Approximating Euler's equation on the full diffeomorphism group, and soliton-like solutions

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Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.

Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.

For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend entirely on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous.

The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
    - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
    - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called smooth or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C_c^∞.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection
$$C^\infty(U,F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R},U),\ell\in F^*} C^\infty(\mathbb{R},\mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$
where $C^\infty(\mathbb{R},\mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces.
   Note that this is the main assumption of variational calculus. Here it is a theorem.

7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if
   $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$.
   (*Smooth uniform boundedness theorem*,
   see [KM97], theorem 5.26).

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

$$\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G)\\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to\\
&\qquad\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

This ends our review of the standard results of convenient calculus.

Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]

Manifolds of mappings

Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar{g}$ on N. Then

$$\begin{tikzcd}
 & 0_N \arrow[dl, bend right, "\text{zero section}"'] \arrow[d, hook] & & N \arrow[d, hook] \arrow[dr, bend left, "\text{diagonal}"] & \\
TN & V^N \arrow[l, hook', "\text{open}"] \arrow[rr, "(\pi_N, \exp^{\bar{g}})", "\cong"'] & & V^{N\times N} \arrow[r, hook, "\text{open}"'] & N \times N
\end{tikzcd}$$

$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^* TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; {\mathrm{pr}_2}^* f^*TN)$
By Cartesian Closedness (I am lying a little).

Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.

Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By Cartesian closedness.

Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves

Corollary (of the chart structure):
$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N)$.

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.
A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\operatorname{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\operatorname{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\operatorname{Evol}^X_t := g(t) = \operatorname{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*

Proof: $\mathrm{Diff}(M) \xrightarrow{open} C^\infty(M, M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\mathrm{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t, x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.
Let $X(t, x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\mathrm{Fl}_s^{\partial_t \times X}(t, x) = (t + s, \mathrm{Evol}^X(t, x))$ satisfies the ODE $\quad \partial_t \mathrm{Evol}(t, x) = X(t, \mathrm{Evol}(t, x))$. If $X(s, t, x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism.
QED.

Groups of smooth diffeomorphisms on $\mathbb{R}^n$

If we consider the group of all orientation preserving diffeomorphisms $\mathrm{Diff}(\mathbb{R}^n)$ of $\mathbb{R}^n$, it is not an open subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with the compact C^∞-topology. So it is not a smooth manifold in the usual sense, but we may consider it as a Lie group in the cartesian closed category of Frölicher spaces, see [KM97], section 23, with the structure induced by the injection $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^)$.

We shall now describe regular Lie groups in $\mathrm{Diff}(\mathbb{R}^n)$ which are given by diffeomorphisms of the form $f = \mathrm{Id}_{\mathbb{R}} + g$ where g is in some specific convenient vector spaces of bounded functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Now we discuss these spaces on $\mathbb{R}^n$, we describe the smooth curves in them, and we describe the corresponding groups.

The group $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$

The space $\mathcal{B}(\mathbb{R}^n)$ (called $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ by [L.Schwartz 1966]) consists of all smooth functions which have all derivatives (separately) bounded. It is a Fréchet space. By [Vogt 1983], the space $\mathcal{B}(\mathbb{R}^n)$ is linearly isomorphic to $\ell^\infty \hat{\otimes}\, \mathfrak{s}$ for any completed tensor-product between the projective one and the injective one, where $\mathfrak{s}$ is the nuclear Fréchet space of rapidly decreasing real sequences. Thus $\mathcal{B}(\mathbb{R}^n)$ is not reflexive, not nuclear, not smoothly paracompact. *The space $C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}^n))$ of smooth curves in $\mathcal{B}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:*

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ and each $t \in \mathbb{R}$ the expression $\partial_t^k \partial_x^\alpha c(t,x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally in t.*

Here $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$.
$\mathrm{Diff}^+_{\mathcal{B}}(\mathbb{R}^n) = \left\{ f = \mathrm{Id} + g : g \in \mathcal{B}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) \geq \varepsilon > 0 \right\}$
denotes the corresponding group, see below.

