Whitney manifold germs aka Manifolds with wild bounday

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https://www.mat.univie.ac.at/~michor/lectures.html

Abstract: During the preparation of a foundational chapter on manifolds of mappings for a book on geometric continuum mechanics I found out that the following object behaves surprisingly well as source of a manifold of mappings:

— A Whitney manifold germ $\tilde{M} \supset M$ consists of an open manifold $\tilde{M}$ together with a closed subset $M \subset \tilde{M}$ which is the closure of its open interior, such that there exists a continuous linear extension operator from the space of Whitney jets on M to the space of smooth functions $C^\infty(\tilde{M})$, with their natural locally convex topologies. This concept is local in $\tilde{M}$, due to recent advances for the existence of continuous Whitney extension operators by D. Vogt, M. Tidten, L. Frerick, and J. Wengenroth. This notion is more general than all existing notions: domains with Lipschitz boundary or Hölder boundary, the manifolds with rough boundary of Roberts and Schmeding.

— The following concepts are very well behaved: Smooth mappings into manifolds. Vector bundles. Fiber bundles. The space of vector fields on M tangent to the boundary is a convenient Lie algebra, with "Lie group" (in a weakened sense) the group of diffeomorphisms of M.

Slides: https://www.mat.univie.ac.at/~michor/lectures.html

Based on:

Peter W. Michor: Manifolds of mappings for continuum mechanics. In the book: Geometric Continuum Mechanics. Editors: Reuven Segev, Marcelo Epstein. Series: Advances in Continuum Mechanics, Vol. 42. pp. 3-75. Birkhäuser Basel 2020. [https://www.mat.univie.ac.at/~michor/ManifoldsOfMappings.pdf]

The infinite dimensional geometry used here is based on:
Andreas Kriegl, Peter W. Michor: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, Volume: 53, Amer. Math. Soc., 1997

See also:
Wikipedia [https://en.wikipedia.org/wiki/Convenient_vector_space]

Slides: https://www.mat.univie.ac.at/~michor/lectures.html

Compact Whitney subsets

Let $\tilde{M}$ be an open smooth m-dim. manifold. A closed connected subset $M \subset \tilde{M}$ is called a *Whitney subset*, or $\tilde{M} \supset M$ is called a *Whitney pair*, if

- M is the closure of its open interior in $\tilde{M}$, and:
- There exists a continuous linear extension operator

  $$\mathcal{E} : \mathcal{W}(M) \to C^\infty(\tilde{M}, \mathbb{R})$$

  from the linear space $\mathcal{W}(M)$ of all Whitney jets of infinite order with its natural Fréchet topology (see below) into the space $C^\infty(\tilde{M}, \mathbb{R})$ of smooth functions on $\tilde{M}$ with the Fréchet topology of uniform convergence on compact subsets in all derivatives separately.

If M is compact, we may equivalently require that $\mathcal{E}$ is linear continuous into the Fréchet space $C^\infty_L(\tilde{M}, \mathbb{R}) \subset C^\infty_c(\tilde{M}, \mathbb{R})$ of smooth functions with support in a fixed compact subset L which contains M in its interior, by using a suitable bump function.

More details:

For $\mathbb{R}^m \supset M$, an extension operator $\mathcal{E} : \mathcal{W}(M) \to C^\infty(\tilde{M}, \mathbb{R})$ satisfies $\partial_\alpha \mathcal{E}(F)|_M = F^{(\alpha)}$ for each multi-index $\alpha \in \mathbb{N}^m_{\geq 0}$ and each *Whitney jet* $F \in \mathcal{W}(M)$: If $M \subset \mathbb{R}^m$ is compact, then

$$F = (F^{(\alpha)})_{\alpha \in \mathbb{N}^m_{\geq 0}} \in \prod_\alpha C^0(M) \qquad \text{such that for}$$

$$T^n_y(F)(x) = \sum_{|\alpha| \leq n} \frac{F^{(\alpha)}(y)}{\alpha!}(x-y)^\alpha \qquad \text{the remainder seminorm}$$

$$q_{n,\varepsilon}(F) := \sup\left\{ \frac{|F^{(\alpha)}(x) - \partial^\alpha T^n_y(F)(x)|}{|x-y|^{n-|\alpha|}} : {\scriptstyle |\alpha| \leq n, x,y \in M \atop 0 < |x-y| \leq \varepsilon} \right\} = o(\varepsilon);$$

so $q_{n,\varepsilon}(F)$ goes to 0 for $\varepsilon \to 0$, for each n separately.
The n-th Whitney seminorm is then

$$\|F\|_n = \sup\{|F^{(\alpha)}(x)| : x \in M, |\alpha| \leq n\} + \sup\{q_{n,\varepsilon}(F) : \varepsilon > 0\}.$$

For closed but non-compact M one uses the projective limit over a countable compact exhaustion of M. This describes the natural Fréchet topology on the space of Whitney jets for closed subsets of $\mathbb{R}^m$. The extension to manifolds is obvious.

