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THE CONVENIENT SETTING FOR NON-QUASIANALYTIC DENJOY–CARLEMAN DIFFERENTIABLE MAPPINGS

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Abstract. For Denjoy–Carleman differentiable function classes C^M where the weight sequence $M = (M_k)$ is logarithmically convex, stable under derivations, and non-quasianalytic of moderate growth, we prove the following: A mapping is C^M if it maps C^M-curves to C^M-curves. The category of C^M-mappings is cartesian closed in the sense that $C^M(E, C^M(F, G)) \cong C^M(E \times F, G)$ for convenient vector spaces. Applications to manifolds of mappings are given: The group of C^M-diffeomorphisms is a C^M-Lie group but not better.

1. Introduction

Denjoy–Carleman differentiable functions form spaces of functions between real analytic and C^∞. They are described by growth conditions on the Taylor expansions, see (2.1). Under appropriate conditions the fundamental results of calculus still hold: Stability under differentiation, composition, solving ODEs, applying the implicit function theorem. See Section (2) for a review of Denjoy–Carleman differentiable functions, which is summarized in Table 1.

In [16], [17], [8], [21], [18], see [19] for a comprehensive presentation, convenient calculus was developed for C^∞, holomorphic, and real analytic functions: see appendix (7), (8), (9) for a short overview of the essential results.

In this paper we develop the convenient calculus for Denjoy–Carleman classes C^M where the weight sequence $M = (M_k)$ is logarithmically convex, stable under derivations, and non-quasianalytic of moderate growth (this holds for all Gevrey differentiable functions $G^{1+\delta}$ for $\delta > 0$). By 'convenient calculus' we mean that the following theorems are proved: A mapping is C^M if it maps C^M-curves to C^M-curves, see (3.9); this is wrong in the quasianalytic case, see 3.12. The category of C^M-mappings is cartesian closed in the sense that $C^M(E, C^M(F, G)) \cong C^M(E \times F, G)$ for convenient vector spaces, see (5.3); this is wrong for weight sequences of non-moderate growth, see (5.4). The uniform boundedness principle holds for linear mappings into spaces of C^M-mappings.

For the quasianalytic case we hope for results similar to the real analytic case, but the methods have to be different. This will be taken up in another paper.

In chapter (6) some applications to manifolds of mappings are given: The group of C^M-diffeomorphisms is a C^M-Lie group but not better.

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2. Review of Denjoy–Carleman differentiable functions

2.1. **Denjoy–Carleman classes $C^M(\mathbb{R}^n,\mathbb{R})$ of differentiable functions.** We mainly follow [27] (see also the references therein). We use $\mathbb{N}=\mathbb{N}_{>0}\cup\{0\}$. For each multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we write $\alpha!=\alpha_1!\cdots\alpha_n!$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and $\partial^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$.

Let $M=(M_k)_{k\in\mathbb{N}}$ be an increasing sequence ($M_{k+1}\ge M_k$) of positive real numbers with $M_0=1$. Let $U\subseteq\mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f\in C^\infty(U)$ such that, for all compact $K\subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)|\le C\,\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|} \tag{2.1.1}$$

for all $\alpha\in\mathbb{N}^n$ and $x\in K$. The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k=1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U)\subseteq C^M(U)\subseteq C^\infty(U)$.

We assume that $M=(M_k)$ is *logarithmically convex*, i.e.,

$$M_k^2\le M_{k-1}\,M_{k+1}\quad\text{ for all } k, \tag{2.1.2}$$

or, equivalently, M_{k+1}/M_k is increasing. Considering $M_0=1$, we obtain that also $(M_k)^{1/k}$ is increasing and

$$M_l\,M_k\le M_{l+k}\quad\text{ for all } l,k\in\mathbb{N}. \tag{2.1.3}$$

We also get (see (2.9))

$$M_1^k\,M_k\ge M_j\,M_{\alpha_1}\cdots M_{\alpha_j}\quad\text{ for all }\alpha_i\in\mathbb{N}_{>0},\alpha_1+\cdots+\alpha_j=k. \tag{2.1.4}$$

Let $M=(M_k)$ be logarithmically convex. Then $M'_k=M_k/M_0\,M_1^k\ge 1$ is increasing by (2.1.4), logarithmically convex, and $C^M(U)=C^{M'}(U)$ for all U open in $\mathbb{R}^n$ by (2.1.5). So without loss we assumed at the beginning that M is increasing.

Hypothesis (2.1.2) implies that $C^M(U)$ is a ring, for all open subsets $U\subseteq\mathbb{R}^n$, which can easily be derived from (2.1.3) by means of Leibniz's rule. Note that definition (2.1.1) makes sense also for mappings $U\to\mathbb{R}^p$. For C^M-mappings, (2.1.2) guarantees stability under composition ([23], see also [1, 4.7]; a proof is also contained in the end of the proof of (3.9)).

A further consequence of (2.1.2) is the inverse function theorem for C^M ([14]; for a proof see also [1, 4.10]): Let $f:U\to V$ be a C^M-mapping between open subsets $U,V\subseteq\mathbb{R}^n$. Let $x_0\in U$. Suppose that the Jacobian matrix $(\partial f/\partial x)(x_0)$ is invertible. Then there are neighborhoods U' of x_0, V' of $y_0:=f(x_0)$ such that $f:U'\to V'$ is a C^M-diffeomorphism.

Moreover, (2.1.2) implies that C^M is closed under solving ODEs (due to [15]): Consider the initial value problem

$$\frac{dx}{dt}=f(t,x),\quad x(0)=y,$$

where $f:(-T,T)\times\Omega\to\mathbb{R}^n$, $T>0$, and $\Omega\subseteq\mathbb{R}^n$ is open. Assume that $f(t,x)$ is Lipschitz in x, locally uniformly in t. Then for each relative compact open subset $\Omega_1\subseteq\Omega$ there exists $0<T_1\le T$ such that for each $y\in\Omega_1$ there is a unique solution $x=x(t,y)$ on the interval $(-T_1,T_1)$. If $f:(-T,T)\times\Omega\to\mathbb{R}^n$ is a C^M-mapping then the solution $x:(-T_1,T_1)\times\Omega_1\to\mathbb{R}^n$ is a C^M-mapping as well.

Suppose that $M=(M_k)$ and $N=(N_k)$ satisfy $M_k\le C^k\,N_k$, for all k and a constant C, or equivalently,

$$\sup_{k\in\mathbb{N}_{>0}}\Big(\frac{M_k}{N_k}\Big)^{\frac1k}<\infty. \tag{2.1.5}$$

Then, evidently $C^M(U) \subseteq C^N(U)$. The converse is true as well (if (2.1.2) is assumed): One can prove that there exists $f \in C^M(\mathbb{R})$ such that $|f^{(k)}(0)| \ge k!\, M_k$ for all k (see [27, Theorem 1]). So the inclusion $C^M(U) \subseteq C^N(U)$ implies (2.1.5).

Setting $N_k = 1$ in (2.1.5) yields that $C^\omega(U) = C^M(U)$ if and only if

$$\sup_{k \in \mathbb{N}_{>0}} (M_k)^{\frac{1}{k}} < \infty.$$

Since $(M_k)^{1/k}$ is increasing (by logarithmic convexity), the strict inclusion $C^\omega(U) \subsetneq C^M(U)$ is equivalent to

$$\lim_{k \to \infty} (M_k)^{\frac{1}{k}} = \infty.$$

We shall also assume that C^M is stable under derivation, which is equivalent to the following condition

$$\sup_{k \in \mathbb{N}_{>0}} \Big(\frac{M_{k+1}}{M_k}\Big)^{\frac{1}{k}} < \infty. \tag{2.1.6}$$

Note that the first order partial derivatives of elements in $C^M(U)$ belong to $C^{M^{+1}}(U)$, where M^{+1} denotes the shifted sequence $M^{+1} = (M_{k+1})_{k \in \mathbb{N}}$. So the equivalence follows from (2.1.5), by replacing M with M^{+1} and N with M.

Definition. By a *DC-weight sequence* we mean a sequence $M = (M_k)_{k \in \mathbb{N}}$ of positive numbers with $M_0 = 1$ which is monotone increasing ($M_{k+1} \ge M_k$), logarithmically convex (2.1.2), and satisfies (2.1.6). Then $C^M(U, \mathbb{R})$ is a differential ring, and the class of C^M-functions is stable under compositions. DC stands for Denjoy-Carleman and also for derivation closed.

2.2. **Quasianalytic function classes.** Let $\mathcal{F}_n$ denote the ring of formal power series in n variables (with real or complex coefficients). For a sequence $M_0 = 1, M_1, M_2, \dots > 0$, we denote by $\mathcal{F}_n^M$ the set of elements $F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha\, x^\alpha$ of $\mathcal{F}_n$ for which there exist positive constants C and ρ such that

$$|F_\alpha| \le C\, \rho^{|\alpha|}\, M_{|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$. A class C^M is called *quasianalytic* if, for open connected $U \subseteq \mathbb{R}^n$ and all $a \in U$, the Taylor series homomorphism

$$T_a : C^M(U) \to \mathcal{F}_n^M,\ f \mapsto T_a f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!}\, \partial^\alpha f(a)\, x^\alpha$$

is injective. By the Denjoy–Carleman theorem ([5], [4]), *the following statements are equivalent:*

1. C^M *is quasianalytic.*
2. $\sum_{k=1}^\infty \frac{1}{m_k} = \infty$ *where* $m_k = \inf\{(j!\, M_j)^{1/j} : j \ge k\}$ *is the increasing minorant of* $(k!\, M_k)^{1/k}$.
3. $\sum_{k=1}^\infty (\frac{1}{M_k^*})^{1/k} = \infty$ *where* $M_k^* = \inf\{(j!\, M_j)^{(l-k)/(l-j)} (l!\, M_l)^{(k-j)/(l-j)} : j \le k \le l, j < l\}$ *is the logarithmically convex minorant of* $k!\, M_k$.
4. $\sum_{k=0}^\infty \frac{M_k^*}{M_{k+1}^*} = \infty$.

For contemporary proofs see for instance [10, 1.3.8] or [24, 19.11].

Suppose that $C^\omega(U) \subsetneq C^M(U)$ and $C^M(U)$ is quasianalytic and logarithmically convex. Then $T_a : C^M(U) \to \mathcal{F}_n^M$ is not surjective. This is due to Carleman [4]; an elementary proof can be found in [27, Theorem 3].

2.3. **Non-quasianalytic function classes.** If M is a DC-weight sequence which is not quasianalytic, then there are C^M partitions of unity. Namely, there exists a C^M function f on $\mathbb{R}$ which does not vanish in any neighborhood of 0 but which has vanishing Taylor series at 0. Let $g(t) = 0$ for $t \le 0$ and $g(t) = f(t)$ for $t > 0$. From g we can construct C^M bump functions as usual.

2.4. **Strong non-quasianalytic function classes.** Let M be a DC-weight sequence with $C^\omega(U,\mathbb{R}) \subsetneq C^M(U,\mathbb{R})$. Then the mapping $T_a : C^M(U,\mathbb{R}) \to \mathcal{F}_n^M$ is surjective, for all $a \in U$, if and only if there is a constant C such that

$$\sum_{k=j}^{\infty} \frac{M_k}{(k+1)\,M_{k+1}} \le C\frac{M_j}{M_{j+1}} \quad \text{for any integer } j \ge 0. \tag{2.4.1}$$

See [22] and references therein. (2.4.1) is called *strong non-quasianalyticity* condition.

2.5. **Moderate growth.** A DC-weight sequence M has *moderate growth* if

$$\sup_{j,k\in\mathbb{N}_{>0}} \Big(\frac{M_{j+k}}{M_j\, M_k}\Big)^{\frac{1}{j+k}} < \infty. \tag{2.5.1}$$

Moderate growth implies derivation closed.

Moderate growth together with strong non-quasianalyticity (2.4.1) is called *strong regularity*: Then a version of Whitney's extension theorem holds for the corresponding function classes (e.g. [3]).

2.6. **Gevrey functions.** Let $\delta > 0$ and put $M_k = (k!)^\delta$, for $k \in \mathbb{N}$. Then $M = (M_k)$ is strongly regular. The corresponding class C^M of functions is the *Gevrey class* $G^{1+\delta}$.

2.7. **More examples.** Let $\delta > 0$ and put $M_k = (\log(k+e))^{\delta\, k}$, for $k \in \mathbb{N}$. Then $M = (M_k)$ is quasianalytic for $0 < \delta \le 1$ and non-quasianalytic (but not strongly) for $\delta > 1$. In any case M is of moderate growth.

Let $q > 1$ and put $M_k = q^{k^2}$, for $k \in \mathbb{N}$. The corresponding C^M-functions are called *q-Gevrey regular*. Then $M = (M_k)$ is strongly non-quasianalytic but not of moderate growth, thus not strongly regular. It is derivation closed.

2.8. **Spaces of C^M-functions.** Let $U \subseteq \mathbb{R}^n$ be open and let M be a DC-weight sequence. For any $\rho > 0$ and $K \subseteq U$ compact with smooth boundary, define

$$C^M_\rho(K) := \{f \in C^\infty(K) : \|f\|_{\rho,K} < \infty\}$$

with

$$\|f\|_{\rho,K} := \sup\Big\{\frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha \in \mathbb{N}^n, x \in K\Big\}.$$

It is easy to see that $C^M_\rho(K)$ is a Banach space. In the description of $C^M_\rho(K)$, instead of compact K with smooth boundary, we may also use open $K \subset U$ with $\overline{K}$ compact in U, like [27]. Or we may work with Whitney jets on compact K, like [13].

