APPLICATIONS OF HAMILTON'S INVERSE FUNCTION THEOREM TO MANIFOLDS OF MAPPINGS

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1. Let X and Y be compact smooth manifolds; let Q(X,Y) denote the space of all smooth surjective submersions from X to Y; let Diff(X) denote the C^∞_c-Lie group of diffeomorphisms. Then Q(X,Y) is open in $C^\infty(X,Y)$ (see Michor [2], 5.6), so all spaces are tame smooth spaces in the sense of Hamilton [1]. We have a canonical tame smooth right action $R: Q(X,Y) \times \mathrm{Diff}(X) \to Q(X,Y)$, $R(q,f) = q \circ f$.

<u>Theorem</u>: <u>Each</u> Diff(X)-<u>orbit in</u> Q(X,Y) <u>is open</u>.

<u>Proof</u>: For $q \in Q(X,Y)$ let $R_q: \mathrm{Diff}(X) \to Q(X,Y)$ be given by $R_q(f) = q \circ f$. By (Hamilton, III, Theorem 2.4.1) it suffices to show that for each $q \in Q(X,Y)$ the linear mapping $T_{Id}(R_q): T_{Id}(\mathrm{Diff}(X)) \to T_qQ(X,Y)$ is surjective with a tame linear right inverse. $T_{Id}\mathrm{Diff}(X) = \mathfrak{X}(X)$, the space of all vector fields on X, $T_qQ(X,Y)$ is the space of all vector fields along q $(s: X \to TY$ with $\pi_Y \circ s = Id)$ which is tamely isomorphic to the space of all sections of the pulled back vector bundle q^*TY over X. It is clear that $T_{Id}(R_q).\xi = Tq.\xi$ for $\xi \in \mathfrak{X}(X)$, so $T_{Id}(R_q)$ is just push forward of sections by a fibrewise surjective vector bundle map $\widetilde{Tq} : TX \to q^*TY$. Using partitions of unity one may construct a right inverse vector bundle map $\alpha : q^*TY \to TX$. $\alpha_*: \Gamma(q^*TY) \to \Gamma(TX)$ is then a tame linear right inverse to $T_{Id}(R_q)$. qed.

2. Let (E,p,X,F) and (E',p',X,F') be compact smooth fibre bundles over the base X with typical fibres F, F' respectively, and with general structure groups Diff(F), Diff(F'). Let $C^\infty\{E,E'\} = \bigcup_{x \in X} C^\infty(E_x,E'_x)$ be the fibre bundle over X with typical fibre $C^\infty(F,F')$ and transition functions $C^\infty(g^{-1}_{\alpha\beta}, g'_{\alpha\beta})$, where $(g_{\alpha\beta})$ and $(g'_{\alpha\beta})$ are transition functions for E and E' with respect to a common trivialisation. Likewise we consider the bundles Emb{E,E'} and Diff{E} over X with typical fibres Emb(F,F') and Diff(F) respectively. It is easy to see that for the spaces of smooth sections of these bundles we have $\Gamma(C^\infty\{E,E'\}) = \{f \in C^\infty(E,E'): p' \circ f = p\}$, $\Gamma(\mathrm{Emb}\{E,E'\}) = \{e \in \mathrm{Emb}(E,E'): p' \circ e = p\}$, $\Gamma(\mathrm{Diff}\{E\}) = \{g \in \mathrm{Diff}(E): p \circ g = p\}$. (By Emb(X,Y) we mean the space of all embeddings of X into Y.) So the spaces of sections are tame smooth submanifolds of $C^\infty(E,E')$, Emb(E,E') and Diff(E), respectively, and Γ(Diff{E}) is a tame smooth Fréchet Lie group. A first result in this setting is:

<u>Theorem</u>: <u>Any orbit of the left action</u> $L: \Gamma(\mathrm{Diff}\{E\}) \times \Gamma(E) \to \Gamma(E)$ <u>is open</u>.

Proof: For a section $s \in \Gamma(E)$ consider the mapping $L^s: \Gamma(\mathrm{Diff}\{E\}) \to \Gamma(E)$, $L^s(g) = g.s$. We have $T_g\Gamma(\mathrm{Diff}\{E\}) = T_g\{h \in \mathrm{Diff}(E): p \circ h = p\} = \{s \in \Gamma(g^*TE): Tp.s = 0_X\} = \Gamma(g^*VE)$, where $VE \to E$ is the vertical subbundle of $TE \to E$. Likewise we have $T_s\Gamma(E) = \Gamma(s^*VE)$ (see Michor [2],10.9 ff). $T_{Id}(L^s): T_{Id}\Gamma(\mathrm{Diff}\{E\}) = \Gamma(VE) \to T_s\Gamma(E) = \Gamma(s^*VE)$ is given by $T_{Id}(L^s).\sigma = \sigma \circ s = s^*(\sigma)$. Note that $s^*VE = VE|s(X)$, and any section of $VE|s(X)$ can easily be extended to the whole of E by a partition of unity operation. This gives a tame linear right inverse. By Hamilton's theorem (loc.cit.) L^s is locally open. qed.

