu_{n+1-i} + 1$.

If $\mu_{n-i} = -k \leq 0$ then $\mu_{n+1-i} = -k-1$ and we are in the situation of (1.1). If $\mu_{n-i} = k+1 > 0$ then $\mu_{n+1-i} = k$ and we are in the situation of (1.2).

Finally assume that π begins with a north step. Then $q_1 = 0$ and $\mu_n = 1 - q_1 = 1$. Hence we are in the situation of (1.3). ■

By use of Theorem 6.3.5 one can prove that the combinatorial zeta map ζ_C is equivalent to the uniform zeta map under the identification of the finite torus with vertically labelled lattice paths and of the non-nesting parking functions with diagonally labelled ballot paths.

THEOREM 6.3.6. [**74**, Thm. 4.21] *Let Φ be a root system of type C_n with coroot lattice $\check{Q}$ and zeta map ζ. Then the following diagram commutes.*

$$\begin{tikzcd} \mathrm{Vert}(C_n) \arrow[r, "\zeta_C"] \arrow[d, "\psi_C"'] & \mathrm{Diag}(C_n) \arrow[d, "\varphi_C"] \\ \check{Q}/(2n+1)\check{Q} \arrow[r, "\zeta"'] & \mathrm{Park}(\Phi) \end{tikzcd}$$

Proof. Let $(w,\pi)\in\mathrm{Vert}(C_n)$ and $(v,\beta)=\zeta_C(w,\pi)\in\mathrm{Diag}(C_n)$. Moreover let q be the partition with south-east boundary π, define $s\in\mathfrak{S}_n^B$ as in Lemma 4.4.2 (ii), and let μ be the area vector of π. Choose $u\in\mathfrak{S}_n^B$ such that $t_\mu u^{-1}$ is a Graßmannian affine permutation.

Comparing to Theorem 6.1.1 it suffices to show that $v=wsu^{-1}$ and $A_\beta^C=A$, where

$$A=\big\{us^{-1}(\alpha):\alpha\in\Delta\cup\{-\tilde{\alpha}^C\}\text{ and }s_\alpha(q)=q\big\}.$$

The claim on the diagonal reading word is established by an appeal to Proposition 6.3.2.

To see the claim involving the anti-chains we first prove that $A\subseteq A_\beta^C$. To this end let $i\in[n-1]$. Then $us^{-1}(\alpha_i)\in A$ implies $q_i=q_{i+1}$ and i is a rise of π. By Theorem 6.3.5 (v,β) has a valley (x,y) labelled by $(w(i+1),w(i))$ or $(-w(i),-w(i+1))$. Moreover, a closer look at the proof of Theorem 6.3.5 reveals that the second case only occurs if $y\leq n$. That is,

$$\begin{aligned} &w(i+1)=v(n+1-x) &&\text{or} && -w(i)=v(n+1-x),\\ &w(i)=\begin{cases} v(n+1-y) & \text{if } y\leq n,\\ v(n-y) & \text{if } y>n,\end{cases} && && -w(i+1)=v(n+1-y)\quad\text{and } y\leq n.\end{aligned}$$

Multiplication by w^{-1} in the above identities yields

$$\begin{aligned} &i+1=su^{-1}(n+1-x) &&\text{or} && -i=su^{-1}(n+1-x)\\ &i=\begin{cases} su^{-1}(n+1-y) & \text{if } y\leq n,\\ su^{-1}(n-y) & \text{if } y>n,\end{cases} && && -i-1=su^{-1}(n+1-y)\quad\text{and } y\leq n.\end{aligned}$$

We obtain

$$us^{-1}(\alpha_i^C)=\alpha_{x,y}\in A_\beta^C.$$

Furthermore $us^{-1}(\alpha_0)\in A$ yields $q_1=0$ and thus π begins with a north step. By Theorem 6.3.5 this is equivalent to (v,β) ending with an east step such that the corresponding valley (x,y) has labels $(v(n+1-x),v(n-y))=(v(1),-v(1))$. Thus $su^{-1}(n+1-x)=1=su^{-1}(y-n)$ and we obtain

$$us^{-1}(\alpha_0^C)=\alpha_{x,y}\in A_\beta^C.$$

Finally since we already observed that $us^{-1}(-\tilde{\alpha}^C)\notin A$, the first inclusion holds.

Conversely, let $\alpha_{x,y}\in A_\beta^C$. By similar reasoning as above it follows that $su^{-1}(\alpha_{x,y})$ is of the form $e_{i+1}-e_i$ for a rise i of π, unless the valley comes from a terminal east step of β, in which case π begins with a north step and $su^{-1}(\alpha_{x,y})=2e_1=\alpha_0^C$. Hence, the second inclusion $A_\beta^C\subseteq A$ holds as well and the proof is complete. ■

In the remainder of this section we provide another interpretation of the type C_n zeta map that is in the spirit of the type A_{n-1} sweep map defined in Section 2.7. Recall the concept of the sweep map: Given a path one assigns to each step a label, the labels being distinct integers. To obtain the image of a path under the sweep map, one rearranges the steps such that the labels are in increasing order.

Let $\pi=s_1s_2,\dots,s_{2n}\in\mathcal{L}_{n,n}$ with $s_i\in\{\mathbf{e},\mathbf{n}\}$. Assign a label ℓ_i to each step s_i by setting $\ell_1=0$, and $\ell_{i+1}=\ell_i+2n+1$ if $s_i=\mathbf{n}$, and $\ell_{i+1}=\ell_i-2n$ if $s_i=\mathbf{e}$. Define a collection X of labelled steps as follows. If $\ell_i < 0$ then add (s_i, ℓ_i) to X. If $\ell_i > 0$ then add $(s_{i-1}, -\ell_i)$. Finally, for the step s_1 which is the only step labelled 0, add $(s_{2n}, -n)$. Thus X contains $2n$ labelled steps.

Now draw a path as follows. Choose $(s, \ell) \in X$ such that ℓ is the minimal label among all pairs in X. Draw the step s and remove (s, ℓ) from X. Repeat until X is empty. We denote the path obtained in this way by $\mathrm{sw}(\pi)$. The resulting map $\mathrm{sw} : \mathcal{L}_{n,n} \to \{\mathbf{e}, \mathbf{n}\}^*$ is called the *sweep map of type C_n*. See Figure 6.4.

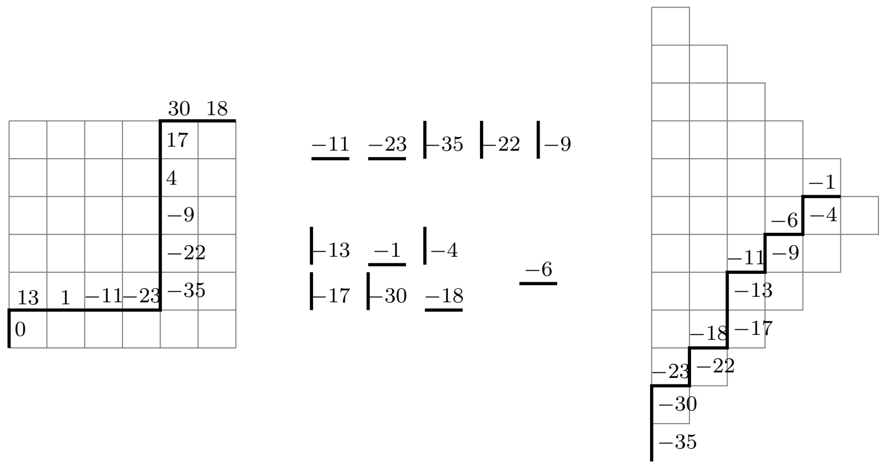


FIGURE 6.4. The labelling of π (left), the set X of labelled steps (middle), and the path $\mathrm{sw}(\pi)$ of steps in increasing order (right).

THEOREM 6.3.7. [**74**, Thm. 4.22] *For each lattice path $\pi \in \mathcal{L}_{n,n}$ we have $\mathrm{sw}(\pi) = \zeta_C(\pi)$. In particular the sweep map $\mathrm{sw} : \mathcal{L}_{n,n} \to \mathcal{B}_{2n}$ is a bijection.*

PROOF. The proof consists of a straightforward but rather tedious case-by-case analysis of the involved labels. Let μ be the type C_n area vector of a path $\pi \in \mathcal{L}_{n,n}$. We use the following notation. The label of the i-th north step of π is denoted by $\ell_i^{\mathbf{n}}$. The corresponding labelled step which is added to X is denoted by $(s_i^{\mathbf{n}}, x_i^{\mathbf{n}})$.

We pair each north step with an east step. If the north step has a non-negative label, this is the next east step in the same diagonal. If the north step has a negative label, this is the previous east step in the same diagonal. We denote by $\ell_i^{\mathbf{e}}$ the label of the east step corresponding to the i-th north step, and by $(s_i^{\mathbf{e}}, x_i^{\mathbf{e}})$ the associated labelled step in X.

For example in Figure 6.4 we have $\ell_4^{\mathbf{n}} = -9$, $(s_4^{\mathbf{n}}, x_4^{\mathbf{n}}) = (\mathbf{n}, -9)$, $\ell_4^{\mathbf{e}} = 1$, and $(s_4^{\mathbf{e}}, x_4^{\mathbf{e}}) = (\mathbf{e}, -1)$. Also $\ell_6^{\mathbf{n}} = 17$, $(s_6^{\mathbf{n}}, x_6^{\mathbf{n}}) = (\mathbf{n}, -17)$, $\ell_6^{\mathbf{e}} = 30$, and $(s_6^{\mathbf{e}}, x_6^{\mathbf{e}}) = (\mathbf{n}, -30)$.

The label of the i-th north step is

$$\ell_i^{\mathbf{n}} = (i-1)(2n+1) - (i - \mu_{n-i+1})(2n) = 2n(\mu_{n-i+1} - 1) + i - 1. \tag{6.1}$$

First consider the case $\mu_{n-i+1} > 0$. Then

$$2n|\mu_{n-i+1}| - 2n \leq \ell_i^{\mathbf{n}} < 2n|\mu_{n-i+1}| - n.$$

If $i > 1$ then $x_i^{\mathbf{n}} = -\ell_i^{\mathbf{n}}$. If $i = 1$ then $x_i^{\mathbf{n}} = -n$. Hence

$$-2n|\mu_{n-i+1}| + n \leq x_i^{\mathbf{n}} < -2n|\mu_{n-i+1}| + 2n.$$

On the other hand, if $\mu_{n-i+1} \leq 0$ then

$$-2n|\mu_{n-i+1}| - 2n \leq \ell_i^{\mathbf{n}} = x_i^{\mathbf{n}} < -2n|\mu_{n-i+1}| - n.$$

Now, let us treat the east steps. We start with the case $\mu_{n-i+1} > 0$. Then

$$\ell_i^{\mathbf{e}} = \ell_i^{\mathbf{n}} + 2n + k_i = 2n|\mu_{n-i+1}| + i - 1 + k_i,$$

for some $k_i \in [n-i+1]$. Since $x_i^{\mathbf{e}} = -\ell_i^{\mathbf{e}}$, we obtain

$$-2n|\mu_{n-i+1}| - n \leq x_i^{\mathbf{e}} < -2n|\mu_{n-i+1}|. \tag{6.2}$$

If $\mu_{n-i+1} \leq 0$ then

$$\ell_i^{\mathbf{e}} = \ell_i^{\mathbf{n}} + 2n - k_i = -2n|\mu_{n-i+1}| + i - 1 - k_i,$$

for some $k_i \in \{0, \dots, i-1\}$. If $\mu_{n+i-1} < 0$ then

$$-2n|\mu_{n-i+1}| \leq \ell_i^{\mathbf{e}} = x_i^{\mathbf{e}} < -2n|\mu_{n-i+1}| + n.$$

Finally assume $\mu_{n-i+1} = 0$. If $\ell_i^{\mathbf{e}} = 0$ then $x_i^{\mathbf{e}} = -n$. Otherwise $x_i^{\mathbf{e}} = -\ell_i^{\mathbf{e}}$. Combined this yields

$$-n \leq x_i^{\mathbf{e}} < 0,$$

which is a special case of (6.2).

We make the following observation. If $-2nk \leq x_i^s < -2nk + n$, where $k = 1, 2, \dots$, then either $s = \mathbf{e}$ and $\mu_{n-i+1} = -k$, or $s = \mathbf{n}$ and $\mu_{n-i+1} = -k+1$. Similarly, if $-2nk+n \leq x_i^s < -2nk+2n$, where $k = 1, 2, \dots$, then either $s = \mathbf{n}$ and $\mu_{n-i+1} = k$ or $s = \mathbf{e}$ and $\mu_{n-i+1} = k-1$.

