General Sobolev metrics on the manifold of all Riemannian metrics

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arxiv:1810.03169]

Based on collaborations with M.Bauer, M.Bruveris, P.Harms.

For a compact manifold M^m equipped with a smooth fixed background Riemannian metric $\hat{g}$ we consider the space $\mathrm{Met}_{H^s}(M)$ of all Riemannian metrics of Sobolev class H^s for real $s < \frac{m}{2}$ with respect to $\hat{g}$. The L^2-metric on $\mathrm{Met}_{C^\infty}(M)$ was considered by DeWitt, Ebin, Freed and Groisser, Gil-Medrano and Michor, Clarke. Sobolev metrics of integer order on $\mathrm{Met}_{C^\infty}(M)$ were considered in [M.Bauer, P.Harms, and P.W. Michor: Sobolev metrics on the manifold of all Riemannian metrics. J. Differential Geom., 94(2):187-208, 2013.] In this talk we consider variants of these Sobolev metrics which include Sobolev metrics of any positive real (not integer) order $s < \frac{m}{2}$. We derive the geodesic equations and show that they are well-posed under some conditions and induce a locally diffeomorphic geodesic exponential mapping.

The diagram

$$
\begin{tikzcd}
\mathrm{Diff}(M) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] \arrow[dd, two heads, "\mathrm{Diff}(M{,}\mu)"] & & \mathrm{Imm}(M,N) \arrow[dl, "\text{needs } \bar{g}"'] \arrow[dr, "\mathrm{Diff}(M)"] & & \mathrm{Diff}_{\mathcal{A}}(N) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dd, "\text{r-acts}"] \\
& \mathrm{Met}(M) \arrow[dr, two heads, "\mathrm{Diff}(M)"'] & & B_i(M,N) \arrow[dl, "\text{needs } \bar{g}"] & \\
\mathrm{Vol}^1_+(M) & & \dfrac{\mathrm{Met}(M)}{\mathrm{Diff}(M)} & & \mathrm{Met}_{\mathcal{A}}(N)
\end{tikzcd}
$$

M compact , N possibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar{g}$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities

Convenient calculus

We will convenient calculus as developed in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997]. A locally convex vector space E is called *convenient* if each Mackey Cauchy sequence has a limit; equivalently, if for each smooth curve $c : \mathbb{R} \to E$ the Riemann integral $\int_0^1 c(t)\,dt$ converges. This property and those mentioned below depend only on the system of bounded sets in E.
Mappings are smooth if they map smooth curves to smooth curves. Smooth curves can be recognized by applying bounded linear functionals in a subset of the dual which is large enough to recognize bounded subsets. Smooth maps are real analytic if they are real analytic along each affine line. Up to Fréchet spaces convenient smoothness coincides with all other notions of C^∞. Up to Banach spaces convenient real analyticity coincides with all other notions of C^ω.

Firstly, consider $\mathrm{Met}(M)$

Let $\mathrm{Met}(M) = \Gamma(S^2_+ T^*M)$ be the space of all smooth Riemannian metrics on a compact manifold M.

Let $\mathrm{Met}_{H^s}(M) = \Gamma_{H^s}(S^2_+ T^*M)$ the space of all Sobolev H^s sections of the bundle of Riemannian metrics, where $s > \frac{m}{2} = \frac{\dim(M)}{2}$; by the Sobolev inequality then it makes sense to speak of positive definite metrics.

Weak Riemann metrics on $\mathrm{Met}(M)$

All of them are $\mathrm{Diff}(M)$-invariant; natural, tautological.

$$\begin{aligned}
G_g(h,k) &= \int_M g_2^0(h,k)\,\mathrm{vol}(g) = \int \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g), \quad L^2\text{-metr.}\\
\text{or } &= \Phi(\mathrm{Vol(g)}) \int_M g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{conformal}\\
\text{or } &= \int_M \Phi(\mathrm{Scal}^g).g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{curvature modified}\\
\text{or } &= \int_M \Big(g_2^0(h,k) + g_3^0(\nabla^g h, \nabla^g k) + \dots + g_p^0((\nabla^g)^p h, (\nabla^g)^p k)\Big)\,\mathrm{vol}(g)\\
\text{or } &= \int_M g_2^0((1+\Delta^g)^p h, k)\,\mathrm{vol}(g) \qquad \text{Sobolev order } p \in \mathbb{R}_{>0}\\
\text{or } &= \int_M g_2^0\Big(f(1+\Delta^g)h, k\Big)\,\mathrm{vol}(g)
\end{aligned}$$

where $\Phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is total volume of (M,g), Scal is scalar curvature, and g_2^0 is the induced metric on $\binom{0}{2}$-tensors. Here f is a suitable spectral function; see below.

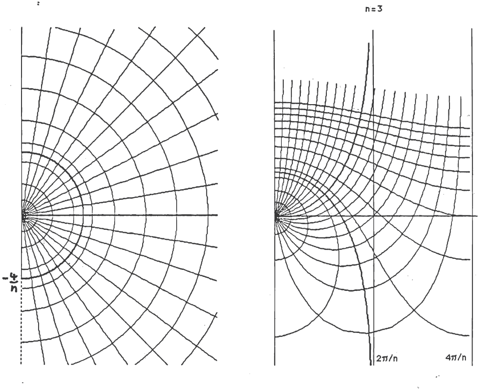
$\Delta^g h := (\nabla^g)^{*,g}\nabla^g h = -\operatorname{Tr}^{g^{-1}}((\nabla^g)^2 h)$ is the Bochner-Laplacian. It can act on all tensor fields h, and it respects the degree of the tensor field it is acting on.

