Sumset Phenomenon in Countable Amenable Groups

Mathias Beiglböck[a,1], Vitaly Bergelson[b,2], Alexander Fish[b]

[a] *Fakultät für Mathematik, Universität Wien*
Nordbergstraße 15, 1090 Wien, Austria
[b] *Department of Mathematics, Ohio State University*
Columbus, OH 43210

Abstract

Jin proved that whenever A and B are sets of positive upper density in $\mathbb{Z}$, $A + B$ is piecewise syndetic. Jin's theorem was subsequently generalized by Jin and Keisler to a certain family of abelian groups, which in particular contains $\mathbb{Z}^d$. Answering a question of Jin and Keisler, we show that this result can be extended to countable amenable groups. Moreover we establish that such sumsets (or — depending on the notation — "productsets") are piecewise Bohr, a result which for $G = \mathbb{Z}$ was proved by Bergelson, Furstenberg and Weiss. In the case of an abelian group G, we show that a set is piecewise Bohr if and only if it contains a sumset of two sets of positive upper Banach density.

Key words: amenable group, Banach density, Bohr set, piecewise syndetic, sumset phenomenon

---

1. Introduction

1.1. Jin's theorem

For a set $A \subseteq \mathbb{Z}$, the upper Banach density, $d^*(A)$, is defined as

$$d^*(A) = \limsup_{b-a\to\infty} \frac{|A \cap \{a, a+1, \dots, b\}|}{b-a+1}. \tag{1}$$

It is well known and not hard to show that if $d^*(A) > 0$ then the set of differences $A - A = \{a - a' : a, a' \in A\}$ is *syndetic*, i.e. has bounded gaps. To see this, one can, for example, argue as follows. First, notice that $n \in A - A$ if and only if $A \cap (A - n) \neq \emptyset$. Second, observe that for any sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{Z}$, the set $A - A$ has to contain an element of the form $n_i - n_j$ for some $i > j$. (This follows from the fact that for some $i > j$ one has to have $(A - n_i) \cap (A - n_j) \neq \emptyset$). Now, if $A - A$ is not syndetic, its complement, $\mathbb{Z} \setminus A$, is *thick*, that is, it contains arbitrarily long intervals. It is easy to see that any thick set in $\mathbb{Z}$ contains a set of differences $D = \{n_i - n_j, i > j\}$ for some sequence $(n_i)_{i\in\mathbb{N}}$. This implies $(A - A) \cap D = \emptyset$ which gives a contradiction.

[1]Supported by the Austrian Science Fund (FWF) under grants S9612 and P21209.
[2]Supported by NSF under grant DMS-0600042

One cannot expect, of course, that the above fact about the syndeticity of $A - A$ extends to the "sumset" $A + B = \{a + b : a \in A, b \in B\}$ of two arbitrary sets of positive upper Banach density. For example, one can easily construct a thick set C which has unbounded gaps, and such that for some thick sets A and B, $A + B \subseteq C$. In this case $d^*(A) = d^*(B) = 1$ but $A + B$ is not syndetic. The following surprising result of Jin shows that, nevertheless, the sumset of any two sets of positive upper Banach density is always *piecewise syndetic*, that is, is the intersection of a syndetic set with a thick set.

Theorem 1 ((Jin02)). *Assume that $A, B \subseteq \mathbb{Z}$ have positive upper Banach density. Then there exist a thick set C and a syndetic set S such that $S \cap C \subseteq A + B$.*

It is not hard to see that not every set of positive upper Banach density is piecewise syndetic. Moreover, one can show that not every set A for which the *density*, $d(A) = \lim_{N\to\infty} \frac{|A\cap\{-N,\ldots,N\}|}{2N+1}$, exists and is positive, is piecewise syndetic. The following remarks show that any piecewise syndetic set contains a highly structured infinite set of a special type.

Note first that any piecewise syndetic set S in $\mathbb{Z}$ has the property that the union of finitely many shifts of S is a thick set. Now, it is easy to verify that any thick set contains an *IP set*, that is, a set of the form $\{x_{n_1} + \ldots + x_{n_k} : n_1 < \ldots < n_k,\ k \in \mathbb{N}\}$, where $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$, which contains infinitely many non-zero elements. Applying Hindman's finite sums theorem, (Hin74), which states that, for any finite partition of an IP set, one of the cells contains an IP set, we see that any piecewise syndetic set contains a shift of an IP set. On the other hand, one can show that there are sets having density arbitrarily close to 1 which do not have this property. (This fact was first observed by E. Strauss, see (BBHS06, Theorem 2.20).)

1.2. Amenable groups

It is natural to ask whether Jin's theorem is valid in a more general setting where the notion of density can be naturally formulated. In (JK03, Application 2.5) it is proved that $A + B$ is piecewise syndetic if A and B are sets which have positive upper Banach density in $\mathbb{Z}^d$ and recently Jin extended this result to $\oplus_{d=1}^{\infty}\mathbb{Z}$ (Jin08). Jin and Keisler (JK03, Question 5.2) ask whether Theorem 1 can be extended to countable amenable groups. In this paper we answer this Question affirmatively. Before stating our results we review in this subsection some basic facts about amenable groups. (A very readable introduction focusing mainly on discrete groups is given in (Wag93, Chapter 10). For a more comprehensive treatment see (Gre69; Pat88; Pie84).)

A definition of amenability which is convenient for our purposes uses the notion of Følner sequence. A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of a countable group G is a (*left*) *Følner sequence* if

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0 \tag{2}$$

for every $g \in G$. Equivalently, $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence if for every finite set K and any $\varepsilon > 0$ all but finitely many F_n are (K, ε)-invariant in the sense that $|gF_n \triangle F_n|/|F_n| < \varepsilon$ for all $g \in K$.

A countable group G is *amenable* if it admits a (left) Følner sequence.[3] The basic example of an amenable group is the group of integers, an example of a Følner sequence being an arbitrary sequence of intervals $\{a_n, \ldots, b_n\}, n \in \mathbb{N}$ with $b_n - a_n \to \infty$. The class of amenable groups is quite rich, and, in particular, contains all solvable groups and is closed under the operations of forming directed unions, subgroups and extensions (see for instance (Gre69, Section 2.3) or (Wag93, Theorem 10.4)). The basic, but not the only examples of non-amenable groups are groups containing the free group on two generators as a subgroup (see for instance the discussion in (Wag93, p 147)).