Thus by definition the path $\mathrm{sw}(\pi)$ is composed of segments $T(-k, -k+1)$, $T(k-1, k)$, where $k = 1, 2, \dots$, such that each step of $T(-k, -k+1)$ corresponds to an entry of the area vector $\mu_{n-i+1} \in \{-k, -k+1\}$, and each step of $T(k-1, k)$ corresponds an entry $\mu_{n-i+1} \in \{k-1, k\}$.

This is a good sign because the path $\zeta_C(\pi)$ is also composed of segments with the same property. Indeed we will carry out the proof by showing that $T(-k, -k+1) = \overleftarrow{w}_{k-1}^{-}(\mu)$ and $T(k-1, k) = \overrightarrow{w}_{k-1}^{+}(\mu)$.

We first prove that $T(-k, -k+1) = \overleftarrow{w}_{k-1}^{-}(\mu)$ for $k \geq 1$. As $\mu_{n-i+1} = -k < 0$ implies $x_i^{\mathbf{e}} = \ell_i^{\mathbf{e}} < 0$ and $\mu_{n-i+1} = -k+1 \leq 0$ implies $x_i^{\mathbf{n}} = \ell_i^{\mathbf{n}} < 0$, we have $s_i^{\mathbf{e}} = \mathbf{e}$ and $s_j^{\mathbf{n}} = \mathbf{n}$. That is, every entry $\mu_{n-i+1} = -k$ will contribute an east step while each entry $\mu_{n-i+1} = -k+1$ contributes a north step. This is consistent with the definition of the zeta map. Therefore, it suffices to check that $i < j$ implies

$$x_i^{\mathbf{e}} < x_j^{\mathbf{n}} \qquad \text{if } \mu_{n-j+1} = -k+1, \mu_{n-i+1} = -k, \tag{6.3}$$

$$x_i^{\mathbf{n}} < x_j^{\mathbf{e}} \qquad \text{if } \mu_{n-j+1} = -k, \mu_{n-i+1} = -k+1. \tag{6.4}$$

Inequality (6.3) is trivial as

$$x_i^{\mathbf{e}} = -2nk + i - 1 - k_i < 2n(-k+1-1) + j - 1 = x_j^{\mathbf{n}}.$$

To see (6.4) note that $\mu_{n-j+1} = -k$ and $\mu_{n-i+1} = -k+1$ imply that the path π has an east step in the same diagonal as its i-th north step somewhere between its i-th and j-th north steps. That is, $k_j \leq j - i - 1$ and therefore

$$x_i^{\mathbf{n}} = 2n(-k+1-1) + i - 1 < -2nk + j - 1 - k_j = x_j^{\mathbf{e}}.$$

Next we show that $T(k-1, k) = \overrightarrow{w}_{k-1}^{+}(\mu)$ for $k \geq 2$. In this case $\mu_{n-i+1} = k > 1$ implies $\ell_i^{\mathbf{n}} > 0$ and $\mu_{n-i+1} = k-1 > 0$ implies $\ell_i^{\mathbf{e}} > 0$. This case is more difficult (confusing) because we do not necessarily have $s_i^{\mathbf{e}} = \mathbf{e}$ and $s_j^{\mathbf{n}} = \mathbf{n}$. Instead, if $\mu_{n-i+1} = k-1$ and $\mu_{n-j+1} = k$ then

$$s_i^{\mathbf{e}} = \begin{cases} \mathbf{e} & \text{if } \mu_{n-i} = k, \\ \mathbf{n} & \text{if } \mu_{n-i} \leq k-1 \text{ or } i = n, \end{cases} \qquad s_j^{\mathbf{n}} = \begin{cases} \mathbf{n} & \text{if } \mu_{n-j+2} = k-1, \\ \mathbf{e} & \text{if } \mu_{n-j+2} \geq k. \end{cases}$$

Thus an entry $\mu_{n-i+1} = k-1$ contributes an east step instead of a north step if and only if the previous entry $\mu_{n-i} = k$ contributes a north step instead of an east step. We see that the number of east and north steps in $T(k-1, k)$ is consistent with the definition of the zeta map.

To see that also the relative orders of the steps in $T(k-1,k)$ and $\overrightarrow{w}^+_{k-1}(\mu)$ agree, it suffices to prove that $i<j$ implies

$$x_j^{\mathbf{e}} < x_i^{\mathbf{e}} \qquad \text{if } \mu_{n-j+1} = \mu_{n-i+1} = k-1, \tag{6.5}$$

$$x_j^{\mathbf{n}} < x_i^{\mathbf{n}} \qquad \text{if } \mu_{n-j+1} = \mu_{n-i+1} = k, \tag{6.6}$$

$$x_j^{\mathbf{e}} < x_i^{\mathbf{n}} \qquad \text{if } \mu_{n-j+1} = k-1, \mu_{n-i+1} = k. \tag{6.7}$$

Moreover, let $i<j$, $\mu_{n-j+1} = k$ and $\mu_{n-i+1} = k-1$. Then we require that

$$x_i^{\mathbf{e}} < x_j^{\mathbf{n}} \tag{6.8}$$

if and only if there exists no r such that $i<r<j$ and $\mu_{n-r+1} = k-1$.

From (6.5)–(6.8) it follows that the order of the steps is (almost) obtained by reading the area vector from left to right drawing $s_i^{\mathbf{e}}$ whenever $\mu_{n-i+1} = k-1$, and $s_i^{\mathbf{n}}$ whenever $\mu_{n-i+1} = k$. The only exception to this rule is when $\mu_{n-i+1} = k-1$ and $\mu_{n-i} = k$. In this case one has to draw the step $s_i^{\mathbf{e}} = \mathbf{e}$ before the step $s_{i+1}^{\mathbf{n}} = \mathbf{n}$. [1]

We now prove the claims (6.5)–(6.8). If $i<j$ and $\mu_{n-i+1} = \mu_{n-j+1} = k-1$ then there must be an east step on the diagonal between the i-th and j-th north steps of π. Hence $k_i \leq j-i$ and we obtain (6.5).

$$x_i^{\mathbf{e}} = -2n(k-1) - i + 1 - k_i \geq -2n(k-1) - j + 1 > -2n(k-1) - j + 1 - k_j = x_j^{\mathbf{e}}$$

The inequalities (6.6) and (6.7) are trivial as

$$x_i^{\mathbf{n}} = -2n(k-1) - i + 1 > \begin{cases} -2n(k-1) - j + 1 = x_j^{\mathbf{n}}, \\ -2n(k-1) - j + 1 - k_j = x_j^{\mathbf{e}}. \end{cases}$$

To see claim (6.8), first assume that there is no r with $i<r<j$ and $\mu_{n-r+1} = k-1$. Then $k_i \geq j-i+1$ and

$$x_i^{\mathbf{e}} = -2n(k-1) - i + 1 - k_i \leq -2n(k-1) - j < -2n(k-1) - j + 1 = x_j^{\mathbf{n}}.$$

On the other hand, if there is such an r then $k_i \leq r - i < j - i$ and we obtain

$$x_i^{\mathbf{e}} = -2n(k-1) - i + 1 - k_i > -2n(k-1) - j + 1 = x_j^{\mathbf{n}}.$$

Finally, we need to show $T(0,1) = \overrightarrow{w}^+_0(\mu)$. Let $\mu_{n-j+1} = 0$ and assume $\ell_j^{\mathbf{e}} \neq 0$. Then $x_j^{\mathbf{e}} = -j + 1 + k_j$. Choose i maximal such that $i<j$ and $\mu_{n-i+1} \in \{0,1\}$. Note that such an i always exists. Then

$$s_j^{\mathbf{e}} = \begin{cases} \mathbf{n} & \text{if } \mu_{n-i+1} = 0, \\ \mathbf{e} & \text{if } \mu_{n-i+1} = 1. \end{cases}$$

If $\ell_j^{\mathbf{e}} = 0$ then $x_j^{\mathbf{e}} = -n$. In this case $\mu_{n-i+1} < 0$ for all $i<j$. Instead choose $i \leq n$ maximal such that $\mu_{n-i+1} \in \{0,1\}$. Then

$$s_j^{\mathbf{e}} = \begin{cases} \mathbf{n} & \text{if } \mu_{n-i+1} = 0, \\ \mathbf{e} & \text{if } \mu_{n-i+1} = 1. \end{cases}$$

Now let $\mu_{n-j+1} = 1$ and assume $j>1$. Then $x_j^{\mathbf{n}} = -j+1$. Choose i maximal such that $i<j$ and $\mu_{n-i+1} \in \{0,1\}$. Again such an i always exists. Then

$$s_j^{\mathbf{n}} = \begin{cases} \mathbf{n} & \text{if } \mu_{n-i+1} = 0, \\ \mathbf{e} & \text{if } \mu_{n-i+1} = 1. \end{cases}$$

[1] In fact, one also draws $s_i^{\mathbf{e}} = \mathbf{e}$ before all the east steps coming from entries of the area vector equal to k occurring between the $(n-i+1)$-th and the $(n-t+1)$-th entry, where t is minimal such that $i<t$ and $\mu_{n-t+1} = k-1$. However, permuting east steps clearly has no effect on the resulting path.

If $j = 1$ then $\ell_j^{\mathbf{n}} = 0$ and $x_j^{\mathbf{n}} = -n$. In this case choose $i \leq n$ maximal such that $\mu_{n-i+1} \in \{0, 1\}$. Then

$$s_j^{\mathbf{n}} = \begin{cases} \mathbf{n} & \text{if } \mu_{n-i+1} = 0, \\ \mathbf{e} & \text{if } \mu_{n-i+1} = 1. \end{cases}$$

We see that every entry of the area vector equal to zero contributes a north step and every entry equal to one contributes an east step. Again this is consistent with the zeta map. To see that the relative orders of north and east steps in $T(0,1)$ and $\overrightarrow{w}_0^+(\mu)$ are the same, it suffices to show that for all $i < j$

$$\begin{aligned} -n &< x_i^{\mathbf{e}} && \text{if } \ell_i^{\mathbf{e}} \neq 0, \\ -n &< x_i^{\mathbf{n}} && \text{if } i > 1, \\ x_j^{\mathbf{e}} &< x_i^{\mathbf{e}} && \text{if } \mu_{n-j+1} = \mu_{n-i+1} = 0, \ell_i^{\mathbf{e}} \neq 0, \\ x_j^{\mathbf{n}} &< x_i^{\mathbf{n}} && \text{if } \mu_{n-j+1} = \mu_{n-i+1} = 1, i > 1, \\ x_j^{\mathbf{e}} &< x_i^{\mathbf{n}} && \text{if } \mu_{n-j+1} = 0, \mu_{n-i+1} = 1, i > 1, \\ x_j^{\mathbf{n}} &< x_i^{\mathbf{e}} && \text{if } \mu_{n-j+1} = 0, \mu_{n-i+1} = 1, \ell_i^{\mathbf{e}} \neq 0. \end{aligned}$$

■

6.4. Combinatorics in type D_n

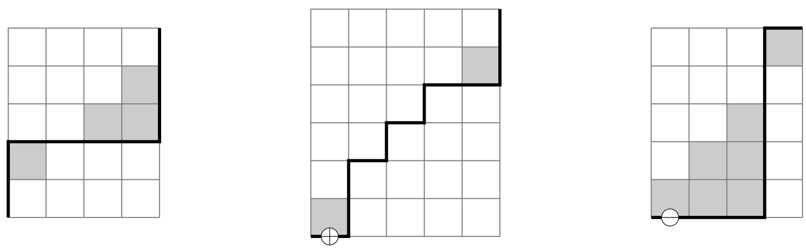


FIGURE 6.5. Signed lattice paths with area vectors $(0, -1, 2, 1, 0)$, $(-1, 0, 0, 0, 1, 0)$ and $(-3, 2, 1, 0, 2)$.

Let $\pi \in \mathcal{L}_{n-1,n}^{\bullet}$ be a signed lattice path, and define $q = \psi_D(\pi)$ as in Proposition 4.3.5, and $x \in \check{Q}$ and $s \in \mathfrak{S}_n^D$ as in Lemma 4.4.2 (iii). We define the type D_n *area vector* of π as $\mu = s(q - x)$.