Whitney proved in 1934 that a linear extension operator always exists for a closed subset $M \subset \mathbb{R}^m$, but not always a continuous one, for example for M a point. For a finite differentiability class C^n there exists always a continuous extension operator.

The property of being a Whitney pair is obviously invariant under diffeomorphisms (of $\tilde{M}$ respecting M) which act linearly and continuously both on $\mathcal{W}(M)$ and $C^\infty(\tilde{M}, \mathbb{R})$ in a natural way.

Proposition. *For a Whitney pair $\tilde{M} \supset M$, the space of $\mathcal{W}(M)$ of Whitney jets on M is linearly isomorphic to the space*

$$C^\infty(M, \mathbb{R}) := \{f|_M : f \in C^\infty(\tilde{M}, \mathbb{R})\} .$$

Proof. Frerick proved in 2007 (3.11): If $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ vanishes on a Whitney subset $M \subset \mathbb{R}^m$, then $\partial^\alpha f|_M = 0$ for each multi-index α. Thus any continuous extension operator is injective. □

The property of being a Whitney pair is local in the following sense: If $\tilde{M}_i \supset M_i$ covers $\tilde{M} \supset M$, then $\tilde{M} \supset M$ is a Whitney pair if and only if each $\tilde{M}_i \supset M_i$ is a Whitney pair, see the Localization Theorem below.

Examples and counterexamples of Whitney pairs

(a) If M is a manifold with corners, then $\tilde{M} \supset M$ is a Whitney pair. This follows from Mitjagin 1961 or Seeley 1964.
(b) If M is closed in $\mathbb{R}^m$ with dense interior and with Lipschitz boundary, then $\mathbb{R}^m \supset M$ is a Whitney pair; Stein 1970. Bierstone 1978 proved that a closed subset $M \subset \mathbb{R}^n$ with dense interior is a Whitney pair, if it has Hölder $C^{0,\alpha}$-boundary for $0 < \alpha \leq 1$ which may be chosen on each $M \cap \{x : N \leq |x| \leq N+2\}$ separately. A fortiori, each subanalytic subset in $\mathbb{R}^n$ gives a Whitney pair.
(c) If $f \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ is flat at 0, and if M is a closed subset containing 0 of $\{(x,y) : x \geq 0, |y| \leq |f(x)|\} \subset \mathbb{R}^2$, then $\mathbb{R}^2 \supset M$ is *not* a Whitney pair; Tidten 1979.
(d) For $r \geq 1$, the set $\{x \in \mathbb{R}^m : 0 \leq x_1 \leq 1, x_2^2 + \cdots + x_m^2 \leq x_1^{2r}\}$ is called the parabolic cone of order r. Then (Tidten 1979):
A closed subset $M \subset \mathbb{R}^m$ is a Whitney subset, if: For each compact $K \subset \mathbb{R}^m$ there exists a parabolic cone S and a family $\varphi_i : S \to \phi_i(S) \subset M \subset \mathbb{R}^m$ of diffeos such that $K \cap M \subseteq \overline{\bigcup_i \varphi_i(S)}$ and $\sup_i |\varphi_i|_k < \infty$, $\sup_i |\varphi_i^{-1}|_k < \infty$ for each k separately.

(e) A characterization is due to Frerick 2007, using local Markov inequalities, which however, is very difficult to check directly.
Let $M \subset \mathbb{R}^m$ be closed. Then the following are equivalent:

(e1) *M admits a continuous linear Whitney extension operator*

$$\mathcal{E} : \mathcal{W}(M) \to C^\infty(\mathbb{R}^m, \mathbb{R}) .$$

(e2) *For each compact $K \subset M$ and $\theta \in (0,1)$ there is $r \geq 0$ and $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}_{\geq 1}$ there is $C \geq 1$ such that*

$$|dp(x_0)| \leq \frac{C}{\varepsilon^r} \sup_{\substack{|y-x_0|\leq\varepsilon \\ y\in\mathbb{R}^m}} |p(y)|^\theta \sup_{\substack{|x-x_0|\leq\varepsilon \\ x\in M}} |p(x)|^{1-\theta}$$

for all $p \in \mathbb{C}[x_1, \dots, x_m]$ of degree $\leq k$, for all $x_0 \in K$, and for all $\varepsilon_0 > \varepsilon > 0$.

(e3) *For each compact $K \subset M$ there exists a compact L in $\mathbb{R}^m$ containing K in its interior, such that for all $\theta \in (0,1)$ there is $r \geq 1$ and $C \geq 1$ such that*

$$\sup_{x\in K} |dp(x)| \leq C \deg(p)^r \sup_{y\in L} |p(y)|^\theta \sup_{z\in L\cap M} |p(z)|^{1-\theta}$$

for all $p \in \mathbb{C}[x_1, \dots, x_m]$.