The space $C^M(U)$ carries the projective limit topology over compact $K \subseteq U$ of the inductive limit over $\rho \in \mathbb{N}_{>0}$:

$$C^M(U) = \varprojlim_{K\subseteq U} \big(\varinjlim_{\rho\in\mathbb{N}_{>0}} C^M_\rho(K)\big).$$

One can prove that, for $\rho < \rho'$, the canonical injection $C^M_\rho(K) \to C^M_{\rho'}(K)$ is a compact mapping; it is even nuclear (see [13], [12, p. 166]). Hence $\varinjlim_\rho C^M_\rho(K)$ is a Silva space, i.e., an inductive limit of Banach spaces such that the canonical mappings are compact; therefore it is complete, webbed, and ultrabornological, see

[7], [11, 5.3.3], also [19, 52.37]. We shall use this locally convex topology below only for $n = 1$ – in general it is stronger than the one which we will define in (3.1), but it has the same system of bounded sets, see (4.6).

2.9. **Lemma.** *For a logarithmically convex sequence M_k with $M_0 = 1$ we have*

$$M_1^k\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j} \quad \text{for all } \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k.$$

Proof. We use induction on k. The assertion is trivial for $k = j$. Assume that $j < k$. Then there exists i such that $\alpha_i \ge 2$. Put $\alpha_i' := \alpha_i - 1$. By induction hypothesis,

$$M_j\, M_{\alpha_1} \cdots M_{\alpha_i'} \cdots M_{\alpha_j} \le M_1^{k-1} M_{k-1}.$$

Since M_{k+1}/M_k is increasing by (2.1.2), we obtain

$$M_j\, M_{\alpha_1} \cdots M_{\alpha_j} = M_j\, M_{\alpha_1} \cdots M_{\alpha_i'} \cdots M_{\alpha_j} \cdot \frac{M_{\alpha_i}}{M_{\alpha_i'}} \le$$

$$\le M_1^{k-1}\, M_{k-1} \cdot \frac{M_k}{M_{k-1}} \le M_1^k\, M_k. \quad \square$$

Table 1: Let $M = (M_k)$ and $N = (N_k)$ be increasing ($\le$) sequences of real numbers with $M_0 = N_0 = 1$. By U we denote an open subset of $\mathbb{R}^n$. The mapping $T_a : C^M(U) \to \mathcal{F}_n^M$ is the Taylor series homomorphism for $a \in U$ (see (2.2)). Recall that M is a DC-weight sequence if it is logarithmically convex and stable under derivation.

Properties of M		**Properties of C^M**
M increasing, $M_0 = 1$, (always assumed below this line)	$\Rightarrow$	$C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$
M is logarithmically convex (always assumed below this line), i.e., $M_k^2 \le M_{k-1}\, M_{k+1}$ for all k. Then: $(M_k)^{1/k}$ is increasing, $M_l\, M_k \le M_{l+k}$ for all l, k, and $M_1^k\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j}$ for $\alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k$.	$\Rightarrow$	$C^M(U)$ is a ring. C^M is closed under composition. C^M is closed under applying the inverse function theorem. C^M is closed under solving ODEs.
$\sup_{k\in\mathbb{N}_{>0}} (M_k/N_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^M(U) \subseteq C^N(U)$
$\sup_{k\in\mathbb{N}_{>0}} (M_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^\omega(U) = C^M(U)$
$\lim_{k\to\infty} (M_k)^{1/k} = \infty$	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^M(U)$
$\sup_{k\in\mathbb{N}_{>0}} (M_{k+1}/M_k)^{1/k} < \infty$ (always assumed below this line)	$\Leftrightarrow$	C^M is closed under derivation.
$\sum_{k=0}^\infty \frac{M_k}{(k+1)M_{k+1}} = \infty$ or, equivalently, $\sum_{k=1}^\infty (\frac{1}{k!\, M_k})^{1/k} = \infty$	$\Leftrightarrow$	C^M is quasianalytic, i.e., $T_a : C^M(U) \to \mathcal{F}_n^M$ is injective (not surjective if $C^\omega(U) \subsetneq C^M(U)$).
$\sum_{k=0}^\infty \frac{M_k}{(k+1)M_{k+1}} < \infty$	$\Leftrightarrow$	C^M is non-quasianalytic. Then C^M partitions of unity exist.

$\lim_{k\to\infty}(M_k)^{1/k}=\infty$ and $\sum_{k=j}^\infty \frac{M_k}{(k+1)M_{k+1}} \le C\frac{M_j}{M_{j+1}}$ for all $j\in\mathbb{N}$ and some C	$\Leftrightarrow$	$C^\omega(U)\subsetneq C^M(U)$ and $T_a: C^M(U)\to \mathcal{F}_n^M$ is surjective, i.e., C^M is strongly non-quasianalytic.
M has moderate growth, i.e., $\sup_{j,k\in\mathbb{N}_{>0}}(\frac{M_{j+k}}{M_j\,M_k})^{1/(j+k)}<\infty$	$\Rightarrow$	C^M is cartesian closed will be proved in (5.3)
M is strongly regular, i.e., it is strongly non-quasianalytic and has moderate growth.	$\Rightarrow$	Whitney's extension theorem holds in C^M.
$\delta>0$ and $M_k=(k!)^\delta$ for $k\in\mathbb{N}$. Then M is strongly regular.	$\Leftrightarrow$	C^M is the Gevrey class $G^{1+\delta}$.

3. C^M-mappings

3.1. **Definition: C^M-mappings.** Let M be a DC-weight sequence, and let E be a locally convex vector space. A curve $c:\mathbb{R}\to E$ is called C^M if for each continuous linear functional $\ell\in E^*$ the curve $\ell\circ c:\mathbb{R}\to\mathbb{R}$ is of class C^M. The curve c is called *strongly* C^M if c is smooth and for all compact $K\subset\mathbb{R}$ there exists $\rho>0$ such that

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k}:k\in\mathbb{N},x\in K\right\}\text{ is bounded in }E.$$

The curve c is called *strongly uniformly* C^M if c is smooth and there exists $\rho>0$ such that

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k}:k\in\mathbb{N},x\in \mathbb{R}\right\}\text{ is bounded in }E.$$

Now let M be a non-quasianalytic DC-weight sequence. Let U be a c^∞-open subset of E, and let F be another locally convex vector space. *A mapping $f:U\to F$ is called C^M if f is smooth in the sense of* (7.3) *and if $f\circ c$ is a C^M-curve in F for every C^M-curve c in U.* Obviously, *the composite of C^M-mappings is again a C^M-mapping, and the chain rule holds.* This notion is equivalent to the expected one on Banach spaces, see 3.9 below.

We equip the space $C^M(U,F)$ with the initial locally convex structure with respect to the family of mappings

$$C^M(U,F)\xrightarrow{C^M(c,\ell)}C^M(\mathbb{R},\mathbb{R}),\quad f\mapsto \ell\circ f\circ c,\quad \ell\in E^*,c\in C^M(\mathbb{R},U)$$

where $C^M(\mathbb{R},\mathbb{R})$ carries the locally convex structure described in (2.8) and where E^* is the space of all continuous linear functionals on E.

For $U\subseteq\mathbb{R}^n$, this locally convex topology differs from the one described in (2.8), but they have the same bounded sets, see (4.6) below.

If F is convenient, then by standard arguments, the space $C^M(U,F)$ is c^∞-closed in the product $\prod_{\ell,c}C^M(\mathbb{R},\mathbb{R})$ and hence is *convenient.* If F is convenient, then a mapping $f:U\to F$ is C^M if and only if $\ell\circ f$ is C^M for all $\ell\in F^*$.

3.2. **Example: There are weak C^M-curves which are not strong.** By [27, Theorem 1], for each DC-weight sequence M there exists $f\in C^M(\mathbb{R},\mathbb{R})$ such that $|f^{(k)}(0)|\ge k!\,M_k$ for all $k\in\mathbb{N}$. Then $g:\mathbb{R}\to\mathbb{R}^\mathbb{N}$ given by $g(t)_n=f(nt)$ is C^M but not strongly C^M. Namely, each bounded linear functional ℓ on $\mathbb{R}^\mathbb{N}$ depends only on finitely many coordinates, so we take the maximal ρ for the finitely many coordinates of g being involved. On the other hand, for each ρ and any compact neighborhood L of 0 the set

$$\left\{\frac{g^{(k)}(t)}{\rho^k\,k!\,M_k} : t \in L, k \in \mathbb{N}\right\}$$

has n-th coordinate unbounded if $n > \rho$.

3.3. **Lemma.** *Let E be a convenient vector space such that there exists a Baire vector space topology on the dual E^* for which the point evaluations ev_x are continuous for all $x \in E$. Then a curve $c : \mathbb{R} \to E$ is C^M if and only if c is strongly C^M, for any DC-weight sequence M.*

See (5.2) for a more general version.

Proof. Let K be compact in $\mathbb{R}$. We consider the sets

$$A_{\rho,C} := \left\{\ell \in E^* : \frac{|(\ell \circ c)^{(k)}(x)|}{\rho^k\,k!\,M_k} \le C \text{ for all } k \in \mathbb{N}, x \in K\right\}$$

which are closed subsets in E^* for the Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = E^*$. By the Baire property there exists ρ and C such that the interior U of $A_{\rho,C}$ is non-empty. If $\ell_0 \in U$ then for all $\ell \in E^*$ there is an $\epsilon > 0$ such that $\epsilon\ell \in U - \ell_0$ and hence for all $x \in K$ and all k we have

$$|(\ell \circ c)^{(k)}(x)| \le \tfrac{1}{\epsilon}\Big(|((\epsilon\ell + \ell_0) \circ c)^{(k)}(x)| + |(\ell_0 \circ c)^{(k)}(x)|\Big) \le \tfrac{2C}{\epsilon}\,\rho^k\,k!\,M_k.$$

So the set

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k} : k \in \mathbb{N}, x \in K\right\}$$

is weakly bounded in E and hence bounded. □

3.4. **Lemma.** *Let M be a DC-weight sequence, and let E be a Banach space. For a curve $c : \mathbb{R} \to E$ the following are equivalent.*

(1) *c is C^M.*
(2) *For each sequence (r_k) with $r_k\,t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\,c^{(k)}(a)\,r_k : a \in K, k \in \mathbb{N}\}$ is bounded in E.*
(3) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\,t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, there exists an $\epsilon > 0$ such that $\{\frac{1}{k!M_k}\,c^{(k)}(a)\,r_k\,\epsilon^k : a \in K, k \in \mathbb{N}\}$ is bounded in E.*

Proof. (1) $\implies$ (2) For K, there exists $\rho > 0$ such that

$$\left\|\frac{c^{(k)}(a)}{k!\,M_k} r_k\right\|_E = \left\|\frac{c^{(k)}(a)}{k!\,\rho^k\,M_k}\right\|_E \cdot |r_k \rho^k|$$

is bounded uniformly in $k \in \mathbb{N}$ and $a \in K$ by (3.3).

(2) $\implies$ (3) Use $\epsilon = 1$.

(3) $\implies$ (1) Let $a_k := \sup_{a\in K} \|\frac{1}{k!\,M_k}\,c^{(k)}(a)\|_E$. Using [19, 9.2.(4⇒1)] these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. □

3.5. **Lemma.** *Let M be a DC-weight sequence. Let E be a convenient vector space, and let $\mathcal{S}$ be a family of bounded linear functionals on E which together detect bounded sets (i.e., $B \subseteq E$ is bounded if and only if $\ell(B)$ is bounded for all $\ell \in \mathcal{S}$). Then a curve $c : \mathbb{R} \to E$ is C^M if and only if $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is C^M for all $\ell \in \mathcal{S}$.*

Proof. For smooth curves this follows from [19, 2.1 and 2.11]. By (3.4), for any $\ell \in E'$, the function $\ell \circ c$ is C^M if and only if:

(1) For each sequence (r_k) with $r_k\,t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set $\{\frac{1}{k!M_k}\,(\ell \circ c)^{(k)}(a)\,r_k : a \in K, k \in \mathbb{N}\}$ is bounded.

By (1) the curve c is C^M if and only if the set $\{\frac{1}{k!M_k}c^{(k)}(a)\,r_k : a\in K, k\in\mathbb{N}\}$ is bounded in E. By (1) again this is in turn equivalent to $\ell\circ c\in C^M$ for all $\ell\in\mathcal{S}$, since $\mathcal{S}$ detects bounded sets. □

3.6. **C^M curve lemma.** A sequence x_n in a locally convex space E is said to be *Mackey convergent* to x, if there exists some $\lambda_n\nearrow\infty$ such that $\lambda_n(x_n-x)$ is bounded. If we fix $\lambda=(\lambda_n)$ we say that x_n is λ-converging.

Lemma. *Let M be a non-quasianalytic DC-weight sequence. Then there exist sequences $\lambda_k\to 0$, $t_k\to t_\infty$, $s_k>0$ in $\mathbb{R}$ with the following property: For $1/\lambda=(1/\lambda_n)$-converging sequences x_n and v_n in a convenient vector space E there exists a strongly uniformly C^M-curve $c:\mathbb{R}\to E$ with $c(t_k+t)=x_k+t.v_k$ for $|t|\le s_k$.*

Proof. Since C^M is not quasianalytic we have $\sum_k 1/(k!M_k)^{1/k}<\infty$. We choose another non-quasianalytic DC-weight sequence $\bar M=(\bar M_k)$ with $(M_k/\bar M_k)^{1/k}\to\infty$. By (2.3) there is a $C^{\bar M}$-function $\varphi:\mathbb{R}\to[0,1]$ which is 0 on $\{t:|t|\ge\frac12\}$ and which is 1 on $\{t:|t|\le\frac13\}$, i.e. there exist $\bar C,\rho>0$ such that

$$|\varphi^{(k)}(t)|\le\bar C\,\rho^k\,k!\,\bar M_k\qquad\text{for all } t\in\mathbb{R}\text{ and } k\in\mathbb{N}.$$

For x,v in a absolutely convex bounded set $B\subseteq E$ and $0<T\le 1$ the curve $c:t\mapsto\varphi(t/T)\cdot(x+t\,v)$ satisfies (cf. [2, Lemma 2]):

$$\begin{aligned}
c^{(k)}(t)&=T^{-k}\varphi^{(k)}(\tfrac{t}{T}).(x+t.v)+k\,T^{1-k}\,\varphi^{(k-1)}(\tfrac{t}{T}).v\\
&\in T^{-k}\bar C\,\rho^k\,k!\,\bar M_k(1+\tfrac{T}{2}).B+k\,T^{1-k}\,\bar C\,\rho^{k-1}\,(k-1)!\,\bar M_{k-1}.B\\
&\subseteq T^{-k}\bar C\,\rho^k\,k!\,\bar M_k(1+\tfrac{T}{2}).B+T\,T^{-k}\,\bar C\tfrac{1}{\rho}\,\rho^k\,k!\,\bar M_k.B\\
&\subseteq\bar C(\tfrac32+\tfrac1\rho)\,T^{-k}\,\rho^k\,k!\,\bar M_k.B
\end{aligned}$$

So there are $\rho, C:=\bar C(\frac32+\frac1\rho)>0$ which do not depend on x,v and T such that $c^{(k)}(t)\in C\,T^{-k}\,\rho^k\,k!\,\bar M_k.B$ for all k and t.

