PATHWISE VERSIONS OF THE BURKHOLDER-DAVIS-GUNDY INEQUALITY

M. BEIGLBÖCK AND P. SIORPAES

Abstract. We present a new proof of the Burkholder-Davis-Gundy inequalities for $1 \leq p < \infty$. The novelty of our method is that these martingale inequalities are obtained as consequences of elementary *deterministic* counterparts. The latter have a natural interpretation in terms of robust hedging.

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1. Introduction

In this paper we derive estimates which compare the running maximum of a martingale with its quadratic variation. Given real numbers x_n, h_n, $n \in \mathbb{N}$ we write

$$x_n^* := \max_{k \leq n} |x_k|, \quad [x]_n := x_0^2 + \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2, \quad (h \cdot x)_n := \sum_{k=0}^{n-1} h_k (x_{k+1} - x_k).$$

We will derive pathwise versions of the famous Burkholder-Davis-Gundy inequalties.

Theorem 1.1. *For $1 \leq p < \infty$, there exist constants $a_p, b_p < \infty$ such that the following holds: for every $N \in \mathbb{N}$ and every martingale $(X_k)_{k=0}^N$*

$$\mathbb{E}[X]_N^{p/2} \leq a_p \mathbb{E}\left[(X_N^*)^p\right], \qquad \mathbb{E}\left[(X_N^*)^p\right] \leq b_p \mathbb{E}[X]_N^{p/2}. \tag{BDG}$$

For $p \in (1, \infty)$ this was established by Burkholder [Bur66]. Under additional assumptions, Burkholder and Gundy [BG70] obtain a version for $p \in (0, 1]$, while the case $p = 1$ of (BDG) without restrictions is due to Davis [Dav70].

For a modern account see for instance [CT03].

Trajectorial inequalities. The novelty of this note is that the above martingale inequalities are established as consequences of *deterministic* counterparts. We postpone the general statements and first state the *trajectorial version* of Davis' inequality.

Theorem 1.2. *Let $x_0, \ldots, x_N$ be real numbers and set*[1] $h_n := \frac{x_n}{\sqrt{[x]_n + (x_n^*)^2}}$, $n \leq N$. *Then*

$$\sqrt{[x]_N} \leq 3x_N^* - (h \cdot x)_N \qquad x_N^* \leq 6\sqrt{[x]_N} + 2(h \cdot x)_N \tag{1.1}$$

While the proof of Theorem 1.2 is not trivial, we emphasize that the inequalities in (1.1) are completely elementary in nature. The significance of the result lies in the fact that it implies Davis' inequalities: indeed, if $(X_n)_{n=0}^N$ is a martingale, we may apply (1.1) to each trajectory of X and obtain a bounded and adapted process H. The decisive observation is that, by the martingale property,

$$\mathbb{E}[(H \cdot X)_N] = 0, \tag{1.2}$$

so Davis' inequalities (with $a_1 = 3, b_1 = 6$) follow from (1.1) by taking expectations.

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[1] Throughout this paper we use the convention $0/0 = 0$.

We recall that the BDG inequalities also apply if $X = (X_t)_t$ is a cadlag local martingale, and that this follows from a straightforward limiting procedure. Moreover, the inequalities are considerably simpler to prove for *continuous* local martingales (see for example [RW00]); in this case, they also hold for $p \in (0, 1)$, as proved by Burkholder and Gundy [BG70].

The problem of finding the optimal values of the constants a_p, b_p is delicate, and has been open for 47 years and counting; we refer to Adams [Ose10] for a discussion of the current state of research.

2. History of the Trajectorial Approach

The inspiration of the pathwise approach to martingale inequalities used in this paper comes from mathematical finance, more specifically, the theory of model-independent pricing. The starting point of the field is the paper [Hob98] of Hobson, which introduces the idea to study option-prices by means of *semi-static hedging*; we explain the concepts using the inequality

$$\sqrt{[x]_N} \leq 3x^*_N - (h \cdot x)_N \tag{2.1}$$

appearing in Theorem 1.2. If the process $x = (x_n)_{n=0}^N$ describes the price evolution of a financial asset, the functions $\Phi(x) = \sqrt{[x]_N}$ and $\Psi(x) = 3x^*_N$ have the natural financial interpretation of being exotic options; specifically, here Φ is an option on realized variance, while Ψ is a look-back option. The seller of the option Φ pays the buyer the amount $\Phi(x_0, \ldots, x_N)$ after the option's expiration at time N, and $(h \cdot x)_N$ corresponds to the gains or losses accumulated while trading in x according to the portfolio $h = (h_k)_k$.