Given a set A in an amenable group G, denote the relative density of A with respect to a finite set F by $d_F(A) := \frac{|A \cap F|}{|F|}$. The *upper density of A with respect to a Følner sequence* $(F_n)_{n\in\mathbb{N}}$ is defined by

$$\overline{d}_{(F_n)}(A) := \limsup_{n\to\infty} d_{F_n}(A), \tag{3}$$

and we write $d_{(F_n)}(A)$ and call it density with respect to $(F_n)_{n\in\mathbb{N}}$ if in formula (3) $\limsup_{n\to\infty} d_{F_n}(A) := \lim_{n\to\infty} d_{F_n}(A)$. The *upper Banach density* in amenable groups is defined by

$$d^*(A) := \sup\left\{\overline{d}_{(F_n)}(A) : (F_n)_{n\in\mathbb{N}} \text{ is a Følner sequence}\right\}. \tag{4}$$

Remark 1.1. For $G = \mathbb{Z}$ the above definition differs from original definition of upper Banach density in Subsection 1.1 (see formula (1)) where the supremum was taken only over intervals instead of arbitrary Følner sets. However the two notions are equivalent. For example, this follows from the following general fact which is a simple corollary of Lemma 3.3 below:

Given a subset B of an amenable group G and any Følner sequence $(F_n)_{n\in\mathbb{N}}$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$d^*(B) = d_{(F_n t_n)}(B). \tag{5}$$

Given two sets A, B in a discrete group G we let $AB = \{ab : a \in A, b \in B\}$. A set $S \subseteq G$ is (*left*) *syndetic* if there is a finite set F such that $FS = G$. A set $T \subseteq G$ is called (*right*) *thick* if for each finite set F there exists some $t \in G$ such that $Ft \subseteq T$[4]. A set $C \subseteq G$ is *piecewise syndetic* if there exist a thick set T and a syndetic set S such that $C \supseteq S \cap T$. It is not hard to see that $C \subseteq G$ is piecewise syndetic if and only if there exists a finite set K such that for each finite set F there is some $t \in G$ such that $Ft \subseteq KC$. Piecewise syndetic sets are partition regular: if $C_1 \cup \ldots \cup C_r$ is piecewise syndetic, then some $C_i, i \in \{1, \ldots, r\}$ is piecewise syndetic. This is not hard to see combinatorially and follows also from the ultrafilter characterization of piecewise syndeticity (cf. (HS98, Section 4.4)).

We are now able to state one of the main results of this paper.

[3] One can show that every amenable group admits also right- and indeed two-sided analogues of left Følner sequences. Throughout this paper we deal only with left Følner sequences; therefore we will routinely omit the adjective "left".

[4] When dealing with non-commutative structures one has at his disposal a "left/right" choice of notions. For brevity, we just write "syndetic" resp. "thick" for what should rigorously be called "left syndetic" resp. "right thick". The choice of left/right is implicitly present in the definitions of piecewise syndetic and piecewise Bohr below.

Theorem 2. *Let G be a countable amenable group and let $A, B \subseteq G$ have positive upper Banach density. Then AB is piecewise syndetic.*

1.3. Bohr sets.

The Bohr compactification bG of a countable discrete group G is defined (up to an isomorphism) as the largest compact group with the property that there exists a (not necessarily 1-1) homomorphism $\iota : G \to bG$ which has dense image. While this object exists for very general reasons, it is not always possible to give a useful down-to-earth description of it. Anyway, we will say that a set $B \subseteq G$ is a *Bohr set* if there exists a non-empty open set $U \subseteq bG$ such that $B \supseteq \iota^{-1}[U]$.[5] If, in the addition, U contains the identity of bG then B will be called Bohr_0 set. If G is abelian, we can consider the embedding

$$\begin{aligned} \iota : G &\to \mathbb{T}^{\hat{G}} && (6) \\ g &\mapsto (\gamma(g))_{\gamma \in \hat{G}}, && (7) \end{aligned}$$

where $\hat{G}$ is the dual group of G. Endowed with the product topology, $\mathbb{T}^{\hat{G}}$ is a compact group, $\overline{\iota[G]}$ is a compact subgroup and it can be shown that it is a "model" for the Bohr compactification of G. This implies that $B \subseteq G$ is a Bohr set if and only if there exist $\gamma_1, \dots, \gamma_n \in \hat{G}$ and an open set $U \subseteq \mathbb{T}^n$ such that $\{g \in G : \gamma_1(g), \dots, \gamma_n(g) \in U\}$ is non-empty and contained in B. (For more information see for instance (Rud62, Section 1.8) or (HR79, Chapter 6).)

Call a set $A \subseteq G$ *piecewise Bohr* if it is the intersection of a Bohr set and a thick set. Since every Bohr set is syndetic, piecewise Bohr sets are piecewise syndetic.

By (BFW06, Theorem 4.3) there exists a syndetic set of integers which is not piecewise Bohr. Note that this also implies that there exists a partition of the integers into finitely many cells none of which is piecewise Bohr.

Given a Bohr set B there exist a Bohr_0 set B_0 and a Bohr set B_1 such that $B \supseteq B_0 B_1$. This is a trivial consequence of the fact that the Bohr-topology is a group topology on G. Also, given a thick set T, it is not difficult to see that there exist thick sets T_0 and T_1 such that $T \supseteq T_0 T_1$ provided that G is abelian. (See Lemma 6.1 below.) It follows that for every piecewise Bohr set A there exist piecewise Bohr sets A_0, A_1 such that $A_0 A_1 \subseteq A$. In particular every piecewise Bohr set contains the product of two sets of positive upper Banach density. This puts an upper bound on the amount of structure which can be expected in the productset of two sets of positive upper Banach density. Somewhat surprisingly, it is in fact always possible to get this much:

Theorem 3. *Let G be a countable amenable group and assume that $A, B \subseteq G$ have positive upper Banach density. Then AB is piecewise Bohr.*

In the case $G = \mathbb{Z}$, Theorem 3 is proved in (BFW06).

Summarizing the above discussion, we have the following characterization of sumsets in the abelian case.

[5]The sets $\iota^{-1}[U]$, where $U \subseteq bG$ is open define the *Bohr-topology* on G. Hence $B \subseteq G$ is Bohr if and only if it contains a non-empty open set.

Theorem 4. *Let $(G, +)$ be a countable abelian group and let $C \subseteq G$. Then C is piecewise Bohr if and only if there exist sets A, B of positive upper Banach density such that $A + B \subseteq C$.*

We will show in Section 6 that Theorem 4 does not extend to the non-commutative setup.

1.4. Organization of the paper

In Section 2 we provide a simple proof of Jin's Theorem for $G = \mathbb{Z}$. In Section 3 we explain how this proof can be modified to extend Jin's result to the amenable setting (Theorem 2). The results in Section 4 allow us to give yet another proof of Theorem 2 and will also be utilized in Section 5 in the proof of Theorem 3. Finally, in Section 6 we prove Theorem 4 and provide an example which demonstrates that Theorem 4 does not extend to the non-commutative setup.

Throughout this paper, G will denote a countable discrete amenable group. We call $(X, \mathfrak{B}, \mu)$ a Borel probability space if $(X, \mathfrak{B})$ is a measurable space isomorphic to the unit interval equipped with the σ-algebra of Borel sets and μ is a Borel probability measure on $(X, \mathfrak{B})$. If $(X, \mathfrak{B}, \mu)$ is a Borel probability space and $T : X \to X$ is an invertible measure preserving transformation, $(X, \mathfrak{B}, \mu, T)$ will be called a measure preserving system.

2. Jin's theorem in the integers

Jin's original proof of Theorem 1 in (Jin02) utilized non-standard analysis. Jin also provided a purely combinatorial proof of Theorem 1 ((Jin04)). The purpose of this "warm-up" section is to give another proof of Theorem 1. While our proof is shorter than the original one, most of the ideas we use can be found, at least implicitly, in Jin's work.

Our proof of Jin's theorem will be based on the following two lemmas:

Lemma 2.1. *Assume that A, B are sets of integers such that $d^*(A) + d^*(B) > 1$. Then $d^*(A + B) = 1$, i.e. $A + B$ is thick.*

Lemma 2.2.[6] *If A is a set of integers then $\sup_{k \geq 0} d^*(\{-k, \ldots, k\} + A)$ is either 0 or 1.*

Taking Lemmas 2.1 and 2.2 for granted, Theorem 1 is almost trivial: By Lemma 2.2 there is some integer k such that $d^*(\{-k, \ldots, k\} + A) + d^*(B) > 1$. Hence by Lemma 2.1, $\{-k, \ldots, k\} + A + B$ is thick. Thus $A + B$ is piecewise syndetic.