Let $0<T_j\le 1$ with $\sum_j T_j<\infty$ and $t_k:=2\sum_{j<k}T_j+T_k$. We choose the λ_j such that $0<\lambda_j/T_j^k\le M_k/\bar M_k$ (note that $T_j^k\,M_k/\bar M_k\to\infty$ for $k\to\infty$) for all j and k, and that $\lambda_j/T_j^k\to 0$ for $j\to\infty$ and each k.

Without loss we may assume that $x_n\to 0$. By assumption there exists a closed bounded absolutely convex subset B in E such that $x_n,v_n\in\lambda_n\cdot B$. We consider $c_j:t\mapsto\varphi\big((t-t_j)/T_j\big)\cdot\big(x_j+(t-t_j)\,v_j\big)$ and $c:=\sum_j c_j$. The c_j have disjoint support $\subseteq[t_j-T_j,t_j+T_j]$, hence c is C^∞ on $\mathbb{R}\setminus\{t_\infty\}$ with

$$c^{(k)}(t)\in C\,T_j^{-k}\,\rho^k k!\bar M_k\,\lambda_j\cdot B\qquad\text{for } |t-t_j|\le T_j.$$

Then

$$\|c^{(k)}(t)\|_B\le C\,\rho^k\,k!\bar M_k\,\frac{\lambda_j}{T_j^k}\le C\rho^k k!\bar M_k\,\frac{M_k}{\bar M_k}=C\,\rho^k\,k!M_k$$

for $t\ne t_\infty$. Hence $c:\mathbb{R}\to E_B$ (see [19, 2.14.6] or (7.1)) is smooth at t_∞ as well, and is strongly C^M by the following lemma. □

3.7. **Lemma.** *Let $c:\mathbb{R}\setminus\{0\}\to E$ be strongly C^M in the sense that c is smooth and for all bounded $K\subset\mathbb{R}\setminus\{0\}$ there exists $\rho>0$ such that*

$$\left\{\frac{c^{(k)}(x)}{\rho^k\,k!\,M_k}:k\in\mathbb{N},x\in K\right\}\ \textit{is bounded in } E.$$

Then c has a unique extension to a strongly C^M-curve on $\mathbb{R}$.

Proof. The curve c has a unique extension to a smooth curve by [19, 2.9]. The strong C^M condition extends by continuity. □

3.8. **Corollary.** *Let M be a non-quasianalytic DC-weight sequence. Then we have:*

(1) *The final topology on E with respect to all strongly C^M-curves equals the Mackey closure topology.*
(2) *A locally convex space E is convenient (7.2) if and only if for any (strongly) C^M-curve $c : \mathbb{R} \to E$ there exists a (strongly) C^M-curve $c_1 : \mathbb{R} \to E$ with $c_1' = c$.*

Proof. (1) For any Mackey converging sequence there exists a C^M-curve passing through a subsequence in finite time by (3.6). So the final topologies generated by the Mackey converging sequences and by the C^M-curves coincide.

(2) In order to show that a locally convex space E is convenient, we have to prove that it is c^∞-closed in its completion. So let $x_n \in E$ converge Mackey to x_∞ in the completion. Then by (3.6) there exists a strongly C^M-curve c in the completion passing in finite time through a subsequence of the x_n with velocity $v_n = 0$. The form of c (in the proof of (3.6)) shows that its derivatives $c^{(k)}(t)$ for $k > 0$ are multiples of the x_n and hence have values in E. Then c' is a C^M-curve and so the antiderivative c of c' lies in E by assumption. In particular $x_\infty \in c(\mathbb{R}) \subseteq E$.

Conversely, if E is convenient, then every smooth curve c has a smooth antiderivative c_1 in E by [19, 2.14]. Since

$$\frac{1}{\rho^{k+1}(k+1)!\, M_{k+1}} c_1^{(k+1)}(t) = \frac{M_k}{\rho(k+1)M_{k+1}} \frac{1}{\rho^k\, k!\, M_k} c^{(k)}(t)$$

and since

$$\frac{M_k}{\rho(k+1)M_{k+1}} \le \frac{1}{\rho M_1}$$

by (2.1.2) the antiderivative c_1 is (strongly) C^M if c is so. □

3.9. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let $U \subseteq E$ be c^∞-open in a convenient vector space, and let F be a Banach space. For a mapping $f : U \to F$, the following assertions are equivalent.*

(1) *f is C^M.*
(2) *f is C^M along strongly C^M curves.*
(3) *f is smooth, and for each closed bounded absolutely convex B in E and each $x \in U \cap E_B$ there are $r > 0$, $\rho > 0$, and $C > 0$ such that*
$$\frac{1}{k!\, M_k} \|d^k(f \circ i_B)(a)\|_{L^k(E_B,F)} \le C \rho^k$$
for all $a \in U \cap E_B$ with $\|a - x\|_B \le r$ and all $k \in \mathbb{N}$.
(4) *f is smooth, and for each closed bounded absolutely convex B in E and each compact $K \subseteq U \cap E_B$ there are $\rho > 0$ and $C > 0$ such that*
$$\frac{1}{k!\, M_k} \|d^k(f \circ i_B)(a)\|_{L^k(E_B,F)} \le C \rho^k$$
for all $a \in K$ and all $k \in \mathbb{N}$.

Proof. (1) $\implies$ (2) is clear.

(2) $\implies$ (3) Without loss let $E = E_B$ be a Banach space. For each $v \in E$ and $x \in U$ the iterated directional derivative $d_v^k f(x)$ exists since f is C^M along affine lines. To show that f is smooth it suffices to check that $d_{v_n}^k f(x_n)$ is bounded for each $k \in \mathbb{N}$ and each Mackey convergent sequences x_n and $v_n \to 0$, by [19, 5.20]. For contradiction let us assume that there exist k and sequences x_n and v_n with $\|d_{v_n}^k f(x_n)\| \to \infty$. By passing to a subsequence we may assume that x_n and v_n are $(1/\lambda_n)$-converging for the λ_n from (3.6). Hence there exists a strongly C^M-curve c in E and with $c(t + t_n) = x_n + t.v_n$ for t near 0 for each n separately, and for t_n

from (3.6). But then $\|(f\circ c)^{(k)}(t_n)\| = \|d^k_{v_n} f(x_n)\| \to \infty$, a contradiction. So f is smooth.

Assume for contradiction that the boundedness condition in (3) does not hold. Then there exists $x \in U$ such that for all $r,\rho,C>0$ there is an $a = a(r,\rho,C) \in U$ and $k = k(r,\rho,C) \in \mathbb{N}$ with $\|a-x\| \le r$ but

$$\frac{1}{k!\, M_k}\|d^k f(a)\|_{L^k(E,F)} > C\,\rho^k.$$

By [19, 7.13] we have

$$\|d^k f(a)\|_{L^k(E,F)} \le (2e)^k \sup_{\|v\|\le 1} \|d^k_v f(a)\|.$$

So for each ρ and n take $r = \frac{1}{n\rho}$ and $C = n$. Then there are $a_{n,\rho} \in U$ with $\|a_{n,\rho} - x\| \le \frac{1}{n\rho}$, moreover $v_{n,\rho}$ with $\|v_{n,\rho}\| = 1$, and $k_{n,\rho} \in \mathbb{N}$ such that

$$\frac{(2e)^{k_{n,\rho}}}{k_{n,\rho}!\, M_{k_{n,\rho}}\,\rho^{k_{n,\rho}}}\|d^{k_{n,\rho}}_{v_{n,\rho}} f(a_{n,\rho})\| > n.$$

Since $K := \{a_{n,\rho} : n,\rho \in \mathbb{N}\} \cup \{x\}$ is compact, this contradicts the following

Claim. *For each compact $K \subseteq E$ there are $C,\rho \ge 0$ such that for all $k \in \mathbb{N}$ and $x \in K$ we have $\sup_{\|v\|\le1}\|d^k_v f(x)\| \le C\,\rho^k k! M_k$.*

Otherwise, there exists a compact set $K \subseteq E$ such that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$, $x_n \in K$, and v_n with $\|v_n\| = 1$ such that

$$\|d^{k_n}_{v_n} f(x_n)\| > k_n!\, M_{k_n}\,\Big(\frac{1}{\lambda_n^2}\Big)^{k_n+1},$$

where we used $C = \rho := 1/\lambda_n^2$ with the λ_n from (3.6). By passing to a subsequence (again denoted n) we may assume that the x_n are $1/\lambda$-converging, thus there exists a strongly C^M-curve $c : \mathbb{R} \to E$ with $c(t_n + t) = x_n + t.\lambda_n.v_n$ for t near 0 by (3.6). Since

$$(f\circ c)^{(k)}(t_n) = \lambda_n^k d^k_{v_n} f(x_n),$$

we get

$$\Big(\frac{\|(f\circ c)^{(k_n)}(t_n)\|}{k_n! M_{k_n}}\Big)^{\frac{1}{k_n+1}} = \Big(\lambda_n^{k_n}\frac{\|d^{k_n}_{v_n} f(x_n)\|}{k_n! M_{k_n}}\Big)^{\frac{1}{k_n+1}} > \frac{1}{\lambda_n^{\frac{k_n+2}{k_n+1}}} \to \infty,$$

a contradiction to $f\circ c \in C^M$.

(3) $\implies$ (4) is obvious since the compact set K is covered by finitely many balls.

(4) $\implies$ (1) We have to show that $f\circ c$ is C^M for each C^M-curve $c : \mathbb{R} \to E$. By (3.4.2) it suffices to show that for each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$, there exists an $\epsilon > 0$ such that $\{\frac{1}{k! M_k}(f\circ c)^{(k)}(a)\, r_k\, \epsilon^k : a \in I, k \in \mathbb{N}\}$ is bounded.

By (3.4.2) applied to $r_k 2^k$ instead of r_k, for each $\ell \in E^*$, each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact interval I in $\mathbb{R}$ the set $\{\frac{1}{k! M_k}(\ell\circ c)^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. Thus $\{\frac{1}{k! M_k} c^{(k)}(a)\, r_k\, 2^k : a \in I, k \in \mathbb{N}\}$ is contained in some closed absolutely convex $B \subseteq E$. Consequently, $c^{(k)} : I \to E_B$ is smooth and hence $K_k := \{\frac{1}{k! M_k} c^{(k)}(a)\, r_k\, 2^k : a \in I\}$ is compact in E_B for each k. Then each sequence (x_n) in the set

$$K := \Big\{\frac{1}{k! M_k} c^{(k)}(a)\, r_k : a \in I, k \in \mathbb{N}\Big\} = \bigcup_{k\in\mathbb{N}} \frac{1}{2^k} K_k$$

has a cluster point in $K \cup \{0\}$: either there is a subsequence in one K_k, or $2^{k_n} x_{k_n} \in K_{k_n} \subseteq B$ for $k_n \to \infty$, hence $x_{k_n} \to 0$ in E_B. So $K \cup \{0\}$ is compact.

By Faà di Bruno ([6] for the 1-dimensional version)

$$\frac{(f\circ c)^{(k)}(a)}{k!}=\sum_{j\ge 0}\sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}}\frac{1}{j!}d^jf(c(a))\Big(\frac{c^{(\alpha_1)}(a)}{\alpha_1!},\dots,\frac{c^{(\alpha_j)}(a)}{\alpha_j!}\Big)$$

and (2.1.4) for $a\in I$ and $k\in\mathbb{N}$ we have

$$\begin{aligned}\Big\|\frac{1}{k!M_k}\,(f\circ c)^{(k)}(a)\,r_k\Big\| &\le\\ &\le M_1^k\sum_{j\ge 0}\sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=k}}\frac{\|d^jf(c(a))\|_{L^j(E_B,F)}}{j!M_j}\prod_{i=1}^{j}\frac{\|c^{(\alpha_i)}(a)\|_B\,r_{\alpha_i}}{\alpha_i!M_{\alpha_i}}\\ &\le M_1^k\sum_{j\ge 0}\binom{k-1}{j-1}C\,\rho^j\,\frac{1}{2^k}=M_1^k\rho(1+\rho)^{k-1}C\,\frac{1}{2^k}.\end{aligned}$$

So $\Big\{\frac{1}{k!M_k}\,(f\circ c)^{(k)}(a)\,\Big(\frac{2}{M_1(1+\rho)}\Big)^k r_k : a\in I, k\in\mathbb{N}\Big\}$ is bounded as required. □

3.10. **Corollary.** *Let M and N be non-quasianalytic DC-weight sequences with* (2.1.5)

$$\sup_{k\in\mathbb{N}_{>0}}\Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}}<\infty.$$

Then $C^M(U,F)\subseteq C^N(U,F)$ for all convenient vector spaces E and F and each c^∞-open $U\subseteq E$. Moreover $C^\omega(U,F)\subseteq C^M(U,F)\subseteq C^\infty(U,F)$. All these inclusions are bounded.