Indeed note that for $1 < i < n$ the entry μ_i counts the number of unit squares in the i-th row between the path π and the alternating path $\mathbf{n}(\mathbf{en})^{n-1} \in \mathcal{L}_{n-1,n}$ (the number being negative while π is above $\mathbf{n}(\mathbf{en})^{n-1}$). Furthermore μ_1 counts the number of unit squares in the first row up to a sign, while μ_n is a little mysterious if one only looks at the picture of π.

The description of the representatives for the orbits of the Weyl group action on the finite torus imposes certain restrictions on the area vector of a signed lattice path. The following lemma captures some of these properties.

LEMMA 6.4.1. [**74**, Lem. 5.16] *Let $\pi \in \mathcal{L}_{n-1,n}^{\bullet}$ be a signed lattice path with area vector μ.*

(i) Let $i, j \in [n]$ with $i < j$ such that $\mu_j = \mu_i - 1$ and $\mu_\ell \notin \{\mu_i - 1, \mu_i\}$ for all ℓ with $i < \ell < j$. Then $j = i + 1$.

(ii) Let $i \in [n]$ such that $\mu_i \leq 0$ and $\mu_\ell \notin \{\mu_i, \mu_i + 1\}$ for all ℓ with $1 \leq \ell < i$. Then $i = 1$ or $i = 2, \mu_2 = 0$.

(iii) If $\mu_1 < 0$ then there exists $i \in [n]$ with $\mu_i \in \{-\mu_1 - 1, -\mu_1\}$.

(iv) *Assume $\mu_1 < 0$ and let $i \in [n]$ such that $\mu_i = -\mu_1 - 1$ and $\mu_\ell \notin \{-\mu_1, -\mu_1 - 1\}$ for all ℓ with $1 < \ell < i$. Then $i = 2$.*

(v) *Let $i \in [n]$ such that $\mu_i > 0$ and $\mu_\ell \notin \{\mu_i - 1, \mu_i\}$ for all ℓ with $i < \ell \leq n$. Then $i = n$.*

(vi) *If $\mu_n > 0$ then there exists $i \in [n]$ with $\mu_i \in \{-\mu_n - 1, -\mu_n\}$.*

(vii) *Assume $\mu_n > 0$ and let $i \in [n-1]$ such that $\mu_i = -\mu_n + 2$ and $\mu_\ell \notin \{-\mu_n + 1, -\mu_n + 2\}$ for all ℓ with $i < \ell < n$. Then $i = n - 1$.*

PROOF. We first prove claim (i). From $q = s(\mu) + x$ we obtain $q_\ell = \mu_\ell + \ell - 1$ for all ℓ with $1 < \ell < n$. Thus, in the described situation we have $q_\ell \notin \{\mu_i + \ell - 2, \mu_i + \ell - 1\}$ for all ℓ with $i < \ell < j$. If $n - 1 \equiv 0, 3$ modulo 4 or if $j < n$ then $q_j = \mu_i + j - 2$. Hence $q_\ell \leq q_{\ell+1}$ implies $q_\ell < \mu_i + \ell - 2$ for all ℓ with $i < \ell < j$. On the other hand if $n - 1 \equiv 1, 2$ modulo 4 and $j = n$, then $q_n = -\mu_i + n + 1$. Since $q_{n-1} + q_n \leq 2n - 1$ we obtain that $q_{n-1} \leq \mu_i + n - 2$, and again we have $q_\ell < \mu_i + \ell - 2$ for all ℓ with $i < \ell < j$. But for $\ell = i + 1$ this yields a contradiction, namely $|q_i| = |\mu_i + i - 1| \leq q_{i+1} < \mu_i + i - 1$. Therefore we must have $j = i + 1$.

To see (ii) note that by the same argument as in the proof of (i) we obtain $q_\ell < \mu_i + \ell - 1$ for all ℓ with $1 < \ell < i$. Thus if $i > 2$ then $0 \leq q_2 < \mu_i + 1$ implies $\mu_i = 0$, $q_2 = 0$ and $\mu_1 = q_1 = 0$, which is a contradiction. If $i = 2$ and $\mu_2 = -1$ then $q_2 = 0$ and $\mu_1 = q_1 = 0$, which is again a contradiction. Hence either $i = 1$ or $i = 2$ and $\mu_2 = 0$ as claimed.

To see claim (iii) assume that $\mu_i \notin \{-\mu_1 - 1, -\mu_1\}$ for all $i \in [n]$. From $q_i \geq |q_{i-1}|$ we obtain $q_i > i - 1 - \mu_1 \geq i$ for all i with $1 < i < n$. In particular $q_{n-1} \geq n$, which is a contradiction.

Similarly in the situation of (iv) we have $q_\ell > \ell - 1 - \mu_1$ for all ℓ with $1 < \ell < i$. If $2 < i < n$, then we obtain $q_{i-1} > i - 2 - \mu_1 = q_i$ which is a contradiction. If $i = n > 2$ then $q_{n-1} \geq n$ is the same contradiction as in the proof of (iii). Thus $i = 2$ by elimination.

In the situation of (v) we have $q_\ell > \ell - 1 + \mu_i \geq \ell$ for all ℓ with $i < \ell < n$ by the same argument as in the proof of (iii). Hence, if $i < n - 1$ then $q_{n-1} \geq n$ is a contradiction. If $i = n - 1$ and $\mu_{n-1} = 1$, then $q_{n-1} = n - 1$. Therefore $q_n \in \{n - 1, n\}$ and $\mu_n \in \{0, 1\}$, which is a contradiction. The only remaining possibility is $i = n$.

To see (vi) assume that $\mu_\ell \notin \{-\mu_n - 1, -\mu_n\}$ for all $\ell \in [n]$. Then, by similar reasoning as in the proof of (i), we have $q_\ell < \ell - \mu_n - 2$ for all ℓ with $1 < \ell < n$. This yields a contradiction for $\ell = 2$.

Next consider claim (vii). If $n - 1 \equiv 0, 3$ modulo 4 then $2n - 1 - q_n = n - \mu_n$. If $n - 1 \equiv 1, 2$ modulo 4 then $q_n = n - \mu_n$. In both cases we have $q_{n-1} \leq n - \mu_n$. Hence $q_\ell \notin \{\ell - \mu_n, \ell - \mu_n + 1\}$ implies $q_\ell < \ell - \mu_n$ for all ℓ with $i < \ell < n$. This yields a contradiction for $\ell = i + 1$, namely $q_{i+1} < i + 1 - \mu_n = |q_i|$. Therefore we must have $i = n - 1$. ■

Let $(w, \pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n-1,n}^\bullet$ be a pair of a signed permutation and a signed lattice path, and let μ denote the type D_n area vector of π. Define the type D_n *diagonal reading word* $\mathrm{drw}_D(w, \pi)$ of (w, π) as follows: For each $i = 0, 1, 2, \ldots$ first write down the labels $w(j)$ of the rows with $\mu_j = -i$ from bottom to top, then write down the negative labels $-w(j)$ of rows with $\mu_j = i + 1$ from top to bottom. Finally we need to adjust some signs: Multiply the label coming from the top row by $(-1)^{1+\mu_{n-1}+\mu_n}$ and the label coming from the bottom row by $\epsilon(\pi)(-1)^{1+x_{n-1}+x_n}$. Then change the sign of $\mathrm{drw}_D(w, \pi)(1)$ if the number of positive entries of μ is odd.

Except for some necessary twists, the definition of the diagonal reading word of type D_n follows very similar rules as the counterparts in types A_{n-1} and C_n. Note that $\mathrm{drw}_D(w, \pi) \in \mathfrak{S}_n^D$, that is, the diagonal reading word has an even number of sign changes, if and only if $w \in \mathfrak{S}_n^B u_\pi$, where u_π is defined as in (4.1).

The following proposition asserts that our definition always yields the desired signed permutation. We denote $\mathrm{Vert}(D_n) = \mathrm{Vert}(D_n, h + 1)$.

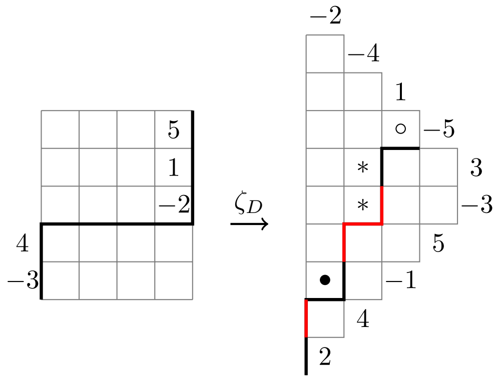


FIGURE 6.6

PROPOSITION 6.4.2. [**74**, Prop. 5.19] *Let $(w,\pi) \in \mathrm{Vert}(D_n)$ be a vertically labelled signed lattice path with area vector μ and choose $u \in \mathfrak{S}_n^D$ such that $t_\mu u^{-1}$ is Graßmannian. Moreover let $u_\pi \in \mathfrak{S}_n^B$ be defined as in* (4.1)*, and $s \in \mathfrak{S}_n^D$ as in Lemma 4.4.2* (iii)*. Then $\mathrm{drw}_D(w,\pi) = wu_\pi s u^{-1}$.*

PROOF. By Lemma 1.5.4 we have $\left|u^{-1}(i)\right| = j$ if and only if

$$\begin{aligned} i = \#\bigl\{r \in [n], |\mu_r| < |\mu_j|\,\bigr\} + \#\bigl\{r \in [n] : j \le r \le n, \mu_r = \mu_j > 0\bigr\} \\ + \#\bigl\{r \in [j], \mu_r = \mu_j \le 0\bigr\} + \#\bigl\{r \in [n], \mu_r = -\mu_j > 0\bigr\}. \end{aligned}$$

Comparing this to the definition of $\mathrm{drw}_D(w,\pi)$ we obtain

$$|\mathrm{drw}_D(w,\pi)(i)| = |w(j)| = \left|wu^{-1}(i)\right| = \left|wu_\pi s u^{-1}(i)\right|.$$

Comparing Lemma 1.5.4 to the description of u_π, s and the definition of the diagonal reading word, one can check that all signs work out and we may indeed drop the absolute value in the above identity. Compare with the proof of Proposition 6.5.1. ■

We are now in a position to present the definition of the combinatorial zeta map.

Define the type D_n *zeta map* on lattice paths $\zeta_D : \mathcal{L}^\bullet_{n-1,n} \to \mathcal{B}^\bullet_{2n-1}$ by mapping be a signed lattice π path with area vector μ to the path obtained from

$$\overleftarrow{w}^-_{2n-1}(\mu)\overrightarrow{w}^+_{2n-1}(\mu)\overleftarrow{w}^-_{2n-2}(\mu)\overrightarrow{w}^+_{2n-2}(\mu)\cdots\overleftarrow{w}^-_1(\mu)\overrightarrow{w}^+_1(\mu)\overleftarrow{w}^-_0(\mu)\overrightarrow{w}^+_0(\mu),$$

by deleting the last step and, if its n-th north step is followed by an east step, adding a sign to this east step such that

$$\epsilon(\zeta_D(\pi)) = (-1)^{\#\{r\in[n]:\mu_r>0\}}.$$

Moreover, define the type D_n *zeta map* on labelled objects $\zeta_D : \mathfrak{S}_n^B \times \mathcal{L}^\bullet_{n-1,n} \to \mathfrak{S}_n^B \times \mathcal{B}^\bullet_{2n-1}$ via

$$\zeta_D(w,\pi) = (\mathrm{drw}_D(w,\pi), \zeta_D(\pi)).$$

Note that by definition the image $\zeta_D(\pi)$ of a signed lattice path is a signed ballot path, that is, we really obtain a map $\zeta_D : \mathcal{L}^\bullet_{n-1,n} \to \mathcal{B}^\bullet_{2n-1}$. See Figures 6.6, 6.7 and 6.8 for examples.

The following theorem is the main result of this section.

THEOREM 6.4.3. [**74**, Thm. 5.22] *The zeta map restricts to a bijection $\zeta_D : \mathrm{Vert}(D_n) \to \mathrm{Diag}(D_n)$ from vertically labelled signed lattice paths to diagonally labelled signed ballot paths.*

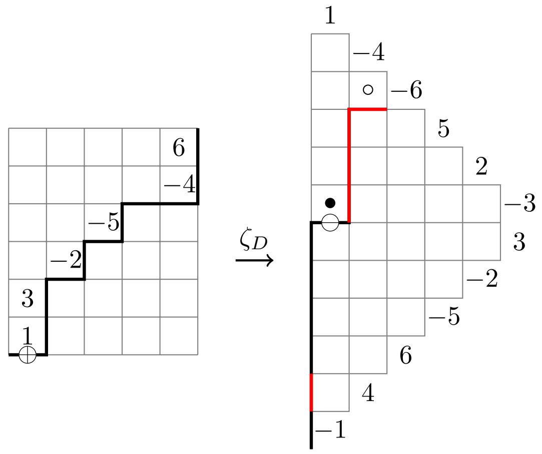


FIGURE 6.7

Our strategy for proving Theorem 6.4.3 is to demonstrate that the type D_n zeta map with domain $\mathrm{Vert}(D_n)$ is equivalent to the uniform zeta map defined in Section 6.1. More precisely, Theorem 6.4.3 follows from Theorems 6.4.8 and 6.4.9 below.

