Uniqueness of the Fisher–Rao metric on the space of smooth densities on a closed manifold

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Based on:

[M.Bauer, M.Bruveris, P.Michor: Uniqueness of the Fisher–Rao metric on the space of smooth densities, Bull. London Math. Soc. 48, 3 (2016), 499-506, arXiv:1411.5577]
[M.Bruveris, P.Michor: Geometry of the Fisher-Rao metric on the space of smooth densities, Mathematische Nachrichten 292 (2019), 511-523, arxiv:1607.04550]
[M.Bruveris, P. Michor, A.Parusinski, A. Rainer: Moser's Theorem for manifolds with corners, Proc. AMS 146 (2018), pp. 4889-4897, arxiv:1604.07787]

The infinite dimensional geometry used here is based on:
[Andreas Kriegl, Peter W. Michor: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, Volume: 53, Amer. Math. Soc., 1997]
Wikipedia [https://en.wikipedia.org/wiki/Convenient_vector_space]
I will use also [Peter W. Michor: Topics in Differential Geometry, Grad. Studies in Math. 93, 2008]

Abstract

For a smooth compact manifold M without boundary, any weak Riemannian metric on the space of smooth positive densities which is invariant under the right action of the diffeomorphism group $Diff(M)$ is of the form

$$G_\mu(\alpha, \beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta$$

for smooth functions C_1, C_2 of the total volume $\mu(M) = \int_M \mu$. This implies uniqueness up to a constant for the Fisher-Rao metric G^{FR} on the space of smooth positive probability densities.

In this talk I prove this, and investigate the geometry. If time permits, I conjecturally extend the result to compact smooth manifolds with corners (for example, a simplex).

The Fisher–Rao metric on the space Prob(M) of probability densities is of importance in the field of information geometry. Restricted to finite-dimensional submanifolds of Prob(M), so-called statistical manifolds, it is called Fisher's information metric [Amari: Differential-geometrical methods in statistics, 1985]. The Fisher–Rao metric is invariant under the action of the diffeomorphism group. A uniqueness result was established [Čencov: Statistical decision rules and optimal inference, 1982, p. 156] for Fisher's information metric on finite sample spaces and [Ay, Jost, Le, Schwachhöfer, 2014] extended it to infinite sample spaces.

See also [Ay, Jost, Le, Schwachhöfer: Information Geometry, 2017].

The Fisher–Rao metric on the infinite-dimensional manifold of all positive smooth probability densities was studied in [Friedrich: Die Fisher-Information und symplektische Strukturen, 1991], including the computation of its curvature.

The space of densities

Let M^m be a smooth manifold. Let $(M \supseteq U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subseteq \mathbb{R}^m)$ be a smooth atlas for it. The *volume bundle* $(\mathrm{Vol}(M), \pi_M, M)$ of M is the 1-dimensional vector bundle (line bundle) which is given by the following cocycle of transition functions:

$$\psi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to \mathbb{R} \setminus \{0\} = GL(1, \mathbb{R}),$$

$$\psi_{\alpha\beta}(x) = |\det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x))| = \frac{1}{|\det d(u_\alpha \circ u_\beta^{-1})(u_\beta(x))|}.$$

Vol(M) is a trivial line bundle over M. But there is no natural trivialization. There is a natural order on each fiber. Since $\mathrm{Vol}(M)$ is a natural bundle of order 1 on M, there is a natural action of the group $\mathrm{Diff}(M)$ on $\mathrm{Vol}(M)$, given by

$$\begin{array}{ccc} \mathrm{Vol(M)} & \xrightarrow{|\det(T\varphi^{-1})| \circ \varphi} & \mathrm{Vol(M)} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & M \end{array}.$$

If M is orientable, then $\mathrm{Vol}(M) = \Lambda^m T^*M$. If M is not orientable, let $\tilde{M}$ be the orientable double cover of M with its deck-transformation $\tau : \tilde{M} \to \tilde{M}$. Then $\Gamma(\mathrm{Vol}(M))$ is isomorphic to the space $\{\omega \in \Omega^m(\tilde{M}) : \tau^*\omega = -\omega\}$. These are the 'formes impaires' of de Rham. See [M 2008, 13.1] for this.

Sections of the line bundle $\mathrm{Vol}(M)$ are called densities. The space $\Gamma(\mathrm{Vol}(M))$ of all smooth sections is a Fréchet space in its natural topology; see [Kriegl-M, 1997]. For each section α of $\mathrm{Vol}(M)$ of compact support the integral $\int_M \alpha$ is invariantly defined as follows: Let (U_α, u_α) be an atlas on M with associated trivialization $\psi_\alpha : \mathrm{Vol}(M)|_{U_\alpha} \to \mathbb{R}$, and let f_α be a partition of unity with $\mathrm{supp}(f_\alpha) \subset U_\alpha$. Then we put

$$\int_M \mu = \sum_\alpha \int_{U_\alpha} f_\alpha \mu := \sum_\alpha \int_{u_\alpha(U_\alpha)} f_\alpha(u_\alpha^{-1}(y)).\psi_\alpha(\mu(u_\alpha^{-1}(y)))\, dy.$$

The integral is independent of the choice of the atlas and the partition of unity.

The Fisher–Rao metric

Let M^m be a smooth compact manifold without boundary. Let $\mathrm{Dens}_+(M)$ be the space of smooth positive densities on M, i.e., $\mathrm{Dens}_+(M) = \{\mu \in \Gamma(\mathrm{Vol}(M)) \,:\, \mu(x) > 0 \;\forall x \in M\}$.
Let $\mathrm{Prob}(M)$ be the subspace of positive densities with integral 1.
For $\mu \in \mathrm{Dens}_+(M)$ we have $T_\mu \mathrm{Dens}_+(M) = \Gamma(\mathrm{Vol}(M))$ and for $\mu \in \mathrm{Prob}(M)$ we have
$T_\mu \mathrm{Prob}(M) = \{\alpha \in \Gamma(\mathrm{Vol}(M)) : \int_M \alpha = 0\}$.
The Fisher–Rao metric on $\mathrm{Prob}(M)$ is defined as:

$$G^{\mathrm{FR}}_\mu(\alpha, \beta) = \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu.$$

It is invariant for the action of $\mathrm{Diff}(M)$ on $\mathrm{Prob}(M)$:

$$\Big((\varphi^*)^* G^{\mathrm{FR}}\Big)_\mu(\alpha, \beta) = G^{\mathrm{FR}}_{\varphi^*\mu}(\varphi^*\alpha, \varphi^*\beta) =$$

$$= \int_M \Big(\frac{\alpha}{\mu} \circ \varphi\Big)\Big(\frac{\beta}{\mu} \circ \varphi\Big)\varphi^*\mu = \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu\,.$$

Theorem [BBM, 2016]

Let M be a compact manifold without boundary of dimension ≥ 2. Let G be a smooth (equivalently, bounded) bilinear form on $\mathrm{Dens}_+(M)$ *which is invariant under the action of* $\mathrm{Diff}(M)$*. Then*

$$G_\mu(\alpha,\beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta$$

for smooth functions C_1, C_2 of the total volume $\mu(M)$.