Proof. The inclusions $C^M\subseteq C^N\subseteq C^\infty$ follow from (3.9) since this is true for condition (3.9.3) applied to $\ell\circ f$ for $\ell\in F^*$.

Without loss let $F=\mathbb{R}$. If f is C^ω then for each closed absolutely convex bounded $B\subseteq E$ the mapping $f\circ i_B: U\cap E_B\to\mathbb{R}$ is given by its locally converging Taylor series by [19, 10.1]. So (3.9.3) is satisfied for $M_k=1$ and thus for each DC-weight sequence M. So f is C^M. All inclusions are bounded by the uniform boundedness principle 4.1 below for C^M and [19, 5.26] for C^∞. □

3.11. **Corollary.** *Let $M=(M_k)$ be a non-quasianalytic DC-weight sequence. Then we have:*

(1) *Multilinear mappings between convenient vector spaces are C^M if and only if they are bounded.*
(2) *If $f: E\supseteq U\to F$ is C^M, then the derivative $df: U\to L(E,F)$ is C^M, and also $\widehat{df}: U\times E\to F$ is C^M, where the space $L(E,F)$ of all bounded linear mappings is considered with the topology of uniform convergence on bounded sets.*
(3) *The chain rule holds.*

Proof. (1) If f is multilinear and C^M then it is smooth by (3.9) and hence bounded by (7.3.2). Conversely, if f is multilinear and bounded then it is smooth by (7.3.2). Furthermore, $f\circ i_B$ is multilinear and continuous and all derivatives of high order vanish. Thus condition (3.9.3) is satisfied, so f is C^M.

(2) Since f is smooth, by (7.3.3) the map $df: U\to L(E,F)$ exists and is smooth. Let $c:\mathbb{R}\to U$ be a C^M-curve. We have to show that $t\mapsto df(c(t))\in L(E,F)$ is C^M. By [19, 5.18] and (3.5) it suffices to show that $t\mapsto c(t)\mapsto \ell(df(c(t)).v)\in\mathbb{R}$ is C^M for each $\ell\in F^*$ and $v\in E$. We are reduced to show that $x\mapsto\ell(df(x).v)$ satisfies the conditions of (3.9). By (3.9) applied to $\ell\circ f$, for each closed bounded

absolutely convex B in E and each $x \in U \cap E_B$ there are $r > 0$, $\rho > 0$, and $C > 0$ such that
$$\frac{1}{k!\,M_k}\|d^k(\ell \circ f \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} \le C\,\rho^k$$
for all $a \in U \cap E_B$ with $\|a - x\|_B \le r$ and all $k \in \mathbb{N}$. For $v \in E$ and those B containing v we then have
$$\begin{aligned}
\|d^k(d(\ell\circ f)(\quad)(v)) \circ i_B)(a)\|_{L^k(E_B,\mathbb{R})} &= \|d^{k+1}(\ell \circ f \circ i_B)(a)(v,\dots)\|_{L^k(E_B,\mathbb{R})}\\
&\le \|d^{k+1}(\ell \circ f \circ i_B)(a)\|_{L^{k+1}(E_B,\mathbb{R})}\|v\|_{E_B} \le C\,\rho^{k+1}\,(k+1)!\,M_{k+1}\\
&\le C\,\rho^k k! M_k\Big((k+1)\rho\frac{M_{k+1}}{M_k}\Big)\\
&\le C\,\bar\rho^k k! M_k \quad \text{ for } \bar\rho > \rho \sup_{k\ge 1}\Big((k+1)\rho\frac{M_{k+1}}{M_k}\Big)^{1/k},
\end{aligned}$$
the latter quantity being finite by (2.1.6). By (4.2) below also $\widehat{df}$ is C^M.

(3) This is valid for all smooth f. □

3.12. **Remark.** For a quasianalytic DC-weight sequence M Theorem 3.9 is wrong. In fact, take any rational function, e.g. $\frac{xy^2}{x^2+y^2}$. Let $t \mapsto x(t), y(t)$ be in $C^M(\mathbb{R},\mathbb{R})$ with $x(0) = 0 = y(0)$. Then $x(t) = t^r\bar x(t)$ and $y(t) = t^r\bar y(t)$ for $r > 0$ and for C^M-functions $\bar x$ and $\bar y$ since C^M is derivation closed. If (x,y) is not constant we may choose r such that $\bar x(0)^2 + \bar y(0)^2 \ne 0$, since C^M is quasianalytic. Then $t \mapsto \frac{x(t)y(t)^2}{x(t)^2+y(t)^2} = t^r\frac{\bar x(t)\bar y(t)^2}{\bar x(t)^2+\bar y(t)^2}$ is C^M near 0, but the rational function is not smooth.

4. C^M-uniform boundedness principles

4.1. **Theorem.** *(Uniform boundedness principle) Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E, F, G be convenient vector spaces and let $U \subseteq F$ be c^∞-open. A linear mapping $T : E \to C^M(U,G)$ is bounded if and only if $\operatorname{ev}_x \circ T : E \to G$ is bounded for every $x \in U$.*

This is the C^M-analogon of (7.3.7). Compare with [19, 5.22–5.26] for the principles behind it. They will be used in the following proof and in (4.6) and (4.10).

Proof. For $x \in U$ and $\ell \in G^*$ the linear mapping $\ell \circ \operatorname{ev}_x = C^M(x,\ell) : C^M(U,G) \to \mathbb{R}$ is continuous, thus ev_x is bounded. So if T is bounded then so is $\operatorname{ev}_x \circ T$.

Conversely, suppose that $\operatorname{ev}_x \circ T$ is bounded for all $x \in U$. For each closed absolutely convex bounded $B \subseteq E$ we consider the Banach space E_B. For each $\ell \in G^*$, each C^M-curve $c : \mathbb{R} \to U$, each $t \in \mathbb{R}$, and each compact $K \subset \mathbb{R}$ the composite given by the following diagram is bounded.

$$\begin{array}{ccccccc}
E & \xrightarrow{T} & C^M(U,G) & & \xrightarrow{\operatorname{ev}_{c(t)}} & & G\\
\uparrow & & \downarrow{\scriptstyle C^M(c,\ell)} & & & & \downarrow{\scriptstyle \ell}\\
E_B & \longrightarrow & C^M(\mathbb{R},\mathbb{R}) & \longrightarrow & \varinjlim_\rho C^M_\rho(K,\mathbb{R}) & \xrightarrow{\operatorname{ev}_t} & \mathbb{R}
\end{array}$$

By [19, 5.24 and 5.25] the map T is bounded. In more detail: Since $\varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is webbed by (2.8), the closed graph theorem [19, 52.10] yields that the mapping $E_B \to \varinjlim_\rho C^M_\rho(K,\mathbb{R})$ is continuous. Thus T is bounded. □

4.2. **Corollary.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence.*

(1) *For convenient vector spaces E and F, on $L(E,F)$ the following bornologies coincide which are induced by:*
   - *The topology of uniform convergence on bounded subsets of E.*

- *The topology of pointwise convergence.*
- *The embedding $L(E,F) \subset C^\infty(E,F)$.*
- *The embedding $L(E,F) \subset C^M(E,F)$.*

(2) *Let E, F, G be convenient vector spaces and let $U \subset E$ be c^∞-open. A mapping $f : U \times F \to G$ which is linear in the second variable is C^M if and only if $f^\vee : U \to L(F,G)$ is well defined and C^M.*

Analogous results hold for spaces of multilinear mappings.

Proof. (1) That the first three topologies on $L(E,F)$ have the same bounded sets has been shown in [19, 5.3 and 5.18]. The inclusion $C^M(E,F) \to C^\infty(E,F)$ is bounded by (3.10) and by the uniform boundedness principle in (7.3.7). It remains to show that the inclusion $L(E,F) \to C^M(E,F)$ is bounded, where the former space is considered with the topology of uniform convergence on bounded sets. This follows from the uniform boundedness principle (4.1).

(2) The assertion for C^∞ is true by (7.3.6).

If f is C^M let $c : \mathbb{R} \to U$ be a C^M-curve. We have to show that $t \mapsto f^\vee(c(t)) \in L(F,G)$ is C^M. By [19, 5.18] and (3.5) it suffices to show that $t \mapsto \ell(f^\vee(c(t))(v)) = \ell(f(c(t),v)) \in \mathbb{R}$ is C^M for each $\ell \in G^*$ and $v \in F$; this is obviously true.

Conversely, let $f^\vee : U \to L(F,G)$ be C^M. We claim that $f : U \times F \to G$ is C^M. By composing with $\ell \in G^*$ we may assume that $G = \mathbb{R}$. By induction we have

$$d^k f(x,w_0)\big((v_k,w_k),\dots,(v_1,w_1)\big) = d^k(f^\vee)(x)(v_k,\dots,v_1)(w_0)+ \\ + \sum_{i=1}^k d^{k-1}(f^\vee)(x)(v_k,\dots,\widehat{v_i},\dots,v_1)(w_i)$$

We check condition (3.9.3) for f:

$$\begin{aligned}
&\|d^k f(x,w_0)\|_{L^k(E_B\times F_{B'},\mathbb{R})} \le \\
&\le \|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))} \\
&\le \|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'},\mathbb{R}))}\|w_0\|_{B'} + \sum_{i=1}^k \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))} \\
&\le C\,\rho^k\,k!\,M_k\|w_0\|_{B'} + \sum_{i=1}^k C\,\rho^{k-1}\,(k-1)!\,M_{k-1} = C\,\rho^k\,k!\,M_k(\|w_0\|_{B'} + \tfrac{M_{k-1}}{\rho\,M_k})
\end{aligned}$$

where we used (3.9.3) for $L(i_{B'},\mathbb{R}) \circ f^\vee : U \to L(F_{B'},\mathbb{R})$. Thus f is C^M. □

4.3. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then we have the bornological identity*

$$C^M(U,F) = \varprojlim_s C^M(\mathbb{R},F),$$

where s runs through the strongly C^M-curves in U and the connecting mappings are given by g^* for all reparametrizations $g \in C^M(\mathbb{R},\mathbb{R})$ of curves s.

Proof. By (3.9) the linear spaces $C^M(U,F)$, $\varprojlim_s C^M(\mathbb{R},F)$ and $\varprojlim_c C^M(\mathbb{R},F)$ coincide, where c runs through the C^M-curves in U: Each element $(f_c)_c$ determines a unique function $f : U \to F$ given by $f(x) := (f \circ \mathrm{const}_x)(0)$ with $f \circ c = f_c$ for all such curves c, and $f \in C^M$ if and only if $f_c \in C^M$ for all such c, by (3.9).

Since $C^M(\mathbb{R},F)$ carries the initial structure with respect to ℓ_* for all $\ell \in F^*$ we may assume $F = \mathbb{R}$. Obviously the identity $\varprojlim_c C^M(\mathbb{R},\mathbb{R}) \to \varprojlim_s C^M(\mathbb{R},\mathbb{R})$ is continuous. As projective limit the later space is convenient, so we may apply the uniform boundedness principle (4.1) to conclude that the identity in the converse direction is bounded. □

4.4. **Proposition.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E and F be convenient vector spaces and let $U \subseteq E$ be c^∞-open. Then the bornology of $C^M(U,F)$ is initial with respect to each of the following families of mappings*

$$i_B^* = C^M(i_B, F) : C^M(U,F) \to C^M(U \cap E_B, F), \tag{1}$$

$$C^M(i_B, \pi_V) : C^M(U,F) \to C^M(U \cap E_B, F_V), \tag{2}$$

$$C^M(i_B, \ell) : C^M(U,F) \to C^M(U \cap E_B, \mathbb{R}), \tag{3}$$

where B runs through the closed absolutely convex bounded subsets of E and $i_B : E_B \to E$ denotes the inclusion, and where ℓ runs through the continuous linear functionals on F, and where V runs through the absolutely convex 0-neighborhoods of F and F_V is obtained by factoring out the kernel of the Minkowsky functional of V and then taking the completion with respect to the induced norm.

Warning: The structure in (2) gives a projective limit description of $C^M(U,F)$ if and only if F is complete since then $F = \varprojlim_V F_V$.

Proof. Since $i_B : E_B \to E$, $\pi_V : F \to F_V$ and $\ell : F \to \mathbb{R}$ are bounded linear the mappings i_B^*, $C^M(i_B, \pi_V)$ and $C^M(i_B, \ell)$ are bounded and linear.

The structures given by (1), (2) and (3) are successively weaker. So let, conversely, $C^M(i_B,\ell)(B)$ be bounded in $C^M(U \cap E_B, \mathbb{R})$ for all B and ℓ. By (4.3) $C^M(U,F)$ carries the initial structure with respect to all $c^* : C^M(U,F) \to C^M(\mathbb{R},F)$, where $c : \mathbb{R} \to U$ are the strongly C^M curves and these factor locally as (strongly) C^M-curves into some E_B. By definition $C^M(\mathbb{R},F)$ carries the initial structure with respect to $C^M(\iota_I, \ell) : C^M(\mathbb{R},F) \to C^M(I,\mathbb{R})$ where $\iota_I : I \hookrightarrow \mathbb{R}$ are the inclusions of compact intervals into $\mathbb{R}$ and $\ell \in F^*$. Thus $C^M(U,F)$ carries the initial structure with respect to $C^M(c|_I, \ell) : C^M(U,F) \to C^M(I,\mathbb{R})$, which is coarser than that induced by $C^M(U,F) \to C^M(U \cap E_B, \mathbb{R})$. □

4.5. **Definition.** Let E and F be Banach spaces and $A \subseteq E$ convex. We consider the linear space $C^\infty(A,F)$ consisting of all sequences $(f^k)_k \in \prod_{k\in\mathbb{N}} C(A, L^k(E,F))$ satisfying

$$f^k(y)(v) - f^k(x)(v) = \int_0^1 f^{k+1}(x + t(y-x))(y-x, v)\, dt$$

for all $k \in \mathbb{N}$, $x,y \in A$, and $v \in E^k$. If A is open we can identify this space with that of all smooth functions $A \to F$ by passing to jets.