Note that Theorem 6.4.3 immediately implies the bijectivity of the map $\zeta_D : \mathcal{L}^\bullet_{n-1} \to \mathcal{B}^\bullet_{n-1}$ since the underlying ballot path of $\zeta_D(w,\pi)$ only depends on π and not on w.

THEOREM 6.4.4. [**74**, Thm. 5.23] *The map $\zeta_D : \mathcal{L}^\bullet_{n-1,n} \to \mathcal{B}^\bullet_{2n-1}$ is a bijection from signed lattice paths to signed ballot paths.* ∎

So far using Theorem 6.4.3 is our only way of proving Theorem 6.4.4. Compare with the proof of Theorem 6.3.3.

OPEN PROBLEM 6.4.5. *Find a combinatorial description of the inverse of the map $\zeta_D : \mathcal{L}^\bullet_{n-1,n} \to \mathcal{B}^\bullet_{2n-1}$ (perhaps using some kind bounce paths).*

Given any signed path π define π^* as the path obtained from π by replacing all signed east steps $\mathbf{e}^+, \mathbf{e}^-$ by simple east steps $\mathbf{e}$. Set $\zeta^*_D(\pi^*) = (\zeta_D(\pi))^*$ for all $\pi \in \mathcal{L}^\bullet_{n-1,n}$. The zeta map of type D_n thereby gives rise to a new bijection between lattice paths in an $n-1 \times n$ rectangle and ballot paths of odd length.

THEOREM 6.4.6. [**74**, Thm. 5.24] *The map $\zeta^*_D : \mathcal{L}_{n-1,n} \to \mathcal{B}_{2n-1}$ is a well-defined bijection.*

PROOF. Suppose $\pi, \rho \in \mathcal{L}^\bullet_{n-1,n}$ differ only by the sign of the initial east step. Then their respective area vectors differ only by the sign of the first entry. It is easy to see that $\zeta_D(\pi)$ and $\zeta_D(\rho)$ can only differ by the sign of an east step. For example, suppose $a = (k, a_2, \dots, a_2)$ and $b = (-k, a_2, \dots, a_n)$, where $k > 0$, then

$$\overleftarrow{w}^-_k(a)\mathbf{n} = \overleftarrow{w}^-_k(b) \text{ and } \overrightarrow{w}^+_k(a) = \mathbf{n}\overrightarrow{w}^+_k(b).$$

Hence

$$\overleftarrow{w}^-_k(a)\overrightarrow{w}^+_k(a) = \overleftarrow{w}^-_k(b)\overrightarrow{w}^+_k(b).$$

Consequently ζ^*_D is well-defined and bijectivity follows from Theorem 6.4.4. ∎

Before we attack the proof of Theorem 6.4.3 we need another auxiliary result.

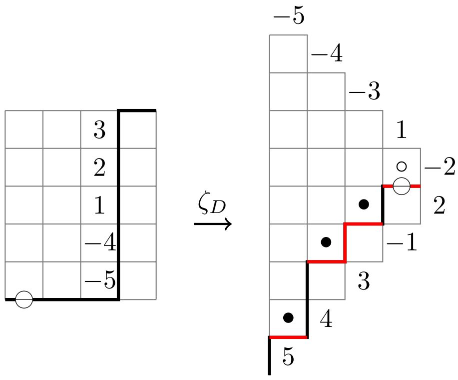


FIGURE 6.8

LEMMA 6.4.7. [**74**, Lem. 5.25] *Let π be a signed lattice path with area vector μ, and let $i \in [n]$ be such that*

$$1 = \#\{r \in [n] : |\mu_r N - r| \leq |\mu_i N - i|\}.$$

Then either i is minimal such that $\mu_i = 0$ or $\mu_j \neq 0$ for all $j \in [n]$, $\mu_n = 1$ and $i = n$.

PROOF. Clearly $\mu_i \neq 0$ implies that $\mu_\ell \neq 0$ for all $\ell \in [n]$. Moreover $\mu_\ell \neq 0$ for all $\ell \in [n]$ and $\mu_n = 1$ implies $i = n$. Thus assume that $\mu_\ell \neq 0$ for all $\ell \in [n]$ and $\mu_n \neq 1$. From $q_{n-1} \leq \min\{q_n, 2n-1-q_n\}$ we obtain $\mu_{n-1} < 0$. Thus $\mu_\ell < 0$ for all ℓ with $1 < \ell < n$ which yields a contradiction for $\ell = 2$. ■

As in types A_{n-1} and C_n there exists a correspondence between the rises of (w,π) and the valleys of $\zeta_D(w,\pi)$.
Let $(w,\pi) \in \mathfrak{S}_n^B \times \mathcal{L}^\bullet_{n-1,n}$ and i be a rise of π. We say i is a *rise* of (w,π) *labelled* by $(w(i), w(i+1))$. In the special case where π begins with two north steps and $i=1$, we say i is labelled by $(\pm|w(i)|, w(i+1))$ instead. Similarly let $(v,\beta) \in \mathfrak{S}_n^D \times \mathcal{B}^\bullet_{2n-1}$ and (i,j) be a valley of β. We say

$$(i,j) \text{ is a } \textit{valley} \text{ of } (v,\beta) \textit{ labelled} \text{ by } \begin{cases} \big(v(n+1-i), v(n+1-j)\big) & \text{if } j < n, \\ \big(v(n+1-i), \epsilon(\beta)v(1)\big) & \text{if } j = n, \\ \big(v(n+1-i), \epsilon(\beta)v(-1)\big) & \text{if } j = n+1, \\ \big(v(n+1-i), v(n-j)\big) & \text{if } j > n+1. \end{cases}$$

Recall that there is a special convention concerning the valleys of signed ballot paths. If $j = n$ and the n-th north step of β is not followed by an east step, then we count both (i,n) and $(i,n+1)$ as valleys. These two valleys are labelled $(v(n+1-i), \pm|v(1)|)$.
Note that by placing the labels $v(i)$, where $i = n, \ldots, 2, \epsilon(\beta), -\epsilon(\beta), -2, \ldots, -n$, in the diagonal, each valley is labelled by the number below it and the number to its right.

THEOREM 6.4.8. [**74**, Thm. 5.26] *Let $(w,\pi) \in \mathfrak{S}_n^B \times \mathcal{L}^\bullet_{n-1,n}$ and $a, b \in w([n])$. Then (w,π) has a rise labelled (a,b) if and only if $\zeta_D(w,\pi)$ has a valley labelled (b,a) or $(-a,-b)$. Moreover (w,π) has a rise labelled $(\pm|a|, b)$ if and only if $\zeta_D(w,\pi)$ has valleys labelled $(b, \pm|a|)$.*

PROOF. Let π_i denote the number of east steps that occur before the i-th north step of π. Let $q = \psi_D(pi) \in \check{Q} \cap (h+1)\overline{A_\circ}$, let μ be the type D_n area vector of π and choose $u \in \mathfrak{S}_n^D$ such that $t_\mu u^{-1}$ is Graßmannian. Define $u_\pi \in \mathfrak{S}_n^B$ as in Theorem 4.3.6 and $s \in \mathfrak{S}_n^D$ as in Lemma 4.4.2 (iii).

(Part 1) We start out by demonstrating the backward implication. Therefore assume that $\zeta_D(\pi)$ has a valley (x,y). Recall that $\zeta_D(\pi)$ is the concatenation of sequences $\overleftarrow{w}_k^-(\mu)$ and $\overrightarrow{w}_k^+(\mu)$, thus there are multiple situations in which a valley can arise: within such a sequence or at the join of two sequences. These cases, while being similar, have to be treated separately.

(1.1) The valley (x,y) appears within a sequence $\overleftarrow{w}_k^-(\mu)$. Then there exist indices $i,j\in[n]$ with $i<j$ such that $\mu_i=-k$, $\mu_j=-k-1$ and $\mu_\ell\notin\{-k,-k-1\}$ for all ℓ with $i<\ell<j$. By Lemma 6.4.1 (i) we have $j=i+1$.

If $i=1$ then $q_2=1+\mu_2=-k\leq 0$ thus $q_1=q_2=0$. It follows that $\pi_1=\pi_2=0$ and i is a rise of π. If $n-1\equiv 1,2$ modulo 4 and $i=n-1$ then $q_n=n-\mu_n=n+k+1$ and $q_{n-1}=n-2-k$. Hence $q_{n-1}+q_n=2n-1$ and i is a rise of π. In all other cases $q_i=i-1-k$ and $q_{i+1}=i-k-1$. Thus $\pi_i=q_i=q_{i+1}=\pi_{i+1}$ and again i is a rise of π.

The number x equals the number of east steps in the sequence

$$\overleftarrow{w}_{2n+1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overleftarrow{w}_{k+1}^-(\mu)\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu_{i+1},\ldots,\mu_n).$$

Hence,

$$\begin{aligned}x&=\#\big\{r\in[n]:|\mu_r|>k+1\big\}+\#\big\{r\in[n]:i+1\leq r,\mu_r=-k-1\big\}\\&=\#\big\{r\in[n]:|\mu_rN-r|\geq|(-k-1)N-(i+1)|\,\big\}.\end{aligned}$$

Note that $|u(i+1)|\neq 1$ by Lemma 6.4.7 because $\mu_{i+1}<0$. Lemma 1.5.4 therefore provides

$$\begin{aligned}n+1-x&=\#\big\{r\in[n]:|\mu_rN-r|\leq|\mu_{i+1}N-(i+1)|\,\big\}\\&=|u(i+1)|=u(i+1).\end{aligned}$$

We conclude that

$$\mathrm{drw}_D(w,\pi)(n+1-x)=wu_\pi su^{-1}(n+1-x)=wu_\pi s(i+1)=w(i+1)$$

because even if $i+1=n$ we have $\mu_{n-1}+\mu_n=-2k-1$, which is odd.

On the other hand, y equals the number of north steps in the sequence

$$\overleftarrow{w}_{2n-1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overleftarrow{w}_{k+1}^-(\mu)\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu_i,\ldots,\mu_n).$$

By Lemma 1.5.4 we therefore have

$$n+1-y=\#\big\{r\in[n]:|\mu_rN-r|\leq|\mu_iN-i|\,\big\}=|u(i)|\,.$$

We first treat the case where $|u(i)|\neq 1$. Note that this implies $i>1$. Then by Lemma 1.5.4 we have $n+1-y=u(i)$ because $\mu_i\leq 0$. We conclude that

$$\mathrm{drw}_D(w,\pi)(n+1-y)=wu_\pi su^{-1}(n+1-y)=wu_\pi s(i)=w(i),$$

and the valley is labelled by $(w(i+1),w(i))$.

Next assume that $|u(i)|=1$, that is, $y=n$. Then $k=0$. If $i=1$ then we have already seen that π begins with two north steps. Since $\overleftarrow{w}_0^-(\mu)$ ends with a valley and $\overrightarrow{w}_0^+(\mu)$ begins with a north step, we are in the special situation that the valleys (x,n) and $(x,n+1)$ are labelled by $(w(2),\pm|w(1)|)$.

Finally if $i>1$ then $\mu_\ell\neq 0$ for all ℓ with $1\leq\ell<i$. It follows that there has to be an index $\ell\in[i-1]$ such that $\mu_\ell\in\{1,-1\}$ and thus the n-th north step of $\zeta_D(\pi)$ is followed by an east step. Consequently $\zeta_D(\pi)$ contains a signed east step whose sign is determined by the number of positive entries of μ. We conclude that

$$\mathrm{drw}_D(w,\pi)(1)=wu_\pi su^{-1}(1)=(-1)^{\#\{r\in[n]:\mu_r>0\}}wu_\pi s(i)=\epsilon(\zeta_D(\pi))w(i),$$

in which case the valley is labelled by $(w(i+1),w(i))$.