(f) The characterization (e) has been generalized to a characterization of compact subsets of $\mathbb{R}^m$ which admit a continuous Whitney extension operator with linear (or even affine) loss of derivatives, by FJW 2011. FJW 2013 gave a similar characterization for an extension operator without loss of derivative, and a sufficient geometric condition is formulated which even implies that there are closed sets with empty interior admitting continuous Whitney extension operators, like the Sierpiński triangle or Cantor subsets. Thus we cannot omit the assumption that M is the closure of its open interior in $\tilde{M}$ in our definition of Whitney pairs.

(g) Theorem 3.15 in [Frerick07] gives an easily verifiable sufficient condition:
Let $K \subset \mathbb{R}^m$ be compact and assume that there exist $\varepsilon_0 > 0$, $\rho > 0$, $r \geq 1$ such that for all $z \in \partial K$ and $0 < \varepsilon < \varepsilon_0$ there is $x \in K$ with $B_{\rho\varepsilon^r}(x) \subset K \cap B_\varepsilon(z)$. Then K admits a continuous linear Whitney extension operator $\mathcal{W}(F) \to C^\infty(\mathbb{R}^m, \mathbb{R})$.
This implies (a), (b), and (d).

Localization Theorem

Let $\tilde{M}$ be an open manifold and let $M \subset \tilde{M}$ be a compact subset that is the closure of its open interior. $M \subset \tilde{M}$ is a Whitney pair if and only if for one (equivalently, every) smooth atlas $(\tilde{M} \supset U_\alpha, u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subset \mathbb{R}^m)_{\alpha \in A}$ of the open manifold $\tilde{M}$, each $u_\alpha(M \cap U_\alpha) \subset u_\alpha(U_\alpha)$ is a Whitney pair.

Consequently, for a Whitney pair $M \subset \tilde{M}$ and $U \subset \tilde{M}$ open, $M \cap U \subset \tilde{M} \cap U$ is also a Whitney pair.

The difficult direction of the proof is inspired by a suggestion of Frerick and Wengenroth.

Proof. (1) The easy direction. We consider a locally finite countable smooth atlas $(\tilde{M} \supset U_\alpha, u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subset \mathbb{R}^m)_{\alpha \in \mathbb{N}}$ of $\tilde{M}$ such that each $u_\alpha(U_\alpha) \supset u_\alpha(M \cap U_\alpha)$ is a Whitney pair.

We use a smooth 'partition of unity' $f_\alpha \in C_c^\infty(U_\alpha, \mathbb{R}_{\geq 0})$ on $\tilde{M}$ with $\sum_\alpha f_\alpha^2 = 1$. The following mappings induce linear embeddings onto closed direct summands of the Fréchet spaces:

$$
\begin{array}{ccc}
C^\infty(\tilde{M}, \mathbb{R}) & \underset{\sum_\alpha f_\alpha . g_\alpha \leftarrow (g_\alpha)_\alpha}{\overset{f \mapsto (f_\alpha . f)_\alpha}{\rightleftarrows}} & \prod_\alpha C^\infty(U_\alpha, \mathbb{R}) \\
\mathcal{W}(M) & \rightleftarrows & \prod_\alpha \mathcal{W}(U_\alpha \cap M)
\end{array}
$$

If each $u_\alpha(U_\alpha) \supset u_\alpha(U_\alpha \cap M)$ is a Whitney pair, then so is $U_\alpha \supset U_\alpha \cap M$, via the isomorphisms induced by u_α, and

$$
\begin{array}{ccc}
\mathcal{W}(M) & \xrightarrow{f \mapsto (f_\alpha . f)_\alpha} & \prod_\alpha \mathcal{W}(U_\alpha \cap M) \\
 & & \Big\downarrow \prod_\alpha \mathcal{E}_\alpha \\
C^\infty(\tilde{M}, \mathbb{R}) & \xleftarrow{\sum_\alpha f_\alpha . g_\alpha \leftarrow (g_\alpha)_\alpha} & \prod_\alpha C^\infty(U_\alpha, \mathbb{R})
\end{array}
$$

is a continuous Whitney extension operator, so that $\tilde{M} \supset M$ is a Whitney pair. This proves the easy direction of the theorem.

(2) (Meise-Vogt 1997) A Fréchet space E is said to have *property* (DN) if for one (equivalently, any) increasing system $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms generating the topology the following holds:

- There exists a continuous seminorm $\|\ \|$ on E (called a *dominating norm*) such that for all (equivalently, one) $0<\theta<1$ and all $m\in\mathbb{N}$ there exist $k\in\mathbb{N}$ and $C>0$ with

$$\|\ \|_m \le C\|\ \|_k^{\theta}\cdot\|\ \|^{1-\theta}.$$

The property (DN) is an isomorphism invariant and is inherited by closed linear subspaces. The Fréchet space $\mathfrak{s}$ of rapidly decreasing sequences has property (DN).

(3) (Tidten 1979) *A closed subset M in $\mathbb{R}^m$ admits a continuous linear extension operator $\mathcal{W}(M)\to C^\infty(\mathbb{R}^m,\mathbb{R})$ if and only if for each $x\in\partial M$ there exists a compact neighborhood K of x in $\mathbb{R}^m$ such that*

$$\mathcal{W}_K(M) := \left\{f\in\mathcal{W}(M) : \operatorname{supp}(f^{(\alpha)})\subset K \textit{ for all } \alpha\in\mathbb{N}^m_{\ge 0}\right\}.$$

has property (DN).

