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THE FLOW COMPLETION OF THE BURGERS EQUATION

Boris A. Khesin, Peter W. Michor

Abstract. For a manifold equipped with vector field there exists the universal completion consisting of a (possibly non-Hausdorff) manifold with a complete vector field on it. We describe the universal completion of the partial differential equations $u_t + F(u)u_x = 0$ viewed as vector fields on infinite dimensional manifolds.

1. Introduction

For a pair (M, X) consisting of a smooth manifold M and a vector field X on it there exists the universal completion $(\bar{M}, \bar{X})$, a possibly non-Hausdorff manifold $\bar{M}$ with a complete vector field $\bar{X}$, where (M, X) is embedded equivariantly as an open subspace. In this note we describe the universal completion of some partial differential equations viewed as vector fields on infinite dimensional manifolds. The equations are $u_t + f(u)u_x = 0$ where $u = u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is some smooth map. A special case is the inviscid Burgers equation $u_t + 3uu_x = 0$ (also called the Hopf equation). The universal completion gives some insight at how solutions of these equations develop shocks. Namely, in the universal completion the solutions are uniquely extended beyond the shocks, and become multivalued functions with infinite derivatives.

Recall that the inviscid Burgers equation can be regarded as the geodesic equation on the infinite dimensional group of diffeomorphisms of $\mathbb{R}^n$ (cf. e.g. [1, 9]). Such a derivation of the geodesic equation in the one-dimensional case, on the manifold of all embeddings $\mathrm{Emb}(\mathbb{R}, \mathbb{R})$, is reminded below, following [7]. The universal completion described in this note requires the consideration of multivalued velocity fields. These fields are solutions in the phase space of the system. In the configuration space, the completion corresponds to an extension of the diffeomorphism group to the semigroup of polymorphisms.

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2. The universal flow completion

2.1. Vector fields on infinite dimensional manifolds. Let M be a connected smooth manifold of possibly infinite dimension, modeled on convenient vector spaces (see [5], section 27 for necessary definitions). Let us assume that M is smoothly Hausdorff, i.e., the global smooth functions on M separate points. Let X be a smooth (kinematic) vector field on M. We say that X *admits a local flow*, if there exists a smooth mapping

$$M \times \mathbb{R} \supset \mathcal{D}(X) \xrightarrow[\mathrm{Fl}^X]{} M$$

defined on a C^∞-open neighborhood $\mathcal{D}(X)$ of $M \times 0$ such that

(1) $\mathcal{D}(X) \cap (\{x\} \times \mathbb{R})$ is a connected open interval.
(2) If $\mathrm{Fl}^X_s(x)$ exists then $\mathrm{Fl}^X_{t+s}(x)$ and $\mathrm{Fl}^X_t(\mathrm{Fl}^X_s(x))$ exist simultaneously and are equal to each other.
(3) $\mathrm{Fl}^X_0(x) = x$ for all $x \in M$.
(4) $\frac{d}{dt}\mathrm{Fl}^X_t(x) = X(\mathrm{Fl}^X_t(x))$.

It is shown in [5], 32.14, that then for each integral curve c of X we have $c(t) = \mathrm{Fl}^X_t(c(0))$ (see [5], 32.14 for the proof, as well as for counterexamples against existence, uniqueness, etc. of integral curves for more general X). Thus there exists a unique maximal flow. Furthermore, X is Fl^X_t-related to itself, i.e., $T(\mathrm{Fl}^X_t) \circ X = X \circ \mathrm{Fl}^X_t$.

2.2. Theorem. *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on a (connected) smooth, possibly infinite-dimensional, manifold M modeled on convenient vector spaces. Let us assume that the vector field X admits a local flow.*

Then there exists a universal flow completion $j : (M,X) \to (\bar M, \bar X)$ of (M,X). Namely, there exists a (connected) smooth not necessarily Hausdorff manifold $\bar M$, a complete vector field $\bar X \in \mathfrak{X}(\bar M)$, and an embedding $j : M \to \bar M$ onto an open submanifold such that X and $\bar X$ are j-related: $Tj \circ X = \bar X \circ j$. Moreover, for any other equivariant morphism $f : (M,X) \to (N,Y)$ for a manifold N and a complete vector field $Y \in \mathfrak{X}(N)$ there exists a unique equivariant morphism $\bar f : (\bar M, \bar X) \to (N,Y)$ with $\bar f \circ j = f$. The leaf spaces M/X and $\bar M/\bar X$ are homeomorphic.