The group $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$

The space $H^\infty(\mathbb{R}^n) = \bigcap_{k\geq 1} H^k(\mathbb{R}^n)$ is the intersection of all Sobolev spaces which is a reflexive Fréchet space. It is called $\mathcal{D}_{L^2}(\mathbb{R}^n)$ in [L.Schwartz 1966]. By [Vogt 1983], the space $H^\infty(\mathbb{R}^n)$ is linearly isomorphic to $\ell^2 \hat{\otimes} \mathfrak{s}$. Thus it is not nuclear, not Schwartz, not Montel, but still smoothly paracompact.
The space $C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n))$ of smooth curves in $H^\infty(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ the expression $\|\partial_t^k \partial_x^\alpha f(t, \quad)\|_{L^2(\mathbb{R}^n)}$ is locally bounded near each $t \in \mathbb{R}$.*

$\mathrm{Diff}^+_{H^\infty}(\mathbb{R}) = \{f = \mathrm{Id} + g : g \in H^\infty(\mathbb{R}), \det(\mathbb{I}_n + dg) > 0\}$ *denotes the correpending group.*

The group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

The algebra $\mathcal{S}(R^n)$ of rapidly decreasing functions is a reflexive nuclear Fréchet space.
The space $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ of smooth curves in $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\geq 0}$ and $\alpha \in \mathbb{N}^n_{\geq 0}$, the expression $(1 + |x|^2)^m \partial_t^k \partial_x^\alpha c(t, x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally uniformly bounded in $t \in \mathbb{R}$.*

$\mathrm{Diff}^+_{\mathcal{S}}(\mathbb{R}^n) = \left\{ f = \mathrm{Id} + g : g \in \mathcal{S}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0 \right\}$ *is the correponding group.*

The group $\mathrm{Diff}_c(\mathbb{R}^n)$

The algebra $C_c^\infty(\mathbb{R}^n)$ of all smooth functions with compact support is a nuclear (LF)-space.
The space $C^\infty(\mathbb{R}, C_c^\infty(\mathbb{R}^n))$ of smooth curves in $C_c^\infty(\mathbb{R}^n)$ consists of all functions $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For each compact interval $[a, b]$ in $\mathbb{R}$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $f(t, x) = 0$ for $(t, x) \in [a, b] \times (\mathbb{R}^n \setminus K)$.*

$\mathrm{Diff}_c(\mathbb{R}^n) = \left\{f = \mathrm{Id} + g : g \in C_c^\infty(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\right\}$ *is the correponding group.*

Diffeomorphism groups on $\mathbb{R}^n$

The function spaces are boundedly mapped into each other:

$$C_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n) \longrightarrow \mathcal{B}(\mathbb{R}^n)$$

and each space is a bounded locally convex algebra and a bounded $\mathcal{B}(\mathbb{R}^n)$-module. Thus each space is an ideal in each larger space.

Theorem. *The sets of diffeomorphisms* $\mathrm{Diff}_c(\mathbb{R}^n)$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *and* $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms*

$$\mathrm{Diff}_c(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{H^\infty}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)\ .$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

Corollary. $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *acts on* Γ_c, $\Gamma_{\mathcal{S}}$ *and* Γ_{H^∞} *of any tensorbundle over* $\mathbb{R}^n$ *by pullback. The infinitesimal action of the Lie algebra* $\mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ *on these spaces by the Lie derivative thus maps each of these spaces into itself. A fortiori,* $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ *acts on* $\Gamma_{\mathcal{S}}$ *of any tensor bundle by pullback.*