Recall that for a finite interval $I \subseteq \mathbb{Z}$ and a set $A \subseteq \mathbb{Z}$, $d_I(A) = \frac{|I \cap A|}{|I|}$ denotes the relative density of A with respect to I.

[6] Lemma 2.2 is originally due to Neil Hindman, see (Hin82, Theorem 3.8). The combinatorial proof given subsequently is based on the same idea as Hindman's proof.

Proof of Lemma 2.1. Note that if $J \subseteq \mathbb{Z}$ is *any* non-empty interval and $d^*(B) > \beta$, then there exists $t \in \mathbb{Z}$ such that $d_{J+t}(B) > \beta$.

Pick $\alpha, \beta > 0$ such that $d^*(A) > \alpha, d^*(B) > \beta, \alpha + \beta = 1$ and fix $n \in \mathbb{N}$. We have to prove that $A + B$ contains a shifted copy of $\{0, 1, \ldots, n\}$. Loosely speaking, long enough intervals are almost invariant with respect to shifts by elements of $\{0, 1, \ldots, n\}$. In particular there exists an interval I such that $d_I(-x + A) > \alpha$ for all $x \in \{0, 1, \ldots, n\}$.

Apply the above observation to the interval $J = -I$ and pick some integer $t \in \mathbb{Z}$ such that $d_{(-I)+t}(B) = d_{-I}(B - t) > \beta$. Let $x \in \{0, 1, \ldots, n\}$. Since $\alpha + \beta = 1$,

$$
\begin{aligned}
d_{-I}(-A + x) + d_{-I}(B - t) > 1 \ &\Rightarrow \ (-A + x) \cap (B - t) \neq \emptyset && (8)\\
&\Rightarrow \ x + t \in A + B. && (9)
\end{aligned}
$$

Since x was arbitrary, we have $\{0, 1, \ldots, n\} + t \subseteq A + B$ as required. □

We will give two proofs of Lemma 2.2. The first one is based on an elementary combinatorial argument, the second one involves more abstract concepts and gives a rigorous meaning to the intuitive fact expressed by Lemma 2.2 that the system

$$(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), n \mapsto n + 1, d^*) \tag{10}$$

is "ergodic".

Combinatorial proof of Lemma 2.2. We will show that for any set $A \subseteq \mathbb{Z}$ with $d^*(A) > 0$ one has $\sup_{n \geq 0} d^*(A + \{-n, \ldots, n\}) = 1$. Assume by way of contradiction that $d^*(A) > 0$, but $\sup_{n \geq 0} d^*(A + \{-n, \ldots, n\}) = \gamma < 1$. Pick $\varepsilon > 0$ such that $(\gamma + \varepsilon)^2 < \gamma - \varepsilon$. For n large enough, $d^*(A + \{-n, \ldots, n\}) > \gamma - \varepsilon$. Hence, replacing A by $A + \{-n, \ldots, n\}$ if necessary, we may assume that $d^*(A) > \gamma - \varepsilon$.

Fix $k \in \mathbb{N}$ such that $d_I(A) < \gamma + \varepsilon$ for any interval $I \subseteq \mathbb{Z}$ of length k. Then pick an interval J such that the following conditions are satisfied:

i. The length of J is $m \cdot k$ for some positive integer m.
ii. $d_J(A + \{-k, \ldots, k\}) < \gamma + \varepsilon$.
iii. $d_J(A) > \gamma - \varepsilon$.

Partition J into intervals $I_1, I_2, \ldots, I_m$ of length k. Assume that A intersects more than $m \cdot (\gamma + \varepsilon)$ of these intervals. Then $A + \{-k, \ldots, k\}$ *covers* more than $m \cdot (\gamma + \varepsilon)$ of these intervals, hence $d_J(A + \{-k, \ldots, k\})$ exceeds $m \cdot (\gamma + \varepsilon)/m = \gamma + \varepsilon$, contradiction. Thus A intersects at most $m \cdot (\gamma + \varepsilon)$ of the intervals I_j, $j \in \{1, 2, \ldots, m\}$. Since the relative density of A in a length k interval is bounded by $\gamma + \varepsilon$ this yields

$$d_J(A) \leq (\gamma + \varepsilon) \cdot m \cdot (\gamma + \varepsilon)/m = (\gamma + \varepsilon)^2 \tag{11}$$

which contradicts $(\gamma + \varepsilon)^2 < \gamma - \varepsilon$. □

Our second proof of Lemma 2.2 is based on the following version of Furstenberg's correspondence principle.

Proposition 2.3. *Assume that $A \subseteq \mathbb{Z}$ has positive upper density. Then there exist an* ergodic *measure preserving system $(X, \mathfrak{B}, \mu, T)$ and a measurable set $B \subseteq X$ such that*

$$d^*(A) = \mu(B) \text{ and } \tag{12}$$

$$d^*(A - n_1 \cup \ldots \cup A - n_k) \geq \mu(T^{-n_1}B \cup \ldots \cup T^{-n_k}B) \tag{13}$$

for all $n_1, \ldots, n_k \in \mathbb{Z}$.

Proposition 2.3 differs from the more familiar forms of Furstenberg's correspondence principle (see (Ber87, Theorem 1.1)) in that we use unions instead of intersections and in that we require that $(X, \mathfrak{B}, \mu, T)$ to be ergodic. One can easily verify that due to the algebraic nature of Furstenberg's correspondence principle, virtually any known proof (see, in particular, the proofs in (Ber87; BM98)) is equally valid for unions. That the system can be chosen to be ergodic follows from (Fur81, Proposition 3.9).

"Dynamical" proof of Lemma 2.2. Assume that $d^*(A) > 0$ and choose $(X, \mathfrak{B}, \mu, T)$ and $B \subseteq X$ according to Proposition 2.3. Since T is ergodic,

$$\sup_{k \geq 0} d^*(\{-k, \ldots, k\} + A) \geq \sup_{k \geq 0} \mu(T^{-k}B \cup \ldots \cup T^k B) = \mu\Big(\bigcup_{k \in \mathbb{Z}} T^{-k}B\Big) = 1. \square$$

Remark 2.4. *For the usual (upper) density the statement of Lemma 2.2 is not true. For example, let $B = \bigcup_{n \in \mathbb{N}} \{n^2, n^2 + 1, \ldots, n^2 + n\}$. Then for $A = B \cup (-B)$ we have*

$$d(A) = \lim_{N \to \infty} \frac{|A \cap [-N, \ldots, N]|}{2N + 1} = 1/2 = \sup_{k \geq 0} \overline{d}(\{-k, \ldots, k\} + A). \tag{14}$$

However, it follows from the proof of Lemma 3.2, that if $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence which satisfies $d_{(F_n)}(A) = d^*(A) > 0$), then we have

$$\sup_{k \geq 0} \overline{d}_{(F_n)}(\{-k, \ldots, k\} + A) = 1. \tag{15}$$

3. Jin's theorem in countable amenable groups

In this section we demonstrate that (with some work) the proof of Jin's theorem which was given in the previous section generalizes to the amenable setting. The proof of the general "amenable" statement is based on the following auxiliary results. (cf. Lemmas 2.1, 2.2)

Lemma 3.1. *Let G be an amenable group and assume that $A, B \subseteq G$, $d^*(A) + d^*(B) > 1$. Then AB is thick.*

Lemma 3.2. *Let G be a countable amenable group and let $A \subseteq G$. Then* $\sup\{d^*(KA) :$ *$K \subseteq G, K$ is finite$\}$ is either* 0 *or* 1.