The decisive observation of Hobson is that inequalities of the type (2.1) can be used to derive *robust bounds* on the relation of the prices of Φ and Ψ: independently of the market model, one should never trade the option Φ at a price higher than the price of Ψ, since the payoff Φ can be *super-hedged* using the option Ψ plus self-financing trading. Here the *hedge* $3x^*_N - (h \cdot x)_N$ is designated *semi-static*: it is made up a static part – the option $3x^*_N$ which is purchased at time 0 and kept during the entire time range – plus a dynamic part which corresponds to the trading in the underlying asset according to the strategy h.

Since the publication of [Hob98] a considerable amount of literature on the topic has evolved (e.g. [Rog93, BHR01, HP02, CHO08, DOR10, CO11a, CO11b, CW11, HN12, HK11]); we refer in particular to the survey by Hobson [Hob11] for a very readable introduction to this area. The most important tool in model-independent finance is the Skorokhod-embedding approach; an extensive overview is given by Obłój in [Obł04]. Starting with the papers [GHLT11, BHLP11] the field has also been linked to the theory of optimal transport, leading to a formal development of the connection between martingale theory and robust hedging ([DM12, ABPS13, DM13]). A benefit for the theory of martingale inequalities is the following guiding principle:

Every martingale inequality which compares expectations of two functionals has a deterministic counterpart.

This idea served as a motivation to derive the Doob-maximal inequalities from deterministic, discrete-time inequalities in [ABP⁺12].[2] In the present article we aim to extend the approach to the case of the Burkholder-Davis-Gundy inequalities.

3. Organisation of the paper

In Section 4 we explain the intuition behind the hedging strategy $h = (h_k)_k$ used in the pathwise version of Davis' inequality. In Section 5 we give a short proof of one Davis' inequality for continuous martingales; notably, this argument leads to a better constant compared to the previous literature, to the best of our knowledge.

[2]Notably, much of the approach of [ABP$^+$12] was already developed earlier by Obłój and Yor [OY06].

In Section 6 we establish Theorem 1.2.

In Section 7 we use Theorem 1.2 to derive trajectorial versions of the BDG-inequalities in the $p > 1$ case; these also lead to their corresponding classical probabilistic counterpart, thus concluding a fully analytic derivation of Theorem 1.1.

4. Heuristics for the pathwise hedging approach

The aim of this section is to explain the basic intuition which lies behind the choice of the integrand in the pathwise Davis inequalities. Arguments are simpler in the case of Brownian motion, which we will now consider.

We focus on one of the two inequalities; according to the pathwise hedging approach, we should be looking for a strategy H and a constant a such that $\sqrt{t} \leq aB^*_t + (H \cdot B)_t$. Indeed, a reasonable ansatz to find a super hedging strategy is to search for a function $f(b, b^*, t)$ such that

$$\sqrt{t} \leq aB^*_t + \Big(f(B, B^*, t) \cdot B\Big)_t, \qquad t \geq 0. \tag{4.1}$$

To make an educated guess for the function f we argue on a *purely heuristic* level and consider paths which evolve in a very particular way. Assume first that the path $(B_t(\omega)_t)_{t\geq 0}$ stays infinitesimally close to the value b for all $t \geq t_0$: we picture BM as a random walk on a time grid with size dt, making alternating up and down steps of height $\sqrt{dt}$. Thus, we assume that B evolves in the form

$$B_{t_0+2ndt} = b, \qquad B_{t_0+(2n+1)dt} = b + \sqrt{dt}, \qquad n \geq 0 \tag{4.2}$$

where necessarily b lies between $-B^*_{t_0}$ and $B^*_{t_0}$. The left side of (4.1) is of course increasing, so we have to ensure the same behavior on the right side. A little calculation reveals that this means that f should have the form

$$f(B, B^*, t) \approx -\frac{B}{\sqrt{t}} \qquad \text{as } t \to \infty\ ; \tag{4.3}$$

to see this, set $H_t := f(B_t, B^*_t, t)$ and compare the value $\sqrt{t + 2dt} - \sqrt{t} \approx dt/\sqrt{t}$ with

$$\begin{aligned}
(H \cdot B)_{t+2dt} - (H \cdot B)_t &\approx f(t, b, b^*)dB_t + f(t + dt, b + \sqrt{dt}, b^*)dB_{t+dt} \\
&\approx f(t, b, b^*)\sqrt{dt} + f(t + dt, b + \sqrt{dt}, b^*)(-\sqrt{dt}) \\
&\approx -\Big[f(t, b + \sqrt{dt}, b^*) - f(t, b, b^*)\Big]\sqrt{dt} + O(dt^{3/2}) \\
&\approx -f_b\, dt.
\end{aligned}$$

To assure that both sides of (4.1) grow at the same speed we thus need to require $dt/\sqrt{t} \approx -f_b\, dt$ which leads to (4.3).