(1.2) Secondly, assume that the valley (x, y) arises within a sequence $\overrightarrow{w}_k^+(\mu)$ for some $k > 0$. Then there exist indices $i, j \in [n]$ with $i < j$ such that $\mu_i = k+1$, $\mu_j = k$ and $\mu_\ell \notin \{k, k+1\}$ for all ℓ with $i < \ell < j$. From Lemma 6.4.1 (i) we obtain $j = i + 1$.

If $n-1 \equiv 1, 2$ modulo 4 and $i = 1$, then $q_1 = -k-1 < 0$ and $q_2 = 1+k$. Hence $\pi_1 = \pi_2 = k+1$, $\epsilon(\pi) = -1$ and i is a rise of π. Note that $i = n-1$ yields a contradiction: Either $n-1 \equiv 0, 3$ modulo 4, then $q_{n-1} = (n-2)+(k+1)$ and $q_n = n-1+k$ thus $q_{n-1}+q_n \geq 2n$, or $n-1 \equiv 1, 2$ modulo 4, then $q_n = n-k$ and $q_{n-1} > q_n$. In all other cases we have $q_i = (i-1)+(k+1) = i+k = q_{i+1}$. Thus $\pi_i = \pi_{i+1}$ and i is a rise of π. Note that if $n-1 \equiv 0, 3$ modulo 4 and $i = 1$ then $q_1 = k+1 > 0$ and $\epsilon(\pi) = 1$.

Similar to the case above, x equals the number of east steps in the sequence

$$\overleftarrow{w}_{2n-1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu)\overrightarrow{w}_k^+(\mu_1, \dots, \mu_i).$$

Using Lemma 1.5.4 we conclude

$$n+1-x = \#\big\{r \in [n] : |\mu_r N - r| \leq |\mu_i N - i|\,\big\} = |u(i)|\,.$$

By Lemma 6.4.7 we have $|u(i)| \neq 1$ because $\mu_i > 1$, thus Lemma 1.5.4 yields $-u(i) = n+1-x$. We obtain

$$\mathrm{drw}_D(w, \pi)(n+1-x) = wu_\pi su^{-1}(n+1-x) = -wu_\pi s(i) = -w(i),$$

where we use that $\epsilon(\pi) = -1$ if and only if $n-1 \equiv 1, 2$ modulo 4 in the case where $i = 1$.

On the other hand, y equals the number of north steps in the sequence

$$\overleftarrow{w}_{2n-1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu)\overrightarrow{w}_k^+(\mu_1, \dots, \mu_{i+1}).$$

It follows that

$$n+1-y = \#\big\{r \in [n] : |\mu_r N - r| \leq |\mu_{i+1} N - (i+1)|\,\big\} = |u(i+1)|\,.$$

Suppose $y = n$, then by Lemma 6.4.7 we see that $\mu_\ell \neq 0$ for all $\ell \in [n]$, $k = 1$ and $i+1 = n$, which is a contradiction as mentioned above. Thus $y < n$. From $\mu_{i+1} > 0$ and Lemma 1.5.4 we obtain $-u(i+1) = n+1-y$. Therefore

$$\mathrm{drw}_D(w, \pi)(n+1-y) = wu_\pi su^{-1}(n+1-y) = -wu_\pi s(i+1) = -w(i+1),$$

and the valley is labelled by $(-w(i), -w(i+1))$.

(1.3) The sequence $\overleftarrow{w}_k^-(\mu)$ ends with an east step and the next non-empty sequence begins with a north step. If $\overleftarrow{w}_k^-(\mu)$ ends with an east step then there exists an index $i \in [n]$ such that $\mu_i = -k-1$ and $\mu_\ell \notin \{-k-1, -k\}$ for all ℓ with $1 \leq \ell < i$. By Lemma 6.4.1 (ii) we have $i = 1$. Consequently by Lemma 6.4.1 (iii) the sequence $\overrightarrow{w}_k^+(\mu)$ is non-empty. By assumption this means that there exists an index $j \in [n]$ such that $\mu_j = k = -\mu_1 - 1$ and $\mu_\ell \notin \{k, k+1\}$ for all ℓ with $1 < \ell < j$. Now Lemma 6.4.1 (iv) implies that $j = 2$.

If $n-1 \equiv 0, 3$ modulo 4 then $q_1 = -k-1$ and $q_2 = k+1$. Hence $\pi_1 = \pi_2 = k+1$ and $\epsilon(\pi) = -1$. On the other hand if $n-1 \equiv 1, 2$ modulo 4 then $\pi_1 = q_1 = k+1 = q_2 = \pi_2$ and $\epsilon(\pi) = 1$. In both cases i is a rise of π.

(1.3.1) Assume that $k > 0$, and let (x, y) be the valley under consideration. Then x equals the number of east steps in the sequence

$$\overleftarrow{w}_{2n-1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overrightarrow{w}_{k+1}^+(\mu)\overleftarrow{w}_k^-(\mu).$$

We deduce that

$$n+1-x = \#\big\{r \in [n] : |\mu_r N - r| \leq |(-k-1)N - 1|\,\big\} = |u(1)|\,,$$

and, since $\mu_1 < 0$ implies $|u(1)| \neq 1$, that $n+1-x = u(1)$. Moreover

$$\mathrm{drw}_D(w, \pi)(n+1-x) = wu_\pi su^{-1}(n+1-x) = wu_\pi s(1) = -w(1),$$

because $\epsilon(\pi) = -1$ if and only if $n - 1 \equiv 0, 3$ modulo 4.

On the other hand, y equals the number of north steps in the sequence

$$\overleftarrow{w}^{-}_{2n-1}(\mu)\overrightarrow{w}^{+}_{2n-1}(\mu) \cdots \overrightarrow{w}^{+}_{k+1}(\mu)\overleftarrow{w}^{-}_{k}(\mu)\overrightarrow{w}^{+}_{k}(\mu_1, \mu_2).$$

As before we obtain

$$n + 1 - y = \#\big\{r \in [n] : |\mu_r N - r| \leq |kN - 2|\,\big\} = |u(2)|\,.$$

We know that $|u(2)| \neq 1$ because of Lemma 6.4.7. Since $\mu_2 = k > 0$ we conclude

$$\mathrm{drw}_D(w, \pi)(n + 1 - y) = wu_\pi su^{-1}(n + 1 - y) = -wu_\pi s(2) = -w(2).$$

(1.3.2) Next assume that $k = 0$. In this case the valley is of the form $(x, n + 1)$, where $n + 1 - x = u(1)$ just as in (1.3.1) above. On the other hand by Lemma 6.4.7 we see that $|u(2)| = 1$. Note that $\overleftarrow{w}^{-}_{0}(\mu)$ ending with an east step implies that the n-th north step of $\zeta_D(\pi)$ is followed by an east step. We obtain

$$\mathrm{drw}_D(w, \pi)(1) = wu_\pi su^{-1}(1) = (-1)^{\#\{r\in[n]:\mu_r>0\}}wu_\pi s(2) = \epsilon(\zeta_D(\pi))w(2),$$

and the valley $(x, n + 1)$ is labelled by

$$\big(\mathrm{drw}_D(w, \pi)(n + 1 - x), -\epsilon(\zeta_D(\pi))\,\mathrm{drw}_D(w, \pi)(1)\big) = \big(-w(1), -w(2)\big).$$

(1.4) For some $k > 0$ the sequence $\overrightarrow{w}^{+}_{k}(\mu)$ ends with an east step and the next non-empty sequence begins with a north step. Then there exists an index $j \in [n]$ such that $\mu_j = k + 1$ and $\mu_\ell \notin \{k, k + 1\}$ for all ℓ with $j < \ell \leq n$. By Lemma 6.4.1 (v) and (vi) we know that $j = n$ and that $\overleftarrow{w}^{-}_{k-1}(\mu)$ is non-empty. By assumption $\overleftarrow{w}^{-}_{k-1}(\mu)$ begins with a north step, thus there exists an index $i \in [n]$ such that $\mu_i = -k + 1$ and $\mu_\ell \notin \{-k, -k + 1\}$ for all ℓ with $i < \ell \leq n$. Using Lemma 6.4.1 (vii) we see that $i = n - 1$.

If $n - 1 \equiv 0, 3$ modulo 4 then $q_{n-1} + q_n = (n - k - 1) + (n + k) = 2n - 1$. If $n - 1 \equiv 1, 2$ modulo 4 then $q_{n-1} = n - k - 1 = q_n$. In both cases i is a rise of π.

Once more let (x, y) be the valley under consideration. Then x is the number of east steps in the sequence

$$\overleftarrow{w}^{-}_{2n-1}(\mu)\overrightarrow{w}^{+}_{2n-1}(\mu) \cdots \overleftarrow{w}^{-}_{k}(\mu)\overrightarrow{w}^{+}_{k}(\mu),$$

and therefore

$$n + 1 - x = \#\big\{r \in [n] : |\mu_r N - r| \leq |(k + 1)N - n|\,\big\} = |u(n)|\,.$$

From $\mu_n > 1$ we obtain $n + 1 - x = -u(n)$ and

$$\mathrm{drw}_D(w, \pi)(n + 1 - x) = wu_\pi su^{-1}(n + 1 - x) = -wu_\pi s(n) = w(n)$$

because $\mu_{n-1} + \mu_n = (k + 1) + (-k + 1) = 2$ is even.

Moreover y equals the number of north steps in the sequence

$$\overleftarrow{w}^{-}_{2n-1}(\mu)\overrightarrow{w}^{+}_{2n-1}(\mu) \cdots \overleftarrow{w}^{-}_{k}(\mu)\overrightarrow{w}^{+}_{k}(\mu)\overleftarrow{w}^{-}_{k-1}(\mu_{n-1}, \mu_n).$$

Thus,

$$n + 1 - y = \#\big\{r \in [n] : |\mu_r N - r| \leq |(-k + 1)N - (n - 1)|\,\big\} = |u(n - 1)|\,.$$

First suppose that $|u(n - 1)| \neq 1$. Then $\mu_{n-1} \leq 0$ implies $n + 1 - y = u(n - 1)$, and we obtain

$$\mathrm{drw}_D(w, \pi)(n + 1 - y) = wu_\pi su^{-1}(n + 1 - y) = wu_\pi s(n - 1) = w(n - 1).$$

On the other hand if $|u(n - 1)| = 1$ then $y = n$, $\mu_{n-1} = 0$, $k = 1$ and $\mu_\ell \neq 0$ for all $\ell \in [n-2]$. We claim that the n-th north step of $\zeta_D(\pi)$, which belongs to $\overleftarrow{w}^{-}_{0}(\mu)$ and corresponds to $\mu_{n-1} = 0$, is followed by an east step. To see this assume that $\mu_\ell \notin \{0, -1\}$ for all $\ell \in [n - 2]$. Then there has to be an index $\ell \in [n - 2]$ with $\mu_\ell = 1$. Thus $\overrightarrow{w}^{+}_{0}(\mu)$ begins with an east step and this east step is not the last step of $\overrightarrow{w}_0^+(\mu)$. Consequently, this east step is replaced by a signed east step such that $\epsilon(\zeta_D(\pi)) = (-1)^{\#\{r\in[n]:\mu_r>0\}}$. From

$$\mathrm{drw}_D(w,\pi)(1) = wu_\pi su^{-1}(1) = (-1)^{\#\{r\in[n]:\mu_r>0\}}wu_\pi s(n-1) = \epsilon(\zeta_D(\pi))w(n-1).$$

we obtain that the valley is labelled by $(w(n), w(n-1))$.

(1.5) The the valley arises within (or at the end of) the sequence $\overrightarrow{w}_0^+(\mu)$. Note that $\overrightarrow{w}_0^+(\mu)$ is non-empty by Lemma 6.4.7, and recall that the last letter of $\overrightarrow{w}_0^+(\mu)$ does not contribute to $\zeta_D(\pi)$.

(1.5.1) There exist indices $i, j \in [n]$ with $i < j$ such that $\mu_i = 1$, $\mu_j = 0$ and $\mu_\ell \notin \{0,1\}$ for all ℓ with $i < \ell < j$. Note that it does not make a difference if the north step corresponding to $\mu_j = 0$ is deleted. In this case $\zeta_D(\pi)$ ends with an east step, which is still counted as a valley. By Lemma 6.4.1 (i) we have $j = i+1$.

If $i = 1$ and $n-1 \equiv 1,2$ modulo 4, then $q_1 = -1$ and $q_2 = 1$. Hence $\pi_1 = \pi_2 = 1$ and $\epsilon(\pi) = -1$. If $i = n-1$ and $n-1 \equiv 1,2$ modulo 4, then $q_{n-1} = n-1$, $q_n = n$ and $q_{n-1}+q_n = 2n-1$. Otherwise $\pi_i = q_i = i = q_{i+1} = \pi_{i+1}$. In all cases i is a rise of π. Also note that $\epsilon(\pi) = 1$ if $i = 1$ and $n-1 \equiv 0,3$ modulo 4.