To see that this theorem implies the uniqueness of the Fisher–Rao metric, note that if G is a $\mathrm{Diff}(M)$-invariant Riemannian metric on $\mathrm{Prob}(M)$, then we can equivariantly extend it to $\mathrm{Dens}_+(M)$ via

$$G_\mu(\alpha,\beta) = G_{\frac{\mu}{\mu(M)}}\left(\alpha - \Big(\int_M \alpha\Big)\frac{\mu}{\mu(M)}, \beta - \Big(\int_M \beta\Big)\frac{\mu}{\mu(M)}\right).$$

Relations to right-invariant metrics on diffeom. groups

Let $\mu_0 \in \mathrm{Prob}(M)$ be a fixed smooth probability density. In [Khesin, Lenells, Misiolek, Preston, 2013] it has been shown, that the degenerate, $\dot{H}^1$-metric $\frac{1}{2}\int_M \mathrm{div}^{\mu_0}(X).\mathrm{div}^{\mu_0}(X).\mu_0$ on $\mathfrak{X}(M)$ is invariant under the adjoint action of $\mathrm{Diff}(M, \mu_0)$. Thus the induced degenerate right invariant metric on $\mathrm{Diff}(M)$ descends to a metric on $\mathrm{Prob}(M) \cong \mathrm{Diff}(M, \mu_0)\backslash \mathrm{Diff}(M)$ via

$$\mathrm{Diff}(M) \ni \varphi \mapsto \varphi^* \mu_0 \in \mathrm{Prob}(M)$$

which is invariant under the right action of $\mathrm{Diff}(M)$. This is the Fisher–Rao metric on $\mathrm{Prob}(M)$. In [Modin, 2014], the $\dot{H}^1$-metric was extended to a non-degenerate metric on $\mathrm{Diff}(M)$, also descending to the Fisher–Rao metric.

Corollary. *Let* $\dim(M) \geq 2$*. If a weak right-invariant (possibly degenerate) Riemannian metric* $\tilde{G}$ *on* $\mathrm{Diff}(M)$ *descends to a metric* G *on* $\mathrm{Prob}(M)$ *via the right action, i.e., the mapping* $\varphi \mapsto \varphi^* \mu_0$ *from* $(\mathrm{Diff}(M), \tilde{G})$ *to* $(\mathrm{Prob}(M), G)$ *is a Riemannian submersion, then* G *has to be a multiple of the Fisher–Rao metric.*

Note that any right invariant metric $\tilde{G}$ on $\mathrm{Diff}(M)$ descends to a metric on $\mathrm{Prob}(M)$ via $\varphi \mapsto \varphi_* \mu_0$; but this is not $\mathrm{Diff}(M)$-invariant in general.

Invariant metrics on $\mathrm{Dens}_+(S^1)$.

$\mathrm{Dens}_+(S^1) = \Omega^1_+(S^1)$, and $\mathrm{Dens}_+(S^1)$ is $\mathrm{Diff}(S^1)$-equivariantly isomorphic to the space of all Riemannian metrics on S^1 via $\Phi = (\quad)^2 : \mathrm{Dens}_+(S^1) \to \mathrm{Met}(S^1)$, $\Phi(fd\theta) = f^2 d\theta^2$.
On $\mathrm{Met}(S^1)$ there are many $\mathrm{Diff}(S^1)$-invariant metrics; see [Bauer, Harms, M, 2013]. For example Sobolev-type metrics. Write $g \in \mathrm{Met}(S^1)$ in the form $g = \tilde{g}\,d\theta^2$ and $h = \tilde{h}\,d\theta^2$, $k = \tilde{k}\,d\theta^2$ with $\tilde{g}, \tilde{h}, \tilde{k} \in C^\infty(S^1)$. The following metrics are $\mathrm{Diff}(S^1)$-invariant:

$$G^l_g(h,k) = \int_{S^1} \frac{\tilde{h}}{\tilde{g}} . (1 + \Delta^g)^n \left(\frac{\tilde{k}}{\tilde{g}}\right) \sqrt{\tilde{g}}\, d\theta\,;$$

here Δ^g is the Laplacian on S^1 with respect to the metric g. The pullback by Φ yields a $\mathrm{Diff}(S^1)$-invariant metric on $\mathrm{Dens}_+(M)$:

$$G_\mu(\alpha,\beta) = 4 \int_{S^1} \frac{\alpha}{\mu} . \left(1 + \Delta^{\Phi(\mu)}\right)^n \left(\frac{\beta}{\mu}\right) \mu\,.$$

For $n = 0$ this is 4 times the Fisher–Rao metric. For $n \geq 1$ we get many $\mathrm{Diff}(S^1)$-invariant metrics on $\mathrm{Dens}_+(S^1)$ and on $\mathrm{Prob}(S^1)$.