Note first, that in complete analogy with the integer setting, Lemma 3.1 and Lemma 3.2 imply that if $d^*(A), d^*(B) > 0$, then there exists a finite set K such that KAB is thick, which, in turn, implies that AB is piecewise syndetic.

The following simple fact is needed in the proof of Lemma 3.1 (and will also be utilized in the next section for the proof of Lemma 4.2.).

Lemma 3.3. *Let $B, K \subseteq G$, K finite and $\beta < d^*(B)$. Then there exists some $t \in G$ such that*

$$d_{Kt}(B) = \frac{|B \cap Kt|}{|K|} \geq \beta. \tag{16}$$

Proof. Pick a Følner set F such that $|B \cap gF|/|F| \geq \beta$ for each $g \in K$. Then

$$\sum_{t \in F} |B \cap Kt| = |\{(g,t) \in K \times F : gt \in B\}| = \sum_{g \in K} |B \cap gF| \geq |K| \cdot |F| \cdot \beta. \tag{17}$$

Dividing by $|K| \cdot |F|$ we see that (16) holds for some $t \in F$. □

Proof of Lemma 3.1. To obtain Lemma 3.1, one just has to rewrite the proof of Lemma 2.1 in terms of Følner sequences. The only part which needs justification is that if $d^*(B) > \beta$ and $F \subseteq G$ is a finite set, then there is some $t \in G$ such that $d_{F^{-1}t}(B) > \beta$. This was proved in Lemma 3.3. □

Lemma 3.2 can be proved in a variety of ways. First, it is possible to prove an appropriate version of Furstenberg's correspondence principle for amenable groups (for instance, one can combine the proof of correspondence principle given in (BM98, Theorem 2.1) or in (Ber00, Theorem 6.4.17) with the amenable analogue of (Fur81, Proposition 3.9)) which then immediately gives the desired result as in the dynamical proof of Lemma 2.2.

Second, one also can prove Lemma 3.2 via an appropriate generalization of the combinatorial proof of Lemma 2.2. There we employed the fact that intervals *tile* the integers. In general, a set T in a countable group G is a *tile* if there exists a set $S \subseteq G$ such that $\{Ts : s \in S\}$ is a partition of G. The group G is called *monotilable* if it admits a Følner sequence consisting of tiles and in this case the proof of Lemma 2.2 can be adapted fairly naturally. Having the construction of Følner sequences in the abelian setting in mind, it is easy to see that every countable abelian group is monotilable and it is shown in (Wei01) that much more general classes of amenable groups share this property. While it is not known whether all amenable groups are monotilable, they do admit so called quasi-tilings (see (OW87)). Those still do allow to push the proof of Lemma 2.2 to the desired generality, but the details become unpleasantly technical.

Since Lemma 3.2 is crucial for a generalization of Jin's theorem to the amenable case, we will give here a self contained proof. While the argument is more involved than that used in the combinatorial proof of Lemma 2.2, it is still entirely elementary.

Proof of Lemma 3.2. It is sufficient to consider the case $d^*(A) > 0$. Pick a Følner sequence $(F_n)_{n \in \mathbb{N}}$ such that $d_{(F_n)}(A) = \alpha > 0$ and $d_{(F_n)}(KA)$ exists for each finite $K \subseteq G$. Let $\beta = \sup\{d_{(F_n)}(KA) : K \subseteq G, K \text{ finite}\}$. We claim that after passing, if necessary, to a subsequence of $(F_n)_{n \in \mathbb{N}}$, there exists a Følner sequence $(G_n)_{n \in \mathbb{N}}, G_n \subseteq F_n$ such that the following hold true:

i. $\lim_{n \to \infty} |G_n|/|F_n| = \beta$.

ii. $d_{(G_n)}(HA) = d_{(F_n)}(HA)\frac{1}{\beta}$ for any finite set $H \subseteq G$.

A particular consequence of (ii) is that $\sup\{d_{(G_n)}(KA) : K \subseteq G, K \text{ finite}\} = \beta/\beta = 1$.

Fix a sequence $(K_n)_{n\in\mathbb{N}}$ of finite subsets of G such that $K_nK_n \subseteq K_{n+1}$, $K_n \uparrow G$ and each K_n contains the identity of G. Passing to subsequences, we can assume that

$$d_{F_m}(K_nA) \in (\beta - 1/n, \beta + 1/m) \text{ for all } m \geq n, \tag{18}$$

and that each F_n is $(K_n, 1/n)$-invariant. Let G_1 be an arbitrary finite non-empty subset of G and, for $n \geq 2$, set $G_n := K_{n-1}A \cap F_n$. Note that $|G_n|/|F_n| \in (\beta - 1/(n-1), \beta + 1/n)$. Let us prove that $(G_n)_{n\in\mathbb{N}}$ is a Følner sequence. For $n \geq 2$ and $t \in K_{n-1}$ we obtain

$$\begin{aligned}
|tG_n \setminus G_n| &= |(tK_{n-1}A \cap tF_n) \setminus (K_{n-1}A \cap F_n)| && (19)\\
&\leq |(K_nA \cap tF_n) \setminus (K_{n-1}A \cap F_n)| && (20)\\
&\leq |(K_nA \cap F_n) \setminus (K_{n-1}A \cap F_n)| + |F_n|/n && (21)\\
&= (d_{F_n}(K_nA) - d_{F_n}(K_{n-1}A) + 1/n) \cdot |F_n| \leq \frac{3|F_n|}{n-1}. && (22)
\end{aligned}$$

Thus for large enough n (so that $\beta - \frac{1}{n-1} > 0$) and for all $t \in K_{n-1}$ we have

$$\frac{|tG_n \triangle G_n|}{|G_n|} \leq 2\frac{3|F_n|}{(n-1)\cdot|G_n|} \leq \frac{6}{(n-1)\cdot(\beta - \frac{1}{n-1})}. \tag{23}$$

Since the latter quantity tends to 0 as n goes to infinity, $(G_n)_{n\in\mathbb{N}}$ is indeed a Følner sequence.

Finally observe that

$$\begin{aligned}
d_{(G_n)}(HA) &= \lim_{n\to\infty} \frac{|HA \cap (K_{n-1}A \cap F_n)|}{|K_{n-1}A \cap F_n|} && (24)\\
&= \lim_{n\to\infty} \frac{|(HA \cap K_{n-1}A) \cap F_n|}{\beta|F_n|} = \frac{1}{\beta} d_{(F_n)}(HA), && (25)
\end{aligned}$$

which gives us (ii).

□

4. Finer structure of product sets.

The following proposition (which is the main result of this section) shows that the product of two sets of positive upper Banach density contains translations of arbitrarily large pieces of the product of a "large set" with its inverse. (This fact will be utilized in the proof of Theorem 3 in the next section.)

Proposition 4.1. *Let G be a countable amenable group and let $A, B \subseteq G$ be such that $d^*(A), d^*(B) > 0$. Then there exists a set $D \subseteq G$ with $d^*(D) > 0$ such that for each finite set $H \subseteq G$, there is some t_H such that*

$$(H \cap DD^{-1})t_H \subseteq AB. \tag{26}$$

Using Lindenstrauss' pointwise ergodic theorem (Lin01) it is possible to show that for any set D which has positive upper Banach density and for any Følner sequence $(F_n)_{n\in\mathbb{N}}$ to which the pointwise ergodic theorem applies, there exists a set E such that $d_{(F_n)}(E) = d^*(D)$ and $EE^{-1} \subseteq DD^{-1}$. Hence it is possible to give a somewhat stronger formulation of Proposition 4.1.