In addition, let $M = (M_k)$ be a non-quasianalytic DC-weight sequence and (r_k) a sequence of positive real numbers. Then we consider the normed spaces

$$C^M_{(r_k)}(A,F) := \Big\{ (f^k)_k \in C^\infty(A,F) : \|(f^k)\|_{(r_k)} < \infty \Big\}$$

where the norm is given by

$$\|(f^k)\|_{(r_k)} := \sup\Big\{ \frac{\|f^k(a)(v_1,\dots,v_k)\|}{k!\, r_k\, M_k\, \|v_1\| \cdot \dots \cdot \|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E \Big\}.$$

If $(r_k) = (\rho^k)$ for some $\rho > 0$ we just write ρ instead of (r_k) as indices. The spaces $C^M_{(r_k)}(A,F)$ are Banach spaces, since they are closed in $\ell^\infty(\mathbb{N}, \ell^\infty(A, L^k(E,F)))$ via $(f^k)_k \mapsto (k \mapsto \frac{1}{k!\, r_k\, M_k} f^k)$.

4.6. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. Let E and F be Banach spaces and let $U \subseteq E$ be open. Then the space $C^M(U,F)$ can* *be described bornologically in the following equivalent ways, i.e. these constructions give the same vector space and the same bounded sets.*

(1) $$\varprojlim_{K} \varinjlim_{\rho,W} C^M_\rho(W,F)$$

(2) $$\varprojlim_{K} \varinjlim_{\rho} C^M_\rho(K,F)$$

(3) $$\varprojlim_{K,(r_k)} C^M_{(r_k)}(K,F)$$

(4) $$\varprojlim_{c,I} \varinjlim_{\rho} C^M_\rho(I,F)$$

Moreover, all involved inductive limits are regular, i.e. the bounded sets of the inductive limits are contained and bounded in some step.

Here K runs through all compact convex subsets of U ordered by inclusion, W runs through the open subsets $K \subseteq W \subseteq U$ again ordered by inclusion, ρ runs through the positive real numbers, (r_k) runs through all sequences of positive real numbers for which $\rho^k/r_k \to 0$ for all $\rho > 0$, c runs through the C^M-curves in U ordered by reparametrization with $g \in C^M(\mathbb{R},\mathbb{R})$ and I runs through the compact intervals in $\mathbb{R}$.

Proof. Note first that all four descriptions describe smooth functions $f : U \to F$, which are given by $x \mapsto f^0(x)$ in (1)–(3) for appropriately chosen K with $x \in K$ where $f^0 : K \to F$ and by $x \mapsto f_c(t)$ in (4) for c with $x = c(t)$, $t \in I$ and $f_c : I \to F$. Smoothness of f follows, since we may test with C^M-curves and these factor locally into some K.

By (3.9) all four descriptions describe $C^M(U,F)$ as vector space.

Obviously the identity is continuous from (1) to (2) and from (2) to (3).

The identity from (3) to (1) is continuous, since the space given by (3) is as inverse limit of Banach spaces convenient and the inductive limit in (1) is by construction an (LB)-space, hence webbed, and thus we can apply the uniform $\mathcal{S}$-boundedness principle [19, 5.24], where $\mathcal{S} = \{\mathrm{ev}_x : x \in U\}$.

So the descriptions in (1)–(3) describe the same complete bornology on $C^M(U,F)$ and satisfy the uniform $\mathcal{S}$-boundedness principle.

Moreover, the inductive limits involved in (1) and (2) are regular: In fact the bounded sets $\mathcal{B}$ therein are also bounded in the structure of (3), i.e., for every compact $K \subseteq U$ and sequence (r_k) of positive real numbers for which $\rho^k/r_k \to 0$ for all $\rho > 0$:

$$\sup\Big\{\frac{\|f^k(a)(v_1,\dots,v_k)\|}{k!\, r_k\, M_k\, \|v_1\| \cdot \dots \cdot \|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E, f \in \mathcal{B}\Big\} < \infty$$

and so the sequence

$$a_k := \sup\Big\{\frac{\|f^k(a)(v_1,\dots,v_k)\|}{k!\, M_k\, \|v_1\| \cdot \dots \cdot \|v_k\|} : a \in A, v_i \in E, f \in \mathcal{B}\Big\} < \infty$$

satisfies $\sup_k a_k/r_k < \infty$ for all (r_k) as above. By [19, 9.2] these are the coefficients of a power series with positive radius of convergence. Thus a_k/ρ^k is bounded for some $\rho > 0$. This means that $\mathcal{B}$ is contained and bounded in $C^M_\rho(K,F)$.

That also (4) describes the same bornology follows again by the $\mathcal{S}$-uniform boundedness principle, since the inductive limit in (4) is regular by what we said before for the special case $E = \mathbb{R}$ and hence the structure of (4) is convenient. □

4.7. **Lemma.** *Let M be a non-quasianalytic DC-weight sequence. For any convenient vector space E the flip of variables induces an isomorphism $L(E, C^M(\mathbb{R},\mathbb{R})) \cong C^M(\mathbb{R}, E')$ as vector spaces.*

Proof. For $c \in C^M(\mathbb{R}, E')$ consider $\tilde{c}(x) := \mathrm{ev}_x \circ c \in C^M(\mathbb{R}, \mathbb{R})$ for $x \in E$. By the uniform boundedness principle (4.1) the linear mapping $\tilde{c}$ is bounded, since $\mathrm{ev}_t \circ \tilde{c} = c(t) \in E'$.

If conversely $\ell \in L(E, C^M(\mathbb{R}, \mathbb{R}))$, we consider $\tilde{\ell}(t) = \mathrm{ev}_t \circ \ell \in E' = L(E, \mathbb{R})$ for $t \in \mathbb{R}$. Since the bornology of E' is generated by $\mathcal{S} := \{ev_x : x \in E\}$, $\tilde{\ell} : \mathbb{R} \to E'$ is C^M, for $\mathrm{ev}_x \circ \tilde{\ell} = \ell(x) \in C^M(\mathbb{R}, \mathbb{R})$, by (3.5). □

4.8. **Lemma.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence. By $\lambda^M(\mathbb{R})$ we denote the c^∞-closure of the linear subspace generated by $\{\mathrm{ev}_t : t \in \mathbb{R}\}$ in $C^M(\mathbb{R}, \mathbb{R})'$ and let $\delta : \mathbb{R} \to \lambda^M(\mathbb{R})$ be given by $t \mapsto \mathrm{ev}_t$. Then $\lambda^M(\mathbb{R})$ is the free convenient vector space over C^M, i.e. for every convenient vector space G the C^M-curve δ induces a bornological isomorphism*

$$L(\lambda^M(\mathbb{R}), G) \cong C^M(\mathbb{R}, G).$$

We expect $\lambda^M(\mathbb{R})$ to be equal to $C^M(\mathbb{R}, \mathbb{R})'$ as it is the case for the analogous situation of smooth mappings, see [19, 23.11], and of holomorphic mappings, see [25] and [26].

Proof. The proof goes along the same lines as in [19, 23.6] and in [8, 5.1.1]. Note first that $\lambda^M(\mathbb{R})$ is a convenient vector space since it is c^∞-closed in the convenient vector space $C^M(\mathbb{R}, \mathbb{R})'$. Moreover, δ is C^M by (3.5), since $\mathrm{ev}_h \circ \delta = h$ for all $h \in C^M(\mathbb{R}, \mathbb{R})$, so $\delta^* : L(\lambda^M(\mathbb{R}), G) \to C^M(\mathbb{R}, G)$ is a well-defined linear mapping. This mapping is injective, since each bounded linear mapping $\lambda^M(\mathbb{R}) \to G$ is uniquely determined on $\delta(\mathbb{R}) = \{\mathrm{ev}_t : t \in \mathbb{R}\}$. Let now $f \in C^M(\mathbb{R}, G)$. Then $\ell \circ f \in C^M(\mathbb{R}, \mathbb{R})$ for every $\ell \in G^*$ and hence $\tilde{f} : C^M(\mathbb{R}, \mathbb{R})' \to \prod_{G^*} \mathbb{R}$ given by $\tilde{f}(\varphi) = (\varphi(\ell \circ f))_{\ell \in G^*}$ is a well-defined bounded linear map. Since it maps $\mathrm{ev_t}$ to $\tilde{f}(\mathrm{ev}_t) = \delta(f(t))$, where $\delta : G \to \prod_{G^*} \mathbb{R}$ denotes the bornological embedding given by $x \mapsto (\ell(x))_{\ell \in G^*}$, it induces a bounded linear mapping $\tilde{f} : \lambda^M(\mathbb{R}) \to G$ satisfying $\tilde{f} \circ \delta = f$. Thus δ^* is a linear bijection. That it is a bornological isomorphism (i.e. δ^* and its inverse are both bounded) follows from the uniform boundedness principles (4.1) and (4.2). □

4.9. **Corollary.** *Let $M = (M_k)$ and $N = (N_k)$ be non-quasianalytic DC-weight sequences. We have the following isomorphisms of linear spaces*

(1) $C^\infty(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, C^\infty(\mathbb{R}, \mathbb{R}))$
(2) $C^\omega(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, C^\omega(\mathbb{R}, \mathbb{R}))$
(3) $C^N(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) \cong C^M(\mathbb{R}, C^N(\mathbb{R}, \mathbb{R}))$

Proof. For $\alpha \in \{\infty, \omega, N\}$ we get

$$\begin{aligned} C^M(\mathbb{R}, C^\alpha(\mathbb{R}, \mathbb{R})) &\cong L(\lambda^M(\mathbb{R}), C^\alpha(\mathbb{R}, \mathbb{R})) && \text{by (4.8)} \\ &\cong C^\alpha(\mathbb{R}, L(\lambda^M(\mathbb{R}), \mathbb{R})) && \text{by (4.7), [19, 3.13.4, 5.3, 11.15]} \\ &\cong C^\alpha(\mathbb{R}, C^M(\mathbb{R}, \mathbb{R})) && \text{by (4.8).} \quad \square \end{aligned}$$

4.10. **Theorem.** *(Canonical isomorphisms) Let $M = (M_k)$ and $N = (N_k)$ be non-quasianalytic DC-weight sequences. Let E, F be convenient vector spaces and let W_i be c^∞-open subsets in such. We have the following natural bornological isomorphisms:*

(1) $C^M(W_1, C^N(W_2, F)) \cong C^N(W_2, C^M(W_1, F))$,
(2) $C^M(W_1, C^\infty(W_2, F)) \cong C^\infty(W_2, C^M(W_1, F))$.
(3) $C^M(W_1, C^\omega(W_2, F)) \cong C^\omega(W_2, C^M(W_1, F))$.
(4) $C^M(W_1, L(E, F)) \cong L(E, C^M(W_1, F))$.
(5) $C^M(W_1, \ell^\infty(X, F)) \cong \ell^\infty(X, C^M(W_1, F))$.

(6) $C^M(W_1, \mathcal{L}\mathrm{ip}^k(X,F)) \cong \mathcal{L}\mathrm{ip}^k(X, C^M(W_1,F))$.

In (5) *the space X is an ℓ^∞-space, i.e. a set together with a bornology induced by a family of real valued functions on X, cf.* [8, 1.2.4]. *In* (6) *the space X is a $\mathcal{L}\mathrm{ip}^k$-space, cf.* [8, 1.4.1]. *The spaces $\ell^\infty(X,F)$ and $\mathcal{L}\mathrm{ip}^k(W,F)$ are defined in* [8, 3.6.1 and 4.4.1].

Proof. All isomorphisms, as well as their inverse mappings, are given by the flip of coordinates: $f \mapsto \tilde{f}$, where $\tilde{f}(x)(y) := f(y)(x)$. Furthermore, all occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations, by (4.1), [19, 11.11, 11.14, 11.12], and by [8, 3.6.1, 4.4.2, 3.6.6, and 4.4.7].

That $\tilde{f}$ has values in the corresponding spaces follows from the equation $\tilde{f}(x) = ev_x \circ f$. One only has to check that $\tilde{f}$ itself is of the corresponding class, since it follows that $f \mapsto \tilde{f}$ is bounded. This is a consequence of the uniform boundedness principle, since

$$(\mathrm{ev}_x \circ (\tilde{\ }))(f) = \mathrm{ev}_x(\tilde{f}) = \tilde{f}(x) = \mathrm{ev}_x \circ f = (\mathrm{ev}_x)_*(f).$$

That $\tilde{f}$ is of the appropriate class in (1) and in (2) follows by composing with the appropriate curves $c_1 : \mathbb{R} \to W_1$, $c_2 : \mathbb{R} \to W_2$ and $\lambda \in F^*$ and thereby reducing the statement to the special case in (4.9).

That $\tilde{f}$ is of the appropriate class in (3) follows by composing with $c_1 \in C^M(\mathbb{R}, W_1)$ and $C^{\beta_2}(c_2, \lambda) : C^\omega(W_2, F) \to C^{\beta_2}(\mathbb{R},\mathbb{R})$ for all $\lambda \in F^*$ and $c_2 \in C^{\beta_2}(\mathbb{R}, W_2)$, where β_2 is in $\{\infty, \omega\}$. Then $C^{\beta_2}(c_2,\lambda) \circ \tilde{f} \circ c_1 = (C^M(c_1,\lambda) \circ f \circ c_2)^\sim : \mathbb{R} \to C^{\beta_2}(\mathbb{R},\mathbb{R})$ is C^M by (4.9), since $C^M(c_1,\lambda) \circ f \circ c_2 : \mathbb{R} \to W_2 \to C^M(W_1,F) \to C^M(\mathbb{R},\mathbb{R})$ is C^{β_2}.