Side remark: On the proof of (3)

A Fréchet space E is said to have *property* (Ω) if the following holds for one (equivalently, any) decreasing basis of neighborhoods $U_1 \supset U_2 \supset \dots$ of 0 consisting of closed absolutely convex sets U_n:

- For each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ with $U_q \subset Cr^n U_k + \frac{1}{r} U_p$ for all $r > 0$.

The property (Ω) is invariant under isomorphisms and is inherited by quotients over closed linear subspaces. A finite product $\prod_{i=1}^{N} E_i$ of Frechet spaces has property (Ω) if and only if each E_i has property (Ω).

The space $\mathfrak{s}$ of rapidly decreasing sequences has properties (DN) and (Ω)

Splitting theorem [Meise-Vogt 1997, Thm 30.1], [Vogt-Wagner 1980, 1981] *Let E, F, G be be Fréchet-Hilbert spaces (i.e., having a fundamental system of Hilbert-seminorms) and let*

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

be a short exact sequence of continuous linear mappings. If E has property (DN) and F has property (Ω), then the sequence splits, i.e., q has a continuous linear right inverse and j has a continuous linear left inverse.

Tidten used this for the sequence

$$0 \to \text{kernel} \to C_K^\infty(M) \to \mathcal{W}_K(M) \to 0$$

and showed that the kernel always has property (Ω).

Still in the proof of the Localization Theorem

We assume from now on that $\tilde{M} \supset M$ is a Whitney pair.

(4) Given a compact set $K \subset \tilde{M}$, let $L \subset \tilde{M}$ be a compact smooth manifold with smooth boundary which contains K in its interior. Let $\tilde{L}$ be the double of L, i.e., L smoothly glued to another copy of L along the boundary; $\tilde{L}$ is a compact smooth manifold containing L as a submanifold with boundary.

Then $C^\infty(\tilde{L}, \mathbb{R})$ is isomorphic to the space $\mathfrak{s}$ of rapidly decreasing sequences: This is due to Vogt 1983. In fact, using a Riemannian metric g on $\tilde{L}$, the expansion in an orthonormal basis of eigenvectors of $1 + \Delta^g$ of a function $h \in L^2$ has coefficients in $\mathfrak{s}$ if and only if $h \in C^\infty(\tilde{L}, \mathbb{R})$, because $1 + \Delta^g : H^{k+2}(\tilde{L}) \to H^k(\tilde{L})$ is an isomorphism for Sobolev spaces H^k with $k \geq 0$, and since the eigenvalues μ_n of Δ^g satisfy $\mu_n \sim C_{\tilde{L}} \cdot n^{2/\dim(\tilde{L})}$ for $n \to \infty$, by Weyl's asymptotic formula. Thus $C^\infty(\tilde{L}, \mathbb{R})$ has property (DN).

Moreover, $C^\infty_L(\tilde M, \mathbb{R}) = \{f \in C^\infty(\tilde M, \mathbb{R}) : \operatorname{supp}(f) \subset L\}$ is a closed linear subspace of $C^\infty(\tilde L, \mathbb{R})$, by extending each function by 0. Thus also $C^\infty_L(\tilde M, \mathbb{R})$ has property (DN).

We choose now a function $g \in C^\infty_L(\tilde M, \mathbb{R}_{\geq 0})$ with $g|_K = 1$ and consider:

$$\begin{array}{ccc}
\mathcal{W}_K(M) & \xrightarrow{\ \mathcal{E}_K\ } & C^\infty_L(\tilde M, \mathbb{R}) \\
\big\downarrow & & \big\uparrow {\scriptstyle f \mapsto g.f} \\
\mathcal{W}(M) & \xrightarrow{\ \mathcal{E}_M\ } & C^\infty(\tilde M, \mathbb{R})
\end{array}$$

The resulting composition $\mathcal{E}_K$ is a continuous linear embedding onto a closed linear subspace of the space $C^\infty_L(\tilde M, \mathbb{R})$ which has (DN). Thus we proved:

(5) **Claim.** *If $\tilde M \supset M$ is a Whitney pair and K is compact in $\tilde M$, the Fréchet space $\mathcal{W}_K(M)$ has property* (DN).

(6) We consider now a smooth chart $\tilde{M} \supset U \xrightarrow{u} u(U) = \mathbb{R}^m$. For $x \in \partial u(M)$ let K be a compact neighborhood of x in $\mathbb{R}^m$. The chart u induces a linear isomorphism

$$\mathcal{W}_K(u(M \cap U)) \xrightarrow{u^*} \mathcal{W}_{u^{-1}(K)}(U \cap M) \cong \mathcal{W}_{u^{-1}(K)}(M),$$

where the right-hand side mapping is given by extending each $f^{(\alpha)}$ by 0. By claim (5) the Fréchet space $W_{u^{-1}(K)}(M)$ has property (DN); consequently also the isomorphic space $\mathcal{W}_K(u(M \cap U))$ has property (DN). By (3) we conclude that $\mathbb{R}^m = u(U) \supset u(M \cap U)$ is a Whitney pair.