EPDiff

On the Lie algebra of VF $\mathfrak{X}_{H^\infty}(\mathbb{R}^n) = H^\infty(\mathbb{R}^n)^n$ we consider a weak inner product of the form

$$\|v\|_L^2 = \int_{\mathbb{R}^n} \langle Lv, v\rangle \, dx$$

where L is a positive L^2-symmetric (pseudo-) differential operator (inertia operator). Leads to a right invariant metric on $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ whose geodesic equation is

$$\partial_t \varphi = u \circ \varphi, \quad \partial_t u = -\operatorname{ad}_u^\top u, \text{ where}$$

$$\int \langle L(\operatorname{ad}_u^\top u), v\rangle \, dx = \int \langle L(u), -[u,v]\rangle \, dx$$

Condsider the *momentum* $m = L(v)$ of a vector field, so that $\langle v, w\rangle_L = \int \langle m, w\rangle dx$. Then the geodesic equation is of the form:

$$\partial_t m = -(v\cdot\nabla)m - \operatorname{div}(v)m - m\cdot(Dv)^t$$

$$\partial_t m_i = -\sum_j (v_j \partial_{x_j} m_i + \partial_{x_j} v_j \cdot m_i + m_j \partial_{x_i} v_j)$$

$$v = K * m, \quad \text{where } K \text{ is the matrix-valued Green function of } L.$$

Suppose, the time dependent vector field v integrates to a flow φ via

$$\partial_t \varphi(x,t) = v(\varphi(x,t),t)$$

and we describe the momentum by a *measure-valued 1-form*

$$\widetilde{m} = \sum_i m_i dx_i \otimes (dx_1 \wedge \cdots \wedge dx_n)$$

so that $\|v\|_L^2 = \int (v, \widetilde{m})$ makes intrinsic sense. Then the geodesic equation is equivalent to: $\widetilde{m}$ is *invariant* under the flow φ, that is,

$$\widetilde{m}(\cdot,t) = \varphi(\cdot,t)_* \widetilde{m}(\cdot,0),$$

whose infinitesimal version is the following, using the Lie derivative:

$$\partial_t \widetilde{m}(\cdot,t) = -\mathcal{L}_{v(\cdot,t)} \widetilde{m}(\cdot,t).$$

Because of this invariance, if a geodesic begins with momentum of compact support, its momentum will always have compact support; and if it begins with momentum which, along with all its derivatives, has 'rapid' decay at infinity, that is it is in $O(\|x\|^{-n})$ for every n, this too will persist, because $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a normal subgroup.

Moreover this invariance gives us a Lagrangian form of EPDiff:

$$\begin{aligned}\partial_t\varphi(x,t) &= \int K^{\varphi(\cdot,t)}(x,y)(\varphi(y,t)_*\widetilde{m}(y,0)) \\ &= K^{\varphi(\cdot,t)} * (\varphi(\cdot,t)_*\widetilde{m}(\cdot,0)) \\ &\qquad \text{where } K^{\varphi}(x,y) = K(\varphi(x),\varphi(y))\end{aligned}$$

Aim of this talk: Solutions of Euler's equation are limits of solutions of equations in the EPDiff class with the operator:

$$L_{\varepsilon,\eta} = (I - \tfrac{\eta^2}{p}\triangle)^p \circ (I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div}), \qquad \text{for any } \varepsilon > 0, \eta \geq 0.$$

All solutions of Euler's equation are limits of solutions of these much more regular EPDiff equations and *give a bound on their rate of convergence*. In fact, so long as $p > n/2 + 1$, these EPDiff equations have a well-posed initial value problem with unique solutions for all time. Moreover, although $L_{0,\eta}$ does not make sense, the analog of its Green's function $K_{0,\eta}$ does make sense as do the geodesic equations in momentum form. These are, in fact, geodesic equations on the group of volume preserving diffeomorphisms SDiff and become Euler's equation for $\eta = 0$. An important point is that so long as $\eta > 0$, the equations have *soliton* solutions (called *vortons*) in which the momentum is a sum of delta functions.