Geometry of the Fisher-Rao metric on $\mathrm{Dens}_+(M)$

$$G_\mu(\alpha,\beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta$$

This metric will be studied in different representations.

$$\mathrm{Dens}_+(M) \xrightarrow{R} C^\infty(M,\mathbb{R}_{>0}) \xrightarrow{\Phi} \mathbb{R}_{>0} \times S \cap C^\infty_{>0} \xrightarrow{W\times \mathrm{Id}} (W_-, W_+) \times S \cap C^\infty_{>0}\,.$$

We fix $\mu_0 \in \mathrm{Prob}(M)$ and consider the mapping

$$R : \mathrm{Dens}_+(M) \to C^\infty(M,\mathbb{R}_{>0})\,, \qquad R(\mu) = f = \sqrt{\frac{\mu}{\mu_0}}\,.$$

The map R is a diffeomorphism and we will denote the induced metric by $\tilde{G} = \left(R^{-1}\right)^* G$; it is given by the formula

$$\tilde{G}_f(h,k) = 4C_1(\|f\|^2_{L^2(\mu_0)})\langle h,k\rangle_{L^2(\mu_0)} + 4C_2(\|f\|^2_{L^2(\mu_0)})\langle f,h\rangle_{L^2(\mu_0)}\langle f,k\rangle_{L^2(\mu_0)}\,,$$

and this formula makes sense for $f \in C^\infty(M,\mathbb{R}) \setminus \{0\}$.
Consequently, for $(\mathrm{Prob}(M), G^{\mathrm{FR}})$ is isometric to the $2\sqrt{C_1(1)}$-sphere in $L^2(\mu_0)$ intersected with $C^\infty(M,\mathbb{R}_{>0})$.

Remark on R^{-1}

The map R is inspired by [B. Khesin, J. Lenells, G. Misiolek, S. C. Preston: Geometry of diffeomorphism groups, complete integrability and geometric statistics. Geom. Funct. Anal., 23(1):334-366, 2013.]

$$R^{-1} : C^\infty(M,\mathbb{R}) \to \Gamma_{\geq 0}(\mathrm{Vol}(M)), \quad f \mapsto f^2\mu_0$$

makes sense on the whole space $C^\infty(M,\mathbb{R})$ and its image is stratified (loosely speaking) according to the rank of TR^{-1}. The image looks somewhat like the orbit space of a discrete reflection group. Geodesics are mapped to curves which are geodesics in the interior $\Gamma_{>0}(\mathrm{Vol}(M))$, and they are reflected following Snell's law at some hyperplanes in the boundary.

Polar coordinates

on the pre-Hilbert space $(C^\infty(M,\mathbb{R}), \langle\ ,\ \rangle_{L^2(\mu_0)})$. Let $S = \{\varphi \in L^2(M,\mathbb{R}) : \int_M \varphi^2\mu_0 = 1\}$ denote the L^2-sphere. Then

$$\Phi : C^\infty(M,\mathbb{R})\backslash\{0\} \to \mathbb{R}_{>0}\times(S\cap C^\infty)\,, \qquad \Phi(f) = (r,\varphi) = \left(\|f\|, \frac{f}{\|f\|}\right)$$

is a diffeomorphism. We set $\bar{G} = \left(\Phi^{-1}\right)^* \tilde{G}$; the metric has the expression

$$\bar{G}_{r,\varphi} = g_1(r)\langle d\varphi, d\varphi\rangle + g_2(r)dr^2\,,$$

with $g_1(r) = 4C_1(r^2)r^2$ and $g_2(r) = 4\left(C_1(r^2) + C_2(r^2)r^2\right)$. Finally we change the coordinate r diffeomorphically to

$$s = W(r) = 2\int_1^r \sqrt{g_2(\rho)}\,d\rho\,.$$

Then, defining $a(s) = 4C_1(r(s)^2)r(s)^2$, we have

$$\boxed{\bar{G}_{s,\varphi} = a(s)\langle d\varphi, d\varphi\rangle + ds^2\,.}$$

Let $W_- = \lim_{r\to 0+} W(r)$ and $W_+ = \lim_{r\to\infty} W(r)$. Then $W : \mathbb{R}_{>0} \to (W_-, W_+)$ is a diffeomorphism.

This completes the first row in Fig. 1.

$$
\begin{array}{ccccccc}
\mathrm{Dens}_+(M) & \xrightarrow{R} & C^\infty(M,\mathbb{R}_{>0}) & \xrightarrow{\Phi} & \mathbb{R}_{>0}\times S\cap C^\infty_{>0} & \xrightarrow{W\times \mathrm{Id}} & (W_-,W_+)\times S\cap C^\infty_{>0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Dens}(M)\setminus\{0\} & \xrightarrow{R} & C^0(M,\mathbb{R})\setminus\{0\} & \xrightarrow{\Phi} & \mathbb{R}_{>0}\times S\cap C^0 & \xrightarrow{W\times \mathrm{Id}} & \mathbb{R}\times S\cap C^0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma_{L^1}(\mathrm{Vol}(M))\setminus\{0\} & \xrightarrow{R} & L^2(M,\mathbb{R})\setminus\{0\} & \xrightarrow{\Phi} & \mathbb{R}_{>0}\times S & \xrightarrow{W\times \mathrm{Id}} & \mathbb{R}\times S
\end{array}
$$

Figure: Representations of $\mathrm{Dens}_+(M)$ and its completions. In the second and third rows we assume that $(W_-, W_+) = (-\infty, +\infty)$ and we note that R is a diffeomorphism only in the first row.

Geodesic equation:

$$
\boxed{
\begin{aligned}
\nabla^S_{\partial_t}\varphi_t &= \partial_t\left(\log g_1(r)\right)\varphi_t \\
r_{tt} &= \frac{C_0^2}{2}\,\frac{g_1'(r)}{g_1(r)^2 g_2(r)} - \frac{1}{2}\partial_t\left(\log g_2(r)\right) r_t
\end{aligned}
}
$$

Since $\bar{G}$ induces the canonical metric on (W_-, W_+), a necessary condition for $\bar{G}$ to be complete is $(W_-, W_+) = (-\infty, +\infty)$. Rewritten in terms of the functions C_1, C_2 this becomes

$$W_+ = \infty \Leftrightarrow \left(\int_1^\infty r^{-1/2} \sqrt{C_1(r)}\, dr = \infty \text{ or } \int_1^\infty \sqrt{C_2(r)}\, dr = \infty \right),$$

and similarly for $W_- = -\infty$, with the limits of the integration being 0 and 1.