Next we consider a path which exhibits a different kind of extreme evolution: assume that $B_t(\omega) \approx Mt$ for some number $M > 0$. Simply setting $f(B, B^*, t) \approx -B/\sqrt{t}$ would lead to $\big(f(B, B^*, t) \cdot B\big)_t \approx -2M^2t^{3/2}/3$. Taking t sufficiently large, this quantity would eventually supersede $aB^*_t \approx aMt$ independent of the choice of a, and thus (4.1) would fail. So, this argument suggest to choose a function which is bounded (at least for fixed (t, B^*)). Moreover, dealing with a bounded integrand would conveniently allow to follow the explanation after Theorem 1.1 and obtain Davis' inequalities from the pathwise Davis' inequalities. Thus, we could consider the function

$$f(B, B^*, t) = -\frac{B_t}{\sqrt{t} \vee B^*_t}. \tag{4.4}$$

Thanks to the additional term aB^*_t in (4.1), it is not a problem if $f(B, B^*, t) \approx -2B/\sqrt{t}$ is violated for "small" values of t; and, if t is large compared to B^*, $f(B, B^*, t) \approx -2B_t/\sqrt{t}$

holds, thus satisfying (4.3). Another similar possibility would be to use the function

$$f(B, B^*, t) = -\frac{B_t}{\sqrt{t + (B_t^*)^2}}, \tag{4.5}$$

as in Theorem 1.2; the latter turns out to lead to easier computations in the discrete time case. We choose however f given by (4.4) when dealing with continuous martingales, since this allows us to obtain Davis' inequality with a better constant than the values we could find in the literature.

5. Davis inequality for continuous local martingales

We now derive one pathwise Davis' inequality for continuous local martingales. We notice that Theorem 5.1 provides the constant $3/2$, which is smaller than the optimal constant for general cadlag martingales (which is known to be $\sqrt{3}$, see [**?**]). We do not address here the opposite pathwise Davis' inequality, since its optimal constant in the case of continuous martingales is known (see [Ose10]).

Theorem 5.1. *If M is a continuous local martingale such that $M_0 = 0$ then*

$$\sqrt{[M]_t} \le \frac{3}{2} M_t^* - \Big(\frac{M_t}{\sqrt{[M]_t} \vee M_t^*} \cdot M_t\Big)_t \qquad \textit{for all } t \ge 0\,. \tag{5.1}$$

Proof. By the Dambis-Dubins-Schwarz time change result, it is enough to consider the case where M is a Brownian Motion, which we will denote by B. From Ito's formula applied to the semi-martingales B_t^2 and $\sqrt{t} \vee B_t^*$ we find

$$d\frac{B_t^2}{\sqrt{t} \vee B_t^*} = -\frac{B_t^2}{t \vee B_t^{*2}} d\Big(\sqrt{t} \vee B_t^*\Big) + \frac{1}{\sqrt{t} \vee B_t^*}\Big(2B_t\, dB_t + \, dt\Big).$$

We may thus replace the integral in (5.1) and arrive at the equivalent formulation

$$\frac{B_t^2}{\sqrt{t} \vee B_t^*} + \int_0^t \frac{B_s^2}{s \vee B_s^{*2}}\, d\Big(\sqrt{s} \vee B_s^*\Big) - \int_0^t \frac{1}{\sqrt{s} \vee B_s^*}\, ds \le 3B_t^* - 2\sqrt{t}. \tag{5.2}$$

Inequality (5.2) gets stronger if we replace each occurrence of B by B^*; thus, setting $f(t) = \sqrt{t}$, $g(t) = B_t^*$, it is enough to prove the following claim:

Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous increasing functions such that $f(0) = g(0) = 0$ and $(f \vee g)(a) > 0$ if $a > 0$. Then, for all $a > 0$

$$\left(\frac{g^2}{f \vee g}\right)(a) + \int_0^a \frac{g^2}{f^2 \vee g^2}\, d(f \vee g) - \int_0^a \frac{1}{f \vee g}\, df^2 \le (3g - 2f)(a). \tag{5.3}$$

To show this, observe that, by a change of variables $\int \frac{g^2}{f^2 \vee g^2}\, d(f \vee g) = -\int g^2\, d\frac{f \vee g}{f^2 \vee g^2}$. Hence, integrating by parts on the interval (ε, a) and taking the limit $\varepsilon \to 0$, we see that the left hand side of (5.3) equals

$$\int_0^a \frac{dg^2 - df^2}{f \vee g}.$$

By a change of variables and applying trivial inequalities we obtain

$$\int_0^a \frac{dg^2}{f \vee g} = \int_0^a 1_{\{g>0\}} \frac{dg^2}{f \vee g} \le \int_0^a \frac{1_{\{g>0\}} dg^2}{g} = 2g(a)\,, \qquad \int_0^a \frac{df^2}{f \vee g} \ge \int_0^a \frac{df^2}{f(\cdot) \vee g(a)}\,.$$