Relation to Euler's equ. Oseledetz 1988

We use the kernel

$$K_{ij}(x) = \delta_{ij}\delta_0(x) + \partial_{x_i}\partial_{x_j}H$$

where H is the Green's function of $-\Delta$. But K now has a rather substantial pole at the origin. If $V_n = \mathrm{Vol}(S^{n-1})$,

$$H(x) = \begin{cases} \frac{1}{(n-2)V_n}(1/|x|^{n-2}) & \text{if } n > 2, \\ \frac{1}{V_2}\log(1/|x|) & \text{if } n = 2 \end{cases}$$

so that, *as a function*

$$(M_0)_{ij}(x) := \partial_{x_i}\partial_{x_j}H(x) = \frac{1}{V_n} \cdot \frac{n x_i x_j - \delta_{ij}|x|^2}{|x|^{n+2}}, \quad \text{if } x \neq 0.$$

Convolution with any $(M_0)_{ij}$ is still a Calderon-Zygmund singular integral operator defined by the limit as $\varepsilon \to 0$ of its value outside an ε-ball, so it is reasonably well behaved. *As a distribution* there is another term:

$$\partial_{x_i}\partial_{x_j}H \overset{\text{distribution}}{=} (M_0)_{ij} - \frac{1}{n}\delta_{ij}\delta_0$$

$$P_{\mathrm{div}=0} : m \mapsto v = \big(m + \partial^2(H)_{\mathrm{distr}}\big) = \big(\tfrac{n-1}{n} \cdot m + M_0 * m\big)$$

is the *orthogonal projection* of the space of vector fields m onto the subspace of divergence free vector fields v, orthogonal in each Sobolev space H^p, $p \in \mathbb{Z}_{\geq 0}$. (Hodge alias Helmholtz projection).

The matrix $M_0(x)$ has $\mathbb{R}x$ as an eigenspace with eigenvalue $(n-1)/V_n|x|^n$ and $\mathbb{R}x^\perp$ as an eigenspace with eigenvalue $-1/V_n|x|^n$. Let $P_{\mathbb{R}x}$ and $P_{\mathbb{R}x^\perp}$ be the orthonormal projections onto the eigenspaces, then

$$P_{\mathrm{div}=0}(m)(x) = \tfrac{n-1}{n} \cdot m(x) +$$
$$+ \frac{1}{V_n} \cdot \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{1}{|y|^n} \big((n-1) P_{\mathbb{R}y}(m(x-y)) - P_{\mathbb{R}y^\perp}(m(x-y)\big)\, dy.$$

With this K, EPDiff in the variables (v, m) is the Euler equation in v with pressure a function of (v, m). *Oseledets's form for Euler*:

$$\boxed{\begin{aligned} v &= P_{\text{div}=0}(m) \\ \partial_t m &= -(v \cdot \nabla)m - m \cdot (Dv)^t \end{aligned}}$$

Let $\widetilde{m} = \sum_i m_i dx_i$ be the 1-form associated to m. Since $\operatorname{div} v = 0$, we can use $\widetilde{m}$ instead of $\sum_i m_i dx_i \otimes dx_1 \wedge \ldots dx_n$. Integrated form:

$$\boxed{\begin{aligned} \partial_t \varphi &= P_{\text{div}=0}(m) \circ \varphi \\ \widetilde{m}(\cdot, t) &= \varphi(\cdot, t)_* \widetilde{m}(\cdot, 0) \end{aligned}}$$

This uses the variables v, m instead of v and pressure.
Advantage: m, like vorticity, is constant when transported by the flow. m determines the vorticity the 2-form $\omega = d(\sum_i v_i dx_i)$, because v and m differ by a gradient, so $\omega = d\widetilde{m}$ also. Thus: vorticity is constant along flows follows from the same fact for momentum 1-form $\widetilde{m}$.

However, these equ. are not part of the true EPDiff framework because the operator $K = P_{\text{div}=0}$ is not invertible and there is no corresponding differential operator L.