Relation to hypersurfaces of revolution in the (pre-) Hilbert space

We consider the metric on $(W_-, W_+) \times S \cap C^\infty$ in the form $\tilde{G}_{r,\varphi} = a(s)\langle d\varphi, d\varphi\rangle + ds^2$ where $a(s) = 4C_1(r(s)^2)r(s)^2$. Then we consider the isometric embedding (remember $\langle \varphi, d\varphi\rangle = 0$ on $S \cap C^\infty$)

$$\Psi : ((W_-, W_+) \times S \cap C^\infty, \tilde{G}) \to \left(\mathbb{R} \times C^\infty(M, \mathbb{R}), du^2 + \langle df, df\rangle\right),$$

$$\Psi(s, \varphi) = \left(\int_0^s \sqrt{1 - \frac{a'(\sigma)^2}{4a(\sigma)}}\, d\sigma \ , \ \sqrt{a(s)}\varphi \right),$$

which is defined and smooth only on the open subset

$$R := \{(s, \varphi) \in (W_-, W_+) \times S \cap C^\infty : a'(s)^2 < 4a(s)\}.$$

Fix some $\varphi_0 \in S \cap C^\infty$ and consider the generating curve

$$s \mapsto \left(\int_0^s \sqrt{1 - \frac{a'(\sigma)^2}{4a(\sigma)}}\, d\sigma \ , \ \sqrt{a(s)} \right) \in \mathbb{R}^2 .$$

Then s is an arc-length parameterization of this curve!

Given any arc-length parameterized curve $I \ni s \mapsto (c_1(s), c_2(s))$ in $\mathbb{R}^2$ and its generated hypersurface of rotation

$$\{(c_1(s), c_2(s)\varphi) : s \in I, \varphi \in S \cap C^\infty\} \subset \mathbb{R} \times C^\infty(M, \mathbb{R}),$$

the induced metric in the (s, φ)-parameterization is $ds^2 + c_2(s)^2 \langle d\varphi, d\varphi \rangle$.

This suggests that the moduli space of hypersurfaces of revolution is naturally embedded in the moduli space of all metrics of the form $\tilde{G}_{r,\varphi} = a(s)\langle d\varphi, d\varphi \rangle + ds^2$.

Example: In the case of $S = S^1$ and the tractrix (c_1, c_2), the surface of revolution is the pseudosphere (curvature -1) whose universal cover is only part of the hyperbolic plane. But in polar coordinates we get a space whose universal cover is the hyperbolic plane. In detail:

$$c_1(s) = \int_0^s \sqrt{1 - e^{-2\sigma}}\, d\sigma = \operatorname{Arcosh}\left(e^s\right) - \sqrt{1 - e^{-2s}}$$

$$c_2(s) = e^{-s}, \quad s > 0$$

$$a(s)\, d\varphi^2 + ds^2 = e^{-2s} d\varphi^2 + ds^2, \qquad s \in \mathbb{R}\,.$$

Theorem

If $(W_-, W_+) = (-\infty, +\infty)$*, then any two points* (s_0, φ_0) *and* (s_1, φ_1) *in* $\mathbb{R} \times S$ *can be joined by a minimal geodesic. If* φ_0 *and* φ_1 *lie in* $S \cap C^\infty$*, then the minimal geodesic lies in* $\mathbb{R} \times S \cap C^\infty$*.*

Proof. If φ_0 and φ_1 are linearly independent, we consider the 2-space $V = V(\varphi_0, \varphi_1)$ spanned by φ_0 and φ_1 in L^2. Then $\mathbb{R} \times V \cap S$ is totally geodesic since it is the fixed point set of the isometry $(s, \varphi) \mapsto (s, \mathfrak{s}_V(\varphi))$ where $\mathfrak{s}_V$ is the orthogonal reflection at V. Thus there is exists a minimizing geodesic between (s_0, φ_0) and (s_1, φ_1) in the complete 3-dimensional Riemannian submanifold $\mathbb{R} \times V \cap S$. This geodesic is also length-minimizing in the strong Hilbert manifold $\mathbb{R} \times S$ by the following arguments:

Given any smooth curve $c = (s, \varphi) : [0,1] \to \mathbb{R} \times S$ between these two points, there is a subdivision $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the piecewise geodesic c_1 which first runs along a geodesic from $c(t_0)$ to $c(t_1)$, then to $c(t_2)$, ..., and finally to $c(t_N)$, has length $\mathrm{Len}(c_1) \le \mathrm{Len}(c)$. This piecewise geodesic now lies in the totally geodesic $(N+2)$-dimensional submanifold $\mathbb{R} \times V(\varphi(t_0), \dots, \varphi(t_N)) \cap S$. Thus there exists a geodesic c_2 between the two points (s_0, φ_0) and s_1, φ_1 which is length minimizing in this $(N+2)$-dimensional submanifold. Therefore $\mathrm{Len}(c_2) \le \mathrm{Len}(c_1) \le \mathrm{Len}(c)$. Moreover, $c_2 = (s \circ c_2, \varphi \circ c_2)$ lies in $\mathbb{R} \times V(\varphi_0, (\varphi \circ c_2)'(0)) \cap S$ which also contains φ_1, thus c_2 lies in $\mathbb{R} \times V(\varphi_0, \varphi_1) \cap S$.

If $\varphi_0 = \varphi_1$, then $\mathbb{R} \times \{\varphi_0\}$ is a minimal geodesic. If $\varphi_0 = -\varphi_0$ we choose a great circle between them which lies in a 2-space V and proceed as above. $\square$

Covariant derivative

On $\mathbb{R} \times S$ (we assume that $(W_-, W_+) = \mathbb{R}$) with metric $\bar{G} = ds^2 + a(s)\langle d\varphi, d\varphi\rangle$ we consider smooth vector fields $f(s,\varphi)\partial_s + X(s,\varphi)$ where $X(s,\ ) \in \mathfrak{X}(S)$ is a smooth vector field on the Hilbert sphere S. We denote by ∇^S the covariant derivative on S and get

$$\begin{aligned}\nabla_{f\partial_s+X}(g\partial_s + Y) = \big(f.g_s + dg(X) - \frac{a_s}{2}\langle X, Y\rangle\big)\partial_s \\ + \frac{a_s}{2a}(fY + gX) + fY_s + \nabla^S_X Y\end{aligned}$$

Curvature:

$$\begin{aligned}&\mathcal{R}(f\partial_s + X, g\partial_s + Y)(h\partial_s + Z) = \\ &= \Big(\frac{a_{ss}}{2} - \frac{a_s^2}{4a}\Big)\langle gX - fY, Z\rangle\partial_s + \mathcal{R}^S(X,Y)Z \\ &\quad - \Big(\Big(\frac{a_s}{2a}\Big)_s + \frac{a_s^2}{4a^2}\Big)h(gX - fY) + \frac{a_s^2}{4a}\big(\langle X, Z\rangle Y - \langle Y, Z\rangle X\big)\,.\end{aligned}$$

Sectional Curvature

Let us take $X, Y \in T_\varphi S$ with $\langle X, Y \rangle = 0$ and $\langle X, X \rangle = \langle Y, Y \rangle = 1/a(s)$, then

$$\boxed{\begin{aligned} \mathrm{Sec}_{(s,\varphi)}(\mathrm{span}(X, Y)) &= \frac{1}{a} - \frac{a_s^2}{4a^2}, \\ \mathrm{Sec}_{(s,\varphi)}(\mathrm{span}(\partial_s, Y)) &= -\frac{a_{ss}}{2a} + \frac{a_s^2}{4a^2} \end{aligned}}$$

are all the possible sectional curvatures.