If $f(a) \le g(a)$, the last integral equals $f^2(a)/g(a)$; otherwise there exists some $b \in [0, a)$ such that $f(b) = g(a)$, and then evaluating separately the integral on $(0, b)$ and on $[b, a)$ we obtain that

$$\int_0^a \frac{df^2}{f(\cdot) \vee g(a)} = \frac{f^2(b)}{g(a)} + 2(f(a) - f(b)) = 2f(a) - g(a).$$

Since $2y - x^2/y \le 3y - 2x$ holds for $y > 0$, either way (5.3) follows. ◻

6. Davis inequality

In this section we prove Theorem 1.2; in fact, we will establish that

$$\sqrt{[x]_n} \le (\sqrt{2}+1)\, x_n^* + (-h\cdot x)_n \tag{6.1}$$

$$x_n^* \le 6\sqrt{[x]_n} + (2h\cdot x)_n, \tag{6.2}$$

where the dynamic hedging strategy is defined by $h_n = \frac{x_n}{\sqrt{[x]_n+(x_n^*)^2}}$ as in Theorem 1.2.

To prove (6.1), (6.2) we introduce the convention, used throughout the paper, that any sequence $(y_i)_{i\ge 0}$ is defined to be 0 at time $i=-1$, and we define the auxiliary functions f, g for $m>0, q\ge 0, |x|\le m$ by

$$g(x,m,q) := -2m + \sqrt{m^2+q} + \frac{m^2-x^2}{2\sqrt{m^2+q}} \tag{6.3}$$

$$f(x,m,q) := -2\sqrt{q} + \sqrt{m^2+q} - \frac{m^2-x^2}{2\sqrt{m^2+q}}. \tag{6.4}$$

and continuously extend them to $(x,m,q)=(0,0,0)$ by setting $f(0,0,0)=g(0,0,0)=0$. We will need the following lemma, whose proof is a somewhat tedious exercise in calculus.

Lemma 6.1. *For $d\in\mathbb{R}, |x|\le m, q\ge 0, m\ge 0$ we have, with $c=\sqrt{2}-1$,*

$$g(x+d, m\vee|x+d|, q+d^2) - g(x,m,q) \le -\frac{xd}{\sqrt{m^2+q}} + c\Big((m\vee|x+d|)-m\Big), \tag{6.5}$$

$$f(x+d, m\vee|x+d|, q+d^2) - f(x,m,q) \le \frac{xd}{\sqrt{m^2+q}} + \Big(\sqrt{q+d^2}-\sqrt{q}\Big). \tag{6.6}$$

Before proving Lemma 6.1 we explain why it implies (6.1) and (6.2).

Proof of Theorem 1.2. Since $f(x_0,|x_0|,x_0^2)\le 0$, (6.6) gives[3]

$$-2\sqrt{[x]_n} + x_n^*/2 \le f(x_n,x_n^*,[x]_n) \le \sum_{k=0}^{n-1} f(x_{k+1},x_{k+1}^*,[x]_{k+1}) - f(x_k,x_k^*,[x]_k) \le (l\cdot x)_n + \sqrt{[x]_n},$$

which implies (6.1); and since $g(x_0,|x_0|,x_0^2)\le 0$, we get (6.2) from (6.5) as follows

$$-2x_n^* + \sqrt{[x]_n} \le g(x_n,x_n^*,[x]_n) \le \sum_{k=0}^{n-1} g(x_{k+1},x_{k+1}^*,[x]_{k+1}) - g(x_k,x_k^*,[x]_k) \le -(l\cdot x)_n + cx_n^*.$$

□

Now we prove Lemma 6.1.

Proof of Inequality (6.5). It is enough to consider the case $m>0$, as the one where $m=0$ then follows by continuity. Then, we can assume that $m=1$ through normalization. Define $h(x,q,d)$ to be the LHS minus the RHS of (6.5); since $h(x,q,d)=h(-x,q,-d)$, it is sufficient to deal with the case $d\ge 0$.

Case I $[1\ge|x+d|]$: Here we have to show that

$$h = \sqrt{1+q+d^2} + \frac{1-(x+d)^2}{2\sqrt{1+q+d^2}} - \sqrt{1+q} - \frac{1-x^2}{2\sqrt{1+q}} + \frac{xd}{\sqrt{1+q}} \le 0. \tag{6.7}$$

Since $h_{xx}\ge 0$, h is convex, so it is sufficient to treat the boundary cases $x=-1$ and $x=1-d$. To simplify notation, we set $r=\sqrt{1+q}$; notice that $r\ge 1$ and $0\le d\le 2$.