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With the notation of section 2 consider the following diagram:

$$\begin{array}{ccc} \Gamma(\mathrm{Emb}\{E,E'\}) = \{f \in \mathrm{Emb}(E,E'): p' \circ f = p\} & \xrightarrow{\quad i \quad} & \mathrm{Emb}(E,E') \\ \downarrow \pi & & \downarrow u \\ \Gamma(\mathrm{Emb}\{E,E'\})/\Gamma(\mathrm{Diff}\{E\}) & \xrightarrow{\quad \tilde{i} \quad} & \mathrm{Emb}(E,E')/\mathrm{Diff}(E) = U(E,E') \end{array}$$

Let us explain the diagram first: $(\mathrm{Emb}(E,E'), u, \mathrm{Emb}(E,E')/\mathrm{Diff}(E) = U(E,E'))$ is a tame smooth principal bundle with structure group Diff(E) (see Michor, §13). π is the projection onto the orbit space with quotient topology. i is the embedding as a tame smooth splitting submanifold mentioned in section 2. It is clear that $u \circ i$ factors over π to a mapping $\tilde{i}$. Obviously the diagram above is a pushout in the category of sets and $\tilde{i}$ is injective.

<u>Theorem</u>: $\tilde{i}$ <u>is open</u>.

<u>Proof</u>: Consider $f \in \mathrm{Emb}(E,E')$ with $p' \circ f = p$. Let $g \in \mathrm{Emb}(E,E')$ be near f. Then $p' \circ g$ is near $p' \circ f$ in $C^\infty(E,X)$, $p' \circ f = p$ is in the open subset $Q(E,X)$, so $p' \circ g$ is a surjective submersion near $p' \circ f = p$ in $Q(E,X)$. By Theorem 1 there is a diffeomorphism $h \circ \mathrm{Diff}(E)$ near Id_E such that $p' \circ g \circ h = p' \circ f = p$, so $g \circ h \in \Gamma(\mathrm{Emb}\{E,E'\})$. Since u is open, we can conclude that i is open. qed.

<u>Corollary</u> (<u>G. Kainz</u>): $(\Gamma(\mathrm{Emb}\{E,E'\}), \pi, \Gamma(\mathrm{Emb}\{E,E'\})/\Gamma(\mathrm{Diff}\{E\}))$ <u>is a tame smooth principal bundle with structure group</u> $\Gamma(\mathrm{Diff}\{E\})$.

This has been proved by G. Kainz before directly along the lines of Binz-Fischer as used in Michor [2] §13.

3. Let X be a compact manifold, let $\mathfrak{M}(X)$ be the space of positive smooth measures (densities) on X with total mass one. $\mathfrak{M}(X)$ is an open convex set in a linear subspace of codimension one in a tame Fréchet space. The following result has been proved by Hamilton ([1], III, 2.5.3, p. 2o3):

<u>Theorem</u>: Diff(X) <u>acts transitively on</u> $\mathfrak{M}(X)$. <u>For each</u> $\mu \in \mathfrak{M}(X)$ <u>the subgroup</u> $\mathrm{Diff}_\mu(X)$ <u>of</u> μ-<u>preserving diffeomorphisms is a closed smooth tame Lie subgroup</u>, <u>and</u> Diff(X) <u>is a smooth tame principal bundle over</u> $\mathfrak{M}(X)$ <u>with fibre</u> $\mathrm{Diff}_\mu(X)$ <u>under the projection</u> $Pf = f_*\mu$. <u>So</u> $\mathrm{Diff}(X)/\mathrm{Diff}_\mu(X) = \mathfrak{M}(X)$.

4. With the notation of section 3 we have:

<u>Theorem: There is a smooth tame diffeomorphism</u> $\mathrm{Diff}(X) = \mathrm{Diff}_\mu(X) \times \mathfrak{M}(X)$ <u>for each</u> μ.

It seems that one can get a rather elementary direct proof of this theorem by adapting the proof of J. Moser [4]. He gives a global section of the bundle $\mathrm{Diff}(X) \to \mathfrak{M}(X)$, which probably can be arranged to be smooth and tame.
We need several steps for the proof.