That $\tilde{f}$ is of the appropriate class in (4) follows, since $L(E,F)$ is the c^∞-closed subspace of $C^M(E,F)$ formed by the linear C^M-mappings.

That $\tilde{f}$ is of the appropriate class in (5) or (6) follows from (4), using the free convenient vector spaces $\ell^1(X)$ or $\lambda^k(X)$ over the ℓ^∞-space X or the the $\mathcal{L}\mathrm{ip}^k$-space X, see [8, 5.1.24 or 5.2.3], satisfying $\ell^\infty(X,F) \cong L(\ell^1(X),F)$ or satisfying $\mathcal{L}\mathrm{ip}^k(X,F) \cong L(\lambda^k(X),F)$. Existence of these free convenient vector spaces can be proved in a similar way as in (4.8). □

5. Exponential law

5.1. **Difference quotients.** For the following see [8, 1.3]. For a subset $K \subseteq \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, a linear space E, and $f : K \to E$ let:

$$\mathbb{R}^{\langle k\rangle} = \big\{(x_0,\dots,x_k) \in \mathbb{R}^{k+1} : x_i \neq x_j \text{ for } i \neq j\big\}$$

$$K^\alpha = \big\{(x^1,\dots,x^n) \in \mathbb{R}^{\alpha_1+1} \times \dots \times \mathbb{R}^{\alpha_n+1} : (x^1_{i_1},\dots,x^n_{i_n}) \in K \text{ for } 0 \le i_j \le \alpha_j\big\}$$

$$K^{\langle\alpha\rangle} = K^\alpha \cap (\mathbb{R}^{\langle\alpha_1\rangle} \times \dots \times \mathbb{R}^{\langle\alpha_n\rangle})$$

$$\beta_i(x) = k! \prod_{\substack{0 \le j \le k\\ j \neq i}} \frac{1}{x_i - x_j} \text{ for } x = (x_0,\dots,x_k) \in \mathbb{R}^{\langle k\rangle}$$

$$\delta^\alpha f(x^1,\dots,x^n) = \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_n=0}^{\alpha_n} \beta_{i_1}(x^1)\dots\beta_{i_n}(x^n) f(x^1_{i_1},\dots,x^n_{i_n})$$

Note that $\delta^0 f = f$ and $\delta^\alpha = \delta^{\alpha_n}_n \circ \dots \circ \delta^{\alpha_1}_1$ where

$$\delta^k_i g(x^1,\dots,x^n) = \delta^k(g(x^1,\dots,x^{i-1},\quad,x^{i+1},\dots,x^n))(x^i).$$

Lemma. *Let E be a convenient vector space. Let $U \subseteq \mathbb{R}^n$ be open. For $f : U \to E$ the following conditions are equivalent:*

(1) *$f : U \to E$ is C^M.*
(2) *For every compact convex set K in U and every $\ell \in E^*$ there exists $\rho > 0$ such that*

$$\Big\{ \frac{\delta^\alpha(\ell\circ f)(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha\in\mathbb{N}^n, x\in K^{\langle\alpha\rangle}\Big\}$$

is bounded in $\mathbb{R}$.

Furthermore, the norm on the space $C^M_\rho(K,\mathbb{R})$ from (2.8) (for convex K) is also given by

$$\|f\|_{\rho,K} := \sup\Big\{\frac{|\delta^\alpha f(x)|}{\rho^{|\alpha|}|\alpha|!M_{|\alpha|}} : \alpha\in\mathbb{N}^n, x\in K^{\langle\alpha\rangle}\Big\}.$$

Proof. By composing with bounded linear functionals we may assume that $E=\mathbb{R}$.

(1) $\implies$ (2) If f is C^M then for each compact convex set K in U there exists $\rho>0$ such that

$$\Big\{\tfrac{\partial^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!M_{|\alpha|}} : \alpha\in\mathbb{N}^n, x\in K\Big\}$$

is bounded in $\mathbb{R}$.

For a differentiable function $g:\mathbb{R}\to\mathbb{R}$ and $t_0<\cdots<t_j$ there exist s_i with $t_i<s_i<t_{i+1}$ such that

$$\delta^j g(t_0,\dots,t_j) = \delta^{j-1}g'(s_0,\dots,s_{j-1}).$$

This follows by Rolle's theorem, see [19, 12.4]. Recursion, for $g=\partial^\alpha f$, shows that $\delta^\alpha f(x^0,\dots,x^n)=\partial^\alpha f(s)$ for some $s\in K$.

(2) $\implies$ (1) f is C^∞ by [8, 1.3.29] since each difference quotient $\delta^\alpha f$ is bounded on bounded sets.

For $g\in C^\infty(\mathbb{R},\mathbb{R})$, using (see [8, 1.3.6])

$$g(t_j)=\sum_{i=0}^{j}\tfrac{1}{i!}\prod_{l=0}^{i-1}(t_j-t_l)\;\delta^i g(t_0,\dots,t_i),$$

induction on j and differentiability of g shows that

$$\delta^j g'(t_0,\dots,t_j)=\tfrac{1}{j+1}\sum_{i=0}^{j}\delta^{j+1}g(t_0,\dots,t_j,t_i),$$

where $\delta^{j+1}g(t_0,\dots,t_j,t_i):=\lim_{t\to t_i}\delta^{j+1}g(t_0,\dots,t_j,t)$. If the right hand side divided by $\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}$ is bounded, then also $\delta^j g'/(\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|})$ is bounded.

By recursion, applied to $g=\delta^\beta\partial^{\alpha-\beta}f$, we conclude that $f\in C^M$. □

5.2. **Lemma.** *Let E be a convenient vector space such that there exists a Baire vector space topology on the dual E^* for which the point evaluations ev_x are continuous for all $x\in E$. For a mapping $f:\mathbb{R}^n\to E$ the following are equivalent:*

(1) *$\ell\circ f$ is C^M for all $\ell\in E^*$.*
(2) *For every convex compact $K\subseteq\mathbb{R}^n$ there exists $\rho>0$ such that*

$$\Big\{\frac{\partial^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha\in\mathbb{N}^n, x\in K\Big\}\ \textit{is bounded in } E.$$

(3) *For every convex compact $K\subseteq\mathbb{R}^n$ there exists $\rho>0$ such that*

$$\Big\{\frac{\delta^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha\in\mathbb{N}^n, x\in K^{\langle\alpha\rangle}\Big\}\ \textit{is bounded in } E.$$

Proof. (2) $\implies$ (1) is obvious.

(1) $\implies$ (2) Let K be compact convex in $\mathbb{R}^n$. We consider the sets

$$A_{\rho,C} := \Big\{\ell \in E^* : \frac{|\partial^\alpha(\ell\circ f)(x)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} \le C \text{ for all } \alpha\in\mathbb{N}^n, x\in K\Big\}$$

which are closed subsets in E^* for the Baire topology. We have $\bigcup_{\rho,C} A_{\rho,C} = E^*$. By the Baire property there exists ρ and C such that the interior U of $A_{\rho,C}$ is non-empty. If $\ell_0 \in U$ then for all $\ell\in E^*$ there is an $\epsilon>0$ such that $\epsilon\ell\in U-\ell_0$ and hence for all $x\in K$ and all α we have

$$|\partial^\alpha(\ell\circ f)(x)| \le \tfrac{1}{\epsilon}\left(|\partial^\alpha((\epsilon\ell+\ell_0)\circ f)(x)| + |\partial^\alpha(\ell_0\circ f)(x)|\right) \le \tfrac{2C}{\epsilon}\,\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}.$$

So the set

$$\Big\{\frac{\partial^\alpha f(x)}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha\in\mathbb{N}^n, x\in K\Big\}$$

is weakly bounded in E and hence bounded.

(3) $\implies$ (1) follows by Lemma (5.1). (1) $\implies$ (3) follows as above for the difference quotients instead of the partial differentials. □

5.3. **Theorem.** *(Cartesian closedness) Let $M=(M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth* (2.5.1). *Then the category of C^M-mappings between convenient real vector spaces is cartesian closed. More precisely, for convenient vector spaces E, F and G and c^∞-open sets $U\subseteq E$ and $W\subseteq F$ a mapping $f: U\times W\to G$ is C^M if and only if $f^\vee: U\to C^M(W,G)$ is C^M.*

Proof. We first show the result for $U=\mathbb{R}$, $W=\mathbb{R}$, $G=\mathbb{R}$.

If $f\in C^M(\mathbb{R}^2,\mathbb{R})$ then clearly for any $x\in\mathbb{R}$ the function $f^\vee(x) = f(x,\quad)\in C^M(\mathbb{R},\mathbb{R})$. To show that $f^\vee:\mathbb{R}\to C^M(\mathbb{R},\mathbb{R})$ is C^M it suffices to check (5.1.2) for all $\ell\in C^M(\mathbb{R},\mathbb{R})^*$. Such an ℓ factors over $\varinjlim_\rho C^M_\rho(L)$ for some compact $L\subset\mathbb{R}$. Let $K\subset\mathbb{R}$ be compact. Since f is C^M there exists $C>0$ and $\rho>0$ by lemma (5.1) such that

$$\frac{|\delta^\alpha f(x,y)|}{\rho^{|\alpha|}|\alpha|!M_{|\alpha|}} \le C \qquad \text{for } \alpha\in\mathbb{N}^2, (x,y)\in(K\times L)^{\langle\alpha\rangle}.$$

Since M is of moderate growth (2.5.1) we have $M_{j+k}\le\sigma^{j+k}M_jM_k$ for some $\sigma>0$. Let $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$. Then:

$$\begin{aligned}
\Big\|\frac{\delta^{\alpha_1}f^\vee(x)}{\rho_1^{\alpha_1}\,\alpha_1!\,M_{\alpha_1}}\Big\|_{\rho_2,L} &= \sup\Big\{\frac{|\delta_2^{\alpha_2}\delta_1^{\alpha_1}f(x,y)|}{\rho_1^{\alpha_1}\,\alpha_1!\,M_{\alpha_1}\,\rho_2^{\alpha_2}\,\alpha_2!\,M_{\alpha_2}} : \alpha_2\in\mathbb{N}, y\in L^{\langle\alpha_2\rangle}\Big\}\\
&\le \sup\Big\{\frac{|\delta_2^{\alpha_2}\delta_1^{\alpha_1}f(x,y)|}{\rho_1^{\alpha_1}\,\rho_2^{\alpha_2}\,\frac{\alpha_1!\,\alpha_2!}{(\alpha_1+\alpha_2)!}\,(\alpha_1+\alpha_2)!\,\sigma^{-\alpha_1-\alpha_2}M_{\alpha_1+\alpha_2}} : \alpha_2\in\mathbb{N}, y\in L^{\langle\alpha_2\rangle}\Big\}\\
&\le \sup\Big\{\frac{|\delta^\alpha f(x,y)|}{\rho_1^{\alpha_1}\,\rho_2^{\alpha_2}\,\sigma^{-|\alpha|}2^{-|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha_2\in\mathbb{N}, y\in L^{\langle\alpha_2\rangle}\Big\}\\
&\le \sup\Big\{\frac{|\delta^\alpha f(x,y)|}{\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}} : \alpha_2\in\mathbb{N}, y\in L^{\langle\alpha_2\rangle}\Big\} \le C \text{ for } \alpha_1\in\mathbb{N}, x\in K^{\langle\alpha_1\rangle}
\end{aligned}$$

for $\rho_1=\rho_2=2\sigma\rho$. So $f^\vee: K\to C^M_{\rho_2}(L,\mathbb{R})$ is C^M. Thus $\ell\circ f^\vee$ is C^M.

Conversely, let $f^\vee:\mathbb{R}\to C^M(\mathbb{R},\mathbb{R})$ be C^M. Then $f^\vee:\mathbb{R}\to\varinjlim_{\rho_2} C^M_{\rho_2}(L,\mathbb{R})$ is C^M for all compact subsets $L\subset\mathbb{R}$. The dual space $(\varinjlim_{\rho_2} C^M_{\rho_2}(L,\mathbb{R}))^*$ can be equipped with the Baire topology of the countable limit $\varprojlim_{\rho_2} C^M_{\rho_2}(L,\mathbb{R})^*$ of Banach spaces.

