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CHARACTERIZING ALGEBRAS OF SMOOTH FUNCTIONS ON MANIFOLDS

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Abstract. Among all C^∞-algebras we characterize those which are algebras of smooth functions on smooth separable Hausdorff manifolds.

1. C^∞-algebras. An $\mathbb{R}$-algebra is a commutative ring A with unit together with a ring homomorphism $\mathbb{R} \to A$. Then every map $p : \mathbb{R}^n \to \mathbb{R}^m$ which is given by an m-tuple of real polynomials $(p_1, \dots, p_m)$ can be interpreted as a mapping $A(p) : A^n \to A^m$ in such a way that projections, composition, and identity are preserved, by just evaluating each polynomial p_i on an n-tuple $(a_1, \dots, a_n) \in A^n$.

A C^∞-algebra A is a real algebra in which we can moreover interpret all smooth mappings $f : \mathbb{R}^n \to \mathbb{R}^m$. There is a corresponding map $A(f) : A^n \to A^m$, and again projections, composition, and the identity mapping are preserved.

More precisely, a C^∞-algebra A is a product preserving functor from the category C^∞ to the category of sets, where C^∞ has as objects all spaces $\mathbb{R}^n$, $n \geq 0$, and all smooth mappings between them as arrows. Morphisms between C^∞-algebras are then natural transformations: they correspond to those algebra homomorphisms which preserve the intepretation of smooth mappings.

This definition of C^∞-algebras is due to Lawvere [2], for a thorough account see Moerdijk-Reyes [3], for a discussion from the point of view of functional analysis see [1]. In [1], 6.6 one finds a method to recognize C^∞-algebras among locally-m-convex algebras.

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2. Theorem. *Let A be a C^∞-algebra. Then A is the algebra of smooth functions on some finite dimensional paracompact Hausdorff second countable manifold M if and only if the following conditions are satisfied:*

(1) *A is point determined (*[3]*, 4.1), so A can be embedded as algebra into a power $\prod_{x\in X}\mathbb{R}$ of copies of $\mathbb{R}$. Equivalently the intersection of all ideals of codimension 1 in A is 0.*
(2) *A is finitely generated, so $A = C^\infty(\mathbb{R}^n)/I$ for some ideal $I \subset C^\infty(\mathbb{R}^n)$.*
(3) *For each ideal $\mathfrak{m}_x$ of codimension 1 in A the localization $A_{\mathfrak{m}_x}$ is isomorphic to the C^∞-algebra $C^\infty_0(\mathbb{R}^m)$ consisting of all germs at 0 of smooth functions on $\mathbb{R}^m$, for some m.*

Proof. By condition (2) A is finitely generated, $A = C^\infty(\mathbb{R}^n)/I$; so by [3], 4.2 the C^∞-algebra A is point determined (1) if and only if the ideal I has the following property:

$$\text{For } f \in C^\infty(\mathbb{R}^n), \quad f|Z(I) = 0 \text{ implies } f \in I, \tag{4}$$

where $Z(I) = \bigcap\{f^{-1}(0) : f \in I\} \subset \mathbb{R}^n$. Let us denote by $\{\mathfrak{m}_x : x \in M\}$ the set of all ideals $\mathfrak{m}_x$ of codimension 1 in A. Then $A/\mathfrak{m}_x \cong \mathbb{R}$ and we write $a(x)$ for the projection of $a \in A$ in $A/\mathfrak{m}_x$. In particular we indentify the elements of A with functions on M. Let $c_1, \dots, c_n \in A$ by a set of generators. Then we may view $c = (c_1, \dots, c_n) : M \to \mathbb{R}^n$ as a mapping such that the pullback $c^*(f) = f \circ c = A(f)(c)$ is the quotient mapping $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/I = A$. By condition (1) $c : M \to \mathbb{R}^n$ is injective, and the image $c(M)$ equals $Z(I) = \bigcap\{f^{-1}(0) : f \in M\}$, by (4). In particular, $c(M)$ is closed. The initial topology on M with respect to all functions in A coincides with the subspace topology induced via the embedding $c : M \to \mathbb{R}^n$, so this topology is metrizable and locally compact.

Let us fix a 'point' $x \in M$. The codimension 1 ideal $\mathfrak{m}_x$ is a prime ideal, so the subset $A \setminus \mathfrak{m}_x \subset A$ is closed under multiplication and without divisors of 0, thus the localization $A_{\mathfrak{m}_x}$ may be viewed as the set of fractions $\frac{a}{b}$ with $a \in A$, $b \in A \setminus \mathfrak{m}_x$; it is a local algebra with maximal ideal $\tilde{\mathfrak{m}}_x = \{\frac{a}{b} : a \in \mathfrak{m}_x, b \in A \setminus \mathfrak{m}_x\}$. Note that $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \cong T_0^*\mathbb{R}^m = \mathbb{R}^m$ by condition (3). Now choose $a_1, \dots, a_m \in \mathfrak{m}_x$ such that $\frac{a_1}{1}, \dots, \frac{a_m}{1} \in A_{\mathfrak{m}_x}$ form a basis of $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \mathbb{R}^m$, and choose $g_1, \dots, g_m \in C^\infty(\mathbb{R}^n)$ with $c^*(g_i) = a_i$. Then $g_i(c(x)) = 0$, so g_i is in the codimension 1 ideal $\mathfrak{m}_{c(x)} = \{f \in C^\infty(\mathbb{R}^n) : f(c(x)) = 0\}$. Since $c^* : C^\infty(\mathbb{R}^n) \to A$ induces in turn homomorphisms

$$C^\infty_{c(x)}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)_{\mathfrak{m}_{c(x)}} \to A_{\mathfrak{m}_x}$$
$$\mathbb{R}^n = T^*_{c(x)}\mathbb{R}^n = \tilde{\mathfrak{m}}_{c(x)}/\tilde{\mathfrak{m}}^2_{c(x)} \to \tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \mathbb{R}^m$$

and since $\mathfrak{m}_{c(x)} \cong \mathfrak{m}_x \oplus I$ as vector spaces, we may find functions $g_{m+1}, \dots, g_n \in I$ such that the quotients $\frac{g_1}{1}, \dots, \frac{g_n}{1} \in C^\infty_{c(x)}(\mathbb{R}^n)$ map to a basis of $\tilde{\mathfrak{m}}_{c(x)}/\tilde{\mathfrak{m}}^2_{c(x)} = T^*_{c(x)}\mathbb{R}^n$. By the implicit function theorem on $\mathbb{R}^n$ the functions $g_{m+1}, \dots, g_n$ are near $c(x)$ an equation of maximal rank for $c(M) = Z(I)$, and the functions $g_1, \dots, g_m$ restrict to smooth coordinates near $c(x)$ on the closed submanifold $c(M) = Z(I)$ of $\mathbb{R}^n$, and the number m turns out to be a locally constant function on M. Also the functions $a_1, \dots, a_m$ restrict to smooth coordinates near x of M. □

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