For $p \in \mathbb{N}_{\geq 1}$ the Sobolev order p-metric was introduced in the paper
[M. Bauer, P. Harms, and P. W. Michor. Sobolev metrics on the manifold of all Riemannian metrics. J. Differential Geom. 94.2 (2013), 187–208.]
where we also claimed that the geodesic equation is well-posed. The proof contained a gap, which is repaired now.

The L^2-metric on the space of all Riemann metrics

[DeWitt 1969]. [Ebin 1970]. Geodesics and curvature [Freed Groisser 1989]. [Gil-Medrano Michor 1991] for non-compact M. [Clarke 2009] showed that geodesic distance for the L^2-metric is positive, and he determined the metric completion of $\mathrm{Met}(M)$. The geodesic equation is completely decoupled from space, it is an ODE:

$$g_{tt} = g_t g^{-1} g_t + \tfrac{1}{4} \operatorname{Tr}(g^{-1} g_t g^{-1} g_t)\, g - \tfrac{1}{2} \operatorname{Tr}(g^{-1} g_t)\, g_t$$



$$
\begin{aligned}
A &= g^{-1}a \qquad \text{for } a \in T_g\mathrm{Met}(M) \\
\exp_0(A) &= \tfrac{2}{n}\log\left(\left(1+\tfrac{1}{4}\,\mathrm{Tr}(A)\right)^2 + \tfrac{n}{16}\,\mathrm{Tr}(A_0^2)\right) Id \\
&\quad + \frac{4}{\sqrt{n\,\mathrm{Tr}(A_0^2)}}\arctan\left(\frac{\sqrt{n\,\mathrm{Tr}(A_0^2)}}{4+\mathrm{Tr}(A)}\right) A_0 .
\end{aligned}
$$

Back to the the general metric on $\mathrm{Met}(M)$.

We describe all these metrics uniformly as

$$G_g^P(h,k) = \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$
$$\text{where } P_g : \Gamma(S^2 T^*M) \to \Gamma(S^2 T^*M)$$

is a positive, symmetric, bijective pseudo-differential operator of order $2p$, $p \geq 0$, depending smoothly on the metric g, and also $\mathrm{Diff}(M)$-equivariantly: $\varphi^* \circ P_g = P_{\varphi^* g} \circ \varphi^*$.
The geodesic equation in this notation:

$$\begin{aligned} g_{tt} = P_g^{-1}\Big[&\frac{1}{2}(D_{(g,.)}P_g g_t)^*(g_t) + \frac{1}{4}.g.\,\mathrm{Tr}(g^{-1}.(P_g g_t).g^{-1}.g_t) \\ &+ \frac{1}{2} g_t.g^{-1}.(P_g g_t) + \frac{1}{2}(P_g g_t).g^{-1}.g_t - (D_{(g,g_t)}P_g)g_t \\ &- \frac{1}{2}\,\mathrm{Tr}(g^{-1}.g_t).(P_g g_t)\Big] \end{aligned}$$

Conserved Quantities on $\mathrm{Met}(M)$.

Right action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ given by

$$(g, \phi) \mapsto \phi^* g.$$

Fundamental vector field (infinitesimal action):

$$\zeta_X(g) = \mathcal{L}_X g = -2\,\mathrm{Sym}\,\nabla(g(X)).$$

If metric G^P is invariant, we have the following conserved quantities

$$\begin{aligned}
\text{const} &= G^P(g_t, \zeta_X(g)) \\
&= -2 \int_M g_1^0\big(\nabla^* \,\mathrm{Sym}\, P g_t, g(X)\big)\,\mathrm{vol}(g) \\
&= -2 \int_M g\big(g^{-1} \nabla^* P g_t, X\big)\,\mathrm{vol}(g)
\end{aligned}$$

Since this holds for all vector fields X,

$$\boxed{(\nabla^* P g_t)\,\mathrm{vol}(g) \in \Gamma(T^*M \otimes_M \mathrm{vol}(M)) \text{ is const. in } t.}$$

On $\mathbb{R}^n$: The pullback of the Ebin metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider here the right action
$r : \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n) \times \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n) \to \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ which is given by
$r(g, \varphi) = \varphi^* g$, together with its partial mappings
$r(g, \varphi) = r^{\varphi}(g) = r_g(\varphi) = \mathrm{Pull}^g(\varphi)$.

Theorem. *If $n \geq 2$, the image of $\mathrm{Pull}^{\bar{g}}$, i.e., the $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$-orbit through $\bar{g}$, is the set $\mathrm{Met}_{\mathcal{A}}^{\mathrm{flat}}(\mathbb{R}^n)$ of all flat metrics in $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.*

The pullback of the Ebin metric to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X, Y) = 4 \int_{\mathbb{R}^n} \mathrm{Tr}\big((\mathrm{Sym}\, dX).(\mathrm{Sym}\, dY)\big)\, dx = \int_{\mathbb{R}^n} \langle X, PY \rangle dx$$

Using the inertia operator P we can write the metric as $\int_{\mathbb{R}^n} \langle X, PY \rangle dx$, with

$$P = -2(\mathrm{grad}\,\mathrm{div} + \Delta)\,.$$

The pullback of the general metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider now a weak Riemannian metric on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ in its general form

$$G^P_g(h,k) = \int_M g^0_2(P_g h, k)\,\mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is as described above. *If the operator P is equivariant for the action of $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$, then the induced pullback metric $(\mathrm{Pull}^{\bar{g}})^* G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is right invariant:*

$$G_{\mathrm{Id}}(X,Y) = -4\int_{\mathbb{R}^n} \partial_j(P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j . Y^i\,dx \tag{1}$$

Thus we we get the following formula for the corresponding inertia operator $(\tilde{P}X)^i = \sum_j \partial_j(P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j$. Note that the pullback metric $(\mathrm{Pull}^{\bar{g}})^* G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is always of one order higher then the metric G^P on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.