An equivariant morphism $f : (M,X) \to (N,Y)$ is a smooth mapping $f : M \to N$ satisfying $Tf \circ X = Y \circ f$. It follows that then $f \circ \mathrm{Fl}^X_t = \mathrm{Fl}^Y_t \circ f$ wherever Fl^X_t is defined.

Sketch of Proof. The finite dimensional version of this theorem is due to Palais [8]. The formulation here is from [4] and the proof given in [4] goes through in the infinite-dimensional case as well.

Since we shall need the construction, we sketch it here: Consider the manifold $\mathbb{R} \times M$ with a coordinate function s on $\mathbb{R}$, the vector field $\tilde X := \partial_s \times X \in \mathfrak{X}(\mathbb{R} \times M)$, and let $\bar M := \mathbb{R} \times_{\tilde X} M$ be the orbit space (or leaf space) of the vector field $\tilde X$. We consider the flow mapping $\mathrm{Fl}^{\tilde X} : \mathcal{D}(\tilde X) \to \mathbb{R} \times M$ given by $\mathrm{Fl}^{\tilde X}_t(s,x) = (s+t, \mathrm{Fl}^X_t(x))$.

For each $s \in \mathbb{R}$ we have the injective mapping

$$j_s : M \xrightarrow{\mathrm{ins}_t} \{s\} \times M \subset \mathbb{R} \times M \xrightarrow{\pi} \mathbb{R} \times_{\tilde X} M = \bar M$$

which is a homeomorphism on its open image $j_s(M)$ in $\bar{M}$ in the quotient topology. We use the mappings $j_s : M \to \bar{M}$ as charts. The chart change for $r < s$ are then $(j_s)^{-1} \circ j_r = \mathrm{Fl}^X_{s-r}$ restricted to $(j_s)^{-1}(j_r(M)) \subset M$.

The flow $(t, (s, x)) \mapsto (s + t, x)$ on $\mathbb{R} \times M$ commutes with the flow of $\tilde{X}$ and thus induces a flow on the leaf space $\bar{M} = \mathbb{R} \times_{\tilde{X}} M$. Differentiating this flow we get a vector field $\bar{X}$ on $\bar{M}$.

The construction $(M, X) \mapsto (\bar{M}, \bar{X})$ is a functor from the category of smooth convenient smoothly Hausdorff manifolds with vector fields admitting local flows and smooth mappings intertwining the vector fields into the category of possibly non-Hausdorff manifolds with smooth vector fields with global flows and smooth mappings intertwining these fields. For a pair (M, X) with a complete vector field X the flow completion $(\bar{M}, \bar{X})$ is equivariantly diffeomorphic to (M, X) since then any of the charts $j_s : M \to \bar{M}$ is also surjective. From this the universal property follows. $\square$

2.3. Example. Consider $M = \mathbb{R}$, $X = -x^2\partial_x$. The solutions of the ordinary differential equation $\dot{x} = -x^2$ are $x(t) = 1/(t + 1/x(0))$ which are all incomplete, and 0. The foliation in $\mathbb{R} \times M$ is given by the graphs of the functions $x(t) = 1/(t + 1/x(0))$. Consider the following illustration of $\mathbb{R} \times M$ and its foliation.

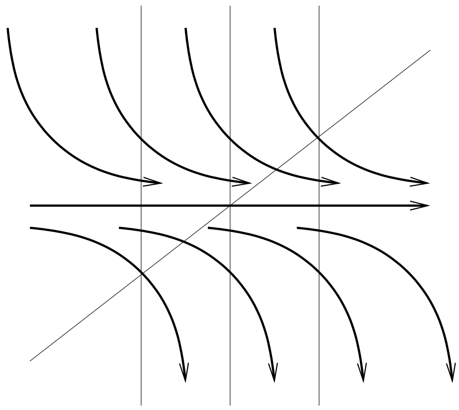

FIG. 1. The flow of the field $\dot{x} = -x^2$. The embeddings j_t are induced by the vertical slices. The completion is $\bar{M} = \mathbb{R}$, the identification is given by the inclined line, for example.