5. $\mathfrak{M}(X)$ is open and convex in the affine space of all smooth measures with total mass one, which is modelled on a nuclear Fréchet space. So by Michor[2], 8.6, $\mathfrak{M}(X)$ admits smooth partitions of unity subordinated to any open cover. So we may construct a principal connection on the principal bundle $(\mathrm{Diff}(X), P, \mathfrak{M}(X), \mathrm{Diff}_\mu(X))$. Since the structure group acts smoothly on X we may consider the associated bundle over $\mathfrak{M}(X)$ with typical fibre X and we may induce the principal connection onto the associated bundle $E :=$ $\mathrm{Diff}(X) \times_{\mathrm{Diff}_\mu(X)} X$ (see Michor [3], §5).

6. Lemma: Any connection on the associated bundle $E \to \mathfrak{M}(X)$ admits a unique global parallel transport: $Pt^E(c,t): E_{c(0)} \to E_{c(t)}$ for each smooth curve c in $\mathfrak{M}(X)$ is given by the condition, that $(d/dt)(Pt(c,t)y)$ is horizontal. Pt is smooth in all appearing variables and we have $Pt(c,0) = Id$, $Pt(c,f(t)) = Pt(c \circ f,t) \circ Pt(c,f(0))$ for each smooth $f: R \to R$.

Sketch of proof: In a local trivialisation the differential equation for the parallel transport looks like the "flow"-equation for a time dependent vector field on the typical fibre X, the vector field being the value of the so called "Christoffel form" $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(X))$ along the curve c (see Michor [3]). Since X is compact, there is a global solution in each local chart, and these solutions fit together by uniqueness.

7. Let $Q: Diff(X) \times X \to E := Diff(X) \times_{Diff_\mu(X)} X$ be the quotient mapping, then for any f in Diff(X) the map $Q(f,.): X \to E_{f_*\mu}$ is a diffeomorphism - this can be seen as in the finite dimensional case. Now we consider the smooth positive measure $Q(f,.)_*\mu$ on $E_{f_*\mu}$. By the definition of the action of $Diff_\mu(X)$ on $Diff(X) \times X$ this measure does in fact depend only on $f_*\mu$ and not on the choice of f. So for each $\nu \in \mathfrak{M}(X)$ we have a unique measure M_ν on E_ν , positive and of mass one. In a trivialisation induced from the trivialisation of the principal bundle M_ν is just the const measure μ on each fibre X, so $\nu \to M_\nu$ is smooth, a kind of measure field on the bundle E.

Lemma: A connection on E is induced from a principal connection on $Diff(X) \to \mathfrak{M}(X)$ if and only if its parallel transport respects the smooth measure field M, that is $Pt(c,t)_* M_{c(0)} = M_{c(t)}$ for all c and t.

The proof is looking at local expressions and using the criterion for lifting of connections from associated bundles to principal bundles (Michor [3], 5.4, 5.5).

8. For $f \in Diff(X)$ consider the mapping $A(f_*\mu): X \to E_{f_*\mu}$, given by $A(f_*\mu) = Q(f,.) \circ f^{-1}$. It is easily seen that this mapping depends only on f_* , not on the choice of f. In fact, A defines a global section of the principal $Diff_\mu(X)$-bundle $Diff\{\mathfrak{M}(X) \times X, E\}$ over $\mathfrak{M}(X)$, in the notation of section 2. The latter bundle can be viewed as a sort of "frame bundle" for E.

Now consider a principal connection on $Diff(X) \to \mathfrak{M}(X)$, its induced connection on E and the global parallel transport Pt^E on E for the induced connection. For a smooth curve $c: R \to \mathfrak{M}(X)$ consider $Pt^P(c,t) := A(c(t))^{-1} \circ Pt^E(c,t) \circ A(c(0))$ in Diff(X). Then $Pt^P(c,t)_*$ defines a global parallel transport on the principal bundle $(Diff(X), P, \mathfrak{M}(X), Diff_\mu(X))$, which is $Diff_\mu(X)$ - equivariant and satisfies the transport equation for the given principal connection. This can be seen by looking at the local trivialisations. Furthermore Pt^P is uniquely given by these requirements, since a parallel transport can easily be pushed down onto the associated bundle E, and there it is unique.

9. Now it is easy to prove theorem 5: we use parallel transport along convex lines connecting each measure in $\mathfrak{M}(X)$ with μ to get a smooth tame global trivialisation of $Diff(X) \to \mathfrak{M}(X)$.

References:

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