(7) If we are given a general chart $\tilde{M} \supset U \xrightarrow{u} u(U) \subset \mathbb{R}^m$, we cover U by a locally finite atlas
$(U \supset U_\alpha, u_\alpha : U_\alpha \to u_\alpha(U_\alpha) = \mathbb{R}^m)_{\alpha \in \mathbb{N}}$. By (6) each
$\mathbb{R}^m = u_\alpha(U_\alpha) \supset u_\alpha(M \cap U_\alpha)$ is a Whitney pair, and by the argument in (1) the pair $U \supset M \cap U$ is a Whitney pair, and thus the diffeomorphic $u(U) \supset u(U \cap M)$ is also a Whitney pair. $\square$

Our use of Whitney pairs

We consider an equivalence class of Whitney pairs $\tilde{M}_i \supset M_i$ for $i = 0, 1$ where $\tilde{M}_0 \supset M_0$ is equivalent to $\tilde{M}_1 \supset M_1$ if there exist an open submanifolds $\tilde{M}_i \supset \hat{M}_i \supset M_i$ and a diffeomorphism $\varphi : \hat{M}_0 \to \hat{M}_1$ with $\varphi(M_0) = M_1$. By a *germ of a Whitney manifold* we mean an equivalence class of Whitney pairs as above. Given a Whitney pair $\tilde{M} \supset M$ and its corresponding germ, we may keep M fixed and equip it with all open connected neighborhoods of M in $\tilde{M}$; each neighborhood is then a representative of this germ; called an *open neighborhood manifold* of M. In the following we shall speak of a *Whitney manifold germ* M and understand that it comes with open manifold neighborhoods $\tilde{M}$. If we want to stress a particular neighborhood we will write $\tilde{M} \supset M$.

The *boundary* ∂M of a Whitney manifold germ is the topological boundary of M in $\tilde{M}$. It can be a quite general set as seen from the examples. But infinitely flat cusps cannot appear.

Other approaches

There are other settings, like the concept of a *manifold with rough boundary*; see Roberts-Schmeding 2018. The main idea there is to start with closed subsets $M \subset \mathbb{R}^m$ with dense interior, to use the space of functions which are C^n in the interior of M such that all partial derivatives extend continuously to M. Then one looks for sufficient conditions (in particular for $n = \infty$) on M such that there exists a continuous Whitney extension operator on the space of these functions, and builds manifolds from that. The condition in Roberts-Schmeding 2018 are in the spirit of (d). By extending these functions and restricting their jets to M we see that manifolds with rough boundary are Whitney manifold germs.

Another possibility is to consider closed subsets $M \subset \mathbb{R}^m$ with dense interior such there exists a continuous linear extension operator on the space $C^\infty(M) = \{f|_M : f \in C^\infty(\mathbb{R}^m)\}$ with the quotient locally convex topology. These are exactly the Whitney manifold pairs $\mathbb{R}^m \supset M$, by [Frerick 2007, line before 3.9 and 3.10, 3.11]; see the proposition above.

In this setting, for C^n with $n < \infty$ there exist continuous extension operators $C^n_b(M) \to C^n_b(\mathbb{R}^m)$ (where the subscript b means bounded for all derivatives separately) for arbitrary subsets $M \subset \mathbb{R}^m$; see Feffermann 2007.

We believe that our use of Whitney manifold germs is quite general, simple, and avoids many technicalities. But it is aimed at the case C^∞; for C^k or $W^{k,p}$ other approaches, like the one in Roberts-Schmeding 2018, might be more appropriate.

Tangent vectors and vector fields on Whitney manifold germs

Just like for vector bundles below, we define the tangent bundle TM of a Whitney manifold germ M as the restriction $TM = T\tilde{M}|_M$. For $x \in \partial M$, a tangent vector $X_x \in T_xM$ is said to be *interior pointing* if there exist a curve $c : [0, 1) \to M$ which is smooth into $\tilde{M}$ with $c'(0) = X_x$. And $X_x \in T_xM$ is called *tangent to the boundary* if there exists a curve $c : (-1, 1) \to \partial M$ which is smooth into $\tilde{M}$ with $c'(0) = X_x$. The *space of vector fields on M* is given as

$$\mathfrak{X}(M) = \{X|_M : X \in \mathfrak{X}(\tilde{M})\}.$$

Using a continuous linear extension operator, $\mathfrak{X}(M)$ is isomorphic to a locally convex direct summand in $\mathfrak{X}(\tilde{M})$. If M is a compact Whitney manifold germ, $\mathfrak{X}(M)$ is a direct summand even in $\mathfrak{X}_L(\tilde{M}) = \{X \in \mathfrak{X}(\tilde{M}) : (X) \subseteq L\}$ where $L \subset \tilde{M}$ is a compact set containing M in its interior.