Before proving Proposition 4.1 we formulate and prove a few auxiliary results.

Lemma 4.2. *Let $A_0, B \subseteq G$, A_0 finite and $\beta < d^*(B)$. There exist $C \subseteq A_0$ and $t \in G$ such that $CC^{-1}t \subseteq A_0B$ and $|C| \geq \beta|A_0|$.*

Proof. Applying Lemma 3.3 to A_0^{-1} we find t such that

$$\beta|A_0| \leq |A_0^{-1}t \cap B| = |A_0 \cap (Bt^{-1})^{-1}|. \tag{27}$$

And for all $x, y \in C := A_0 \cap (Bt^{-1})^{-1}$ we have $xy^{-1} \in A_0(Bt^{-1})$. □

While the formulation of Lemma 4.2 appears to be somewhat technical, it allows to show that AB contains arbitrary large sets of the form $C_nC_n^{-1}t_n$. The remaining ingredient in the proof of Proposition 4.1 is the following statement.

Lemma 4.3. *Let $(F_n)_{n\in\mathbb{N}}$, $(G_n)_{n\in\mathbb{N}}$ be Følner sequences, let $C_n \subseteq F_n$ and set $\gamma := \limsup d_{F_n}(C_n)$. Then there exists a set D such that the following hold.*

i. $\overline{d}_{(G_n)}(D) = \gamma$.
ii. *For each finite set $D_0 \subseteq D$ there exist $c \in G$ and $n \in \mathbb{N}$ such that $D_0c \subseteq C_n$.*

The proof of Lemma 4.3 relies on the following Fubini-type Lemma.

Lemma 4.4. *Let $(X, \mathfrak{A}, m)$ be some space equipped with a finitely additive measure, assume that $(A_g)_{g\in G}$ is a sequence of sets in $\mathfrak{A}$ such that $m(A_g) \geq \gamma$ for all $g \in G$ and let $(G_n)_{n\in\mathbb{N}}$ be a Følner sequence. Then there exists a set D such that $\overline{d}_{(G_n)}(D) \geq \gamma$ and $m(\bigcap_{t\in D_0} A_t) > 0$ for every finite set $D_0 \subseteq D$.*

Lemma 4.4 is essentially (Ber06, Lemma 5.10), the only difference being that here we only require that m is finitely additive. The following argument shows that the case of finitely additive measures follows from the σ-additive setup. Indeed, set $Y := \{0,1\}^{\mathbb{N}}$, let $B_n = \{(x_k)_{k\in\mathbb{N}} \in Y : x_n = 1\}$ for $n \in \mathbb{N}$ and put

$$\mu_0\Big(\bigcap_{k\in S} B_k \cap \bigcap_{n\in T}(Y \setminus B_n)\Big) := m\Big(\bigcap_{k\in S} A_k \cap \bigcap_{n\in T}(Y \setminus A_n)\Big) \tag{28}$$

for finite sets $S, T \subseteq \mathbb{N}$. Then μ_0 naturally extends to a σ-additive Borel probability measure μ on Y and it is sufficient to prove Lemma 4.4 for the sets $B_1, B_2, \ldots$ in $(Y, \mathfrak{B}, \mu)$.

Proof of Lemma 4.3. Passing to a subsequence if necessary, we can assume that $\gamma = \lim d_{F_n}(C_n)$ exists. Consider $C := \bigcup_n C_n \times \{n\} \subseteq G \times \mathbb{N} =: X$. Let $\mathfrak{A}$ be the algebra of subsets of X generated by all sets of the form $gC := \bigcup_n (gC_n) \times \{n\}, g \in G$. Since $\mathfrak{A}$ is countable, we can pick a sequence $k_1 < k_2 < \dots$ in $\mathbb{N}$ such that

$$m(A) = \lim_{k\to\infty} \frac{|A \cap (F_{n_k} \times \{n_k\})|}{|F_{n_k}|} \tag{29}$$

exists for all $A \in \mathfrak{A}$. Note that

$$m(gC) = \lim_{k\to\infty} \frac{|gC_{n_k} \cap F_{n_k}|}{|F_{n_k}|} = \gamma \tag{30}$$

for all $g \in G$. Let D be the "outcome" of applying Lemma 4.4 to the space $(X, \mathfrak{A}, m)$ and the sets $g^{-1}C, g \in G$. Given a finite set $D_0 \subseteq D$, $m\left(\bigcap_{g\in D_0} g^{-1}C\right) > 0$. Hence for k large enough, $\bigcap_{g\in D_0} g^{-1}C_{n_k}$ has positive relative density with respect to F_{n_k}, so pick $c \in \bigcap_{g\in D_0} g^{-1}C_{n_k}$. Then $D_0c \subseteq C_{n_k}$ as required. □

We are now in the position to prove the main result of this section.

Proof of Proposition 4.1. Pick a Følner sequence $(F_n)_{n\in\mathbb{N}}$ and $\alpha > 0$ such that $d_{F_n}(A) \geq \alpha > 0$ for all $n \in \mathbb{N}$. Pick $\beta > 0$ such that $d^*(B) > \beta$. Applying Lemma 4.2 to the sets $A_n := A \cap F_n$, we find sequences $(C_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $\bigcup_{k=1}^{\infty} C_kC_k^{-1}t_k \subseteq AB$ and $d_{F_n}(C_n) \geq \alpha\beta > 0$ for each $n \in \mathbb{N}$. Pick a set D guaranteed by Lemma 4.3. Given an arbitrary finite set $H \subseteq G$, there is a finite set $D_0 \subseteq D$ such that $H \cap DD^{-1} \subseteq D_0D_0^{-1}$. By Lemma 4.3, there exist $c \in G$ and $n \in \mathbb{N}$ such that $D_0c \subseteq C_n$. Hence

$$(DD^{-1} \cap H)t_n \subseteq D_0D_0^{-1}t_n = D_0c(D_0c)^{-1}t_n \subseteq \bigcup_{k=1}^{\infty} C_kC_k^{-1}t_k \subseteq AB. \tag{31}$$

□

In the next section we will use Proposition 4.1 together with Lemma 4.5 to prove that AB is piecewise Bohr if $d^*(A), d^*(B) > 0$.

Lemma 4.5. *Let $A \subseteq G$ and assume that $d^*(A) > 0$. Then there exist a Borel probability space $(X, \mathfrak{B}, \mu)$, a measure preserving G action $(T_g)_{g\in G}$ on X and a set $B \subseteq X, \mu(B) = d^*(A)$ such that*

$$\{g \in G : \mu(T_g^{-1}B \cap B) > 0\} \subseteq AA^{-1}. \tag{32}$$

In particular AA^{-1} is syndetic.

In a certain sense Lemma 4.5 can be reversed. Indeed, using the ergodic theorem it is not difficult to see that for any set R of return times there exists a set A of positive upper Banach density such that $AA^{-1} \subseteq R$.

We will derive Lemma 4.5 from the following amenable version of Furstenberg's correspondence principle (see for instance (BM98, Theorem 2.1), (Ber00, Theorem 6.4.17)).