[3] Recall that, by our convention, $x_{-1}=x_{-1}^*=[x]_{-1}=0$.

Sub-case I.A $[1 \geq |x+d|, x=-1]$: Then (6.7) follows from

$$\begin{aligned}
& \sqrt{r^2+d^2} + \frac{1-(d-1)^2}{2\sqrt{r^2+d^2}} - r - \frac{d}{r} && \leq 0 \\
\Leftarrow \quad & r^2 + d^2 + d - d^2/2 && \leq (r+d/r)\sqrt{r^2+d^2} \\
\Leftarrow \quad & r^4 + d^4/4 + d^2 + r^2d^2 + d^3 + 2dr^2 && \leq r^4 + 2dr^2 + d^2 + r^2d^2 + 2d^3 + d^4/r^2 \\
\Leftarrow \quad & d^4/4 && \leq d^3 + d^4/r^2,
\end{aligned}$$

which is true since $0 \leq d \leq 2$.

Sub-case I.B $[1 \geq |x+d|, x = 1-d]$: Here (6.7) amounts to

$$\begin{aligned}
& \sqrt{r^2+d^2} - r - \frac{1-(1-d)^2}{2r} + \frac{(1-d)d}{r} && \leq 0 \\
\Leftarrow \quad & \sqrt{r^2+d^2} && \leq r + d^2/2r \\
\Leftarrow \quad & r^2+d^2 && \leq r^2 + d^2 + d^4/4r^2.
\end{aligned}$$

Case II $[1 \leq |x+d|]$: Since $|x| \leq 1$ and $d \geq 0$, we find that $|x+d| \geq 1$ implies $x+d = |x+d| \geq 1$. In this case h equals

$$-(2+c)(x+d-1) + \sqrt{(x+d)^2+q+d^2} - \sqrt{1+q} - \frac{1-x^2}{2\sqrt{1+q}} + \frac{xd}{\sqrt{1+q}}. \tag{6.8}$$

Since $s \mapsto \sqrt{s^2+1}$ is convex, $h \leq 0$ holds iff it holds for all x on the boundary. Moreover if $-1 \leq 1-d = x \leq 1$ then we already know that $h \leq 0$ from the corresponding sub-case $1 \geq |x+d|$; so we only need to show that $h \leq 0$ for $x = 1, q, d \geq 0$ and for $x = -1, q \geq 0, d \geq 2$, respectively.

Sub-case II.A $[1 \leq |x+d|, x=1]$: We have to show that, for all $q, d \geq 0$,

$$h(1,q,d) = -(2+c)d + \sqrt{(1+d)^2+q+d^2} - \sqrt{1+q} + \frac{d}{\sqrt{1+q}} \leq 0.$$

Since $(1+d)^2 + d^2 = 2(d+1/2)^2 + 1/2$ and $s \mapsto \sqrt{1+s^2}$ is convex, it follows that $h(1,q,d)$ is convex in d; hence, the inequality has to be checked only for $d=0$ and for $d \to \infty$. The first case is trivial, and in the latter, after dividing both sides by d, we arrive at $\sqrt{2}+1-2 \leq c$, which holds by our choice of $c = \sqrt{2}-1$.

Sub-case II.B $[1 \leq x+d, x=-1]$: We have to show that, for all $q \geq 0, d \geq 2$,

$$h(-1,q,d) = -(2+c)(d-2) + \sqrt{(-1+d)^2+q+d^2} - \sqrt{1+q} + \frac{d}{\sqrt{1+q}} \leq 0.$$

As above, by convexity in d it suffices to consider the cases $d=2$ and $d \to \infty$. The first one amounts to $\sqrt{5+q} \leq \sqrt{1+q} + 2/\sqrt{1+q}$, which is easily proved taking the squares. The second one, after dividing by d, amounts to $-(2+c)+\sqrt{2}-1/\sqrt{1+q} \leq 0$; by monotonicity in q it is enough to consider the case $q \to \infty$, which yields $\sqrt{2}-2 \leq c$, which holds by our choice of c. □

Proof of Inequality (6.6). As before we can assume w.l.o.g. that $m=1$ and $d \geq 0$. Define $k(x,q,d)$ to be the LHS minus the RHS of (6.6).