We define the *space of vector fields on M which are tangent to the boundary* as

$$\mathfrak{X}_\partial(M) = \big\{X|_M : X \in \mathfrak{X}(\tilde{M}), x \in \partial M \implies \mathrm{Fl}^X_t(x) \in \partial M \text{ for all } t \text{ for which } \mathrm{Fl}^X_t(x) \text{ exists in } \tilde{M}\big\}\,,$$

where Fl^X_t denotes the flow mapping of the vector field X up to time t which is locally defined on $\tilde{M}$. Obviously, for $X \in \mathfrak{X}_\partial(M)$ and $x \in \partial M$ the tangent vector $X(x)$ is tangent to the boundary in the sense defined above. I have no proof that the converse is true:

Question. *Suppose that $X \in \mathfrak{X}(\tilde{M})$ has the property that for each $x \in \partial M$ the tangent vector $X(x)$ is tangent to the boundary. Is it true that then $X|_M \in \mathfrak{X}_\partial(M)$?*

A related question for which I have no answer is:

Question. *For each $x \in \partial M$ and tangent vector $X_x \in T_xM$ which is tangent to the boundary, is there a smooth vector field $X \in \mathfrak{X}_{c,\partial}(M)$ with $X(x) = X_x$?*

Lemma

For a Whitney manifold germ M, the space $\mathfrak{X}_\partial(M)$ of vector field tangent to the boundary is a closed linear sub Lie algebra of $\mathfrak{X}(M)$. The space $\mathfrak{X}_{c,\partial}(M)$ of vector fields with compact support tangent to the boundary is a closed linear sub Lie algebra of $\mathfrak{X}_c(M)$.

Proof. By definition, for $X \in \mathfrak{X}(\tilde{M})$ the restriction $X|_M$ is in $\mathfrak{X}_\partial(M)$ if and only if $x \in \partial M$ implies that $\mathrm{Fl}^X_t(x) \in \partial M$ for all t for which $\mathrm{Fl}^X_t(x)$ exists in $\tilde{M}$. These conditions describe a set of continuous equations, since $(X, t, x) \mapsto \mathrm{Fl}^X_t(x)$ is smooth. Thus $X \in \mathfrak{X}(\tilde{M})$ is closed.

The formulas (see, e.g., Nelson 1969)

$$\lim_{n\to\infty} (Fl^X_{t/n} \circ \mathrm{Fl}^Y_{t/n})^n(x) = \mathrm{Fl}^{X+Y}_t(x)$$

$$\lim_{n\to\infty} \Big(\mathrm{Fl}^Y_{-(t/n)^{1/2}} \circ \mathrm{Fl}^X_{-(t/n)^{1/2}} \circ \mathrm{Fl}^Y_{(t/n)^{1/2}} \circ \mathrm{Fl}^X_{(t/n)^{1/2}} \Big)^n(x) = \mathrm{Fl}^{[X,Y]}_t(x)$$

shows that $\mathfrak{X}_\partial(M)$ is a Lie subalgebra. □

Smooth mappings

Whitney jets on M naively make sense only if they take values in a vector space or, more generally, in a vector bundle. We use a closed embedding $i : N \to \mathbb{R}^p$, a tubular neighborhood $p : U \to i(N)$ and a bump function $g \in C^\infty(\mathbb{R}^p, [0,1])$ such that $g = 1$ near $i(N)$ with support in U.
Consider a Whitney jet F on M with values in $\mathbb{R}^p$ such that the 0-order part lies in $i(N)$. Using a continuous Whitney extension operator, extend F to a smooth map $\tilde{f} : \tilde{M} \to \mathbb{R}^p$. Using fiber scalar multiplication we get $f := (g \circ \tilde{f}) \cdot \tilde{f} : \tilde{M} \to U$, and and a smooth extension $p \circ f : \tilde{M} \to N$ of F. (The proof of this is wrong in my paper).

Hence, the space of Whitney jets is isomorphic to the space

$$C^\infty(M, N) = \{f|_M : f \in C^\infty(\tilde{M}, N)\}.$$

This describes a nonlinear extension operator $C^\infty(M, N) \to C^\infty(\tilde{M}, N)$ which is continuous, and is even smooth in the manifold structures.

Bundles and sections

By a (vector or fiber) bundle $E \to M$ over a germ of a Whitney manifold M we mean the restriction to M of a (vector or fiber) bundle $\tilde{E} \to \tilde{M}$, i.e., of a (vector or fiber) bundle over an open manifold neighborhood. By a smooth section of $E \to M$ we mean the restriction of a smooth section of $\tilde{E} \to \tilde{M}$ for a neighborhood $\tilde{M}$. Using classifying smooth mappings into a suitable Grassmannian for vector bundles over M and using the discussion above one could talk about Whitney jets of vector bundles and extend them to a manifold neighborhood of M.

