On the Hunter-Saxton equation on $\text{Diff}_{\mathcal{A}}(\mathbb{R})$

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Groups related to $\mathrm{Diff}_c(\mathbb{R})$

The reflexive nuclear (LF) space $C_c^\infty(\mathbb{R})$ of smooth functions with compact support leads to the well-known regular Lie group $\mathrm{Diff}_c(\mathbb{R})$. We will now define an extension of this group which will play a major role in the later parts of this article.

Define $C_{c,2}^\infty(\mathbb{R}) = \{f : f' \in C_c^\infty(\mathbb{R})\}$ to be the space of antiderivatives of smooth functions with compact support. It is a reflexive nuclear (LF) space. We also define the space $C_{c,1}^\infty(\mathbb{R}) = \left\{ f \in C_{c,2}^\infty(\mathbb{R}) \ : \ f(-\infty) = 0 \right\}$ of antiderivatives of the form $x \mapsto \int_{-\infty}^x g\, dy$ with $g \in C_c^\infty(\mathbb{R})$.

$\mathrm{Diff}_{c,2}(\mathbb{R}) = \left\{ \varphi = \mathrm{Id} + f : f \in C_{c,2}^\infty(\mathbb{R}), f' > -1 \right\}$ *is the corresponding group.*

Define the two functionals $\mathrm{Shift}_\ell, \mathrm{Shift}_r : \mathrm{Diff}_{c,2}(\mathbb{R}) \to \mathbb{R}$ by

$$\mathrm{Shift}_\ell(\varphi) = \mathrm{ev}_{-\infty}(f) = \lim_{x\to-\infty} f(x), \quad \mathrm{Shift}_r(\varphi) = \mathrm{ev}_{\infty}(f) = \lim_{x\to\infty} f(x)$$

for $\varphi(x) = x + f(x)$.

Then the short exact sequence of smooth homomorphisms of Lie groups

$$\mathrm{Diff}_c(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{c,2}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell, \mathrm{Shift}_r)} (\mathbb{R}^2, +)$$

describes a semidirect product, where a smooth homomorphic section $s : \mathbb{R}^2 \to \mathrm{Diff}_{c,2}(\mathbb{R})$ is given by the composition of flows $s(a, b) = \mathrm{Fl}_a^{X_\ell} \circ \mathrm{Fl}_b^{X_r}$ for the vectorfields $X_\ell = f_\ell \partial_x$, $X_r = f_r \partial_x$ with $[X_\ell, X_r] = 0$ where $f_\ell, f_r \in C^\infty(\mathbb{R}, [0, 1])$ satisfy

$$f_\ell(x) = \begin{cases} 1 & \text{for } x \le -1 \\ 0 & \text{for } x \ge 0, \end{cases} \qquad f_r(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x \ge 1. \end{cases} \tag{1}$$

The normal subgroup $\mathrm{Diff}_{c,1}(\mathbb{R}) = \ker(\mathrm{Shift}_\ell) = \{\varphi = \mathrm{Id} + f : f \in C^\infty_{c,1}(\mathbb{R}), f' > -1\}$ of diffeomorphisms which have no shift at $-\infty$ will play an important role later on.

Some diffeomorphism groups on $\mathbb{R}$

We have the following smooth injective group homomorphisms:

$$
\begin{array}{ccccccc}
 & & & & \mathrm{Diff}_{H^\infty}(\mathbb{R}) & & \\
 & & & \nearrow & \downarrow & & \\
\mathrm{Diff}_c(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{H_0^\infty}(\mathbb{R}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Diff}_{c,1}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{H_1^\infty}(\mathbb{R}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Diff}_{c,2}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{H_2^\infty}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{B}}(\mathbb{R})
\end{array}
$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$.

For $\mathcal{S}$ this works the same as for C_c^∞. For H^∞ it is slightly mote subtle.

The setting

We will denote by $\mathcal{A}(\mathbb{R})$ any of the spaces $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $H^\infty(\mathbb{R})$ and by $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_c(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ or $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ as defined in Sections **??**, **??** and **??**.

Similarly $\mathcal{A}_1(\mathbb{R})$ will denote any of the spaces $C_{c,1}^\infty(\mathbb{R})$, $\mathcal{S}_1(\mathbb{R})$ or $H_1^\infty(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}\,1}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_{c,1}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R})$ or $\mathrm{Diff}_{H_1^\infty}(\mathbb{R})$ as defined in Sections **??**, **??** and **??**.