The Sobolev metric of order $p \in \mathbb{N}$.

The Sobolev metric G^P

$$G_g^P(h,k) = \int_{\mathbb{R}^n} \mathrm{Tr}(g^{-1}.((1+\Delta)^p h).g^{-1}.k)\,\mathrm{vol}(g)\,.$$

The pullback of the Sobolev metric G^P to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X,Y) = -2\int_{\mathbb{R}^n} \Big\langle (\mathrm{grad}\,\mathrm{div} + \Delta)(1-\Delta)^p X, Y \Big\rangle dx\,.$$

Thus the inertia operator is given by

$$\tilde{P} = -2(1-\Delta)^p(\Delta + \mathrm{grad}\,\mathrm{div}) = -2(1-\Delta)^p(\Delta + \mathrm{grad}\,\mathrm{div})\,.$$

It is a linear isomorphism $H^s(\mathbb{R}^n)^n \to H^{s-2p-2}(\mathbb{R}^n)^n$ for every s.

Sobolev spaces of sections of vector bundles.

For $s \in \mathbb{R}$ let $H^s(\mathbb{R}^m, \mathbb{R}^n)$ be the Sobolev space of order s described via Fourier transform $\|f\|_{H^s} = \|\hat{f}(\xi)(1+|\xi|^2)^{s/2}\|_{L^2}$.
Let $E \to M$ be a vector bundle, M compact. Choose a finite vector bundle atlas and a subordinate partition of unity in the following way: Let $(u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq \mathbb{R}^m)_{\alpha \in A}$ be a finite atlas for M, let $(\varphi_\alpha)_{\alpha \in A}$ be a smooth partition of unity subordinated to $(U_\alpha)_{\alpha \in A}$, and let $\psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^n$ be vector bundle charts. Choose open sets U_α° such that $\operatorname{supp}(\psi_\alpha) \subset U_\alpha^\circ \subset \overline{U_\alpha^\circ} \subset U_\alpha$ such that each $u_\alpha(U_\alpha^\circ)$ is an open set in $\mathbb{R}^m$ with Lipschitz boundary.
Then we define for each $s \in \mathbb{R}$ and $f \in \Gamma_{C^\infty}(E)$

$$\|f\|^2_{\Gamma_{H^s}(E)} := \sum_{\alpha \in A} \| \operatorname{pr}_{\mathbb{R}^n} \circ \psi_\alpha \circ (\varphi_\alpha \cdot f) \circ u_\alpha^{-1} \|^2_{H^s(\mathbb{R}^m, \mathbb{R}^n)}.$$

Then $\|\cdot\|_{\Gamma_{H^s}(E)}$ is a norm, which comes from a scalar product, and we write $\Gamma_{H^s}(E)$ for the Hilbert completion of $\Gamma_{C^\infty}(E)$ under the norm. Then $\Gamma_{H^s}(E)$ is independent of the choice of atlas and partition of unity, up to equivalence of norms.

C. Schneider and N. Grosse. Sobolev spaces on Riemannian manifolds with bounded geometry: General coordinates and traces, 2013
H. Triebel. Theory of functions spaces II

Theorem. Module properties of Sobolev spaces. *Let E_1, E_2 be vector bundles over M, and let $s_1, s_2, s \in \mathbb{R}$ satisfy*

(i) $s_1 + s_2 \geq 0$, $\min(s_1, s_2) \geq s$, *and* $s_1 + s_2 - s > \frac{m}{2}$, *or*

(ii) $s \in \mathbb{N}$, $\min(s_1, s_2) > s$, *and* $s_1 + s_2 - s \geq \frac{m}{2}$, *or*

(iii) $-s_1 \in \mathbb{N}$ *or* $-s_2 \in \mathbb{N}$, $s_1 + s_2 > 0$, $\min(s_1, s_2) > s$, $s_1 + s_2 - s \geq \frac{m}{2}$.

Then the tensor product of smooth sections extends to a bounded bilinear mapping

$$\Gamma_{H^{s_1}}(E_1) \times \Gamma_{H^{s_2}}(E_2) \to \Gamma_{H^s}(E_1 \otimes E_2).$$

A. Behzadan and M. Holst. On certain geometric operators between Sobolev spaces of sections of tensor bundles on compact manifolds equipped with rough metrics, 2017.

Invariance under multiplication and adjoints. If $p(s_1, s) = \{s_2 : (s_1, s_2, s) \text{ satisfies } (i) \text{ or } (ii) \text{ or } (iii) \text{ above}\}$ then for all $r, s, t \in \mathbb{R}$:

- If $\alpha \in p(r, s)$ and $\beta \in p(s, t)$, then $\min(\alpha, \beta) \in p(r, t)$, and the tensor product of smooth sections extends to a bounded bilinear mapping $\Gamma_{H^\alpha}(E_1) \times \Gamma_{H^\beta}(E_2) \to \Gamma_{H^{\min(\alpha,\beta)}}(E_1 \otimes E_2)$.
- If $\beta \in p(r, s)$, then $\beta \in p(-s, -r)$.