Case I $[1 \geq |x+d|]$: In this case k equals

$$\sqrt{1+q+d^2} - \frac{1-(x+d)^2}{2\sqrt{1+q+d^2}} - \sqrt{1+q} + \frac{1-x^2}{2\sqrt{1+q}} - \frac{xd}{\sqrt{1+q}} - 3\left(\sqrt{q+d^2} - \sqrt{q}\right).$$

Let us first isolate the terms that depend on x. Define $k_0 := (1+q+d^2)^{-1/2} - (1+q)^{-1/2}$, and $k_2 := k - k_0(x+d)^2/2$, so that

$$k_2 = \sqrt{1+q+d^2} - \sqrt{1+q} - \frac{1}{2\sqrt{1+q+d^2}} + \frac{1+d^2}{2\sqrt{1+q}} - 3\Big(\sqrt{q+d^2} - \sqrt{q}\Big).$$

Notice that we can write

$$k_0 = \int_0^{d^2} k_1(s)\,ds \quad \text{for} \quad k_1(s) := \frac{d}{ds}(1+q+s)^{-1/2},$$

and similarly $k_2 = \int_0^{d^2} k_3(s,d^2)\,ds$ for

$$k_3(s,d^2) := \frac{d}{ds}\left(\sqrt{1+q+s} - \frac{1-s+d^2}{2\sqrt{1+q+s}} - 3\sqrt{q+s}\right) \tag{6.9}$$

$$= \frac{1}{2\sqrt{1+q+s}} + \frac{2(1+q+s)+1-s+d^2}{4(1+q+s)^{3/2}} - \frac{3}{2\sqrt{q+s}}. \tag{6.10}$$

Since the $(k_i)_i$ do not depend on x and $k_0 \le 0$, $\max_x k = k_2 + k_0 \min_x (x+d)^2/2$. Since $\min_{-1\le x\le 1}(x+d)^2$ equals 0 if $0 \le d \le 1$ and equals $(-1+d)^2$ if $1 \le d$, to show $k \le 0$ we are lead to study the following two sub-cases.

Sub-case I.A $[1 \ge |x+d|,\ d \le 1]$: In this case $k = k_2$; to show that $k_2 \le 0$ it is enough to show $k_3 \le 0$. Since $0 \le s \le d^2 \le 1$ we get $-s + d^2 \le 1$, and so trivially

$$k_3 \le \frac{2(1+q+s)+1+1}{4(1+q+s)^{3/2}} - \frac{2}{2\sqrt{q+s}}. \tag{6.11}$$

So, calling $y := q+s$, it is enough to prove that for all $y \ge 0$

$$\frac{2y+4}{4(1+y)^{3/2}} - \frac{2}{2\sqrt{y}} \le 0, \text{ i.e. } \sqrt{y}(y+2) \le (1+y)^{3/2}2, \tag{6.12}$$

which is seen to be true by taking squares and bringing everything on the RHS to obtain a polynomial whose coefficients are all positive.

Sub-case I.B $[1 \ge |x+d|,\ d \ge 1]$: In this case $k = k_2 + k_0(1-d)^2/2$, so it is enough to show that $k_3 + k_1(1-d)^2/2 \le 0$. Since from $1 \ge |x+d|, |x| \le 1$ it follows that $d \le 2$, computations entirely similar[4] to the other sub-case establish the desired result.

Case II $[1 \le |x+d|]$: In this case $x + d = |x+d| \ge 1$ and k equals

$$\sqrt{(x+d)^2+q+d^2} - \sqrt{1+q} + \frac{1-x^2}{2\sqrt{1+q}} - \frac{xd}{\sqrt{1+q}} - 3\Big(\sqrt{q+d^2} - \sqrt{q}\Big).$$

Since trivially $dk/dx \le 0$, to show $k \le 0$ we can assume that $x = 1-d$, in which case we can write k as $k = \int_0^{d^2} \tilde{k}(s)\,ds$ for

$$\tilde{k}(s) := \frac{d}{ds}\left(\sqrt{1+q+s} + \frac{s}{2\sqrt{1+q}} - 3\sqrt{q+s}\right) \tag{6.13}$$

$$= \frac{1}{2\sqrt{1+q+s}} + \frac{1}{2\sqrt{1+q}} - \frac{3}{2\sqrt{q+s}}. \tag{6.14}$$

Since $1 - d = x \in [-1,1]$ we have $d^2 \le 4$, and so to get $k \le 0$ it suffices to show that $\tilde{k} \le 0$ for $s \le 4$. This holds since

$$\tilde{k} \le \frac{1}{2\sqrt{1+q}} - \frac{2}{2\sqrt{q+s}} \le 0 \quad \text{for } s \le 4. \quad \square$$

[4] Use that in this case $0 \le s \le d^2 \le 4$ implies $-s + d^2 - (d-1)^2 \le 3$.

7. Pathwise Burkholder-Gundy Inequality

Garsia has given a simple proof of the fact that the BDG inequalities for general $p \geq 1$ are a consequence of Davis inequality ($p = 1$) and of the famous lemma by Garsia and Neveu; in this section we revisit his proof and turn it into pathwise discrete-time arguments.