Note that this incompleteness of a quadratic field is similar to the incompleteness of the Burgers equation described below. Examples leading to non-Hausdorff completions can be found in [4].

2.4. Remark on Hamiltonian systems. Suppose that M is a symplectic or Poisson manifold and that X_f is the Hamiltonian vector field of a smooth function f. Then there exists a unique symplectic or Poisson structure on the flow completion $\bar{M}$ and a unique smooth function $\bar{f}$ such that $\bar{X}$ is again the Hamiltonian vector field of $\bar{f}$. Moreover, if $f = f_1, \dots, f_n$ is a maximal Poisson commuting set of smooth function such that (M, X_f) is a completely integrable system, then there are unique extensions $\bar{f}_1, \dots, \bar{f}_n$ to $\bar{M}$ such that the flow completion $(\bar{M}, \overline{X_f})$ is again a completely integrable system.

In the infinite-dimensional symplectic case (M,ω) should be a weak symplectic manifold and all (possibly, infinitely many) functions f_i have to be taken in the space $C^\infty_\omega(M,\mathbb{R})$ of smooth functions with a smooth ω-gradient, see [5], section 48. All this is an easy consequence of the fact that the symplectic or Poisson structures and the conservation laws f_i are invariant under the flow of X_f, and that restrictions of this flow are the chart transfer mappings for the atlas used to define the flow completion.

3. The Burgers equation as a geodesic equation

3.1. The principal bundle of embeddings. Let M and N be smooth connected finite-dimensional manifolds without boundary, such that $\dim M \le \dim N$. The space $\operatorname{Emb}(M,N)$ of all embeddings (immersions which are homeomorphisms on their images) from M into N is an open submanifold of $C^\infty(M,N)$ which is stable under the right action of the diffeomorphism group of M. Here $C^\infty(M,N)$ is a smooth manifold modeled on spaces of sections $\Gamma_c(f^*TN)$ with compact support. In particular, the tangent space at f is canonically isomorphic to the space of vector fields along f with compact support in M. If f and g differ on a non-compact set then they belong to different connected components of $C^\infty(M,N)$.

Then $\operatorname{Emb}(M,N)$ is the total space of a smooth principal fiber bundle whose structure group is the diffeomorphism group of M. Its base, denoted by $B(M,N)$, is a Hausdorff smooth manifold modeled on nuclear (LF)-spaces. It can be thought of as the "nonlinear Grassmannian" of all submanifolds of N which are of type M. If we take a Hilbert space H instead of N, then $B(M,H)$ is the classifying space for $\operatorname{Diff}(M)$ if M is compact, and the classifying bundle $\operatorname{Emb}(M,H)$ carries also a universal connection, see details in [5], sections 42-44.

3.2. A geodesic equation. Consider the convenient manifold $\operatorname{Emb}(\mathbb{R},\mathbb{R})$ of all embeddings of the real line into itself, which contains the diffeomorphism group $\operatorname{Diff}(\mathbb{R})$ as an open subset. Each connected component is a free orbit of the diffeomorphism group $\operatorname{Diff}(\mathbb{R})$ for the action of composition from the right. The tangent bundle is trivial, $T\operatorname{Emb}(\mathbb{R},\mathbb{R}) = \operatorname{Emb}(\mathbb{R},\mathbb{R}) \times C^\infty_c(\mathbb{R},\mathbb{R})$, tangent vectors are smooth functions with compact support. For our purposes, we may restrict attention to the space of orientation-preserving embeddings, denoted by $\operatorname{Emb}^+(\mathbb{R},\mathbb{R})$. The case S^1 is treated in a similar fashion and the results are also valid in this situation, where $\operatorname{Emb}(S^1,S^1) = \operatorname{Diff}(S^1)$.