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{f^\vee} & C^M(\mathbb{R},\mathbb{R}) & \longrightarrow & \varinjlim_{\rho_2} C^M_{\rho_2}(L,\mathbb{R}) \\ \uparrow & & & & \uparrow \\ K & & \xrightarrow{\qquad f^\vee \qquad} & & C^M_{\rho_2}(L,\mathbb{R}) \end{array}$$

Thus the mapping $f^\vee : \mathbb{R} \to \varinjlim_{\rho_2} C^M_{\rho_2}(L,\mathbb{R})$ is strongly C^M by (5.2). Since the inductive limit $\varinjlim_{\rho_2} C^M_{\rho_2}(L,\mathbb{R})$ is countable and regular ([7, 7.4 and 7.5] or [19, 52.37]), for each compact $K \subset \mathbb{R}$ there exists $\rho_1 > 0$ such that the bounded set

$$\Big\{ \frac{\partial^{\alpha_1} f^\vee(x)}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}} : \alpha_1 \in \mathbb{N}, x \in K \Big\}$$

is contained and bounded in $C^M_{\rho_2}(L,\mathbb{R})$ for some $\rho_2 > 0$. Thus for $\alpha_1 \in \mathbb{N}$ and $x \in K$ we have (using (2.1.3))

$$\begin{aligned} \infty > C := \sup_{\substack{\alpha_1 \in \mathbb{N} \\ y \in K}} \Big\| \frac{\delta^{\alpha_1} f^\vee(y)}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}} \Big\|_{\rho_2, L} &\ge \Big\| \frac{\delta^{\alpha_1} f^\vee(x)}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}} \Big\|_{\rho_2, L} \\ &= \sup \Big\{ \frac{|\delta_2^{\alpha_2} \delta_1^{\alpha_1} f(x,y)|}{\rho_1^{\alpha_1}\, \alpha_1!\, M_{\alpha_1}\, \rho_2^{\alpha_2}\, \alpha_2!\, M_{\alpha_2}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle \alpha_2 \rangle} \Big\} \\ &\ge \sup \Big\{ \frac{|\delta_2^{\alpha_2} \delta_1^{\alpha_1} f(x,y)|}{\rho_1^{\alpha_1}\, \rho_2^{\alpha_2}\, \frac{\alpha_1!\, \alpha_2!}{(\alpha_1+\alpha_2)!}\, (\alpha_1+\alpha_2)! M_{\alpha_1+\alpha_2}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle \alpha_2 \rangle} \Big\} \\ &\ge \sup \Big\{ \frac{|\delta^{\alpha} f(x,y)|}{\rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|}} : \alpha_2 \in \mathbb{N}, y \in L^{\langle \alpha_2 \rangle} \Big\} \end{aligned}$$

where $\rho = \max(\rho_1, \rho_2)$. Thus f is C^M.

Now we consider the general case. Given a C^M-mapping $f : U \times W \to G$ we have to show that $f^\vee : U \to C^M(W,G)$ is C^M. Any continuous linear functional on $C^M(W,G)$ factors over some step mapping $C^M(c_2,\ell) : C^M(W,G) \to C^M(\mathbb{R},\mathbb{R})$ of the cone in (3.1) where c_2 is a C^M-curve in W and $\ell \in G^*$. So we have to check that $C^M(c_2,\ell) \circ f^\vee \circ c_1 : \mathbb{R} \to C^M(\mathbb{R},\mathbb{R})$ is C^M for every C^M-curve c_1 in U. Since $(\ell \circ f \circ (c_1 \times c_2))^\vee = C^M(c_2,\ell) \circ f^\vee \circ c_1$ this follows from the special case proved above.

If $f^\vee : U \to C^M(W,G)$ is C^M then $(\ell \circ f \circ (c_1 \times c_2))^\vee = C^M(c_2,\ell) \circ f^\vee \circ c_1$ is C^M for all C^M-curves $c_1 : \mathbb{R} \to U$, $c_2 : \mathbb{R} \to W$ and $\ell \in G^*$. By the special case, f is then C^M. □

5.4. **Example: Cartesian closedness is wrong in general.** Let M be a DC-weight sequence which is strongly non-quasianalytic but not of moderate growth. For example, $M_k = 2^{k^2}$ satisfies this by (2.7). Then by (2.4) there exists $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^M with $\partial^\alpha f(0,0) = |\alpha|!\, M_{|\alpha|}$. *We claim that* $f^\vee : \mathbb{R} \to C^M(\mathbb{R},\mathbb{R})$ *is not* C^M.

Since M is not of moderate growth there exist $j_n \nearrow \infty$ and $k_n > 0$ such that

$$\Big(\frac{M_{k_n+j_n}}{M_{k_n} M_{j_n}} \Big)^{\frac{1}{k_n+j_n}} \ge n.$$

Consider the linear functional $\ell : C^M(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ given by

$$\ell(g) = \sum_n \frac{g^{(j_n)}(0)}{j_n!\, M_{j_n}\, n^{j_n}}.$$

This functional is continuous since

$$\left|\sum_n \frac{g^{(j_n)}(0)}{j_n!\,M_{j_n}\,n^{j_n}}\right| \le \sum_n \frac{g^{(j_n)}(0)}{j_n!\,\rho^{j_n}\,M_{j_n}}\frac{\rho^{j_n}}{n^{j_n}} \le C(\rho)\,\|g\|_{\rho,[-1,1]} < \infty$$

for suitable ρ where

$$C(\rho) := \sum_n \rho^{j_n}\frac{1}{n^{j_n}} < \infty$$

for all ρ. But $\ell\circ f^\vee$ is not C^M since

$$\begin{aligned}\|\ell\circ f^\vee\|_{\rho_1,[-1,1]} &\ge \sup_k \frac{1}{\rho_1^k\,k!\,M_k}\sum_n \frac{f^{(j_n,k)}(0,0)}{j_n!\,M_{j_n}n^{j_n}}\\ &\ge \sup_n \frac{1}{\rho_1^{k_n}\,k_n!\,M_{k_n}}\frac{f^{(j_n,k_n)}(0,0)}{j_n!\,M_{j_n}\,n^{j_n}}\\ &\ge \sup_n \frac{(j_n+k_n)!\,M_{j_n+k_n}}{\rho_1^{k_n}\,k_n!\,j_n!\,M_{k_n}\,M_{j_n}\,n^{j_n}} \ge \sup_n \frac{n^{j_n+k_n}}{\rho_1^{k_n}\,n^{j_n}} = \infty\end{aligned}$$

for all $\rho_1 > 0$.

5.5. **Theorem.** *Let M be a non-quasianalytic DC-weight sequence which is of moderate growth. Let E, F, etc., be convenient vector spaces and let U and V be c^∞-open subsets of such.*

(1) *The exponential law holds:*

$$C^M(U, C^M(V,G)) \cong C^M(U\times V, G)$$

is a linear C^M-diffeomorphism of convenient vector spaces.

The following canonical mappings are C^M.

(2) $\quad \mathrm{ev} : C^M(U,F)\times U \to F, \quad \mathrm{ev}(f,x) = f(x)$

(3) $\quad \mathrm{ins} : E \to C^M(F, E\times F), \quad \mathrm{ins}(x)(y) = (x,y)$

(4) $\quad (\ \ )^\wedge : C^M(U, C^M(V,G)) \to C^M(U\times V, G)$

(5) $\quad (\ \ )^\vee : C^M(U\times V, G) \to C^M(U, C^M(V,G))$

(6) $\quad \mathrm{comp} : C^M(F,G)\times C^M(U,F) \to C^M(U,G)$

(7) $\quad C^M(\ \ ,\ \ ) : C^M(F,F_1)\times C^M(E_1,E) \to C^M(C^M(E,F), C^M(E_1,F_1))$

$\qquad (f,g) \mapsto (h\mapsto f\circ h\circ g)$

(8) $\quad \prod : \prod C^M(E_i,F_i) \to C^M(\prod E_i, \prod F_i)$

Proof. (2) The mapping associated to ev via cartesian closedness is the identity on $C^M(U,F)$, which is C^M, thus ev is also C^M.

(3) The mapping associated to ins via cartesian closedness is the identity on $E\times F$, hence ins is C^M.

(4) The mapping associated to $(\ \ )^\wedge$ via cartesian closedness is the C^M-composition of evaluations $\mathrm{ev}\circ(\mathrm{ev}\times\mathrm{Id}) : (f;x,y)\mapsto f(x)(y)$.

(5) We apply cartesian closedness twice to get the associated mapping $(f;x;y)\mapsto f(x,y)$, which is just a C^M evaluation mapping.

(6) The mapping associated to comp via cartesian closedness is $(f,g;x)\mapsto f(g(x))$, which is the C^M-mapping $\mathrm{ev}\circ(\mathrm{Id}\times\mathrm{ev})$.

(7) The mapping associated to the one in question by applying cartesian closedness twice is $(f,g;h,x)\mapsto g(h(f(x)))$, which is the C^M-mapping $\mathrm{ev}\circ(\mathrm{Id}\times\mathrm{ev})\circ(\mathrm{Id}\times\mathrm{Id}\times\mathrm{ev})$.

(8) Up to a flip of factors the mapping associated via cartesian closedness is the product of the evaluation mappings $C^M(E_i,F_i)\times E_i\to F_i$.

(1) follows from (4) and (5). □

6. Manifolds of C^M-mappings

6.1. **C^M-manifolds.** Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth. A C^M-manifold is a smooth manifold such that all chart changings are C^M-mappings. Likewise for C^M-bundles and C^M Lie groups.

Note that any finite dimensional (always assumed paracompact) C^∞-manifold admits a C^∞-diffeomorphic real analytic structure thus also a C^M-structure. Maybe, any finite dimensional C^M-manifold admits a C^M-diffeomorphic real analytic structure.

6.2. **Spaces of C^M-sections.** Let $E \to B$ be a C^M vector bundle (possibly infinite dimensional). The space $C^M(B \leftarrow E)$ of all C^M sections is a convenient vector space with the structure induced by

$$C^M(B \leftarrow E) \to \prod_\alpha C^M(u_\alpha(U_\alpha), V)$$
$$s \mapsto \operatorname{pr}_2 \circ \psi_\alpha \circ s \circ u_\alpha^{-1}$$

where $B \supseteq U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subset W$ is a C^M-atlas for B which we assume to be modelled on a convenient vector space W, and where $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$ form a vector bundle atlas over charts U_α of B.

Lemma. *For a C^M vector bundle $E \to B$ a curve $c : \mathbb{R} \to C^M(B \leftarrow E)$ is C^M if and only if $c^\wedge : \mathbb{R} \times B \to E$ is C^M.*

Proof. By the description of the structure on $C^M(B \leftarrow E)$ we may assume that B is c^∞-open in a convenient vector space W and that $E = B \times V$. Then $C^M(B \leftarrow B \times V) \cong C^M(B, V)$. Then the statement follows from the exponential law (5.3). □

An immediate consequence is the following: If $U \subset E$ is an open neighborhood of $s(B)$ for a section s, $F \to B$ is another vector bundle and if $f : U \to F$ is a fiber respecting C^M mapping, then $f_* : C^M(B \leftarrow U) \to C^M(B \leftarrow F)$ is C^M on the open neighborhood $C^M(B \leftarrow U)$ of s in $C^M(B \leftarrow E)$. We have $(d(f_*)(s)v)_x = d(f|_{U \cap E_x})(s(x))(v(x))$.

6.3. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth. Let A and B be finite dimensional C^M manifolds with A compact. Then the space $C^M(A, B)$ of all C^M-mappings $A \to B$ is a C^M-manifold modelled on convenient vector spaces $C^M(A \leftarrow f^*TB)$ of C^M sections of pullback bundles along $f : A \to B$. Moreover, a curve $c : \mathbb{R} \to C^M(A, B)$ is C^M if and only if $c^\wedge : \mathbb{R} \times A \to B$ is C^M.*

Proof. Choose a C^M Riemannian metric on B which exists since we have C^M partitions of unity. C^M-vector fields have C^M-flows by [15]; applying this to the geodesic spray we get the C^M exponential mapping $\exp : TB \supseteq U \to B$ of this Riemannian metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_B, \exp) : U \to B \times B$ is a C^M diffeomorphism onto an open neighborhood V of the diagonal, by the C^M inverse function theorem due to [14].

For $f \in C^M(A, B)$ we consider the pullback vector bundle

$$\begin{array}{ccccc} A \times_B TB & = & f^*TB & \xrightarrow{\pi_B^* f} & TB \\ & & {\scriptstyle f^*\pi_B}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ & & A & \xrightarrow{f} & B \end{array}$$

Then $C^M(A \leftarrow f^*TB)$ is canonically isomorphic to the space $C^M(A,TB)_f := \{h \in C^M(A,TB) : \pi_B \circ h = f\}$ via $s \mapsto (\pi_B^* f) \circ s$ and $(\mathrm{Id}_A, h) \leftarrow\!\shortmid h$. Now let

$$U_f := \{g \in C^M(A,B) : (f(x),\ g(x)) \in V \text{ for all } x \in A\},$$
$$u_f : U_f \to C^M(A \leftarrow f^*TB),$$
$$u_f(g)(x) = (x, \exp_{f(x)}^{-1}(g(x))) = (x, ((\pi_B, \exp)^{-1} \circ (f,g))(x)).$$

Then u_f is a bijective mapping from U_f onto the set $\{s \in C^M(A \leftarrow f^*TB) : s(A) \subseteq f^*U = (\pi_B^* f)^{-1}(U)\}$, whose inverse is given by $u_f^{-1}(s) = \exp \circ (\pi_B^* f) \circ s$, where we view $U \to B$ as fiber bundle. The push forward u_f is C^M since it maps C^M-curves to C^M-curves by lemma (6.2). The set $u_f(U_f)$ is open in $C^M(A \leftarrow f^*TB)$ for the topology described above in (6.2).

Now we consider the atlas $(U_f, u_f)_{f \in C^M(A,B)}$ for $C^M(A,B)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C^M(A \leftarrow g^*TB)$ by

$$\begin{aligned}(u_f \circ u_g^{-1})(s) &= (\mathrm{Id}_A, (\pi_B, \exp)^{-1} \circ (f, \exp \circ (\pi_B^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s),\end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$ is a C^M diffeomorphism $\tau_g : g^*TB \supseteq g^*U \to (g \times \mathrm{Id}_B)^{-1}(V) \subseteq A \times B$ which is fiber respecting over A. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and it is also C^M since it respects C^M-curves.

Finally for the topology on $C^M(A,B)$ we take the identification topology from this atlas (with the c^∞-topologies on the modeling spaces), which is obviously finer than the compact-open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the C^M structure does not depend on the choice of the C^M Riemannian metric on B.

The statement on C^M-curves follows from lemma (6.2). □

6.4. **Corollary.** *Let A_1, A_2 and B be finite dimensional C^M manifolds with A_1 and A_2 compact. Then composition*

$$C^M(A_2, B) \times C^M(A_1, A_2) \to C^M(A_1, B), \quad (f,g) \mapsto f \circ g$$

is C^M. However, if $N = (N_k)$ is another non-quasianalytic DC-weight sequence of moderate growth with $(N_k/M_k)^{1/k} \searrow 0$ then composition is **not** *C^N.*

Proof. Composition maps C^M-curves to C^M-curves, so it is C^M.