Lemma 4.6. *Let G be an amenable group and assume that $A \subseteq G$. Then there exist a Borel probability space $(X, \mathfrak{B}, \mu)$, a measure preserving G action $(T_g)_{g\in G}$ on X and set $B \subseteq X, \mu(B) = d^*(A)$ such that*

$$\mu(T_{g_1}^{-1} B \cap \ldots \cap T_{g_n}^{-1} B) \leq d^*(g_1^{-1} A \cap \ldots \cap g_n^{-1} A) \tag{33}$$

for all $g_1, \ldots, g_n \in G$.

Proof of Lemma 4.5. Let $(X, \mathfrak{B}, \mu), (T_g)_{g\in G}$ and B be as in Proposition 4.6. Then

$$AA^{-1} \supseteq \{g : d^*(g^{-1} A \cap A) > 0\} \supseteq \{g : \mu(T_g^{-1} B \cap B) > 0\} =: S. \tag{34}$$

Set $Y := \bigcup_{g\in G} T_g^{-1} B$. Pick a finite set $K \subseteq G$ such that $\mu(\bigcup_{g\in K} T_g^{-1} B) + \mu(B) > \mu(Y)$. Fix $h \in G$. Then $\mu(\bigcup_{g\in K} T_g^{-1} B \cap T_h^{-1} B) > 0$. Hence for some $g \in K$ we have $\mu(T_{gh}^{-1} B \cap B) > 0$. Equivalently $gh \in S = \{f \in G : \mu(T_f^{-1} B \cap B) > 0\}$. Since $h \in G$ was arbitrary, $G = K^{-1} S$, so AA^{-1} is indeed syndetic. □

We conclude this section with showing how Proposition 4.1 offers yet another way to establish Theorem 2. If A, B have positive upper Banach density, we may choose a set D of positive upper Banach density such that AB contains shifts of arbitrary finite portions of $S = DD^{-1}$. By Lemma 4.5 the set S is syndetic and hence also piecewise syndetic. Thus piecewise syndeticity of AB follows from the following natural property of piecewise syndetic sets.

Lemma 4.7. *Let G be a group, $S, T \subseteq G$ and assume that $S \subseteq G$ is piecewise syndetic and that for each finite set $H \subseteq G$ there is some $t_H \in G$ such that*

$$(H \cap S) t_H \subseteq T. \tag{35}$$

Then T is piecewise syndetic as well.

Proof. Pick a finite set $K \subseteq G$ such that KS is thick. Given an arbitrary finite set $F \subseteq G$, there is some $f \in G$ such that $Ff \subseteq KS$. Choose a finite set H such that $F \subseteq K(S \cap H)$. Since $(S \cap H) \subseteq T t_H^{-1}$, we have $Ff \subseteq KT t_H^{-1}$. As H was arbitrary, KT is thick. □

5. Bohr sets and almost periodic functions

Consider the space $B(G)$ of bounded real-valued functions on G. The group G acts[7] on $B(G)$ by $\sigma_t(f)(g) := f(tg), t, g \in G, f \in B(G)$. Let $AP(G)$ denote the subspace of *almost periodic functions*, namely the set of those $f \in B(G)$ for which $\{\sigma_t(f) : t \in G\} \subseteq B(G)$ is pre-compact in the sup-norm $\|.\|_\infty$ on $B(G)$.

The following statement is presumably well known to experts. However we give a proof to increase the readability of the paper.

[7]To be more precise, $(\sigma_g)_{g\in G}$ is an anti-action.

Proposition 5.1. *Let $(X, \mathfrak{B}, \mu)$ be a Borel probability space, let $(T_g)_{g\in G}$ be a measure preserving G-action on X, $B \in \mathfrak{B}, \mu(B) > 0$. Then there exist functions $\varphi_c, \varphi_{wm} : G \to \mathbb{R}$, where is φ_c is almost periodic and non-negative such that $\mu(B \cap T_g^{-1}B) = \varphi_c(g) + \varphi_{wm}(g)$ and*

$$m := \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \varphi_c(g) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \mu(T_g^{-1}B \cap B) > 0, \tag{36}$$

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} |\varphi_{wm}(g)| = 0 \tag{37}$$

for any Følner sequence $(F_n)_{n\in\mathbb{N}}$.

Proof. Set $\mathcal{H} = L_2(X, \mathfrak{B}, \mu)$. Let $U_g h := h \circ T_g$, $g \in G$, $h \in \mathcal{H}$ be the induced unitary anti-action of G on $\mathcal{H}$. Pick a Følner sequence $(F_n)_{n\in\mathbb{N}}$. Consider now the following $(U_g)_{g\in G}$-invariant subspaces of $\mathcal{H}$.

$$\mathcal{H}_c = \Big\{f \in \mathcal{H} : \{U_g f : g \in G\} \text{ is precompact in the norm topology}\Big\} \tag{38}$$

$$\mathcal{H}_{wm} = \Big\{f \in \mathcal{H} : \frac{1}{|F_n|} \sum_{g\in F_n} |\langle U_g f, f'\rangle| \to 0 \text{ for all } f' \in \mathcal{H}\Big\}. \tag{39}$$

By (BR88, Theorem 1.9) $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$. Since $\mathcal{H}_c$ does not depend on the particular choice of the Følner sequence $(F_n)_{n\in\mathbb{N}}$, $\mathcal{H}_{wm}$ doesn't either. Set $f := 1_B$ and choose $f_c \in \mathcal{H}_c, f_{wm} \in \mathcal{H}_{wm}$ such that $f = f_c + f_{wm}$. Set

$$\varphi_c(g) := \langle U_g f_c, f_c\rangle, \varphi_{wm}(g) := \langle U_g f_{wm}, f_{wm}\rangle, \tag{40}$$

$$\mu(T_g^{-1}B \cap B) = \langle U_g f, f\rangle = \varphi_c(g) + \varphi_{wm}(g). \tag{41}$$

It follows directly from the definition of $\mathcal{H}_{wm}$ that $\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} |\varphi_{wm}(g)| = 0$. Note that for $t_1, t_2 \in G$

$$\|\sigma_{t_1}(\varphi_c) - \sigma_{t_2}(\varphi_c)\|_\infty = \sup_{g\in G} |\varphi_c(t_1 g) - \varphi_c(t_2 g)| = \tag{42}$$

$$\sup_{g\in G} |\langle U_{t_1 g} f_c, f_c\rangle - \langle U_{t_2 g} f_c, f_c\rangle| = \tag{43}$$

$$\sup_{g\in G} |\langle U_g((U_{t_1} - U_{t_2})(f_c)), f_c\rangle| \le \|U_{t_1} f_c - U_{t_2} f_c\|_2, \tag{44}$$

hence pre-compactness of $\{U_t f_c : t \in G\}$ implies pre-compactness of $\{\sigma_t(\varphi_c) : t \in G\}$, thus φ_c is almost periodic. By the mean ergodic theorem

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \varphi_c(g) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \int_G f U_g f \, d\mu = \int f P f \, d\mu, \tag{45}$$

where P denotes the projection from $L_2(\mu)$ onto the subspace of the U_g-invariant functions. Since $\int Pf \, d\mu = \int f \, d\mu = \mu(B)$, $f \neq 0$. Thus

$$0 < \int (Pf)^2 \, d\mu = \int PfPf \, d\mu = \int fP^2 f \, d\mu = \int fPf \, d\mu. \tag{46}$$

Hence also the right hand side of (45) is positive. □

We will need the following alternative characterization of almost periodicity. (See (BJM89) for a proof that these two properties are equivalent.)