Following V.Arnold's approach to Euler equations on diffeomorphism groups, we define the weak Riemannian metric on $\operatorname{Emb}^+(\mathbb{R},\mathbb{R})$ by the formula:

$$G_f(h,k) = \int_{\mathbb{R}} h(x)k(x)|f'(x)|\,dx, \quad f \in \operatorname{Emb}(\mathbb{R},\mathbb{R}), \quad h,k \in C^\infty_c(\mathbb{R},\mathbb{R}).$$

It is invariant under the right action of the diffeomorphism group. The energy of a curve f of embeddings is

$$E(f) = \tfrac{1}{2}\int_a^b G_f(f_t,f_t)dt = \tfrac{1}{2}\int_a^b\int_{\mathbb{R}} f_t^2 f_x\,dxdt.$$

Consider smooth variations of $f(x,t)$ with fixed endpoints. Then variational calculus provides the following form of the geodesic equation with its corresponding initial data:

$$f_{tt} = -2\frac{f_t f_{tx}}{f_x},$$

where

$$f(\,.\,,0) \in \mathrm{Emb}^+(\mathbb{R},\mathbb{R}), \quad f_t(\,.\,,0) \in C_c^\infty(\mathbb{R},\mathbb{R}).$$

The geodesic equation has the following conservation law: if instead of the obvious framing we change variables to $T\,\mathrm{Emb} = \mathrm{Emb} \times C_c^\infty \ni (f,h) \mapsto (f, h f_x^2) =: (f,H)$ then the geodesic equation becomes $H_t = \frac{\partial}{\partial t}(f_t f_x^2) = f_x^2(f_{tt} + 2\frac{f_t f_{tx}}{f_x}) = 0$, so that $H = f_t f_x^2$ is constant in t.

3.3 The geodesic property of the Burgers equation. We restrict our attention from the whole space $\mathrm{Emb}(\mathbb{R},\mathbb{R})$ to the open subset $\mathrm{Diff}(\mathbb{R})$. Consider the trivialization of $T\,\mathrm{Diff}(\mathbb{R})$ by right translation. The derivative of the inversion $\mathrm{Inv} : g \mapsto g^{-1}$ is given by

$$T_g(\mathrm{Inv})h = -T(g^{-1}) \circ h \circ g^{-1} = \frac{h \circ g^{-1}}{g_x \circ g^{-1}} \qquad \text{for} \qquad g \in \mathrm{Diff}(\mathbb{R}),\ h \in C_c^\infty(\mathbb{R},\mathbb{R}).$$

Defining $u := f_t \circ f^{-1}$, or, in more detail, $u(x,t) = f_t(f(\quad,t)^{-1}(x),t)$, we have

$$\begin{aligned}
u_x &= (f_t \circ f^{-1})_x = (f_{tx} \circ f^{-1})\frac{1}{f_x \circ f^{-1}} = \frac{f_{tx}}{f_x} \circ f^{-1},\\
u_t &= (f_t \circ f^{-1})_t = f_{tt} \circ f^{-1} + (f_{tx} \circ f^{-1})(f^{-1})_t\\
&= f_{tt} \circ f^{-1} + (f_{tx} \circ f^{-1})\frac{1}{f_x \circ f^{-1}}(f_t \circ f^{-1})
\end{aligned}$$

which, by the geodesic equation of 3.2 becomes

$$u_t = f_{tt} \circ f^{-1} - \left(\frac{f_{tx} f_t}{f_x}\right) \circ f^{-1} = -3\left(\frac{f_{tx} f_t}{f_x}\right) \circ f^{-1} = -3u_x u.$$

The geodesic equation on $\mathrm{Diff}(\mathbb{R})$ in right trivialization, that is, in Eulerian formulation, is hence

$$u_t = -3u_x u$$

which is just the inviscid Burgers equation. Similarly, one obtains the derivation in the n-dimensional case.