Let (x,y) be the present valley. Then x equals the number of east steps in the sequence

$$\overleftarrow{w}_{2n-1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overleftarrow{w}_0^-(\mu)\overrightarrow{w}_0^+(\mu_1,\ldots,\mu_i).$$

We obtain

$$n+1-x = \#\big\{r\in[n] : |\mu_r N - r| \le |N-i|\,\big\} = |u(i)|\,.$$

Since $\mu_i = 1$ and $\mu_{i+1} = 0$ we have $|u(i)| \neq 1$ and $-u(i) = n+1-x$. Thus

$$\mathrm{drw}_D(w,\pi)(n+1-x) = wu_\pi su^{-1}(n+1-x) = -wu_\pi s(i) = -w(i)$$

as even if $i = 1$ we have $\epsilon(\pi) = -1$ if and only if $n-1 \equiv 1,2$ modulo 4.

On the other hand y is the number of north steps in the sequence

$$\overleftarrow{w}_{2n-1}^-(\mu)\overrightarrow{w}_{2n-1}^+(\mu)\cdots\overleftarrow{w}_0^-(\mu)\overrightarrow{w}_0^+(\mu_1,\ldots,\mu_{i+1}).$$

Hence

$$y-n = \#\big\{r\in[i+1] : \mu_r = 0\big\} = \big\{r\in[n] : |\mu_r N - r| \le i+1\big\} = |u(i+1)|\,.$$

First assume that $y > n+1$, then by Lemma 1.5.4 we have $u(i+1) = y-n$ since $\mu_{i+1} = 0$. It follows that

$$\mathrm{drw}_D(w,\pi)(n-y) = -wu_\pi su^{-1}(y-n) = -wu_\pi s(i+1) = -w(i+1)$$

because even if $i+1 = n$ then $\mu_{n-1}+\mu_n = 1$, which is odd. Thus the valley under consideration is labelled by $(-w(i), -w(i+1))$.

If $y = n+1$ then there are exactly n north steps in $\zeta_D(\pi)$ that occur before the east step corresponding to $\mu_i = 1$. Hence the n-th north step of $\zeta_D(\pi)$ is followed by an east step, and $\zeta_D(\pi)$ contains a signed east step. Thus

$$\mathrm{drw}_D(w,\pi)(y) = wu_\pi su^{-1}(1) = (-1)^{\#\{r\in[n]:\mu_r>0\}}wu_\pi s(i+1) = \epsilon(\zeta_D(\pi))w(i+1)$$

because even if $i+1 = n$ then $\mu_{n-1}+\mu_n = 1$, which is odd. Thus the valley $(x, n+1)$ is labelled by $(-w(i), -w(i+1))$ in this case as well.

(1.5.2) There exist indices $i, j \in [n]$ with $i < j$ such that $\mu_i = \mu_j = 1$ and $\mu_\ell \notin \{0,1\}$ for all ℓ with $i < \ell < j$ or $j < \ell \le n$. In this case the final two steps of $\overrightarrow{w}_0^+(\mu)$ are east steps. The latter one is deleted and $\zeta_D(\pi)$ ends with an east step.

From Lemma 6.4.1 (v) we obtain $j = n$, hence by Lemma 6.4.1 (vii) we have $i = n-1$. If $n-1 \equiv 0,3$ modulo 4 then $q_n = n$ and $q_{n-1} = n-1$ hence $q_{n-1}+q_n = 2n-1$. On the other hand, if $n-1 \equiv 1,2$ modulo 4 then $q_{n-1} = q_n = n-1$. In both cases i is a rise of π.

Let (x, y) be the valley under consideration. Then x equals the number of east steps in the sequence

$$\overleftarrow{w}_{2n-1}^{-}(\mu)\overrightarrow{w}_{2n-1}^{+}(\mu)\cdots\overleftarrow{w}_{0}^{-}(\mu)\overrightarrow{w}_{0}^{+}(\mu_1,\dots,\mu_{n-1}).$$

Therefore

$$n+1-x = \#\big\{r\in[n] : |\mu_r N - r| \le n+2\big\} = |u(n-1)|\,.$$

From Lemmas 1.5.4 and 6.4.7 we obtain $n+1-x = -u(n-1)$ because $\mu_{n-1}=1$. As above we conclude that

$$\mathrm{drw}_D(w,\pi)(n+1-x) = wu_\pi s u^{-1}(n+1-x) = -wu_\pi s(n-1) = -w(n-1).$$

On the other hand y equals one plus the number of north steps in $\zeta_D(\pi)$, that is,

$$y-n = 1+\#\big\{r\in[n] : \mu_r = 0\big\} = \#\big\{r\in[n] : |\mu_r N - r| \le n+1\big\} = |u(n)|\,.$$

First assume that $\mu_\ell = 0$ for some $\ell\in[n]$. Then $|u(n)|\neq 1$. Lemmas 1.5.4 and 6.4.7 therefore imply $y-n=-u(n)$, and we obtain

$$\mathrm{drw}_D(w,\pi)(n-y) = -wu_\pi s u^{-1}(y-n) = wu_\pi s(n) = -w(i)$$

because $\mu_{n-1}+\mu_n = 2$, which is even. The valley is thus labelled by $(-w(n-1),-w(n))$.

On the other hand if $\mu_\ell\neq 0$ for all $\ell\in[n]$, then $y-n=|u(n)|=1$ by Lemma 6.4.7. Note that the n-th north step of $\zeta_D(\pi)$ is followed by an east step, because $\zeta_D(\pi)$ only has n north steps and ends with an east step. Hence $\zeta_D(\pi)$ contains a signed east step. Since $\mu_n=1$, Lemma 1.5.4 then implies that

$$\mathrm{drw}_D(w,\pi)(1) = wu_\pi s u^{-1}(1) = -(-1)^{\#\{r\in[n]:\mu_r=0\}}wu_\pi s(n) = \epsilon(\zeta_D(\pi))w(n),$$

where we again use that $\mu_{n-1}+\mu_n$ is even. The valley under consideration is labelled by

$$\big(\mathrm{drw}_D(w,\pi)(n+1-x), -\epsilon(\zeta_D(\pi))\,\mathrm{drw}_D(w,\pi)(1)\big) = \big(-w(n-1),-w(n)\big).$$

(Part 2) In the second part of the proof we demonstrate the forward implication. Therefore let $i\in[n-1]$ be a rise of π. We have to show that we are in one of the five cases of Part 1 of the proof.

(2.1) Assume $1<i<n-1$. Then $\pi_i = q_i = i-1+\mu_i$ and $\pi_{i+1} = q_{i+1} = i+\mu_{i+1}$, hence $\pi_i=\pi_{i+1}$ implies

$$\mu_i = \mu_{i+1}+1.$$

If $\mu_i = -k\le 0$ then there is a valley in the sequence $\overleftarrow{w}_{k}^{-}(\mu)$ and we are in case (1.1). If $\mu_i = k+1>0$ then there is a valley in the sequence $\overrightarrow{w}_{k}^{+}(\mu)$ and we are in case (1.2) or (1.5.1).

(2.2) Next assume $i=1$. Then $\pi_1 = |q_1| = |\mu_1|$ and $\pi_2=q_2=1+\mu_2$, hence $\pi_1=\pi_2$ implies $|\mu_1| = \mu_2+1$. If $\mu_1\ge 0$ we are in the same situation as in (2.1). On the other hand if $\mu_1<0$ then

$$-\mu_1 = \mu_2+1.$$

Set $-k-1=\mu_1$ then the sequence $\overleftarrow{w}_{k}^{-}(\mu)$ ends with an east step and the sequence $\overrightarrow{w}_{k}^{+}(\mu)$ begins with a north step. We are therefore in case (1.3).

(2.3) Finally assume that $i=n-1$. Then $\pi_{n-1}=q_{n-1}=n-2+\mu_{n-1}$.

(2.3.1) Suppose $n-1\equiv 0,3$ modulo 4, thus $q_n = n-1+\mu_n$. If $\pi_1+\cdots+\pi_{n-2}$ is even, then $\pi_{n-1}=\pi_n$ implies $q_n = 2\pi_n-\pi_{n-1}=\pi_{n-1}$ and $\mu_{n-1}=\mu_n+1$ as in (2.1). Otherwise $\pi_1+\cdots+\pi_{n-2}$ is odd and $\pi_{n-1}=\pi_n$ implies $q_n = 2n-1-2\pi_n+\pi_{n-1} = 2n-1-\pi_{n-1}$ and

$$\mu_{n-1} = -\mu_n+2.$$

If $\mu_{n-1} = -k \leq 0$ then the sequence $\overrightarrow{w}^+_{k+1}(\mu)$ ends with an east step and the sequence $\overleftarrow{w}^-_k(\mu)$ begins with a north step. We are therefore in case (1.4). If $\mu_{n-1} = k+1 > 1$ then the sequence $\overrightarrow{w}^+_k(\mu)$ ends with an east step while the sequence $\overleftarrow{w}^-_{k-1}(\mu)$ begins with a north step. This again corresponds to case (1.4). If $\mu_{n-1} = 1$ then also $\mu_n = 1$ which puts us into case (1.5.2).

(2.3.2) Suppose $n-1 \equiv 1, 2$ modulo 4, thus $q_n = n - \mu_n$. If $\pi_1 + \cdots + \pi_{n-2}$ is even, then $\pi_{n-1} = \pi_n$ implies $q_n = \pi_{n-1}$ and $-\mu_{n-1} = \mu_n + 2$. If $\pi_1 + \cdots + \pi_{n-2}$ is odd, then $\pi_{n-1} = \pi_n$ implies $q_n = 2n-1-\pi_{n-1}$ and $\mu_{n-1} = \mu_n + 1$. Hence we are in the same situation as in (2.3.1). ■

By use of Theorem 6.4.8 it can be shown that the type D_n instance of the uniform zeta map is equivalent to the combinatorial map introduced in this section.

THEOREM 6.4.9. [**74**, Thm. 5.28] *Let Φ be a root system of type D_n with coroot lattice $\check{Q}$ and zeta map ζ, and let ψ_D and φ_D be defined as in Theorem 4.3.6 and Proposition 5.2.8. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Vert}(D_n) & \xrightarrow{\zeta_D} & \mathrm{Diag}(D_n) \\ \psi_D \big\downarrow & & \big\downarrow \varphi_D \\ \check{Q}/(2n-1)\check{Q} & \xrightarrow[\zeta]{} & \mathrm{Park}(\Phi) \end{array}$$

PROOF. Let $(w, \pi) \in \mathrm{Vert}(D_n)$ and $(v, \beta) = \zeta_D(w, \pi) \in \mathrm{Diag}(D_n)$. Furthermore, let $q = \psi_D(\pi)$ and u_π be defined as in Proposition 4.3.5 and (4.1) such that $\psi_D(w, \pi) = wu_\pi(q) + (2n-1)\check{Q}$. Finally, let μ be the area vector of π, choose $u \in \mathfrak{S}^D_n$ such that $t_\mu u^{-1}$ is a Graßmannian affine permutation and fix s as in Lemma 4.4.2(iii). Recall from Theorem 6.1.1 that it suffices to show $v = wu_\pi s u^{-1}$ and $A_\beta = A$, where

$$A = \big\{ us^{-1}(\alpha) : \alpha \in \Delta \cup \{-\tilde{\alpha}^D\} \text{ and } s_\alpha(q) = q \big\}.$$

The first claim is immediate from Proposition 6.4.2. In order to show the second claim, we first prove $A \subseteq A^D_\beta$. Assume $us^{-1}(\alpha^D_i) \in A$ for some $i \in [n-1]$. Then $s^D_i(q) = q$ implies $q_i = q_{i+1}$. Thus $\pi_i = \pi_{i+1}$ and i is a rise of π. By Theorem 6.4.8 $\zeta_D(w, \pi)$ has a valley (x, y) labelled either $(w(i+1), w(i))$ or $(-w(i), -w(i+1))$. That is, either $wu_\pi(i+1) = v(n+1-x)$ and

$$wu_\pi(i) = \begin{cases} v(n+1-y) & \text{if } y < n, \\ v(\epsilon(\beta)) & \text{if } y = n, \\ v(-\epsilon(\beta)) & \text{if } y = n+1, \\ v(n-y) & \text{if } y > n+1, \end{cases}$$

or $-wu_\pi(i) = v(n+1-x)$ and

$$-wu_\pi(i+1) = \begin{cases} v(n+1-y) & \text{if } y < n, \\ v(\epsilon(\beta)) & \text{if } y = n, \\ v(-\epsilon(\beta)) & \text{if } y = n+1, \\ v(n-y) & \text{if } y > n+1. \end{cases}$$

Note that we may replace $w(i), w(i+1)$ by $wu_\pi(i), wu_\pi(i+1)$ by choice of i. Applying $(wu_\pi)^{-1}$ to the above identities and using $v = wu_\pi s u^{-1}$, we see that $su^{-1}(\alpha_{x,y}) = \alpha^D_i$. Hence $us^{-1}(\alpha^D_i) = \alpha_{x,y} \in A^D_\beta$.