Proof of the Main Theorem

First, the main theorem again:

Theorem [BBM, 2016] *Let M be a compact manifold without boundary of dimension ≥ 2. Let G be a smooth (equivalently, bounded) bilinear form on* $\mathrm{Dens}_+(M)$ *which is invariant under the action of* $\mathrm{Diff}(M)$. *Then*

$$G_\mu(\alpha, \beta) = C_1(\mu(M)) \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu + C_2(\mu(M)) \int_M \alpha \cdot \int_M \beta$$

for smooth functions C_1, C_2 of the total volume $\mu(M)$.

Proof of the Main Theorem

Here M is a compact smooth manifold without boundary, possibly non-orientable.

Let us fix a basic probability density μ_0. By Moser's theorem [Moser, 1965], see [M, 2008, 31.13] or the proof of [Kriegl, M, 1997, 43.7] for proofs in the notation used here, there exists for each $\mu \in \mathrm{Dens}_+(M)$ a diffeomorphism $\varphi_\mu \in \mathrm{Diff}(M)$ with $\varphi_\mu^* \mu = \mu(M)\mu_0 =: c.\mu_0$ where $c = \mu(M) = \int_M \mu > 0$. Then

$$\big((\varphi_\mu^*)^* G\big)_\mu(\alpha, \beta) = G_{\varphi_\mu^* \mu}(\varphi_\mu^* \alpha, \varphi_\mu^* \beta) = G_{c.\mu_0}(\varphi_\mu^* \alpha, \varphi_\mu^* \beta)\,.$$

Thus it suffices to show that for any $c > 0$ we have

$$G_{c\mu_0}(\alpha,\beta) = C_1(c).\int_M \frac{\alpha}{\mu_0}\frac{\beta}{\mu_0}\mu_0 + C_2(c)\int_M \alpha\cdot\int_M \beta$$

for some functions C_1, C_2 of the total volume $c = \mu(M)$. Both bilinear forms are still invariant under the action of the group $\mathrm{Diff}(M, c\mu_0) = \mathrm{Diff}(M,\mu_0) = \{\psi \in \mathrm{Diff}(M) : \psi^*\mu_0 = \mu_0\}$. The bilinear form

$$T_{\mu_0}\,\mathrm{Dens}_+(M) \times T_{\mu_0}\,\mathrm{Dens}_+(M) \ni (\alpha,\beta) \mapsto G_{c\mu_0}\Big(\frac{\alpha}{\mu_0}\mu_0, \frac{\beta}{\mu_0}\mu_0\Big)$$

can be viewed as a bilinear form

$$C^\infty(M) \times C^\infty(M) \ni (f,g) \mapsto G_c(f,g)\,.$$

We will consider now the associated bounded linear mapping

$$\check{G}_c : C^\infty(M) \to C^\infty(M)' = \mathcal{D}'(M)\,.$$

(1) The Lie algebra $\mathfrak{X}(M, \mu_0)$ of $\mathrm{Diff}(M, \mu_0)$ consists of vector fields X with

$$0 = \mathrm{div}^{\mu_0}(X) := \frac{\mathcal{L}_X \mu_0}{\mu_0}.$$

On an oriented open subset $U \subset M$, each density is an m-form, $m = \dim(M)$, and $\mathrm{div}^{mu_0}(X) = d i_X \mu_0$.

The mapping $\hat{\imath}_{\mu_0} : \mathfrak{X}(U) \to \Omega^{m-1}(U)$ given by $X \mapsto i_X \mu_0$ is an isomorphism. The Lie subalgebra $\mathfrak{X}(U, \mu_0)$ of divergence free vector fields corresponds to the space of closed $(m-1)$-forms.

Denote by $\mathfrak{X}_{\mathrm{exact}}(M, \mu_0)$ the set (not a vector space) of 'exact' divergence free vector fields $X = \hat{\imath}_{\mu_0}^{-1}(d\omega)$, where $\omega \in \Omega_c^{m-2}(U)$ for an oriented open subset $U \subset M$.

(2) *If for* $f \in C^\infty(M)$ *and a connected open set* $U \subseteq M$ *we have* $(\mathcal{L}_X f)|U = 0$ *for all* $X \in \mathfrak{X}_{exact}(M, \mu_0)$*, then* $f|U$ *is constant.*

Since we shall need some details later on, we prove this well-known fact.

Let $x \in U$. For every tangent vector $X_x \in T_x M$ we can find a vector field $X \in \mathfrak{X}_{\text{exact}}(M, \mu_0)$ such that $X(x) = X_x$; to see this, choose a chart (U_x, u) near x such that $\mu_0|U_x = du^1 \wedge \cdots \wedge du^m$, and choose $g \in C_c^\infty(U_x)$, such that $g = 1$ near x.

Then $X := \hat{\iota}_{\mu_0}^{-1} d(g.u^2.du^3 \wedge \cdots \wedge du^m) \in \mathfrak{X}_{\text{exact}}(M, \mu_0)$ and $X = \partial_{u^1}$ near x. So we can produce a basis for $T_x M$ and even a local frame near x.

Thus $\mathcal{L}_X f|U = 0$ for all $X \in \mathfrak{X}_{\text{exact}}(M, \mu_0)$ implies $df = 0$ and hence f is constant.

(3) *If for a distribution (generalized function)* $A \in \mathcal{D}'(M)$ *and a connected open set* $U \subseteq M$ *we have* $\mathcal{L}_X A|U = 0$ *for all* $X \in \mathfrak{X}_{exact}(M, \mu_0)$, *then* $A|U = C\mu_0|U$ *for some constant* C, *meaning* $\langle A, f\rangle = C\int_M f\mu_0$ *for all* $f \in C_c^\infty(U)$.