4. The flow completion of some hyperbolic systems

4.1 A partial differential equation. Let $f = (f_1, \dots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$ be smooth and consider the partial differential equation

$$u_t + (f(u) \cdot \nabla)u = 0$$

or, which is the same,

$$u_t + f_1(u)u_{x_1} + \dots + f_k(u)u_{x_k} = 0, \qquad \mathbb{R}^k \times \mathbb{R} \supseteq U \xrightarrow{u} \mathbb{R}^n,$$

where U is an open neighborhood of $\mathbb{R}^k \times 0$ in $\mathbb{R}^k \times \mathbb{R}^n$, and u is a smooth $\mathbb{R}^n$-valued function on U. This type of equations are called hyperbolic conservation laws in physics, see [2].

We consider now the manifold $C^\infty(\mathbb{R}^k, \mathbb{R}^n)$ of all smooth $\mathbb{R}^n$-valued functions on $\mathbb{R}^k$ with the manifold structure described in [5], section 42. The tangent bundle is trivial, and the space $C_c^\infty(\mathbb{R}^k, \mathbb{R}^n)$ of functions with compact support serves as the fiber. Note that $C_c^\infty(\mathbb{R}^k, \mathbb{R}^n)$ is an open connected component in $C^\infty(\mathbb{R}^k, \mathbb{R}^n)$. We consider the characteristic vector field

$$X(u) = (f(u) \cdot \nabla)u.$$

It is a vector field on $C^\infty(\mathbb{R}^k, \mathbb{R}^n)$ if $X(u)$ has compact support for each u. This is the case if u has compact support. In the general case one has to leave the realm of manifolds with charts. (See [6] for a setting for infinite dimensional manifolds based on curves instead of charts, which is applicable in this situation.)

For the sake of simplicity, let us restrict attention to $C_c^\infty(\mathbb{R}^k, \mathbb{R}^n)$. There, flow lines of the vector field X are given by solutions of the above partial differential equation (where one has to adapt the domain of definition). We may thus consider $(C_c^\infty(\mathbb{R}^k, \mathbb{R}^n), X)$ as smooth convenient manifolds with vector fields admitting local flows.

4.2. Characteristics and solutions. To describe the universal completion of the quasilinear equation $u_t + (f(u) \cdot \nabla)u = 0$ we apply the characteristic method (see e.g. [3] or [1], where in particular, the case of the Burgers equation is treated).

In the space $\mathbb{R}^{k+n}$ with coordinates (x, y) consider the vector field $Y(x, y) = (f(y), 0) = f_1(y)\partial_{x^1} + \cdots + f_k(y)\partial_{x^k}$ with differential equation $\dot{x} = f(y), \dot{y} = 0$. It has the complete flow $\mathrm{Fl}_t^Y(x, y) = (x + tf(y), y)$.

Let now $u(x, t)$ be a curve of functions. We ask when the graph of u can be reparametrized in such a way that it becomes a solution curve of the push forward vector field $Y_* : f \mapsto Y \circ f$ on the space of embeddings $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^{k+n})$. Thus consider a time dependent reparametrization $z \mapsto x(z, t)$, i.e., $x \in C^\infty(\mathbb{R}^{k+1}, \mathbb{R}^k)$. The curve $t \mapsto (x(z, t), u(x(z, t), t))$ in $\mathbb{R}^{k+n}$ is an integral curve of Y if and only if

$$\begin{aligned} &\begin{pmatrix} f \circ u \circ x \\ 0 \end{pmatrix} = \partial_t \begin{pmatrix} x \\ u \circ x \end{pmatrix} = \begin{pmatrix} x_t \\ u_t \circ x + (\nabla u \circ x) \cdot x_t \end{pmatrix} \\ &\iff \begin{cases} x_t = f \circ u \circ x \\ 0 = (u_t + (f \circ u) \cdot \nabla u) \circ x \end{cases} \end{aligned}$$

This implies that the graph of $u(\cdot, t)$, namely the curve $t \mapsto (x \mapsto (x, u(x, t)))$, may be parameterized as a solution curve of the vector field Y_* on the space of embeddings $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^{k+n})$ starting at $x \mapsto (x, u(x, 0))$ if and only if u is a solution of the partial differential equation $u_t + (f(u) \cdot \nabla)u = 0$. The parameterization $z \mapsto x(z, t)$ is then given by $x_t(z, t) = f(u(x(z, t), t))$ with $x(z, 0) = z \in \mathbb{R}^k$.