Riemannian Metrics of Sobolev order

For any $\alpha \in (\frac{\dim(M)}{2}, \infty]$, we define the space of Riemannian metrics of Sobolev order α as

$$\mathrm{Met}_{H^\alpha}(M) := \Gamma_{H^\alpha}(S^2_+ T^* M).$$

Well-defined: $\alpha > \frac{m}{2} \implies \Gamma_{H^\alpha}(S^2 T^* M) \subset \Gamma_{C^0}(S^2 T^* M)$.

Lemma. *Let $\alpha \in (\frac{\dim(M)}{2}, \infty]$. Let $E \to M$ be a first order natural bundle. Then:*
(1) $g \in \mathrm{Met}_{H^\alpha}(M)$ induces a canonical fiber metric of class H^α on E (up to the choice of some constants).
(2) This gives a real analytic map $\mathrm{Met}_{H^\alpha}(M) \to \Gamma_{H^\alpha}(S^2_+ E^*)$. In particular, for $E = T^* M$ one obtains that g^{-1} is real analytic in g.
(3) If E is trivial, then the fiber metric is of class C^∞ and does not depend on g.

Covariant derivative

Lemma.
Let $\alpha \in (\dim(M)/2, \infty)$ *and* $s \in [1-\alpha, \alpha]$. *Then:*
(1) *For each* $g \in \mathrm{Met}_{H^\alpha}(M)$ *and natural first order vector bundle* E *over* M, *there is a unique bounded linear mapping*

$$\Gamma_{H^s}(E) \ni h \mapsto \nabla^g h \in \Gamma_{H^{s-1}}(T^*M \otimes E)$$

which acts as a derivation with respect to tensor products, commutes with each symmetrization operator, and coincides with the Levi-Civita covariant derivative in the cases $E = TM$ *and* $E = T^*M$.
(2) *The covariant derivative is real analytic as a mapping*

$$\mathrm{Met}_{H^\alpha}(M) \ni g \mapsto \nabla^g \in L(\Gamma_{H^s}(E), \Gamma_{H^{s-1}}(T^*M \otimes E)).$$

for all $s \in [1-\alpha, \alpha]$.
(3) *If* E *is trivial, then this holds for all* $s \in \mathbb{R}$.

Remarks to the proof of the lemma

Using the Levi-Civita covariant derivative $\nabla^{\hat{g}}$ for a smooth background Riemannian metrig $\hat{g}$, we express the Levi-Civita connection of $g \in \mathrm{Met}_{H^\alpha}(M)$ as

$$\nabla^g_X = \nabla^{\hat{g}}_X + A^g(X, \quad)$$

for a suitable
$A^g \in \Gamma_{H^{\alpha-1}}(T^*M \otimes T^*M \otimes TM) = \Gamma_{H^{\alpha-1}}(T^*M \otimes L(TM, TM))$.
This tensor field A has to satisfy the following conditions (for smooth vector fields X, Y, Z):

$$(\nabla^{\hat{g}}_X g)(Y, Z) = g(A(X, Y), Z) + g(Y, A(X, Z)) \quad \Longleftrightarrow \quad \nabla^g_X g = 0,$$
$$A(X, Y) = A(Y, X) \quad \Longleftrightarrow \quad \nabla^g \quad \text{is torsionfree.}$$

We take the cyclic permutations of the first equation, sum them with signs $+, +, -$, and use symmetry of A to obtain

$$2g(A(X, Y), Z) = (\nabla^{\hat{g}}_X g)(Y, Z) + (\nabla^{\hat{g}}_Y g)(Z, X) - (\nabla^{\hat{g}}_Z g)(X, Y);$$

this equation determines A uniquely as a $H^{\alpha-1}$-tensor field. It is easy checked that it satisfies the two requirements above.

Remark on geodesics

The Christoffel symbols are of class $H^{\alpha-1}$. They transform as the last part in the second tangent bundle, and the associated spray S^g is an $H^{\alpha-1}$-section of both $\pi_{TM} : T^2M \to TM$ and $T(\pi_M) : T^2M \to TM$.
If $\alpha > \frac{\dim(M)}{2} + 1$, then the spray S^g is continuous and we have local existence (but not uniqueness) of geodesics in each chart separately, by Peano's theorem.

If $\alpha > \frac{\dim(M)}{2} + 2$, then S^g is C^1 and there is existence and uniqueness of geodesics by Picard-Lindelöf.

Bochner Laplacian

Theorem. *Let* $\alpha \in (\dim(M)/2, \infty)$*, let* $s \in [2-\alpha, \alpha]$*, and let* E *be a natural first order vector bundle over* M*. Then:*
(1) *For each* $g \in \mathrm{Met}_{H^\alpha}(M)$*, the Bochner Laplacian is a bounded Fredholm operator of index zero*

$$\Delta^g : \Gamma_{H^s}(E) \ni h \mapsto -\mathrm{Tr}^{g^{-1}}(\nabla^g \nabla^g h) \in \Gamma_{H^{s-2}}(E).$$

which is self-adjoint as an unbounded linear operator on the space $\Gamma_{H^{s-2}}(E)$ *with the* $H^{s-2}(g)$ *inner product.*
(2) *The Laplacian depends real analytically on the metric, i.e., the following mapping is real analytic:*

$$\mathrm{Met}_{H^\alpha}(M) \ni g \mapsto \Delta^g \in L(\Gamma_{H^s}(E), \Gamma_{H^{s-2}}(E)).$$

(3) *If* E *is trivial then these statements hold for all* $s \in [2-\alpha, \alpha+1]$*.*

Derivative of the Laplacian with respect to the metric

This is an essential step of later proofs, and is not obvious.