Next set $S = \{us^{-1}(\alpha^D_0), us^{-1}(\alpha^D_1)\}$. If $S \cap A = \{us^{-1}(\alpha^D_1)\}$ then $q_1 = q_2 > 0$. We have $\epsilon(\pi) = 1$ and 1 is a rise of π. In particular $wu_\pi(1) = w(1)$. By Theorem 6.4.8 (v, β) has a valley (x, y) labelled by $(wu_\pi(2), wu_\pi(1))$ or $(-wu_\pi(1), -wu_\pi(2))$. We obtain $su^{-1}(\alpha_{x,y}) = e_2 - e_1$ and consequently $us^{-1}(\alpha^D_1) = \alpha_{x,y} \in A^D_\beta$.

If $S \cap A = \{us^{-1}(\alpha_0^D)\}$ then $q_1 = -q_2 > 0$. We have $\epsilon(\pi) = -1$ and 1 is a rise of π. In particular $wu_\pi(1) = -w(1)$. By Theorem 6.4.8 (v, β) has a valley (x, y) labelled by $(wu_\pi(2), -wu_\pi(1))$ or $(wu_\pi(1), -wu_\pi(2))$. We obtain $su^{-1}(\alpha_{x,y}) = e_2 + e_1$ and thus $us^{-1}(\alpha_1^D) = \alpha_{x,y} \in A_\beta^D$.

If $S \subseteq A$ then $q_1 = q_2 = 0$ and π begins with two north steps. By Theorem 6.4.8 (v, β) has special valleys (x, n), $(x, n+1)$ labelled by $(wu_\pi(2), \pm|wu_\pi(1)|)$. As above we see that $su^{-1}(\{\alpha_{x,n}, \alpha_{x,n+1}\}) = \{\alpha_0^D, \alpha_1^D\}$. Hence $S \subseteq A_\beta^D$.

If $us^{-1}(\alpha_{n-1}^D) \in A$ then $q_{n-1} = q_n$. It follows that $\pi_{n-1} = \pi_n$, that is, $n-1$ is a rise of π, and $\pi_1 + \cdots + \pi_{n-2}$ is even. Moreover $wu_\pi(n) = w(n)$. By Theorem 6.4.8 (v, β) has a valley (x, y) labelled $(wu_\pi(n), wu_\pi(n-1))$ or $(-wu_\pi(n-1), -wu_\pi(n))$. We obtain $su^{-1}(\alpha_{x,y}) = e_n - e_{n-1}$ and therefore $us^{-1}(\alpha_{n-1}^D) \in A_\beta^D$.

If $us^{-1}(-\tilde{\alpha}^D) \in A$ then $q_{n-1} + q_n = 2n - 1$. It follows that $\pi_{n-1} = \pi_n$ and $\pi_1 + \cdots + \pi_{n-2}$ is odd. Moreover $wu_\pi(n) = -w(n)$. By Theorem 6.4.8 (v, β) has a valley (x, y) labelled $(-wu_\pi(n), wu_\pi(n-1))$ or $(-wu_\pi(n-1), wu_\pi(n))$. We obtain $su^{-1}(\alpha_{x,y}) = -e_{n-1} - e_n$ and thus $us^{-1}(-\tilde{\alpha}^D) \in A_\beta^D$.

To complete the proof we need to demonstrate $A_\beta^D \subseteq A$. Therefore suppose $\alpha_{x,y} \in A_\beta^D$ for some valley (x, y) of (v, β). Then by Theorem 6.4.8 (x, y) is labelled either $(w(i+1), w(i))$ or $(-w(i), -w(i+1))$ for some rise i of π; or we are in the special case were $y = n$ and the valley (x, n) is not followed by an east step, in which the valleys $(x, n), (x, n+1)$ are labelled $(w(2), \pm|w(1)|)$.

If $wu_\pi(i) = w(i)$ and $wu_\pi(i+1) = w(i+1)$ then the valley (x, y) is labelled by $(wu_\pi(i+1), wu_\pi(i))$ or $(-wu_\pi(i), -wu_\pi(i+1))$, and $su^{-1}(\alpha_{x,y}) = \alpha_i^D$ for a rise i of π. By similar arguments as above we see that $\alpha_{x,y} \in A$.

If $i = 1$ and $wu_\pi(1) = -w(1)$ then $su^{-1}(\alpha_{x,y}) = \alpha_0^D$, and $q_1 = -q_2$. Again we obtain $\alpha_{x,y} \in A$.

If $i = n-1$ and $wu_\pi(n) = -w(n)$ then $su^{-1}(\alpha_{x,y}) = -\tilde{\alpha}^D$ and $q_{n-1} + q_n = 2n - 1$. Again we obtain $\alpha_{x,y} \in A$.

Finally, if the valley is of the special form (x, n) and is not followed by an east step then $su^{-1}(\{\alpha_{x,n}, \alpha_{x,n+1}\}) = \{\alpha_0, \alpha_1\}$, π begins with two north steps and $q_1 = q_2 = 0$. We see that $\alpha_{x,n}, \alpha_{x,n+1} \in A$. Thus $A_\beta^D \subseteq A$ and the proof is complete. ∎

6.5. Combinatorics in type B_n

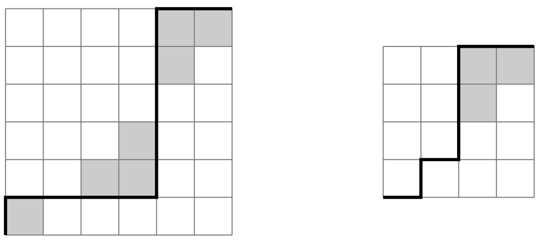


FIGURE 6.9. The lattice paths with type B area vectors $(-1, 2, 1, 0, -1, 3)$ and $(0, 0, -1, 3)$.

Let $\pi \in \mathcal{L}_{n,n}$ be a lattice path, $q = \psi_B(\pi)$ be defined as in Lemma 4.3.2 and $x \in \check{Q}$ and $s \in \mathfrak{S}_n^B$ as in Lemma 4.4.2 (i). Define the type B_n *area vector* of π as

$$\mu = s(q - x) = \begin{cases} (q_1 - 1, \ldots, q_{n-1} - n + 1, q_n - n) & \text{if } n \equiv 0, 3 \mod 4, \\ (q_1 - 1, \ldots, q_{n-1} - n + 1, n + 1 - q_n) & \text{if } n \equiv 1, 2 \mod 4. \end{cases}$$

Note that similarly to the other types, the entry μ_i of the area vector counts the number of boxes in the i-th row that lie between the path π and the alternating path $(\mathbf{en})^n \in \mathcal{L}_{n,n}$, where μ_i is negative as long as π is above $(\mathbf{en})^n$. The only exception to this rule is the top row, where μ_n does not have as nice of an interpretation.

Let $(w,\pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ and μ be the type B_n area vector of π. Define the type B_n *diagonal reading word* $\mathrm{drw}_B(w,\pi)$ as follows: For each $i = 0,1,2,\dots$ first write down the labels $w(j)$ of the rows with $\mu_j = -i$ from bottom to top, then write down the negative labels $-w(j)$ of rows with $\mu_j = i+1$ from top to bottom. Finally, if $\mu_{n-1} + \mu_n$ is even, change the sign of the label coming from the top row.

Note that the diagonal reading word of type B_n is closely related to that of type D_n. Indeed the two definitions almost coincide except that some technical details are less complicated in type B_n.

The following proposition asserts that the diagonal reading word is the correct signed permutation to consider. To simplify notation we denote $\mathrm{Vert}(B_n) = \mathrm{Vert}(B_n, h+1)$.

Proposition 6.5.1. [**74**, Prop. 6.15] *Let $(w,\pi) \in \mathrm{Vert}(B_n)$ be a vertically labelled lattice path with area vector μ, define $u_\pi \in \mathfrak{S}_n^B$ as in Theorem 4.3.3, $s \in \mathfrak{S}_n^B$ as in Lemma 4.4.2* (i) *and $u \in \mathfrak{S}_n^B$ such that $t_\mu u^{-1}$ is a Graßmannian affine permutation. Then $\mathrm{drw}_B(w,\pi) = wu_\pi su^{-1}$.*

Proof. Let $i \in [n]$ and choose $j \in [n]$ such that $|\mathrm{drw}_B(w,\pi)(i)| = |w(j)|$. Then by Lemma 1.5.4

$$\begin{aligned} i &= \#\big\{r \in [n] : |\mu_r| < |\mu_j|\,\big\} + \#\big\{r \in [n] : j \leq r, \mu_r = \mu_j > 0\big\} \\ &\quad + \#\big\{r \in [j], \mu_r = \mu_j \leq 0\big\} + \#\big\{r \in [n], \mu_r = -\mu_j > 0\big\} \\ &= \#\big\{r \in [r] : |\mu_r N + r| \leq |\mu_j N - j|\,\big\} = |u(j)|\,. \end{aligned}$$

Thus $|\mathrm{drw}_B(w,\pi)(i)| = \left|wu^{-1}(i)\right| = \left|wu_\pi su^{-1}(i)\right|$. If $j < n$ then $\mathrm{drw}_B(w,\pi) = w(j) = wu_\pi s(j)$ if and only if $\mu_j \leq 0$, which is the case if and only if $u^{-1}(i) = j$. If $j < n$ and $\mu_j > 0$ then $\mathrm{drw}_B(w,\pi) = -wu_\pi s(j)$ and $u^{-1}(i) = -j$. If $j = n$ and $\mu_n \leq 0$ then

$$\begin{aligned} \mathrm{drw}_B(w,\pi) &= (-1)^{1+\mu_{n-1}+\mu_n} w(n) \\ &= (-1)^{1+\mu_{n-1}+\mu_n}(-1)^{q_{n-1}+q_n} wu_\pi(n) \\ &= (-1)^{1+x_{n-1}+x_n} wu_\pi(n) \\ &= wu_\pi s(n) = wu_\pi su^{-1}(j) \end{aligned}$$

and analogously one treats the case where $j = n$ and $\mu_n > 0$. ∎

As it turns out the zeta map of type B_n is closely related to the zeta map of type D_{n+1}.

Define the type B_n *zeta map* on lattice paths $\zeta_B : \mathcal{L}_{n,n} \to \mathcal{B}_{2n}$ by mapping a lattice path $\pi \in \mathcal{L}_{n,n}$ with the type B_n area vector μ to the path

$$\zeta_B(\pi) = \overleftarrow{w}_{2n}^{-}(\mu)\overrightarrow{w}_{2n}^{+}(\mu)\overleftarrow{w}_{2n-1}^{-}(\mu)\overrightarrow{w}_{2n-1}^{+}(\mu)\cdots\overleftarrow{w}_{1}^{-}(\mu)\overrightarrow{w}_{1}^{+}(\mu)\overleftarrow{w}_{0}^{-}(\mu)\big(\mathbf{n}\overrightarrow{w}_{0}^{+}(\mu)\big)^{\circ},$$

where $(\mathbf{n}\overrightarrow{w}_{0}^{+}(\mu))^{\circ}$ is obtained from $\mathbf{n}\overrightarrow{w}_{0}^{+}(\mu)$ by deleting the last letter. It takes little effort to prove that $\zeta_B(\pi)$ is indeed a ballot path of length $2n$.

Define the type B_n *zeta map* on labelled objects $\zeta_B : \mathfrak{S}_n^B \times \mathcal{L}_{n,n} \to \mathfrak{S}_n^B \times \mathcal{B}_{2n}$ via

$$\zeta_B(w,\pi) = (\mathrm{drw}_B(w,\pi), \zeta_B(\pi)).$$

As expected the type B_n zeta map yields a bijection from square lattice paths to ballot paths. We deduce the respective result from the analogous claim in type D_{n+1}. Also compare with Problem 6.4.5.