The $\dot{H}^1$-metric. For $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the homogeneous H^1-metric is given by

$$G_\varphi(X \circ \varphi, Y \circ \varphi) = G_{\mathrm{Id}}(X, Y) = \int_{\mathbb{R}} X'(x) Y'(x)\, dx \,,$$

where X, Y are elements of the Lie algebra $\mathcal{A}(\mathbb{R})$ or $\mathcal{A}_1(\mathbb{R})$. We shall also use the notation

$$\langle \cdot, \cdot \rangle_{\dot{H}^1} := G(\cdot, \cdot)\,.$$

Theorem

On $\mathrm{Diff}_{\mathcal{A}^1}(\mathbb{R})$ *the geodesic equation is the Hunter-Saxton equation*

$$\boxed{(\varphi_t)\circ\varphi^{-1} = u \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^{x} (u_x(z))^2\, dz\ ,}$$

and the induced geodesic distance is positive.

On the other hand the geodesic equation does not exist on the subgroups $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, since the adjoint* $\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X)$ *does not lie in* $\check{G}_{\mathrm{Id}}(\mathcal{A}(\mathbb{R}))$ *for all* $X\in\mathcal{A}(\mathbb{R})$*.*

One obtains the classical form of the Hunter-Saxton equation by differentiating:

$$u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2\ ,$$

Note that $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ is a natural example of a non-robust Riemannian manifold.

Proof

Note that $\check{G}_{\mathrm{Id}} : \mathcal{A}_1(\mathbb{R}) \to \mathcal{A}_1(\mathbb{R})^*$ is given by $\check{G}_{\mathrm{Id}}(X) = -X''$ if we use the L^2-pairing $X \mapsto (Y \mapsto \int XYdx)$ to embed functions into the space of distributions. We now compute the adjoint of $\mathrm{ad}(X)$:

$$\begin{aligned} \left\langle\, \mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y), Z \right\rangle &= \check{G}_{\mathrm{Id}}(Y, \mathrm{ad}(X)Z) = G_{\mathrm{Id}}(Y, -[X, Z]) \\ &= \int_{\mathbb{R}} Y'(x)\big(X'(x)Z(x) - X(x)Z'(x)\big)'\, dx \\ &= \int_{\mathbb{R}} Z(x)\big(X''(x)Y'(x) - (X(x)Y'(x))''\big)\, dx \,. \end{aligned}$$

Therefore the adjoint as an element of $\mathcal{A}_1^*$ is given by

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y) = X''Y' - (XY')'' \,.$$

For $X = Y$ we can rewrite this as

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(X) = \tfrac{1}{2}\big((X'^2)' - (X^2)'''\big) = \frac{1}{2}\Big(\int_{-\infty}^{x} X'(y)^2\,dy - (X^2)'\Big)''$$
$$= \frac{1}{2}\check{G}_{\mathrm{Id}}\Big(-\int_{-\infty}^{x} X'(y)^2\,dy + (X^2)'\Big)\,.$$

If $X \in \mathcal{A}_1(\mathbb{R})$ then the function $-\frac{1}{2}\int_{-\infty}^{x} X'(y)^2\,dy + \frac{1}{2}(X^2)'$ is again an element of $\mathcal{A}_1(\mathbb{R})$. This follows immediately from the definition of $\mathcal{A}_1(\mathbb{R})$. Therefore the geodesic equation exists on $\mathrm{Diff}_{\mathcal{A}\,1}(\mathbb{R})$ and is as given.

However if $X \in \mathcal{A}(\mathbb{R})$, a neccessary condition for $\int_{-\infty}^{x}(X'(y))^2 dy \in \mathcal{A}(\mathbb{R})$ would be $\int_{-\infty}^{\infty} X'(y)^2 dy = 0$, which would imply $X' = 0$. Thus the geodesic equation does not exist on $\mathcal{A}(\mathbb{R})$.

The positivity of geodesic distance will follow from the explicit formula for geodesic distance below. QED.

Theorem

We define the R-map by:

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}) \\ \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The R-map is invertible with inverse

$$R^{-1} : \begin{cases} \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \\ \gamma \mapsto x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2 + 4\gamma \, dx . \end{cases}$$

The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e.,*

$$R^*\langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{\dot{H}^1} .$$

Thus the space $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is a flat space in the sense of Riemannian geometry.*

Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2-inner product on $\mathcal{A}(\mathbb{R})$ with constant volume dx.