Because $\langle \mathcal{L}_X A, f\rangle = -\langle A, \mathcal{L}_X f\rangle$, the invariance property $\mathcal{L}_X A|U = 0$ implies $\langle A, \mathcal{L}_X f\rangle = 0$ for all $f \in C_c^\infty(U)$. Clearly, $\int_M (\mathcal{L}_X f)\mu_0 = 0$. For each $x \in U$ let $U_x \subset U$ be an open oriented chart which is diffeomorphic to $\mathbb{R}^m$. Let $g \in C_c^\infty(U_x)$ satisfy $\int_M g\mu_0 = 0$; we will show that $\langle A, g\rangle = 0$. Because the integral over $g\mu_0$ is zero, the compact cohomology class $[g\mu_0] \in H_c^m(U_x) \cong \mathbb{R}$ vanishes; thus there exists $\alpha \in \Omega_c^{m-1}(U_x) \subset \Omega^{m-1}(M)$ with $d\alpha = g\mu_0$. Since U_x is diffeomorphic to $\mathbb{R}^m$, we can write $\alpha = \sum_j f_j d\beta_j$ with $\beta_j \in \Omega^{m-2}(U_x)$ and $f_j \in C_c^\infty(U_x)$. Choose $h \in C_c^\infty(U_x)$ with $h = 1$ on $\bigcup_j \operatorname{supp}(f_j)$, so that $\alpha = \sum_j f_j d(h\beta_j)$ and $h\beta_j \in \Omega_c^{m-2}(M) \subset \Omega^{m-2}(M)$. Thus the fields $X_j = \hat{\iota}_{\mu_0}^{-1} d(h\beta_j)$ lie in $\mathfrak{X}_{\text{exact}}(M, \mu_0)$ and we have the identity $\sum_j f_j . i_{X_i}\mu_0 = \alpha$.

This means $\sum_j (\mathcal{L}_{X_j} f_j)\mu_0 = \sum_j \mathcal{L}_{X_j}(f_j\mu_0) = \sum_j di_{X_j}(f_j\mu_0) = d\Big(\sum_j f_j.i_{X_j}\mu_0\Big) = d\alpha = g\mu_0$ or $\sum_j \mathcal{L}_{X_j} f_j = g$, leading to

$$\langle A, g\rangle = \sum_j \langle A, \mathcal{L}_{X_j} f_j\rangle = -\sum_j \langle \mathcal{L}_{X_j} A, f_j\rangle = 0\,.$$

So $\langle A, g\rangle = 0$ for all $g \in C_c^\infty(U_x)$ with $\int_M g\mu_0 = 0$. Finally, choose a function φ with support in U_x and $\int_M \varphi\mu_0 = 1$. Then for any $f \in C_c^\infty(U_x)$, the function defined by $g = f - (\int_M f\mu_0).\varphi$ in $C^\infty(M)$ satisfies $\int_M g\mu_0 = 0$ and so

$$\langle A, f\rangle = \langle A, g\rangle + \langle A, \varphi\rangle \int_M f\mu_0 = C\int_M f\mu_0\,,$$

with $C_x = \langle A, \varphi\rangle$. Thus $A|U_x = C_x\mu_0|U_x$. Since U is connected, the constants C_x are all equal: Choose $\varphi \in C_c^\infty(U_x \cap U_y)$ with $\int \varphi\mu_0 = 1$. Thus (3) is proved.

(4) *The operator* $\check{G}_c : C^\infty(M) \to \mathcal{D}'(M)$ *has the following property: If for* $f \in C^\infty(M)$ *and a connected open* $U \subseteq M$ *the restriction* $f|U$ *is constant, then we have* $\check{G}(f)|U = C_U(f)\mu_0|U$ *for some constant* $C_U(f)$*.*

For $x \in U$ choose $g \in C^\infty(M)$ with $g = 1$ near $M \setminus U$ and $g = 0$ on a neighborhood V of x. Then for any $X \in \mathfrak{X}_{\text{exact}}(M, \mu_0)$, that is $X = \hat{\imath}_{\mu_0}^{-1}(d\omega)$ for some $\omega \in \Omega_c^{m-2}(W)$ where $W \subset M$ is an oriented open set, let $Y = \hat{\imath}_{\mu_0}^{-1}(d(g\omega))$. The vector field $Y \in \mathfrak{X}_{\text{exact}}(M, \mu_0)$ equals X near $M \setminus U$ and vanishes on V. Since f is constant on U, $\mathcal{L}_X f = \mathcal{L}_Y f$. For all $h \in C^\infty(M)$ we have $\left\langle \mathcal{L}_X \check{G}_c(f), h \right\rangle = \left\langle \check{G}_c(f), -\mathcal{L}_X h \right\rangle = -G_c(f, \mathcal{L}_X h) = G_c(\mathcal{L}_X f, h) = \left\langle \check{G}_c(\mathcal{L}_X f), h \right\rangle$, since G_c is invariant. Thus also

$$\mathcal{L}_X \check{G}_c(f) = \check{G}_c(\mathcal{L}_X f) = \check{G}_c(\mathcal{L}_Y f) = \mathcal{L}_Y \check{G}_c(f) \,.$$

Now Y vanishes on V and therefore so does $\mathcal{L}_X \check{G}_c(f)$. By (3) we have $\check{G}_c(f)|V = C_V(f)\mu_0|V$ for some $C_V(f) \in \mathbb{R}$. Since U is connected, all the constants $C_V(f)$ have to agree, giving a constant $C_U(f)$, depending only on U and f. Thus (4) follows.

By the Schwartz kernel theorem, $\check{G}_c$ has a kernel $\hat{G}_c$, which is a distribution (generalized function) in

$$\mathcal{D}'(M \times M) \cong \mathcal{D}'(M) \bar{\otimes} \mathcal{D}'(M) = \\ = (C^\infty(M) \bar{\otimes} C^\infty(M))' \cong L(C^\infty(M), \mathcal{D}'(M)) .$$

Note the defining relations

$$G_c(f, g) = \langle \check{G}_c(f), g \rangle = \langle \hat{G}_c, f \otimes g \rangle.$$

Moreover, $\hat{G}_c$ is invariant under the diagonal action of $\mathrm{Diff}(M, \mu_0)$ on $M \times M$. In view of the tensor product in the defining relations, the infinitesimal version of this invariance is: $\mathcal{L}_{X \times 0 + 0 \times X} \hat{G}_c = 0$ for all $X \in \mathfrak{X}(M, \mu_0)$.