Lemma. *Let $\alpha \in (m/2, \infty)$ with $\alpha \geq 1$, let E be a natural first order vector bundle over M, let $r \in [2-\alpha, \alpha]$, and let $s \in [2-r, \alpha]$. Then the directional derivative of the Laplace operator with respect to the metric*

$$\begin{aligned} d\Delta : g &\mapsto (m \mapsto D_{g,m}\Delta^g) \\ \mathrm{Met}_{H^\alpha}(M) &\to L(\Gamma_{H^\alpha}(S^2T^*M), L(\Gamma_{H^\alpha}(E), \Gamma_{H^{\alpha-2}}(E))) \end{aligned}$$

extends to a real analytic mapping

$$\begin{aligned} \mathrm{Met}_{H^\alpha}(M) \times \Gamma_{H^r}(S^2T^*M) \ni (g, q) \mapsto \\ \mapsto D_{g,q}\Delta^g \in L(\Gamma_{H^s}(E), \Gamma_{H^{r+s-2-\alpha}}(E)). \end{aligned}$$

Functional calculus of the Laplacian

Let $\alpha \in (\dim(M)/2, \infty)$ *with* $\alpha \geq 1$*, let* $g \in \mathrm{Met}_{H^\alpha}(M)$ *and let* E *be a natural first order vector bundle over* M*. Then:*
(1) *Let* $\Gamma_{H^{-1}(g)}(E)$ *be* $\Gamma_{H^{-1}}(E)$ *with scalar product* $\langle h, k, \rangle_{H^{-1}(g)} = \langle (1+\Delta^g)^{-1} h, k\rangle_{H^0(g)}$.
(2) $1+\Delta^g$*, with domain* $\Gamma_{H^1}(E)$*, is unbounded self-adjoint on* $\Gamma_{H^{-1}(g)}(E)$ *and has a compact resolvent. Thus, there exists an* $H^{-1}(g)$*-orthonormal basis of eigenvectors* $(e_i)_{i\in\mathbb{N}}$ *in* $\Gamma_{H^{-1}(g)}(E)$ *and eigenvalues* $(\lambda_i)_{i\in\mathbb{N}}$ *in* $(1,\infty)$ *such that*

$$\forall i \in \mathbb{N}: \quad e_i \in \Gamma_{H^1}(E), \quad (1+\Delta^g)e_i = \lambda_i e_i.$$

(3) *For each function* $f\colon \{\lambda_1, \lambda_2, \dots\} \to \mathbb{R}$ *the following is a densely defined self-adjoint linear operator on* $\Gamma_{H^{-1}(g)}(E)$*:*

$$f(1+\Delta^g)\colon\ \mathrm{Dom}(f(1+\Delta^g)) \ni h \mapsto \sum_{i\in\mathbb{N}} \langle h_i, e_i\rangle f(\lambda_i) e_i \in \Gamma_{H^{-1}}(E),$$

$$\mathrm{Dom}(f(1+\Delta^g)) = \left\{ h \in \Gamma_{H^{-1}(g)}(E); \sum_{i\in\mathbb{N}} \langle h_i, e_i\rangle^2 f(\lambda_i)^2 < \infty \right\}.$$

(4) *Let* $S_\omega := \{z \in \mathbb{C} : z \neq 0$ *and* $|\arg z| < \omega\}$ *be a sector of angle* $\omega \in (0, \pi)$*, let* $\bigcirc$ *be a closed centered ball contained in the resolvent set of* $1 + \Delta^g$*, and let* f *be a holomorphic function on* S_ω *such that* $\sup_{\lambda \in \partial S_\omega} |\lambda^s f(\lambda)| < \infty$ *for some* $s \in (0, \infty)$*. Then the operator* $f(1 + \Delta^g) \in L(\Gamma_{H^{-1}(g)}(E))$ *can be represented as*

$$f(1+\Delta^g) = -\frac{1}{2\pi i} \int_{\partial(S_\omega \setminus \bigcirc)} f(\lambda)(1+\Delta^g-\lambda)^{-1} d\lambda \in L(\Gamma_{\dot{H}^{-1}(g)}(E)),$$

where the resolvent integral converges in $L(\Gamma_{H^{-1}(g)}(E))$*.*

The above result is based on a functional calculus using $1 + \Delta^g$ viewed as an operator from $\Gamma_{H^1}(E)$ to $\Gamma_{H^{-1}}(E)$. Note, that we would obtain the same result using a functional calculus based on the operator $1 + \Delta^g : L(\Gamma_{H^2}(E), \Gamma_{H^0}(E))$. This would, however, require the more stringent condition $2 \leq \alpha \in \dim(M)/2, \infty)$.