Proof

To compute the pullback of the L^2-metric via the R-map we first need to calculate its tangent mapping. For this let $h = X \circ \varphi \in T_\varphi \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and let $t \mapsto \psi(t)$ be a smooth curve in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with $\psi(0) = \mathrm{Id}$ and $\partial_t|_0 \psi(t) = X$. We have:

$$\begin{aligned}
T_\varphi R.h &= \partial_t|_0 R(\psi(t) \circ \varphi) = \partial_t|_0 2\Big(((\psi(t) \circ \varphi)_x)^{1/2} - 1\Big) \\
&= \partial_t|_0 2((\psi(t)_x \circ \varphi)\,\varphi_x)^{1/2} \\
&= 2(\varphi_x)^{1/2} \partial_t|_0((\psi(t)_x)^{1/2} \circ \varphi) = (\varphi_x)^{1/2}\Big(\frac{\psi_{tx}(0)}{(\psi(0)_x)^{-1/2}} \circ \varphi\Big) \\
&= (\varphi_x)^{1/2}(X' \circ \varphi) = (\varphi')^{1/2}(X' \circ \varphi)\,.
\end{aligned}$$

Using this formula we have for $h = X_1 \circ \varphi, k = X_2 \circ \varphi$:

$$R^*\langle h, k\rangle_{L^2} = \langle T_\varphi R.h, T_\varphi R.k\rangle_{L^2} = \int_{\mathbb{R}} X_1'(x) X_2'(x)\,dx = \langle h, k\rangle_{\dot{H}^1} \; QED$$

Corollary

Given $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic* $\varphi(t,x)$ *connecting them is given by*

$$\varphi(t,x) = R^{-1}\Big((1-t)R(\varphi_0) + tR(\varphi_1)\Big)(x)$$

and their geodesic distance is

$$d(\varphi_0,\varphi_1)^2 = 4\int_{\mathbb{R}} \left((\varphi_1')^{1/2} - (\varphi_0')^{1/2}\right)^2 \, dx \; .$$

The methods of Boris and my lecture give local well-posedness of the geodesic equation in $\mathrm{Diff}_{H_1^\infty}(\mathbb{R})$ only. But this construction shows much more: For $\mathcal{S}_1$, C_1^∞, and even for many kinds of Denjoy-Carleman ultradifferentiable model spaces (not explained here). This shows that Sobolev space methods for treating nonlinear PDEs is not the only method.

Corollary: *The metric space* $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is path-connected and geodesically convex but not geodesically complete. In particular, for every* $\varphi_0 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *and* $h \in T_{\varphi_0}\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, $h \neq 0$ *there exists a time* $T \in \mathbb{R}$ *such that* $\varphi(t,\cdot)$ *is a geodesic for* $|t| < |T|$ *starting at* φ_0 *with* $\varphi_t(0) = h$*, but* $\varphi_x(T,x) = 0$ *for some* $x \in \mathbb{R}$.

Theorem: *The square root representation on the diffeomorphism group* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *is a bijective mapping, given by:*

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \to \left(\mathrm{Im}(R), \|\cdot\|_{L^2}\right) \subset \left(\mathcal{A}\left(\mathbb{R}, \mathbb{R}_{>-2}\right), \|\cdot\|_{L^2}\right) \\ \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right). \end{cases}$$

The pull-back of the restriction of the flat L^2*-metric to* $\mathrm{Im}(R)$ *via* R *is again the homogeneous Sobolev metric of order one. The image of the* R*-map is the splitting submanifold of* $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ *given by:*

$$\mathrm{Im}(R) = \left\{\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) : F(\gamma) := \int_{\mathbb{R}} \gamma(\gamma + 4)\, dx = 0\right\}.$$

On the space $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the geodesic equation does not exist. Still:

Corollary: *The geodesic distance* $d^{\mathcal{A}}$ *on* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *coincides with the restriction of* $d^{\mathcal{A}_1}$ *to* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e., for* $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *we have*

$$d^{\mathcal{A}}(\varphi_0, \varphi_1) = d^{\mathcal{A}_1}(\varphi_0, \varphi_1)\ .$$