Let $A_1 = A_2 = S^1$ and $B = \mathbb{R}$. Then by (2.1.5) there exists $f \in C^M(S^1, \mathbb{R}) \setminus C^N(S^1, \mathbb{R})$. We consider $f : \mathbb{R} \to \mathbb{R}$ periodic. The universal covering space of $C^M(S^1, S^1)$ consists of all $2\pi\mathbb{Z}$-equivariant mappings in $C^M(\mathbb{R}, \mathbb{R})$, namely the space of all $g + \mathrm{Id}_{\mathbb{R}}$ for 2π-periodic $g \in C^M$. Thus $C^M(S^1, S^1)$ is a real analytic manifold and $t \mapsto (x \mapsto x + t)$ induces a real analytic curve c in $C^M(S^1, S^1)$. But $f_* \circ c$ is not C^N since:

$$\frac{(\partial_t^k|_{t=0}(f_* \circ c)(t))(x)}{k!\rho^k N_k} = \frac{\partial_t^k|_{t=0} f(x+t)}{k!\rho^k N_k} = \frac{f^{(k)}(x)}{k!\rho^k N_k}$$

which is unbounded for x in a suitable compact set and for all $\rho > 0$ since $f \notin C^N$. □

6.5. **Theorem.** *Let $M = (M_k)$ be a non-quasianalytic DC-weight sequence of moderate growth. Let A be a compact ($\Longrightarrow$ finite dimensional) C^M manifold. Then the group $\mathrm{Diff}^M(A)$ of all C^M-diffeomorphisms of A is an open subset of the C^M manifold $C^M(A,A)$. Moreover, it is a C^M-regular C^M Lie group: Inversion and composition are C^M. Its Lie algebra consists of all C^M-vector fields on A, with the* *negative of the usual bracket as Lie bracket. The exponential mapping is C^M. It is not surjective onto any neighborhood of* Id_A.

Following [20], see also [19, 38.4], a C^M-Lie group G with Lie algebra $\mathfrak{g} = T_eG$ is called C^M-regular if the following holds:

- For each C^M-curve $X \in C^M(\mathbb{R}, \mathfrak{g})$ there exists a C^M-curve $g \in C^M(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^M(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^M also.

Proof. The group $\mathrm{Diff}^M(A)$ is open in $C^M(A, A)$ since it is open in the coarser C^1 compact open topology, see [19, 43.1]. So $\mathrm{Diff}^M(A)$ is a C^M-manifold and composition is C^M by (6.3) and (6.4). To show that inversion is C^M let c be a C^M-curve in $\mathrm{Diff}^M(A)$. By (6.3) the map $c^\wedge : \mathbb{R} \times A \to A$ is C^M and $(\mathrm{inv} \circ c)^\wedge : \mathbb{R} \times A \to A$ satisfies the finite dimensional implicit equation $c^\wedge(t, (\mathrm{inv} \circ c)^\wedge(t, x)) = x$ for $x \in A$. By the finite dimensional C^M implicit function theorem [14] the mapping $(\mathrm{inv} \circ c)^\wedge$ is locally C^M and thus C^M. By (6.3) again, $\mathrm{inv} \circ c$ is a C^M-curve in $\mathrm{Diff}^M(A)$. So $\mathrm{inv} : \mathrm{Diff}^M(A) \to \mathrm{Diff}^M(A)$ is C^M. The Lie algebra of $\mathrm{Diff}^M(A)$ is the convenient vector space of all C^M-vector fields on A, with the negative of the usual Lie bracket (compare with the proof of [19, 43.1]).

To show that $\mathrm{Diff}^M(A)$ is a C^M-regular Lie group, we choose a C^M-curve in the space of C^M-curves in the Lie algebra of all C^M vector fields on A, $c : \mathbb{R} \to C^M(\mathbb{R}, C^M(A \leftarrow TA))$. By lemma (6.2) c corresponds to a $\mathbb{R}^2$-time-dependent C^M vector field $c^{\wedge\wedge} : \mathbb{R}^2 \times A \to TA$. Since C^M-vector fields have C^M-flows and since A is compact, $\mathrm{evol}^r(c^\wedge(s))(t) = \mathrm{Fl}_t^{c^\wedge(s)}$ is C^M in all variables by [15]. Thus $\mathrm{Diff}^M(A)$ is a C^M-regular C^M Lie group.

The exponential mapping is evol^r applied to constant curves in the Lie algebra, i.e., it consists of flows of autonomous C^M vector fields. That the exponential map is not surjective onto any C^M-neighborhood of the identity follows from [19, 43.5] for $A = S^1$. This example can be embedded into any compact manifold, see [9]. □

7. Appendix. Calculus beyond Banach spaces

The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces we sketch here the convenient approach as explained in [8] and [19]. The main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. We use the notation of [19] and this is the main reference for the whole appendix. We list results in the order in which one can prove them, without proofs for which we refer to [19]. This should explain how to use these results.

7.1. **The c^∞-topology.** Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.

(2) The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).
(3) The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

7.2. **Convenient vector spaces.** A locally convex vector space E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(1) For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
(2) Any Lipschitz curve in E is locally Riemann integrable.
(3) A curve $c : \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E^*$, where E^* is the dual consisting of all continuous linear functionals on E. Equivalently, we may use the dual E' consisting of all bounded linear functionals.
(4) Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.
(5) If B is bounded closed absolutely convex, then E_B is a Banach space.
(6) If $f : \mathbb{R} \to E$ is scalarwise $\mathcal{L}\mathrm{ip}^k$, then f is $\mathcal{L}\mathrm{ip}^k$, for $k > 1$.
(7) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.
(8) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.

Here a mapping $f : \mathbb{R} \to E$ is called $\mathcal{L}\mathrm{ip}^k$ if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

7.3. **Smooth mappings.** Let E, F, and G be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. *The main properties of smooth calculus are the following.*

(1) *For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Even on* $\mathbb{R}^2$ *this is non-trivial.*
(2) *Multilinear mappings are smooth if and only if they are bounded.*
(3) *If* $f : E \supseteq U \to F$ *is smooth then the derivative* $df : U \times E \to F$ *is smooth, and also* $df : U \to L(E, F)$ *is smooth where* $L(E, F)$ *denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.*
(4) *The chain rule holds.*
(5) *The space* $C^\infty(U, F)$ *is again a convenient vector space where the structure is given by the obvious injection*

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R},U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ *carries the topology of compact convergence in each derivative separately.*

(6) *The exponential law holds: For c^∞-open $V \subset F$,*

$$C^\infty(U, C^\infty(V,G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces. Note that this is the main assumption of variational calculus.

(7) *A linear mapping $f : E \to C^\infty(V,G)$ is smooth (bounded) if and only if $E \xrightarrow{f} C^\infty(V,G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. This is called the smooth uniform boundedness theorem* [19, 5.26].

(8) *The following canonical mappings are smooth.*

$$\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G)\\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\ \ ,\ \ ) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

7.4. **Remarks.** Note that the conclusion of (7.3.6) is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more. It is also the source of the name convenient calculus. This and some other obvious properties already determines the convenient calculus.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E \times E^* \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

8. Calculus of holomorphic mappings

8.1. **Holomorphic curves.** Let E be a complex locally convex vector space whose underlying real space is convenient – this will be called convenient in the sequel. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and let us denote by $\mathcal{H}(\mathbb{D}, E)$ the space of all mappings $c : \mathbb{D} \to E$ such that $\lambda \circ c : \mathbb{D} \to \mathbb{C}$ is holomorphic for each continuous complex-linear functional λ on E. Its elements will be called the holomorphic curves.

If E and F are convenient complex vector spaces (or c^∞-open sets therein), a mapping $f : E \to F$ is called *holomorphic* if $f \circ c$ is a holomorphic curve in F for each holomorphic curve c in E. Obviously f is holomorphic if and only if $\lambda \circ f : E \to \mathbb{C}$ is holomorphic for each complex linear continuous (equivalently: bounded) functional λ on F. Let $\mathcal{H}(E,F)$ denote the space of all holomorphic mappings from E to F.

8.2. **Lemma.** *(Hartog's theorem) Let E_k for $k = 1,2$ and F be complex convenient vector spaces and let $U_k \subset E_k$ be c^∞-open. A mapping $f : U_1 \times U_2 \to F$ is holomorphic if and only if it is separately holomorphic (i. e. $f(\ \ ,y)$ and $f(x,\ \ )$ are holomorphic for all $x \in U_1$ and $y \in U_2$).*

This implies also that in finite dimensions we have recovered the usual definition.

8.3. **Lemma.** *If $f : E \supset U \to F$ is holomorphic then $df : U \times E \to F$ exists, is holomorphic and $\mathbb{C}$-linear in the second variable.*

A multilinear mapping is holomorphic if and only if it is bounded.

8.4. **Lemma.** *If E and F are Banach spaces and U is open in E, then for a mapping $f : U \to F$ the following conditions are equivalent:*

(1) *f is holomorphic.*
(2) *f is locally a convergent series of homogeneous continuous polynomials.*
(3) *f is $\mathbb{C}$-differentiable in the sense of Fréchet.*

8.5. **Lemma.** *Let E and F be convenient vector spaces. A mapping $f : E \to F$ is holomorphic if and only if it is smooth and its derivative in each point is $\mathbb{C}$-linear.*

An immediate consequence of this result is that $\mathcal{H}(E, F)$ is a closed linear subspace of $C^\infty(E_\mathbb{R}, F_\mathbb{R})$ and so it is a convenient vector space if F is one, by (7.3.5). The chain rule follows from (7.3.4).

8.6. **Theorem.** *The category of convenient complex vector spaces and holomorphic mappings between them is cartesian closed, i. e.*

$$\mathcal{H}(E \times F, G) \cong \mathcal{H}(E, \mathcal{H}(F, G)).$$

An immediate consequence of this is again that all canonical structural mappings as in (7.3.8) are holomorphic.

9. Calculus of real analytic mappings

9.1. We now sketch the cartesian closed setting to real analytic mappings in infinite dimension following the lines of the Frölicher–Kriegl calculus, as it is presented in [19]. Surprisingly enough one has to deviate from the most obvious notion of real analytic curves in order to get a meaningful theory, but again convenient vector spaces turn out to be the right kind of spaces.

9.2. **Real analytic curves.** Let E be a real convenient vector space with continuous dual E^*. A curve $c : \mathbb{R} \to E$ is called *real analytic* if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for each $\lambda \in E^*$. It turns out that the set of these curves depends only on the bornology of E. Thus we may use the dual E' consisting of all bounded linear functionals in the definition.

In contrast a curve is called *strongly real analytic* if it is locally given by power series which converge in the topology of E. They can be extended to germs of holomorphic curves along $\mathbb{R}$ in the complexification $E_\mathbb{C}$ of E. If the dual E^* of E admits a Baire topology which is compatible with the duality, then each real analytic curve in E is in fact topologically real analytic for the bornological topology on E.

9.3. **Real analytic mappings.** Let E and F be convenient vector spaces. Let U be a c^∞-open set in E. A mapping $f : U \to F$ is called *real analytic* if and only if it is smooth (maps smooth curves to smooth curves) and maps real analytic curves to real analytic curves.

Let $C^\omega(U, F)$ denote the space of all real analytic mappings. We equip the space $C^\omega(U, \mathbb{R})$ of all real analytic functions with the initial topology with respect to the families of mappings

$$C^\omega(U, \mathbb{R}) \xrightarrow{c^*} C^\omega(\mathbb{R}, \mathbb{R}), \text{ for all } c \in C^\omega(\mathbb{R}, U)$$
$$C^\omega(U, \mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R}, \mathbb{R}), \text{ for all } c \in C^\infty(\mathbb{R}, U),$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately, and where $C^\omega(\mathbb{R}, \mathbb{R})$ is equipped with the final locally convex topology with respect to the embeddings (restriction mappings) of all spaces of holomorphic mappings from a neighborhood V of $\mathbb{R}$ in $\mathbb{C}$ mapping $\mathbb{R}$ to $\mathbb{R}$, and each of these spaces carries the topology of compact convergence.

Furthermore we equip the space $C^\omega(U,F)$ with the initial topology with respect to the family of mappings

$$C^\omega(U,F) \xrightarrow{\lambda_*} C^\omega(U,\mathbb{R}), \text{ for all } \lambda \in F^*.$$

It turns out that this is again a convenient space.

9.4. **Theorem.** *In the setting of* (9.3) *a mapping $f : U \to F$ is real analytic if and only if it is smooth and is real analytic along each affine line in E.*

9.5. **Lemma.** *The space $L(E,F)$ of all bounded linear mappings is a closed linear subspace of $C^\omega(E,F)$. A mapping $f : U \to L(E,F)$ is real analytic if and only if $\operatorname{ev}_x \circ f : U \to F$ is real analytic for each point $x \in E$.*

9.6. **Theorem.** *The category of convenient spaces and real analytic mappings is cartesian closed. So the equation*

$$C^\omega(U, C^\omega(V,F)) \cong C^\omega(U \times V, F)$$

is valid for all c^∞-open sets U in E and V in F, where E, F, and G are convenient vector spaces.

This implies again that all structure mappings as in (7.3.8) are real analytic. Furthermore the differential operator

$$d : C^\omega(U,F) \to C^\omega(U, L(E,F))$$

exists, is unique and real analytic. Multilinear mappings are real analytic if and only if they are bounded.

9.7. **Theorem** (Real analytic uniform boundedness principle)**.** *A linear mapping $f : E \to C^\omega(V,G)$ is real analytic (bounded) if and only if $E \xrightarrow{f} C^\omega(V,G) \xrightarrow{\operatorname{ev}_v} G$ is real analytic (bounded) for each $v \in V$.*

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