(5) *There exists a constant $C_2 = C_2(c)$ such that the distribution $\hat{G}_c - C_2\mu_0 \otimes \mu_0$ is supported on the diagonal of $M \times M$.*

Namely, if $(x, y) \in M \times M$ is not on the diagonal, then there exist open neighborhoods U_x of x and U_y of y in M such that $\overline{U_x} \times \overline{U_y}$ is disjoint to the diagonal, or $\overline{U_x} \cap \overline{U_y} = \emptyset$. Choose any functions $f, g \in C^\infty(M)$ with $\mathrm{supp}(f) \subset U_x$ and $\mathrm{supp}(g) \subset U_y$. Then $f|(M \setminus \overline{U_x}) = 0$, so by (4), $\check{G}_c(f)|(M \setminus \overline{U_x}) = C_{M\setminus\overline{U_x}}(f).\mu_0$. Therefore,

$$\begin{aligned}
G_c(f,g) &= \langle \hat{G}_c, f \otimes g\rangle = \langle \check{G}_c(f), g\rangle \\
&= \langle \check{G}_c(f)|(M \setminus \overline{U_x}), g|(M \setminus \overline{U_x})\rangle\,, \quad \text{since } \mathrm{supp}(g) \subset U_y \subset M \setminus \overline{U_x}, \\
&= C_{M\setminus\overline{U_x}}(f) \cdot \int_M g\mu_0
\end{aligned}$$

By applying the argument for the transposed bilinear form $G_c^T(g, f) = G_c(f, g)$, which is also $\mathrm{Diff}(M, \mu_0)$-invariant, we arrive at

$$G_c(f,g) = G_c^T(g,f) = C'_{M\setminus\overline{U_y}}(g) \cdot \int_M f\mu_0\,.$$

Fix two functions f_0, g_0 with the same properties as f, g and additionally $\int_M f_0\mu_0 = 1$ and $\int_M g_0\mu_0 = 1$. Then we get $C_{M\setminus\overline{U_x}}(f) = C'_{M\setminus\overline{U_y}}(g_0)\int_M f\mu_0$, and so

$$
\begin{aligned}
G_c(f,g) &= C'_{M\setminus\overline{U_y}}(g_0)\int_M f\mu_0 \cdot \int_M g\mu_0 \\
&= C_{M\setminus\overline{U_x}}(f_0)\int_M f\mu_0 \cdot \int_M g\mu_0 .
\end{aligned}
$$

Since $\dim(M) \geq 2$ and M is connected, the complement of the diagonal in $M \times M$ is also connected, and thus the constants $C_{M\setminus\overline{U_x}}(f_0)$ and $C'_{M\setminus\overline{U_y}}(g_0)$ cannot depend on the functions f_0, g_0 or the open sets U_x and U_y as long as the latter are disjoint. Thus there exists a constant $C_2(c)$ such that for all $f, g \in C^\infty(M)$ with disjoint supports we have

$$
G_c(f,g) = C_2(c)\int_M f\mu_0 \cdot \int_M g\mu_0
$$

Since $C_c^\infty(U_x \times U_y) = C_c^\infty(U_x)\bar{\otimes} C_c^\infty(U_y)$, this implies claim (5).

Now we can finish the proof. We may replace $\hat{G}_c \in \mathcal{D}'(M \times M)$ by $\hat{G}_c - C_2\mu_0 \otimes \mu_0$ and thus assume without loss that the constant C_2 in (5) is 0. Let (U, u) be an oriented chart on M such that $\mu_0|U = du^1 \wedge \cdots \wedge du^m$. The distribution $\hat{G}_c|U \times U \in \mathcal{D}'(U \times U)$ has support contained in the diagonal and is of finite order k. By [Hörmander I, 1983, Theorem 5.2.3], the corresponding operator $\check{G}_c : C_c^\infty(U) \to \mathcal{D}'(U)$ is of the form $\hat{G}_c(f) = \sum_{|\alpha|\le k} A_\alpha.\partial^\alpha f$ for $A_\alpha \in \mathcal{D}'(U)$, so that $G(f, g) = \langle \check{G}_c(f), g\rangle = \sum_\alpha \langle A_\alpha, (\partial^\alpha f).g\rangle$. Moreover, the A_α in this representation are uniquely given, as is seen by a look at [Hörmander I, 1983, Theorem 2.3.5].

For $x \in U$ choose an open set U_x with $x \in U_x \subset \overline{U_x} \subset U$, and choose $X \in \mathfrak{X}_{\text{exact}}(M, \mu_0)$ with $X|U_x = \partial_{u^i}$, as in the proof of (2). For functions $f, g \in C_c^\infty(U_x)$ we then have, by the invariance of G_c,

$$
\begin{aligned}
0 &= G_c(\mathcal{L}_X f, g) + G_c(f, \mathcal{L}_X g) = \langle \hat{G}_c | U \times U, \mathcal{L}_X f \otimes g + f \otimes \mathcal{L}_X g \rangle \\
&= \sum_\alpha \langle A_\alpha, (\partial^\alpha \partial_{u^i} f).g + (\partial^\alpha f)(\partial_{u^i} g) \rangle \\
&= \sum_\alpha \langle A_\alpha, \partial_{u^i}((\partial^\alpha f).g) \rangle = \sum_\alpha \langle -\partial_{u^i} A_\alpha, (\partial^\alpha f).g \rangle \,.
\end{aligned}
$$

Since the corresponding operator has again a kernel distribution which is supported on the diagonal, and since the distributions in the representation are unique, we can conclude that $\partial_{u^i} A_\alpha | U_x = 0$ for each α, and each i.

To see that this implies that $A_\alpha|U_x = C_\alpha\mu_0|U_x$, let $f \in C_c^\infty(U_x)$ with $\int_M f\mu_0 = 0$. Then, as in (3), there exists $\omega \in \Omega_c^{m-1}(U_x)$ with $d\omega = f\mu_0$. In coordinates we have $\omega = \sum_i \omega_i . du^1 \wedge \cdots \wedge \widehat{du^i} \wedge du^m$, and so $f = \sum_i (-1)^{i+1}\partial_{u^i}\omega_i$ with $\omega_i \in C_c^\infty(U_x)$. Thus

$$\langle A_\alpha, f\rangle = \sum_i (-1)^{i+1}\langle A_\alpha, \partial_{u^i}\omega_i\rangle = \sum_i (-1)^i \langle \partial_{u^i} A_\alpha, \omega_i\rangle = 0\,.$$

Hence $\langle A_\alpha, f\rangle = 0$ for all $f \in C_c^\infty(U_x)$ with zero integral and as in the proof of (3) we can conclude that $A_\alpha|U_x = C_\alpha\mu_0|U_x$.