Garsia's proof (for which we refer to [MC76, Chapter 3, Theorem 30 and 32] or to [CS75]) works similarly to how the Doob L^p-inequalities for $p > 1$ follow by writing x^p as an integral, applying the (weak) Doob L^1-inequality, using Fubini's theorem, and finally applying Hölder's inequality (see for example [RY99]). The difference is that for the BDG inequalities one needs to use a different integral expression for x^p, and so one has to consider Davis' inequalities not on the time interval $[0, T]$ but on $[\tau, T]$, where τ is a stopping time.

In the pathwise setting, by the guiding principle stated in Section 2, if L is a functional of a martingale X and τ is a stopping time, a statement of the type $\mathbb{E}[L|F_\tau] \leq 0$ will have to be turned into one of the type $L + (H \cdot X)_T - (H \cdot X)_\tau \leq 0$; moreover, since there will be no expectations involved, Hölder's inequality will have to be replaced by Young's inequality.

We will need to consider discrete time stochastic integrals for which the initial time is different from 0; given $i < n$ and real numbers $(h_j)_{i\leq j\leq n-1}$ and $(x_j)_{i\leq j\leq n}$, we define

$$(h \cdot x)_i^n := \sum_{j=i}^{n-1} h_j (x_{j+1} - x_j). \tag{7.1}$$

Moreover if, for $i \leq j \leq n-1$, h_j is a *function* from $\mathbb{R}^{j+1}$ to $\mathbb{R}$, given real numbers $(x_j)_{0\leq j\leq n}$ we define $(h \cdot x)_i^n$ as

$$\sum_{j=i}^{n-1} h_j(x_0, \dots, x_j)(x_{j+1} - x_j).$$

Either way, we set $(h \cdot x)_i^n := 0$ if $n = i$.

We now deduce pathwise Davis' inequalities on $\{i, i+1, \dots, n\}$ from the ones on $\{0, 1, \dots, n\}$ by a simple time shift. We recall that, by convention, $x_{-1} = x^*_{-1} = [x]_{-1} = 0$.

Lemma 7.1. *Assume that* $\alpha, \beta > 0$ *and* $h_n, k_n : \mathbb{R}^{n+1} \to \mathbb{R}, n \geq 0$ *satisfy*

$$\sqrt{[x]_n} \leq \alpha\, x_n^* + (h \cdot x)_n, \qquad x_n^* \leq \beta \sqrt{[x]_n} + (k \cdot x)_n \tag{7.2}$$

for every sequence $(x_n)_{n\geq 0}$*. Define, for* $i \geq 0$*,* $n \geq i$*, the functions* $f_n^{(i)}, g_n^{(i)} : \mathbb{R}^{n+1} \to \mathbb{R}$ *by*

$$f_n^{(i)}((x_j)_{0\leq j\leq n}) := h_{n-i}((x_l - x_{i-1})_{i\leq l\leq n}), \qquad g_n^{(i)}((x_j)_{j\leq n}) := k_{n-i}((x_l - x_{i-1})_{i\leq l\leq n})$$

Then we have, for $n \geq i \geq 0$*,*

$$\sqrt{[x]_n} - \sqrt{[x]_{i-1}} \leq 2\alpha\, x_n^* + (f^{(i)} \cdot x)_i^n, \qquad x_n^* - x_{i-1}^* \leq \beta \sqrt{[x]_n} + (g^{(i)} \cdot x)_i^n.$$

Proof. Fix $n \geq i \geq 0$, $(x_n)_{n\geq 0}$ and let $y_j^{(i)} := x_{j+i} - x_{i-1}$. Applying (7.2) to $(y_j^{(i)})_{j\geq 0}$ we find

$$\sqrt{[x]_n} - \sqrt{[x]_{i-1}} \leq \sqrt{[x]_n - [x]_{i-1}} = \sqrt{[y^{(i)}]_{n-i}} \leq \alpha\, (y^{(i)})^*_{n-i} + (h \cdot y^{(i)})_{n-i} \leq \alpha\, 2x_n^* + (f^{(i)} \cdot x)_i^n,$$

and (respectively)

$$x_n^* - x_{i-1}^* \leq (y^{(i)})^*_{n-i} \leq \beta \sqrt{[y^{(i)}]_{n-i}} + (k \cdot y^{(i)})_{n-i} \leq \beta \sqrt{[x]_n} + (g^{(i)} \cdot x)_i^n. \qquad \square$$

Here follows the pathwise version of Garsia-Neveu's lemma.