Fractional domain spaces

Let $g \in \mathrm{Met}_{H^\alpha}(M)$ with $\alpha \in (m/2, \infty)$ satisfying $\alpha \geq 1$. Using $1 + \Delta^g : L(\Gamma_{H^1}(E), \Gamma_{H^{-1}}(E))$ we let $\Gamma_{H^s(g)}(E)$ be the space $\Gamma_{H^s}(E)$ with inner product $\langle h, k\rangle_{H^s(g)} = \langle (1+\Delta^g)^{s/2} h, k\rangle_{H^0(g)}$.
For all $s \in [-1, \infty)$ we define the following Hilbert spaces:

$$\Gamma_{H^s(g)}(E) := \mathrm{Dom}((1+\Delta^g)^{\frac{s+1}{2}}) \subseteq \Gamma_{H^{-1}}(E) \quad \text{with norm}$$

$$\|h\|_{\mathcal{D}^s(g)} := \|(1+\Delta^g)^{\frac{s+1}{2}} h\|_{\Gamma_{H^{-1}(g)}(E)}$$

$$\Gamma_{H^{-s}(g)}(E) := \text{ the completion of } \Gamma_{H^{-1}(g)}(E) \text{ with respect to the norm}$$

$$\|h\|_{\mathcal{D}^{-s}(g)} := \|(1+\Delta^g)^{-\frac{s+1}{2}} h\|_{\Gamma_{H^{-1}(g)}(E)}$$

We will show that the identity map extends to an isomorphism $\Gamma_{H^s(g)}(E) \to \Gamma_{H^s}(E)$ for all $s \in [-\alpha, \alpha]$.

Proposition. Fractional Laplacian. *Let* $\alpha \in (\dim(M)/2, \infty)$ *with* $\alpha \geq 1$, *let* $g \in \mathrm{Met}_{H^\alpha}(M)$ *and let* E *be a natural first order vector bundle over* M. *Then:*
(1) *For all* $r, s \in \mathbb{R}$, *the map* $(1 + \Delta^g)^{\frac{s-r}{2}} : \Gamma_{H^s(g)}(E) \to \Gamma_{H^r(g)}(E)$ *is an isometry with the same eigenfunctions* $(e_i) \in \Gamma_{H^\alpha}(E)$ *as* $1 + \Delta^g$ *and with eigenvalues* $(\lambda_i^{(s-r)/2})$.
(2) *For all* $s \in [-\alpha, \alpha]$, *the identity on* $\Gamma(E)$ *extends to a bounded linear map* $\Gamma_{H^s(g)}(E) \to \Gamma_{H^s}(E)$ *with bounded inverse such that the following function is locally bounded:*

$$\mathrm{Met}_{H^\alpha}(M) \ni g \mapsto \| \mathrm{Id} \|_{L(\Gamma_{H^s(g)}(E), \Gamma_{H^s}(E))} + \| \mathrm{Id} \|_{L(\Gamma_{H^s}(E), \Gamma_{H^s(g)}(E))} \in \mathbb{R}.$$

(3) *If* $E = \mathbb{R}$, *then this holds for all* $s \in [-\alpha, \alpha + 1]$, *and the eigenfunctions* e_i *belong to* $\Gamma_{H^{\alpha+1}}(E)$.
Note that $\Gamma_{H^s(g)}(E) \neq \Gamma_{H^s}(E)$ for $s \notin [-\alpha, \alpha]$!

Smoothness and real analycity of the fractional Laplacian

Theorem. *Let $\alpha \in (m/2, \infty)$ with $\alpha > 1$, let E be a natural first order vector bundle over M, let $r, s \in \mathbb{R}$ with $s, s + r \in [-\alpha, \alpha]$, let $\varphi \in (0, \pi)$, and let f be a holomorphic function on S_φ with $\sup_{\lambda \in S_\varphi} |\lambda^{r/2} f(\lambda)| < \infty$. Then the following map is real analytic:*

$$g \mapsto f(1 + \Delta^g), \qquad \mathrm{Met}_{H^\alpha}(M) \to L(\Gamma_{H^s}(E), \Gamma_{H^{r+s}}(E)).$$

If E is trivial, then this holds with $[-\alpha, \alpha]$ replaced by $[-\alpha, \alpha + 1]$.

Lemma *Let $\alpha \in (m/2, \infty)$ with $\alpha > 1$, let E be a natural first order vector bundle over M, let $\varphi \in (0, \pi)$, and let f be a holomorphic function on S_φ which satisfies for some $p \in (1, \alpha]$ that $\sup_{\lambda \in S_\varphi} |\lambda^p f(\lambda)| < \infty$. Then the derivative of $P_g = f(1 + \Delta^g)$ with respect to the metric g extends to a real analytic map*

$$\mathrm{Met}_{H^\alpha}(M) \times \Gamma_{H^{2p-\alpha}}(S^2 T^* M)) \ni (g, q) \mapsto D_{g,q} P_g \in L(\Gamma_{H^\alpha}(E), \Gamma_{H^{-\alpha}}(E))$$

This statement also holds for $f(z) = z^p$ with $p = 1$.

Back to the the general Riemannian metric on $\operatorname{Met}(M)$.

$$G_g^P(h,k) = \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) = \int_M \operatorname{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ with $\varphi^* \circ P_g = P_{\varphi^* g} \circ \varphi^*$.

Conditions on P: There is $p \in \mathbb{R}_{\geq 0}$ and $\alpha_0 \in (m/2, \infty)$ with $\alpha_0 \geq p$ such that $g \mapsto P_g$ satisfies the following for all $\alpha \in [\alpha_0, \infty)$.

(a) The operator field P is smooth as a map

$$\operatorname{Met}_{H^\alpha}(M) \ni g \mapsto P_g \in GL(\Gamma_{H^\alpha}(S^2T^*M), \Gamma_{H^{\alpha-2p}}(S^2T^*M)),$$

(b) P is $\operatorname{Diff}(M)$-equivariant: For all $\varphi \in \operatorname{Diff}(M)$, $g \in \operatorname{Met}_{H^\alpha}(M)$, and $h \in \Gamma_{H^\alpha}(S^2T^*M)$ we have $\varphi^*(P_g h) = P_{\varphi^* g}(\varphi^* h)$.