Lemma 5.2. *A function $\varphi : G \to \mathbb{R}$ is almost periodic if and only if there exists a continuous function $f : bG \to \mathbb{R}$ such that $\varphi = f \circ \iota$.*

As a consequence of Proposition 5.1 and Lemma 5.2 we obtain Følner's Theorem ((Føl54a; Føl54b)) for countable amenable groups:

Corollary 5.3. *Let G be a countable amenable group and let $A \subseteq G$ such that $d^*(A) > 0$. Then there exist a Bohr set B and a set $N \subseteq G$ with $d^*(N) = 0$ such that*

$$B \subseteq AA^{-1} \cup N. \tag{47}$$

Proof. By Lemma 4.5 there exist a Borel probability space $(X, \mathfrak{B}, \mu)$, $B \in \mathfrak{B}$, $\mu(B) > 0$ and a measure preserving action $(T_g)_{g\in G}$ on X, such that $\{g \in G : \mu(T_g^{-1}B \cap B) > 0\} \subseteq AA^{-1}$. Pick m and φ_c, φ_{wm} according to Proposition 5.1 such that $\mu(T_g^{-1}B \cap B) = \varphi_c(g) + \varphi_{wm}(g)$ for $g \in G$. Set $N = \{g : \varphi_{wm} < -m/2\}$ and $\psi = \varphi_c - m/2$. Then $d^*(N) = 0$ and for $g \in G \setminus N, \psi(g) > 0$ implies that $\mu(T_g^{-1}B \cap B) > 0$. Pick a continuous function $f : bG \to \mathbb{R}$ such that $\psi = f \circ \iota$. Since $\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \psi(g) = m/2$, f takes some positive value, in particular $U := \{x \in bG : f(x) > 0\}$ is a non-empty open set. Putting things together we have

$$\iota^{-1}U = \{g : \psi(g) > 0\} \subseteq \{g : \mu(T_g^{-1}B \cap B) > 0\} \cup N \subseteq AA^{-1} \cup N. \tag{48}$$

□

Having Corollary 5.3 at hand, Theorem 3 follows from Proposition 4.1 once we establish the following regularity property of piecewise Bohr sets.

Lemma 5.4. *Let $S, T \subseteq G$. If S is piecewise Bohr and for each finite set $H \subseteq G$ there is some $t_H \in G$ such that $(S \cap H)t_H \subseteq T$ then T is piecewise Bohr as well.*

Proof. There exist a thick set $H \subseteq G$ and an open set $U \subseteq bG$ such that $H \cap \iota^{-1}[U] \subseteq S$. Pick sequences $(H_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ such that $H_n s_n \uparrow G$ and $H_n \subseteq H$. Pick for each $n \in \mathbb{N}$ some t_n such that $(\iota^{-1}[U] \cap H_n)t_n \subseteq T$. Then

$$T \supseteq (\iota^{-1}[U] \cap H_n)t_n = \{g \in H_n : \iota(g) \in U\}t_n = \{gt_n \in H_n t_n : \iota(g) \in U\} = \tag{49}$$

$$\{h \in H_n t_n : \iota(h)\iota(t_n^{-1}) \in U\} = \{h \in H_n t_n : \iota(h) \in U\iota(t_n)\} = \iota^{-1}[U\iota(t_n)] \cap H_n t_n \tag{50}$$

Choose an accumulation point x of $\iota(t_n)^{-1}, n = 1, 2, \ldots$ and open sets U_1, U_2 such that $x \in U_2$ and $U_1 \cdot U_2 \subseteq U$. Then $U_1\iota(t_n)^{-1} \subseteq U$ for infinitely many $n \in \mathbb{N}$ and for each such n

$$\iota^{-1}[U_1] \cap H_n t_n \subseteq T, \tag{51}$$

hence T is piecewise Bohr. □

Proof of Theorem 3. Pick the set D in G of positive upper Banach density guaranteed by Proposition 4.1. Then by Corollary 5.3 the set DD^{-1} is piecewise Bohr. By Lemma 5.4 the set AB is piecewise Bohr. □

6. Abelian versus non-abelian

The following Lemma is the only remaining fact needed for the proof of Theorem 4.

Lemma 6.1. *Assume that $(G,+)$ is a countable abelian group and $T \subseteq G$ is thick. Then there exist thick sets $T_1, T_2 \subseteq G$ such that $T_1 + T_2 \subseteq T$.*

Proof. Pick sequences $(c_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}}$ such that all $K_n \subseteq G$ are finite, $K_n \uparrow G$ and $\bigcup_{n\in\mathbb{N}} K_n + c_n \subseteq T$. We will inductively define sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ such that

$$\bigcup_{l\in\mathbb{N}} K_l + a_l + \bigcup_{m\in\mathbb{N}} K_m + b_m \subseteq T. \tag{52}$$

To start the induction, let $a_1 \in G$ be arbitrary, pick n such that $K_1 + a_1 + K_1 \subseteq K_n$ and set $b_1 = c_n$ such that $K_1 + a_1 + K_1 + b_1 \subseteq K_n + c_n \subseteq T$.

Next assume that after k steps $a_1, \dots, a_k, b_1, \dots b_k \in G$ have been chosen such that $\bigcup_{l\le k} K_l + a_l + \bigcup_{m\le k} K_m + b_m \subseteq T$. Pick n such that $K_{k+1} + \bigcup_{l\le k} K_l + a_l + \bigcup_{m\le k} K_m + b_m \subseteq K_n$ and set $a_{k+1} := c_n$. Choose b_{k+1} analogously. The induction continues. □

Proof of Theorem 4. If $C \subseteq G$ is piecewise Bohr then $C \supseteq B \cap T$, where B is a Bohr set and T is a thick set. As explained in Subsection 1.3 one can find Bohr sets $B_0, B_1 \subset G$ such that $B \supseteq B_0 + B_1$. By Lemma 6.1 we can find thick sets T_1, T_2 in G such that $T_1 + T_2 \subseteq T$. Then $C \supseteq (B_0 \cap T_1) + (B_1 \cap T_2)$. On the other hand, if $A + B \subseteq C$ for A, B of positive upper Banach density then, by Theorem 3, C is piecewise Bohr. □

One may wonder whether given three sets A, B, C of positive upper Banach density in an abelian group the sum $A + B + C$ has stronger properties than the sumset of two sets. The following result, which follows from the familiar by now fact that a piecewise Bohr set contains the sum of two piecewise Bohr sets, shows that there is not much to look for.

Proposition 6.2. *Let G be a countable abelian group and let $A, B \subseteq G$ have positive upper Banach density. Then for every $k \in \mathbb{N}$ there exist piecewise Bohr sets $C_1, \dots, C_k$ such that*

$$C_1 + C_2 + \ldots + C_k \subseteq A + B.$$

The following Proposition 6.3 demonstrates that in Theorem 4 one cannot drop the assumption of commutativity of the group G. However, before formulating Proposition 6.3 we want to introduce some convenient terminology. Note first that the definition of upper Banach density introduced in Subsection 1.2 is based on the notion of left Følner sequence. One could also introduce a "right" version of upper Banach density with the help of the notion of right Følner sequences (that is a sequence satisfying $\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0$). Accordingly, we will say that a set $A \subseteq G$ is *left large* (*right large*) if it has positive upper "left" ("right") Banach density. Finally, let us say that a set $A \subseteq G$ is *large* if it is either left large or right large.