Theorem 6.5.2. [**74**, Thm. 6.18] *The zeta map $\zeta_B : \mathcal{L}_{n,n} \to \mathcal{B}_{2n}$ is a bijection.*

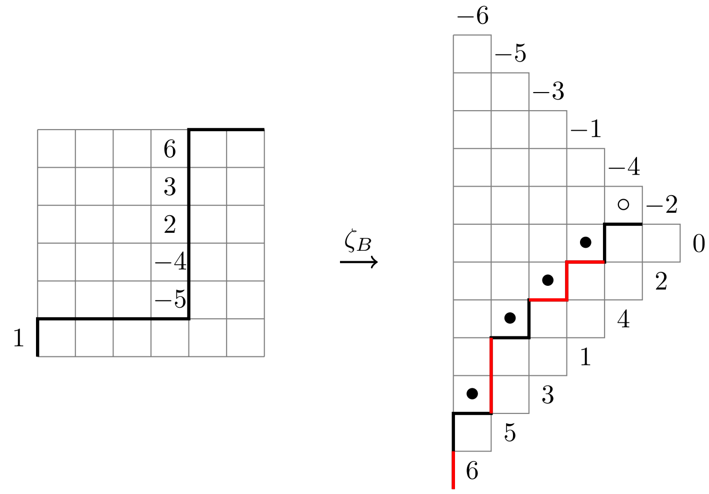


FIGURE 6.10. A vertically labelled lattice path and its image under the Haglund–Loehr-zeta map.

PROOF. Let $\pi \in \mathcal{L}_{n,n}$ be a lattice path with type B_n area vector μ. Consider the path $\mathbf{n}\pi \in \mathcal{L}_{n,n+1}$. Recalling the bijection $\zeta_D^* : \mathcal{L}_{n,n+1} \to \mathcal{B}_{2n+1}$ from Theorem 6.4.6 we have

$$\zeta_D^*(\mathbf{n}\pi) = \overleftarrow{w}^{-}_{2n}(\mu)\overrightarrow{w}^{+}_{2n}(\mu)\overleftarrow{w}^{-}_{2n-1}(\mu)\overrightarrow{w}^{+}_{2n-1}(\mu)\cdots\overleftarrow{w}^{-}_{1}(\mu)\overrightarrow{w}^{+}_{1}(\mu)\overleftarrow{w}^{-}_{0}(\mu)\mathbf{n}\big(\mathbf{n}\overrightarrow{w}^{+}_{0}(\mu)\big)^{\circ}.$$

It follows from the proof of Theorem 6.4.6 that ζ_D^* restricts to a bijection from the set of lattice paths in $\mathcal{L}_{n,n+1}$ that begin with a north step to the set of ballot paths in $\mathcal{B}_{2n+1}$ whose $(n+1)$-st north step is not followed by an east step. Since $\zeta_D^*(\mathbf{n}\pi)$ is easily obtained from $\zeta_B(\pi)$ by inserting a north step, we conclude that ζ_B is also a bijection. ■

The next theorem further exploits the relation between ζ_B and ζ_D to connect the rises of (w,π) to the valleys of $\zeta_B(w,\pi)$.

Let $(w,\pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ and $i \in [n]$ be a rise of π. We say i is a *rise* of (w,π) *labelled* $(w(i), w(i+1))$. Similarly let $(v,\beta) \in \mathfrak{S}_n^B \times \mathcal{B}_{2n}$ and (i,j) be a valley of β. Then we say (i,j) is a *valley* of (v,β) labelled by $(v(n+1-i), v(n+1-j))$.

THEOREM 6.5.3. [**74**, Thm. 6.19] *Let $(w,\pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ and $a,b \in w([n])$. Then (w,π) has a rise labelled (a,b) if and only if $\zeta_B(w,\pi)$ has a valley labelled (b,a) or $(-a,-b)$. Furthermore π begins with a north step if and only if $\zeta_B(w,\pi)$ has a valley labelled $(v(1),0)$.*

PROOF. Let $(w,\pi) \in \mathfrak{S}_n^B \times \mathcal{L}_{n,n}$ and consider the labelled path $(\mathbf{n}\pi, w')$ where we label the initial north step by 0 and π retains the labelling v. This is basically an element of $\mathfrak{S}_n^B \times \mathcal{L}_{n,n+1}$ except that the absolute values of all labels have been decreased by one.

The claim is obtained from Theorem 6.4.8. First notice that $\zeta_B(w,\pi)$ is obtained from $\zeta_D(\mathbf{n}\pi, w')$ simply by deleting the $(n+1)$-st north step and its label, which is always 0. Furthermore let $a,b \in \mathbb{Z} - \{0\}$, then (w,π) has a rise i labelled (a,b) if and only if $(\mathbf{n}\pi, w')$ has a rise $i+1$ labelled (a,b). This is the case if and only if $\zeta_D(\mathbf{n}\pi, w')$ has a valley labelled (b,a) or $(-a,-b)$ and equivalently $\zeta_B(w,\pi)$ has a valley labelled (a,b) or $(-a,-b)$. On the other hand π begins with a north step if and only if $(\mathbf{n}\pi, w')$ has rise $i=1$ labelled by $(\pm 0, v(1))$. This is equivalent to $\zeta_D(\mathbf{n}\pi, w')$ having two valleys labelled $(v(1), \pm 0)$, which is the case if and only if $\zeta_B(w,\pi)$ has a valley labelled $(v(1),0)$. ■

Theorems 6.5.2 and 6.5.3 imply the main result of this section.

THEOREM 6.5.4. [**74**, Thm. 6.20] *The type B_n zeta map restricts to a bijection $\zeta_B : \mathrm{Vert}(B_n) \to \mathrm{Diag}(B_n)$.*

We conclude this section by proving that the combinatorial zeta ζ_B map is indeed the type B_n instance of the uniform zeta map.

THEOREM 6.5.5. [**74**, Thm. 6.21] *Let Φ be a root system of type B_n with coroot lattice $\check{Q}$ and zeta map ζ, and define ψ_B and φ_B as in Theorem 4.3.3 and Proposition 5.2.4 respectively. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Vert}(B_n) & \xrightarrow{\ \zeta_B\ } & \mathrm{Diag}(B_n) \\ \psi_B \big\downarrow & & \big\downarrow \varphi_B \\ \check{Q}/(2n+1)\check{Q} & \xrightarrow[\ \zeta\ ]{} & \mathrm{Park}(\Phi) \end{array}$$

PROOF. Let $(w,\pi) \in \mathrm{Vert}(B_n)$ and set $(v,\beta) = \zeta_B(w,\pi) \in \mathrm{Diag}(B_n)$. Define $q \in \check{Q}$ and u_π as in Lemma 4.3.2 and Theorem 4.3.3 such that $\psi_B(w,\pi) = wu_\pi(q) + (2n+1)\check{Q}$. Let $s \in \mathfrak{S}_n^B$ be defined as in Lemma 4.4.2 (i) and μ be the type B_n area vector of π. Choose $u \in \mathfrak{S}_n^B$ such that $t_\mu u^{-1}$ is a Graßmannian affine permutation.

Recall that by Theorem 6.1.1 it suffices to show that $v = wu_\pi su^{-1}$ and $A_\beta^B = A$, where

$$A = \{(us^{-1}(\alpha) : \alpha \in \Delta \cup \{-\tilde{\alpha}^B\} \text{ and } s_\alpha(q) = q\}.$$

The first claim is taken care of by Proposition 6.5.1. In order to demonstrate the second claim we first show $A \subseteq A_\beta^B$. Therefore let $i \in [n-1]$ and suppose that $us^{-1}(\alpha_i^B) \in A$. Then $q_i = q_{i+1}$, hence $\pi_i = \pi_{i+1}$ and i is a rise of π. By Theorem 6.5.3 (v,β) has a valley (x,y) labelled either $(w(i+1), w(i))$ or $(-w(i), -w(i+1))$. Note that in particular $y \neq n+1$. Moreover $wu_\pi(i) = w(i)$ and $wu_\pi(i+1) = w(i+1)$ for this choice of i. We obtain

$$(v(n+1-x), v(n+1-y)) \in \{(wu_\pi(i+1), wu_\pi(i)), (-wu_\pi(i), -wu_\pi(i+1))\}.$$

Applying $(wu_\pi)^{-1}$ and using the fact that $v = wu_\pi su^{-1}$, yields

$$(su^{-1}(n+1-x), su^{-1}(n+1-y) \in \{(i+1,i), (-i,-i-1)\}.$$

Recalling that

$$\alpha_{x,y} = \begin{cases} e_{n+1-x} + e_{n+1-y} & \text{if } y < n+1, \\ e_{n+1-x} - e_{y-n-1} & \text{if } y < n+1, \end{cases}$$

we compute $us^{-1}(\alpha_i^B) = \alpha_{x,y} \in A_\beta^B$ in all cases.

Next suppose $us^{-1}(\alpha_0^B) \in A$. Then $q_1 = 0$ and thus $\pi_1 = 0$. By Theorem 6.5.3 (v,β) has a valley $(x, n+1)$ labelled $(w(1), 0) = (wu_\pi(1), 0)$. From $wu_\pi su^{-1}(n+1-x) = v(n+1-x) = w(1)$ we obtain $su^{-1}(n+1-x) = 1$, and compute $us^{-1}(\alpha_0^B) = e_{n+1-x} = \alpha_{x,n+1} \in A_\beta^B$.

Similarly suppose $us^{-1}(\alpha_{n-1}^B) \in A$. Then $q_{n-1} = q_n$. Consequently $\pi_{n-1} = \pi_n$, that is, $n-1$ is a rise of π and $\pi_1 + \cdots + \pi_{n-2}$ is even. In particular $wu_\pi(n) = w(n)$. By Theorem 6.5.3 (v,β) has a valley (x,y) labelled $(wu_\pi(n), wu_\pi(n-1))$ or $(-wu_\pi(n-1), -wu_\pi(n))$. As in the cases above we conclude that $us^{-1}(\alpha_{n-1}^B) = \alpha_{x,y} \in A_\beta^B$.

Finally suppose $us^{-1}(-\tilde{\alpha}^B) \in A$. Then $q_{n-1} + q_n = 2n+1$. Here again $\pi_{n-1} = \pi_n$, but contrary to the previous case $\pi_1 + \cdots + \pi_{n-2}$ is now odd. Therefore $wu_\pi(n) = -w(n)$. By Theorem 6.5.3 (v,β) has a valley labelled $(wu_\pi(n-1), -wu_\pi(n))$ or $(wu_\pi(n), -wu_\pi(n-1))$. As before we compute $us^{-1}(-\tilde{\alpha}^B) = \alpha_{x,y} \in A_\beta^B$.

It remains to prove the reverse inclusion $A_\beta^B \subseteq A$. Therefore assume $\alpha_{x,y} \in A_\beta^B$ for some valley (x, y) of (v, β).

If $y < n + 1$ then $\alpha_{x,y} = e_{n+1-x} - e_{n+1-y}$. Furthermore by Theorem 6.5.3 the valley (x, y) is labelled by

$$(v(n + 1 - x), v(n + 1 - y)) \in \big\{(w(i + 1), w(i)), (-w(i), -w(i + 1))\big\}$$

for some rise i of π. If $i < n - 1$ or if $\pi_1 + \cdots + \pi_{n-2}$ is even, then $q_i = \pi_i = \pi_{i+1} = q_{i+1}$ and $w(i) = wu_\pi(i)$ and $w(i + 1) = wu_\pi(i + 1)$. Thus $s_i^B(q) = q$ and we obtain

$$\alpha_{x,y} = us^{-1}(\alpha_i^B) \in A.$$

If $i = n - 1$ and $\pi_1 + \cdots + \pi_{n-2}$ is odd, then $q_{n-1} + q_n = 2n + 1$ and $wu_\pi(n) = -w(n)$. We obtain $s_{\tilde{\alpha}^B}(q) = q$ and $\alpha_{x,y} = us^{-1}(-\tilde{\alpha}^B) \in A$.

If $y = n + 1$ then $\alpha_{x,y} = e_{n+1-x}$ and the valley (x, y) is labelled $(v(n + 1 - x, 0) = (w(1), 0)$ by Theorem 6.5.3. Moreover π begins with a north step, that is, $q_1 = \pi_1 = 0$, and $s_{0^B}(q) = q$. We conclude $\alpha_{x,y} = us^{-1}(\alpha_0^B) \in A$.

Finally the case $y > n+1$ can be treated in a similar fashion as the case $y < n+1$ above, which completes the proof and this thesis. ■

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