For $k = n$ the characteristics have a simple physical meaning. Consider freely flying particles in $\mathbb{R}^n$, and trace a trajectory $x(t)$ of one of the particles. Denote the velocity of a particle at the position x at the moment t by $u(t, x)$, or rather, by $f(u(x, t)) := \dot{x}(t)$. (For the inviscid Burgers equation, $u(x, t) := \dot{x}(t)$.) Due to the absence of interaction, the Newton equation of any particle is $\ddot{x}(t) = 0$.

Example. The inviscid 1D Burgers equation (see [1]). Consider the equation $u_t + 3uu_x = 0$ with $k = n = 1$ and $f(u) = 3u$. There the flow of the vector field $Y = 3u\partial_x$ is tilting the plane to the right with constant speed. The illustration shows how a graph of an honest function is moved through a shock (when the derivatives become infinite) towards the graph of a multivalued function; each piece of it is still a local solution.

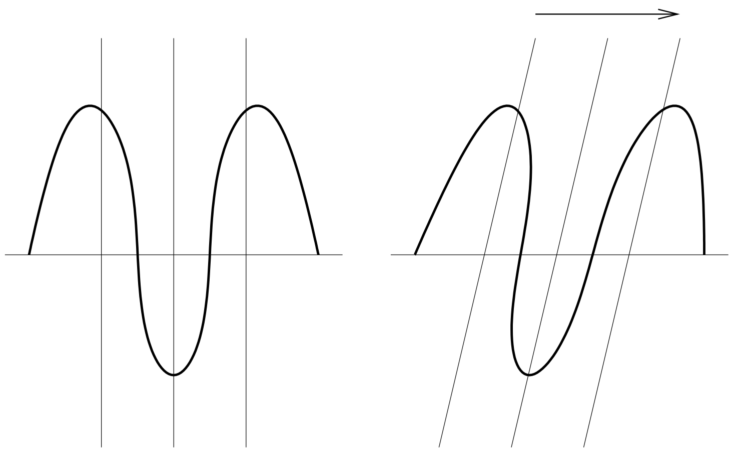

FIG.2. The characteristic flow of the inviscid Burgers equations tilts the plane.

We also refer to [2] for a treatment of more general equations $u_t + A(u)u_x = 0$ (where A is matrix valued with all eigenvalues distinct) as the limits of equations with "viscous" right hand side $\epsilon\Delta u$.

We now interpret the characteristics in the space of graphs of functions. Given a function $u_0 \in C_c^\infty(\mathbb{R}^k, \mathbb{R}^n)$ with compact support we consider the graph of u_0 as the submanifold $\Gamma(u_0) = \{(x, u_0(x)) : x \in \mathbb{R}^k\}$ of $\mathbb{R}^{k+n}$. Let $\mathrm{pr}_1 : \mathbb{R}^{k+n} \to \mathbb{R}^k$ and $\mathrm{pr}_2 : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be the projections. Consider the interval of all $t \in \mathbb{R}$ such that $\mathrm{pr}_1 |Fl_t^Y(\Gamma(u_0)) : Fl_t^Y(\Gamma(u_0)) \to \mathbb{R}^k$ is a diffeomorphism for all $t' \in [0, t]$ or $t' \in [t, 0]$, respectively. Then

$$u(x, t) = \mathrm{pr}_2(\mathrm{pr}_1 |Fl_t^Y(\Gamma(u_0)))^{-1}(x)$$

is a solution of equation 4.1 with initial value $u(x, 0) = u_0(x)$. Thus the vector field X on $C_c^\infty(\mathbb{R}^k, \mathbb{R}^n)$ admits a local flow.

4.3. The flow completion. Now one can see that after some time graphs of functions become graphs of multivalued functions. This explains the following construction of the completion.