Lemma 7.2. *Let* $p > 1$*,* $c_n \in \mathbb{R}, (x_j)_{j\leq n}, (h_n^{(i)})_{i\leq n} \in \mathbb{R}^{n+1}$*, and assume that* $0 = a_{-1} \leq a_0 \leq \dots \leq a_n < \infty$ *and*

$$a_n - a_{i-1} \leq c_n + (h^{(i)} \cdot x)_i^n \quad \textit{for} \quad n \geq i \geq 0\,.$$

Then, if we set

$$w_j := \sum_{i=0}^{j} p\left(a_i^{p-1} - a_{i-1}^{p-1}\right) h_j^{(i)} \quad j \le n,$$

we have that

$$a_n^p \le p c_n a_n^{p-1} + (w \cdot x)_n \,, \tag{7.3}$$
$$a_n^p \le (p-1)^{p-1} c_n^p + (pw \cdot x)_n \,. \tag{7.4}$$

Proof. From $a_n^p = p(p-1)\int_0^{a_n} s^{p-2}(a_n - s)\,ds = p\sum_{i=0}^n \int_{a_{i-1}}^{a_i}(p-1)s^{p-2}(a_n - s)\,ds$ and $a_n - s \le a_n - a_{i-1}$ on $s \in [a_{i-1}, a_i]$ we find (7.3) by writing

$$\begin{aligned}
a_n^p &\le p\textstyle\sum_{i=0}^n (a_i^{p-1} - a_{i-1}^{p-1})(a_n - a_{i-1})\\
&\le p\textstyle\sum_{i=0}^n (a_i^{p-1} - a_{i-1}^{p-1})\Big[c_n + \left(h^{(i)} \cdot x\right)_i^n\Big]\\
&= p c_n a_n^{p-1} + p\textstyle\sum_{i=0}^n \sum_{j=i}^{n-1} (a_i^{p-1} - a_{i-1}^{p-1})\, h_j^{(i)}(x_{j+1} - x_j)\\
&= p c_n a_n^{p-1} + \textstyle\sum_{j=0}^{n-1}\left(\sum_{i=0}^j p(a_i^{p-1} - a_{i-1}^{p-1})\, h_j^{(i)}\right)(x_{j+1} - x_j) = p c_n a_n^{p-1} + (w \cdot x)_n.
\end{aligned}$$

We then obtain (7.4) from (7.3) by applying Young's inequality $ab \le C_\varepsilon a^p/p + \varepsilon b^q/q$ (where $C_\varepsilon^{-1} = p(\varepsilon q)^{p-1}$ and $1/p + 1/q = 1$) with $\varepsilon = 1/p$, $a = c_n$, $b = a_n^{p-1}$. □

Finally, from Theorem 1.2, Lemma 7.1 and Lemma 7.2 we obtain the following discrete-time pathwise BDG inequalities for $p > 1$. We recall that, by convention, $x_{-1} = x^*_{-1} = [x]_{-1} = 0$ and $0/0 = 0$, and in particular the integrand $f_n^{(i)}$ is well defined.

Theorem 7.3. *Let $x_0, \ldots, x_N$ be real numbers, $c_p := 6^p(p-1)^{p-1}$ for $p > 1$, and define*

$$h_n := \sum_{i=0}^n p^2\left(\sqrt{[x]_i^{p-1}} - \sqrt{[x]_{i-1}^{p-1}}\right) f_n^{(i)}, \qquad g_n := \sum_{i=0}^n p^2\left((x_i^*)^{p-1} - (x_{i-1}^*)^{p-1}\right) f_n^{(i)},$$

where

$$f_n^{(i)} := \frac{x_n - x_{i-1}}{\sqrt{[x]_n - [x]_{i-1} + \max_{i \le k \le n}(x_k - x_{i-1})^2}}.$$

Then

$$\sqrt{[x]_N^p} \le c_p (x_N^*)^p - (h \cdot x)_N \qquad (x_N^*)^p \le c_p \sqrt{[x]_N^p} + 2(g \cdot x)_N \tag{7.5}$$

We notice that Theorem 7.3 yields (BDG); indeed, given a finite constant N and a martingale $(X_n)_{n=0}^N$, trivially $\sqrt{[X]_N}$ and X_N^* are in $L^p(\mathbb{P})$ iff X_n is in $L^p(\mathbb{P})$ for every $n \le N$, and in this case the adapted integrands $(H_n)_{n=0}^{N-1}$ and $(G_n)_{n=0}^{N-1}$ which we obtain applying Theorem 7.3 to the paths of X are in $L^q(\mathbb{P})$ for every n (for $q = p/(p-1)$), thus $H \cdot X$ and $G \cdot X$ are martingales and so

$$\mathbb{E}[(H \cdot X)_N] = 0 = \mathbb{E}[(G \cdot X)_N],$$

and the Burkholder-Davis-Gundy inequalities for $p > 1$ (with $a_p = b_p = 6^p(p-1)^{p-1}$) follow from (7.5) by taking expectations, completing the proof of Theorem 1.1.

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