Spaces of sections

We shall use the following spaces of sections of a vector bundle $E \to M$ over a Whitney manifold germ M.

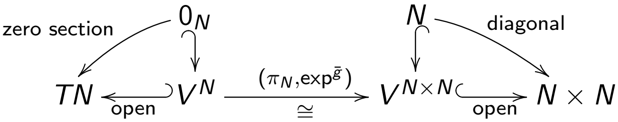
- $\Gamma(E) = \Gamma(M \leftarrow E)$, the space of smooth sections, i.e., restrictions of smooth sections of $\tilde{E} \to \tilde{M}$ for a fixed neighborhood $\tilde{M}$, with the Fréchet space topology of compact convergence on the isomorphic space of Whitney jets of sections.
- $\Gamma_c(E)$, the space of smooth sections with compact support, with the inductive limit (LF)-topology.
- $\Gamma_{C^n}(E)$, the space of C^n-section, with the Fréchet space topology of compact convergence on the space of Whitney n-jets. If M is compact and n finite, $\Gamma_{C^n}(E)$ is a Banach space.

Spaces of sections, II

- $\Gamma_{H^s}(E)$, the space of Sobolev H^s-sections, for $s \in \mathbb{R}_{\geq 0}$. Here M should be a compact Whitney manifold germ. The measure on M is the restriction of the volume density with respect to a Riemannian metric on $\tilde{M}$. One also needs a fiber metric on E. The space $\Gamma_{H^k}(E)$ is independent of all choices, but the inner product depends on them. One way to define $\Gamma_{H^k}(E)$ is to use a finite atlas which trivializes $\tilde{E}|_L$ over a compact manifold with smooth boundary L which is a neighborhood of M in $\tilde{M}$ and a partition of unity, and then use the Fourier transform description of the Sobolev space. For $0 \leq k < s - \dim(M)/2$ we have $\Gamma_{H^s}(E) \subset \Gamma_{C^k}(E)$ continuously. For $s < 0$ one can define $\Gamma_{H^s}(E) = \Gamma_{H^{-s}}(E^*)'$ by duality.
- More generally, for $0 \geq s \in \mathbb{R}$ and $1 < p < \infty$ we also consider $\Gamma_{W^{s,p}}(E)$, the space of $W^{s,p}$-sections: For integral s, all (covariant) derivatives up to order s are in L^p. For $0 \leq k < s - \dim(M)/p$ we have $\Gamma_{H^s}(E) \subset \Gamma_{C^k}(E)$ continuously.

The manifold structure on $C^\infty(M,N)$ and $C^k(M,N)$

Let M be a compact or open finite dimensional smooth manifold or even a Whitney manifold germ, and let N be a smooth manifold. We use an auxiliary Riemannian metric $\bar g$ on N and its exponential mapping $\exp^{\bar g}$; some of its properties are summarized in the following diagram:



Without loss we may assume that $V^{N\times N}$ is symmetric:

$$(y_1, y_2) \in V^{N\times N} \iff (y_2, y_1) \in V^{N\times N}.$$

• If M is compact, then $C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. A chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^*TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

Note that $\tilde{U}_f$ is open in $\Gamma(f^*TN)$ if M is compact.

• If M is open, then the compact C^∞-topology on $\Gamma(f^*TN)$ is not suitable since $\tilde{U}_f$ is in general not open. We have to control the behavior of sections near infinity on M. One solution is to use the space $\Gamma_c(f^*TN)$ of sections with compact support as modeling spaces and to adapt the topology on $C^\infty(M,N)$ accordingly.

• If M is compact Whitney manifold germ with neighborhood manifold $\tilde{M} \supset M$ we use the Fréchet space $\Gamma(f^*TN) = \{s|_M : s \in \Gamma_L(\tilde{M} \leftarrow \tilde{f}^*TN)\}$ where $L \subset \tilde{M}$ is a compact set containing M in its interior and $\tilde{f} : \tilde{M} \to N$ is an extension of f to a suitable manifold neighborhood of M. Via an extension operator the Fréchet space $\Gamma(f^*TN)$ is a direct summand in the Fréchet space $\Gamma_L(\tilde{M} \leftarrow \tilde{f}^*TN)$ of smooth sections with support in L.

• Likewise, for a non-compact Whitney manifold germ we use the convenient (LF)-space

$$\Gamma_c(M \leftarrow f^*TN) = \{s|_M : s \in \Gamma_c(\tilde{M} \leftarrow \tilde{f}^*TN)\}$$

of sections with compact support.

• On the space $C^k(M, N,)$ for $k \in \mathbb{N}_{\geq 0}$ we use only charts as described above with the center $f \in C^\infty(M, N)$, namely

$$C^k(M, N) \supset U_f = \{g : (f, g)(M) \subset V^{N \times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma_{C^k}(f^* TN)\,.$$

We claim that these charts cover $C^k(M, N)$: Since $C^\infty(M, N)$ is dense in $C^k(M, N)$ in the Whitney C^k-topology, for any $g \in C^k(M, N)$ there exists $f \in C^\infty(M, N,) \cap U_g$. But then $g \in U_f$ since $V^{N \times N}$ is symmetric. This is true for compact M. For a compact Whitney manifold germ we can apply this argument in a compact neighborhood L of M in $\tilde{M}$, replacing $\tilde{M}$ by the interior of L after the fact.