We consider the principal bundle of all proper smooth embedded k-surfaces in $\mathbb{R}^{k+n}$ which deviate from $\mathbb{R}^k \times 0$ only in a compact set, with projection $\pi : \mathrm{Emb}_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n}) \to B_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ onto the convenient manifold of k-dimensional submanifolds of $\mathbb{R}^{k+n}$ which deviate from $\mathbb{R}^k \times 0$ only in a compact set. The structure group is the group of $\mathrm{Diff}_c(\mathbb{R}^k)$ of diffeomorphisms with compact support. We have the graph embedding, a smooth mapping

$$\gamma : C_c^\infty(\mathbb{R}^k, \mathbb{R}^n) \to \mathrm{Emb}_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n}), \qquad \gamma(u)(x) = (x, u(x)).$$

Let us assume now that $f(0) = 0$. Then the flow $\mathrm{Fl}^Y_t(x,y) = (x + tf(y), y)$ of the vector field $Y(x,y) = f_1(y)\partial_{x^1} + \dots f_k(y)\partial_{x^k}$ on $\mathbb{R}^{k+n}$ acts on parameterized k-surfaces in $\mathrm{Emb}_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ by $(\mathrm{Fl}^Y_t \circ (c_1, c_2))(x) = (c_1(x) + tf(c_2(x)), c_2(x))$ and is the flow of the vector field Y_* on $\mathrm{Emb}_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ given by $Y_*(c_1, c_2) = (f_1 \circ c_2)\partial_{x^1} + \dots + (f_k \circ c_2)\partial_{x^k} = (f \circ c_2, 0)$. The vector field Y_* is invariant under the principal right action of $g \in \mathrm{Diff}_c(\mathbb{R}^k)$ which is given by $(c_1, c_2) \mapsto (c_1 \circ g, c_2 \circ g)$. Thus Y_* induces a smooth vector field Z on the base manifold $B_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ whose flow is again Fl^Y_t applied to closed submanifolds of $\mathbb{R}^{k+n}$.

We consider now the space $\mathcal{G}$ of all closed non-compact k-dimensional submanifolds $N \in B_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ such that for some $t \in \mathbb{R}$ the mapping $\mathrm{pr}_1 \circ \mathrm{Fl}^Y_t |N : N \to \mathbb{R}^k$ is a diffeomorphism. By the choice of topology on $B_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ the space $\mathcal{G}$ is open, and obviously invariant under the flow of the vector field Z.

Proposition. *Let $f(0) = 0$. Then the flow completion $\overline{C^\infty_c(\mathbb{R}^k, \mathbb{R}^n)}$ of the infinite dimensional manifold with vector field $(C^\infty_c(\mathbb{R}^k, \mathbb{R}^n), X)$ is diffeomorphic to $(\mathcal{G}, Z)$. The mapping $j_t : (C^\infty_c(\mathbb{R}, \mathbb{R}^n), X) \to \mathcal{G}$ is given by $j_t = \mathrm{Fl}^Z_t \circ \pi \circ \gamma$.*

Proof. In the proof of theorem 2.2 we have seen that the completion $\overline{C^\infty_c(\mathbb{R}^k, \mathbb{R}^n)}$ can be described by taking the pieces $j_t(C^\infty_c(\mathbb{R}, \mathbb{R}^n))$ which are all diffeomorphic to $C^\infty_c(\mathbb{R}, \mathbb{R}^n)$ and gluing them via the smooth mappings $(j_s)^{-1} \circ j_r = \mathrm{Fl}^X_{s-r}$ for $r < s$. But this is realized in the open subset $\mathcal{G} \subset B_{\mathbb{R}^k}(\mathbb{R}^k, \mathbb{R}^{k+n})$ by the global flow Fl^Z. Thus we reconstructed the atlas describing the completion in 2.2 as a smooth manifold. □

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P.W. Michor: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria; and Erwin Schrödinger International Institute of Mathematical Physics, Pasteurgasse 6/7, A-1090 Wien, Austria.

E-mail address: peter.michor@esi.ac.at

B.A. Khesin: Department of Mathematics, University of Toronto, 100 St.George Street, Toronto, ON M5S 3G3, Canada.

E-mail address: khesin@math.toronto.edu