(c) P_g is nonnegative and symmetric with respect to the $H^0(g)$ inner product on $\Gamma_{H^\alpha}(S^2T^*M)$, i.e., for all $h, k \in \Gamma_{H^\alpha}(S^2T^*M)$:

$$\int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) = \int_M g_2^0(h, P_g k)\,\mathrm{vol}(g), \ \int_M g_2^0(P_g h, h)\,\mathrm{vol}(g) \geq 0.$$

(d) The $H^0(g)$ adjoint of the derivative of P with respect to the metric is well-defined as a smooth map

$$\begin{aligned} \mathrm{Met}_{H^\alpha}(M) \times \Gamma_{H^\alpha}(S^2T^*M) \ni (g,h) \mapsto (D_{(g,\cdot)}P_g h)^* \\ \in L(\Gamma_{H^\alpha}(S^2T^*M), \Gamma_{H^{\alpha-2p}}(S^2T^*M)) \end{aligned}$$

such that the following relation is satisfied for all $g \in \mathrm{Met}_{H^\alpha}(M)$ and $h, k \in \Gamma_{H^\alpha}(S^2T^*M)$:

$$\int_M g_2^0\big((D_{(g,q)}P_g)h, k\big)\,\mathrm{vol}(g) = \int_M g_2^0\big(q, (D_{(g,\cdot)}P_g h)^*(k)\big)\,\mathrm{vol}(g).$$

Theorem (Conditions on P)

Let $\varphi \in (0,\pi)$, let $p \in (1,\infty)$, and let f be a holomorphic function on the sector S_φ which satisfies for some constant $C > 0$ that

$$\forall z \in S_\varphi : \qquad C^{-1}|z^p| \leq |f(z)| \leq C|z^p|.$$

Then the field of operators

$$\mathrm{Met}(M) \ni g \mapsto P_g := f(1+\Delta^g) \in L(\Gamma(S^2T^*M), \Gamma(S^2T^*M))$$

satisfies the conditions above for any $\alpha_0 \in (m/2,\infty)$ with $\alpha_0 > 1$ and $\alpha_0 \geq p$.

Theorem (Well-posedness of the geodesic equation)

Assume that the operator P satisfies the above conditions for some $p \in \mathbb{R}_{\geq 0}$ and $\alpha_0 \in (m/2, \infty)$. Then for each $\alpha \in [\alpha_0, \infty)$ we have:

1. *The initial value problem for the geodesic equation has unique local solutions in* $\mathrm{Met}_{H^\alpha}(M)$*. The solutions depend smoothly on t and on the initial conditions $g(0) \in \mathrm{Met}^\alpha(M)$ and $g_t(0) \in \Gamma_{H^\alpha}(S^2 T^* M)$.*
2. *The Riemannian exponential map* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in $T\mathrm{Met}_{H^\alpha}(M)$, and $(\pi, \exp^P)$ is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Met}^\alpha(M) \times \mathrm{Met}^\alpha(M)$*.*
3. *The neighborhoods in 1 and 2 are uniform in α and can be chosen open in the H^{α_0} topology. Thus, 1 nd 2 continue to hold for $\alpha = \infty$, i.e., on the Fréchet manifold* $\mathrm{Met}(M)$ *of smooth metrics.*

Proofs are based on **Sectorial operators**

For each $\omega \in [0, \pi]$, the sector S_ω of angle $\pm\omega$ is defined as

$$S_\omega := \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \omega\} & \text{if } \omega \in (0, \pi] \\ (0, \infty) & \text{if } \omega = 0. \end{cases}$$

For $\omega \in (0, \pi]$, let $\mathcal{H}^\infty(S_\omega)$ be the Banach algebra of bounded holomorphic functions on S_ω with supremum norm.
Let A be a (possibly unbounded) closed linear operator on a Banach space X. Its resolvent set $\rho(A)$ is the set of $\lambda \in \mathbb{C}$ such that $A - \lambda$ has a bounded inverse, the resolvent $R_\lambda(A) = (A - \lambda)^{-1}$ for $\lambda \in \rho(A)$. Then A is called *sectorial* of angle $\omega \in [0, \pi)$ if the spectrum of A is contained in $\overline{S_\omega}$ and for all $\omega' \in (\omega, \pi)$, the function $\mathbb{C} \setminus \overline{S_{\omega'}} \ni \lambda \mapsto \lambda R_\lambda(A) \in L(X)$ is bounded [Haase 2006].

Sectorial operators admit a holomorphic functional calculus: let $0 < \omega < \varphi < \pi$, let $r > 0$, let A be an invertible sectorial operator of angle strictly less than ω, let $\bigcirc$ be a closed centered ball contained in $\rho(A)$, and let f be a holomorphic function on S_φ satisfying

$$\sup_{\lambda \in \partial(S_\omega \setminus \bigcirc)} |\lambda^r f(\lambda)| < \infty.$$

Then the following Bochner integral is well-defined by the sectoriality of A:

$$f(A) := \frac{-1}{2\pi i} \int_{\partial(S_\omega \setminus \bigcirc)} f(\lambda) R_\lambda(A) d\lambda \in L(X).$$

This primary functional calculus can be extended to larger classes of functions as described in [Haase 2006]. For any $z \in \mathbb{C}$, the fractional power A^z is well-defined as an invertible sectorial operator. The homogeneous fractional domain space $\dot{X}_r$ of A is defined for any $r \in \mathbb{R}$ as the completion of the domain of A^r with respect to the norm $\|x\|_{\dot{X}_r} := \|A^r x\|_X$.