Proposition 6.3. *Let G be the Heisenberg group over the integers, i.e. the group of 3×3 upper triangular matrices with integer entries and 1's on the diagonal. There exists a thick set $T \subseteq G$ which does not contain the product AB of any two large sets $A, B \subseteq G$.*

Proof. We will view G as $\mathbb{Z}^3$ equipped with the operation given by

$$\left(a^{(x)}, a^{(y)}, a^{(z)}\right) * \left(b^{(x)}, b^{(y)}, b^{(z)}\right) := \left(a^{(x)} + b^{(x)}, a^{(y)} + b^{(y)}, a^{(z)} + b^{(z)} + a^{(x)} b^{(y)}\right). \quad (53)$$

Set $K_n = \{-n, \dots, n\}^3$ for $n \in \mathbb{N}$ and $T = \bigcup_{n \in \mathbb{N}} K_n * (n^2, 0, 0)$. Assume that, contrary to the claim of our Proposition, there exist large sets $A, B \subseteq G$ such that $A * B \subseteq T$. Pick $b_1 = (b_1^{(x)}, b_1^{(y)}, b_1^{(z)})$, $b_2 = (b_2^{(x)}, b_2^{(y)}, b_2^{(z)}) \in B$ such that $b_1^{(y)} \neq b_2^{(y)}$. Set $n_0 = 10\left(|b_1^{(x)}| + |b_1^{(y)}| + |b_1^{(z)}| + |b_2^{(x)}| + |b_2^{(y)}| + |b_2^{(z)}|\right)$. Since A is infinite, Ab_1 is not contained in $\bigcup_{n \leq n_0} K_n * (n^2, 0, 0)$. Hence there exist $a = (a^{(x)}, a^{(y)}, a^{(z)}) \in A$ and $m \geq n_0$ such that $a * b_1 \in K_m * (m^2, 0, 0)$. Note that this implies that $a^{(x)} \in [m^2 - 2m, m^2 + 2m]$. By assumption, $a * b_2 \in T$ and since the difference $\left|(a^{(x)} + b_1^{(x)}) - (a^{(x)} + b_2^{(x)})\right|$ is small compared to m, we have in fact $a * b_2 \in K_m * (m^2, 0, 0)$. This implies that the z-coordinates of $a * b_1$ and $a * b_2$ differ at most by $2m$, hence

$$2m \geq \left|\left(a^{(z)} + b_1^{(z)} + a^{(x)} b_1^{(y)}\right) - \left(a^{(z)} + b_2^{(z)} + a^{(x)} b_2^{(y)}\right)\right| \quad (54)$$

$$= \left|b_1^{(z)} - b_2^{(z)} + a^{(x)}\left(b_1^{(y)} - b_2^{(y)}\right)\right| \quad (55)$$

which is not possible since $\left|b_1^{(y)} - b_2^{(y)}\right| \geq 1$ and $a^{(x)}$ is of order m^2. □

Acknowledgment 6.4. *The authors thank Michael Hochman, Gabriel Maresch and Ilya Shkredov for helpful comments on the topic of this paper.*

References

[BBHS06] M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss. Multiplicative structures in additively large sets. *J. Combin. Theory Ser. A*, 113(7):1219–1242, 2006.

[Ber85] V. Bergelson. Sets of recurrence of $\mathbf{Z}^m$-actions and properties of sets of differences in $\mathbf{Z}^m$. *J. London Math. Soc. (2)*, 31(2):295–304, 1985.

[Ber87] V. Bergelson. Ergodic Ramsey theory. *Amer. J. Math.*, 65(6):63–87, 1987.

[Ber00] V. Bergelson. Ergodic theory and Diophantine problems. In *Topics in symbolic dynamics and applications (Temuco, 1997)*, volume 279 of *London Math. Soc. Lecture Note Ser.*, pages 167–205. Cambridge Univ. Press, Cambridge, 2000.

[Ber06] V. Bergelson. Combinatorial and Diophantine applications of ergodic theory. In *Handbook of dynamical systems. Vol. 1B*, pages 745–869. Elsevier B. V., Amsterdam, 2006. Appendix A by A. Leibman and Appendix B by Anthony Quas and Máté Wierdl.

[BFW06] V. Bergelson, H. Furstenberg, and B. Weiss. Piecewise-Bohr sets of integers and combinatorial number theory. In *Topics in discrete mathematics*, volume 26 of *Algorithms Combin.*, pages 13–37. Springer, Berlin, 2006.

[BJM89] J. Berglund, H. Junghenn, and P. Milnes. *Analysis on semigroups*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1989. Function spaces, compactifications, representations, A Wiley-Interscience Publication.

[BM98] V. Bergelson and R. McCutcheon. Recurrence for semigroup actions and a non-commutative Schur theorem. In *Topological dynamics and applications (Minneapolis, MN, 1995)*, volume 215 of *Contemp. Math.*, pages 205–222. Amer. Math. Soc., Providence, RI, 1998.

[BR88] V. Bergelson and J. Rosenblatt. Mixing actions of groups. *Illinois J. Math.*, 32(1):65–80, 1988.

[Føl54a] E. Følner. Generalization of a theorem of Bogolioùboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups. *Math. Scand.*, 2:5–18, 1954.

[Føl54b] E. Følner. Note on a generalization of a theorem of Bogolioùboff. *Math. Scand.*, 2:224–226, 1954.

[Fur81] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures.

[Gre69] F. P. Greenleaf. *Invariant means on topological groups and their applications*. Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York, 1969.

[HR79] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I*, volume 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1979. Structure of topological groups, integration theory, group representations.

[Hin74] N. Hindman. Finite sums from sequences within cells of a partition of N. *J. Combinatorial Theory Ser. A*, 17:1–11, 1974.

[Hin82] N. Hindman. On density, translates, and pairwise sums of integers. *J. Combin. Theory Ser. A*, 33(2):147–157, 1982.

[HS98] N. Hindman and D. Strauss. *Algebra in the Stone-Čech compactification*, volume 27 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1998. Theory and applications.

[Jin02] R. Jin. The sumset phenomenon. *Proc. Amer. Math. Soc.*, 130(3):855–861, 2002.

[Jin04] R. Jin. Standardizing nonstandard methods for upper Banach density problems. In *Unusual applications of number theory*, volume 64 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 109–124. Amer. Math. Soc., Providence, RI, 2004.

[Jin08] R. Jin. *Private Communication*, 2008.

[JK03] R. Jin and H. Keisler. Abelian groups with layered tiles and the sumset phenomenon. *Trans. Amer. Math. Soc.*, 355(1):79–97, 2003.

[Lin01] E. Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.

[OW87] D. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.

[Pat88] A. L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[Pie84] J.-P. Pier. *Amenable locally compact groups*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1984. A Wiley-Interscience Publication.

[Rud62] W. Rudin. *Fourier analysis on groups*. Interscience Tracts in Pure and Applied Mathematics, No. 12. Interscience Publishers (a division of John Wiley and Sons), New York-London, 1962.

[Wag93] S. Wagon. *The Banach-Tarski paradox*. Cambridge University Press, Cambridge, 1993. With a foreword by Jan Mycielski, Corrected reprint of the 1985 original.

[Wei01] B. Weiss. Monotileable amenable groups. In *Topology, ergodic theory, real algebraic geometry*, volume 202 of *Amer. Math. Soc. Transl. Ser. 2*, pages 257–262. Amer. Math. Soc., Providence, RI, 2001.