• On the space $W^{s,p}(M,N)$ for $\dim(M)/p < s \in \mathbb{R}$ we use only charts as described above with the center $f \in C^\infty(M,N)$, namely:

$$W^{s,p}(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f}$$
$$\xrightarrow{u_f} \tilde{U}_f \subset \Gamma_{W^{s,p}}(M \leftarrow f^* TN)\,.$$

These charts cover $W^{s,p}(M,N)$, by the following argument: Since $C^\infty(M,N)$ is dense in $W^{s,p}(M,N)$ and since $W^{s,p}(M,N) \subset C^k(M,N)$ via a continuous injection for $0 \le k < s - \dim(M)/p$, a suitable C^0 – sup-norm neighborhood of $g \in W^{s,p}(M,N)$ contains a smooth $f \in C^\infty(M,N)$, thus $f \in U_g$ and by symmetry of $V^{N\times N}$ we have $g \in U_f$. This is true for compact M. For a compact Whitney manifold germ we can apply this argument in a compact neighborhood which is a manifold with smooth boundary L of M in $\tilde{M}$ and apply the argument there.

In each case, we equip $C^\infty(M,N)$ or $C^k(M,N)$ or $W^{s,p}(M,N)$ with the initial topology with respect to all chart mappings described above: The coarsest topology, so that all chart mappings u_f are continuous.

For non-compact M we use the direct limit $\Gamma_c(f^*TN) = \varinjlim_L \Gamma_L(f^*TN)$ over a compact exhaustion L of M in the category of Hausdorff topological spaces.

Lemma (1) *If M is a compact smooth manifold or is a compact Whitney manifold germ,*

$$C^\infty(\mathbb{R}, \Gamma(M \leftarrow f^*TN)) = \Gamma(\mathbb{R} \times M \leftarrow {\mathrm{pr}_2}^* f^*TN)\,.$$

For smooth $f \in C^\infty(M, N)$,

$$C^\infty(\mathbb{R}, \Gamma_{C^n}(M \leftarrow f^*TN)) = \Gamma_{C^{\infty,n}}(\mathbb{R} \times M \leftarrow {\mathrm{pr}_2}^* f^*TN)\,.$$

(2) *If M is a non-compact smooth manifold of Whitney manifold germ, one has to add compact support on M, locally in $\mathbb{R}$.*

Lemma *Let M be a smooth manifold or Whitney manifold germ, compact or not, and let N be a manifold. Then the chart changes for charts centered on smooth mappings are smooth (C^∞) on the space $C^\infty(M,N)$, also on $C^k(M,N)$ for $k \in \mathbb{N}_{\geq 0}$, and on $W^{s,p}(M,N)$ for $1 < p < \infty$ and $s > \dim(M)/p$:*

$$\tilde{U}_{f_1} \ni s \mapsto (u_{f_2,f_1})_*(s) := (\exp^{\bar{g}}_{f_2})^{-1} \circ \exp^{\bar{g}}_{f_1} \circ s \in \tilde{U}_{f_2}\,.$$

Lemma (1) *If M is a compact manifold or a compact Whitney manifold germ, then*

$$C^\infty(\mathbb{R}, C^\infty(M,N)) \cong C^\infty(\mathbb{R} \times M, N)\,.$$

(2) *If M is not compact, $C^\infty(\mathbb{R}, C^\infty(M,N))$ consists of all smooth $c : \mathbb{R} \times M \to N$ such that*

- *for each compact interval $[a,b] \subset \mathbb{R}$ there is a compact subset $K \subset M$ such that $c(t,x)$ is constant in $t \in [a,b]$ for each $x \in M \setminus K$.*

Lemma *Composition $(f, g) \mapsto g \circ f$ is smooth as a mapping*

$$C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$$
$$C^k(P, M) \times C^\infty(M, N) \to C^k(P, N)$$
$$W^{s,p}(P, M) \times C^\infty(M, N) \to W^{s,p}(P, N)$$

for P a manifold or a Whitney manifold germ, compact or not, and for M and N manifolds.

Lemma *For M a manifold or a Whitney manifold germ and a manifold N, the tangent bundle of the manifold $C^\infty(M, N)$ of mappings is given by*

$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M,\pi_N)=(\pi_N)_*} C^\infty(M, N)\,,$$

$$TC^k(M, N) = C^k(M, TN) \xrightarrow{C^k(M,\pi_N)=(\pi_N)_*} C^k(M, N)\,,$$

$$TW^{s,p}(M, N) = W^{s,p}(M, TN) \xrightarrow{W^{s,p}(M,\pi_N)=(\pi_N)_*} W^{s,p}(M, N)\,.$$

Thank you for listening.