But then $G_c(f,g) = \int_{U_x}(Lf).g\mu_0$ for the differential operator $L = \sum_{|\alpha|\leq k} C_\alpha \partial^\alpha$ with constant coefficients on U_x. Now we choose $g \in C_c^\infty(U_x)$ such that $g = 1$ on the support of f. By the invariance of G_c we have again

$$
\begin{aligned}
0 = G_c(\mathcal{L}_X f, g) + G_c(f, \mathcal{L}_X g) &= \int_{U_x} L(\mathcal{L}_X f).g\mu_0 + \int_{U_x} L(f).\mathcal{L}_X g.\mu_0 \\
&= \int_{U_x} L(\mathcal{L}_X f)\mu_0 + 0
\end{aligned}
$$

for each $X \in \mathfrak{X}(M,\mu_0)$. Thus the distribution $f \mapsto \int_{U_x} L(f)\mu_0$ vanishes on all functions of the form $\mathcal{L}_X f$, and by (3) we conclude that $L(\quad).\mu_0 = C_x.\mu_0$ in $\mathcal{D}'(U_x)$, or $L = C_x \,\mathrm{Id}$. By covering M with open sets U_x, we see that all the constants C_x are the same. This concludes the proof of the Main Theorem. □

Thank you for listening up to now.

If you are willing to listen more, there is a little more.

Manifolds with corners

A manifold with corners (recently also called a quadrantic manifold) M is a smooth manifold modelled on open subsets of $\mathbb{R}^m_{\geq 0}$. Assume it is connected and second countable; then it is paracompact and it admits smooth partitions of unity. Any manifold with corners M is a submanifold with corners of an open manifold $\tilde{M}$ of the same dim. Restriction $C^\infty(\tilde{M}) \to C^\infty(M)$ is a surjective continuous linear map which admits a continuous linear section.Thus $C^\infty(M)$ is a topological direct summand in $C^\infty(\tilde{M})$ and the same holds for the dual spaces: The space of distributions $\mathcal{D}'(M)$, which we identity with $C^\infty(M)'$, is a direct summand in $\mathcal{D}'(\tilde{M})$. It consists of all distributions with support in M.

We do not assume that M is oriented, but eventually, that M is compact. Diffeomorphisms of M map the boundary ∂M to itself and map the boundary $\partial^q M$ of corners of codimension q to itself; $\partial^q M$ is a submanifold of codimension q in M; in general $\partial^q M$ has finitely many connected components. We shall consider ∂M as stratified into the connected components of all $\partial^q M$ for $q > 0$.

Moser's theorem for manifolds with corners [BMPR18]

Let M be a compact connected smooth manifold with corners, possibly non-orientable. Let μ_0 and μ_1 be two smooth positive densities in $\mathrm{Dens}_+(M)$ *with $\int_M \mu_0 = \int_M \mu_1$. Then there exists a diffeomorphism $\varphi : M \to M$ such that $\mu_1 = \varphi^* \mu_0$. If and only if $\mu_0(x) = \mu_1(x)$ for each corner $x \in \partial^{\geq 2} M$ of codimension ≥ 2, then φ can be chosen to be the identity on ∂M.*

This result is highly desirable even for M a simplex. The proof is essentially contained in [Banyaga1974], who proved it for manifolds with boundary.

Conjecture

Let M be an oriented compact connected manifold with corners, of dimension $m \geq 2$, and let

$$\partial^p M = (\partial^p M)_1 \sqcup (\partial^p M)_2 \sqcup \cdots \sqcup (\partial^p M)_{n_p}$$

be the decomposition of the set of corners of codimension p into its connected components which are manifolds of dimension $m - p$. Then the the associative algebra of bounded $\mathrm{Diff}_0(M)$-invariant tensor fields on $\mathrm{Dens}_+(M)$ is has the following set of generators, where $\mu \in \mathrm{Dens}_+(M)$ is the footpoint and $\alpha_i \in \Gamma(\mathrm{Vol}(M)) = T_\mu \mathrm{Dens}_+(M)$:

$$f(\mu(M)) \quad \textit{where } f \in C^\infty(\mathbb{R}_{>0}, \mathbb{R}), \quad \int_M \frac{\alpha_1}{\mu} \dots \frac{\alpha_n}{\mu}\, \mu \quad n \geq 1,$$

$$\int_{(\partial^p M)_j} \frac{\alpha_1}{\mu} \dots \frac{\alpha_n}{\mu} d\Big(\frac{\alpha_{i_{n+1}}}{\mu}\Big) \wedge \cdots \wedge d\Big(\frac{\alpha_{i_{n+m-p}}}{\mu}\Big), \quad \begin{matrix} p = 0, \dots m-1, \\ j = 0, \dots, n_p \end{matrix}$$

$$\frac{\alpha}{\mu}((\partial^m M)_j), \quad \textit{for } j = 1, \dots, n_m; \ \textit{note that } (\partial^m M)_j \textit{ is a point.}$$

For a non-orientable compact manifold $\overline{M}$ with corners, let $\pi : M \to \overline{M}$ be its orientable double cover with its deck transformation $\tau : M \to M$. We consider the bounded linear isomorphism

$$\frac{1}{2}\pi^* : \mathrm{Dens}_+(\overline{M}) \to \{\alpha \in \mathrm{Dens}_+(M) : \tau^*\alpha = \alpha\} \subset \mathrm{Dens}_+(M).$$

Then the set of generators for the algebra of bounded $\mathrm{Diff}(M)$*-invariant tensor fields on* $\mathrm{Dens}_+(M)$*, applied to* $\frac{1}{2}\pi^*\alpha_i$ *and* $\frac{1}{2}\pi^*\mu$ *for* $\mu \in \mathrm{Dens}_+(\overline{M})$ *and* $\alpha_i \in T_\mu \mathrm{Dens}_+(\overline{M})$*, is a set of generators for the algebra of* $\mathrm{Diff}(\overline{M})$*-invariant bounded tensor fields on* $\mathrm{Dens}_+(\overline{M})$*.*

Really thank you for listening.