Lemma (Perturbations of sectorial operators)

Let A be an invertible sectorial operator of angle $< \omega \in (0, \pi)$ on a complex Banach space X, let $(\dot{X}_r)_{r\in\mathbb{R}}$ be the fractional domain spaces and let $\bigcirc$ be a closed centered ball $\subset \rho(A)$. Then there an open neighbhd. U of A in $L(\dot{X}_1, \dot{X}_0)$ such that for all $r \in (-\infty, 1]$, $\varphi \in (\omega, \pi)$, and holomorphic functions $f\colon S_\varphi \to \mathbb{C}$ with $\sup_{\lambda\in S_\varphi\setminus\bigcirc} |\lambda^r f(\lambda)| < \infty$ we have:

(1) *All $B \in U$ are sectorial of angle $< \omega$, and $\rho(B) \supset \bigcirc$.*

(2) *The following maps are well-defined and holomorphic:*

$$U \ni B \mapsto (\lambda \mapsto \lambda^{1-r} R_\lambda(B)) \in C_b(\partial(S_\omega \setminus \bigcirc), L(\dot{X}_0, \dot{X}_r)).$$

$$U \ni B \mapsto (\lambda \mapsto \lambda^{1-r} R_\lambda(B)) \in C_b(\partial(S_\omega \setminus \bigcirc), L(\dot{X}_{1-r}, \dot{X}_1)).$$

(3) *If A is densely defined and $B\colon \mathbb{D} \to U$ is holomorphic with $\sup_{z\in\mathbb{D}} \|f(B(z))\|_{L(\dot{X}_0,\dot{X}_r)} < \infty$. Then*

$$\mathbb{D} \ni z \mapsto f(B(z)) = \frac{-1}{2\pi i} \int_{\partial(S_\omega\setminus\bigcirc)} f(\lambda) R_\lambda(B(z)) d\lambda \in L(\dot{X}_0, \dot{X}_r),$$

is holomorphic; integral converges in $L(\dot{X}_0, \dot{X}_{<r})$ and $L(\dot{X}_{>1-r}, \dot{X}_1)$.

Lemma (Perturb. of operators with bounded $\mathcal{H}^\infty$ calculus)

Let A be an invertible densely defined R-sectorial operator of positive angle $< \omega \in (0,\pi)$ with bounded $\mathcal{H}^\infty(S_\omega)$ calculus on a complex Banach space X, let $(\dot{X}_r)_{r\in\mathbb{R}}$ be the fractional domain spaces for A, let $\bigcirc \subset \rho(A)$, let $\delta \in \mathbb{R}\setminus\{0\}$, and let $V = L(\dot{X}_1, \dot{X}_0) \cap L(\dot{X}_{\delta+1}, \dot{X}_\delta)$. Then $\exists$ an open neighbhd U of $A \in V$ such that for all $r \in [0,1]$ and $\varphi \in (\omega,\pi)$ we have:

(1) All $B \in U$ are R-sectorial of positive angle $< \omega$ with $\rho(B) \supset \bigcirc$, and admit a bounded $\mathcal{H}^\infty(S_\varphi)$ calculus with uniform bounds

$$\sup_{B\in U} \sup_{g\in\mathcal{H}^\infty(S_\varphi)\setminus\{0\}} \frac{\|g(B)\|_{L(X)}}{\|g\|_{H^\infty(S_\varphi)}} + \|B^{-r}\|_{L(\dot{X}_0,\dot{X}_r)} < \infty.$$

(2) For any holomorphic $f\colon S_\varphi \to \mathbb{C}$ with $\sup_{\lambda\in S_\varphi} |\lambda^r f(\lambda)| < \infty$,

$$U \ni B \mapsto f(B) = \int_{\partial(S_\omega\setminus\bigcirc)} f(\lambda) R_\lambda(B)\, d\lambda \in L(X, \dot{X}_r)$$

is well-defined and holomorphic, where the integral converges in $L(\dot{X}_0, \dot{X}_{<r}) \cap L(\dot{X}_{>1-r}, \dot{X}_1)$.

Theorem (Perturb. of operators with bounded $\mathcal{H}^\infty$ calculus)

Let A be an invertible densely defined R-sectorial operator of positive angle strictly less than $\omega \in (0, \pi)$ with bounded $\mathcal{H}^\infty(S_\omega)$ calculus on a complex Banach space X, let $(\dot{X}_r)_{r\in\mathbb{R}}$ be the fractional domain spaces associated to A, let $\beta, \gamma \in \mathbb{R}$ with $\beta < \gamma$, and let $V = L(\dot{X}_{\beta+1}, \dot{X}_\beta) \cap L(\dot{X}_{\gamma+1}, \dot{X}_\gamma)$. Then there exists an open neighborhood U of $A \in V$ such that for all $r, s \in \mathbb{R}$ with $s, s + r \in [\beta, \gamma + 1]$, $\varphi \in (\omega, \pi)$, and holomorphic functions $f : S_\varphi \to \mathbb{C}$ with $\sup_{\lambda\in S_\varphi} |\lambda^r f(\lambda)| < \infty$, the following map is well-defined and holomorphic:

$$U \ni B \mapsto f(B) \in L(\dot{X}_s, \dot{X}_{s+r}).$$

Thank you for your attention