In fact, v does not determine m as we have rewritten Euler's equation using extra non-unique variables m, albeit ones which obey a conservation law so they may be viewed simply as extra parameters.

Approximationg Euler by EPDiff

Replace the Green's function H of $-\Delta$ by the Green's function H_ε of the positive $\varepsilon^2 I - \triangle$ for $\varepsilon > 0$ (whose dimension is length^{-1}). The Green's function is be given explicitly using the 'K' Bessel function via the formula

$$H_\varepsilon(x) = c_n \varepsilon^{n-2} |\varepsilon x|^{1-n/2} K_{n/2-1}(|\varepsilon x|)$$

for a suitable constant c_n independent of ε. Then we get the modified kernel

$$(K_\varepsilon)_{ij} = \delta_{ij}\delta_0 + (\partial_{x_i}\partial_{x_j} H_\varepsilon)_{\mathrm{distr}}$$

This has exactly the same highest order pole at the origin as K did and the second derivative is again a Calderon-Zygmund singular integral operator minus the same delta function. The main difference is that this kernel has exponential decay at infinity, not polynomial decay. By weakening the requirement that the velocity be divergence free, the resulting integro-differential equation behaves much more locally, more like a hyperbolic equation rather than a parabolic one.

The corresponding inverse is the differential operator

$$L_\varepsilon = I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div}$$

$$v = K_\varepsilon * m, \quad m = L_\varepsilon(v)$$

$$\|v\|^2_{L_\varepsilon} = \int \langle v, v\rangle + \mathrm{div}(v).\,\mathrm{div}(v)dx$$

Geodesic equation:

$$\partial_t(v_i) = (K_\varepsilon)_{ij} * \partial_t(m_j)$$

$$= -(K_\varepsilon)_{ij} * (v_k v_{j,k}) - v_i\,\mathrm{div}(v) - \tfrac{1}{2}(K_\varepsilon)_{ij} * \left(|v(x)|^2 + (\tfrac{\mathrm{div}(v)}{\varepsilon})^2\right)_{,j}$$

Curiously though, the parameter ε can be scaled away. That is, if $v(x,t), m(x,t)$ is a solution of EPDiff for the kernel K_1, then $v(\varepsilon x, \varepsilon t), m(\varepsilon x, \varepsilon t)$ is a solution of EPDiff for K_ε.

Regularizing more

Compose L_ε with a scaled version of the standard regularizing kernel $(I - \triangle)^p$ to get

$$L_{\varepsilon,\eta} = (I - \tfrac{\eta^2}{p}\triangle)^p \circ (I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div})$$
$$K_{\varepsilon,\eta} := {L_{\varepsilon,\eta}}^{-1} = G_\eta^{(p)} * K_\varepsilon$$

where $G_\eta^{(p)}$ is the Green's function of $(I - \frac{\eta^2}{p}\triangle)^p$ and is again given explicitly by a 'K'-Bessel function $d_{p,n}\eta^{-n}|x|^{p-n/2}K_{p-n/2}(|x|/\eta)$. For $p \gg 0$, the kernel converges to a Gaussian with variance depending only on η, namely $(2\sqrt{\pi}\eta)^{-n}e^{-|x|^2/4\eta^2}$. This follows because the Fourier transform takes $G_\eta^{(p)}$ to $\left(1 + \frac{\eta^2|\xi|^2}{p}\right)^{-p}$, whose limit, as $p \to \infty$, is $e^{-\eta^2|\xi|^2}$. These approximately Gaussian kernels lie in C^q if $q \leq p - (n+1)/2$.

So long as the kernel is in C^1, it is known that EPDiff has solutions for all time, as noted first by A.Trouve and L.Younes.

Theorem

Let $F(x) = f(|x|)$ be any integrable C^2 radial function on $\mathbb{R}^n$. Assume $n \geq 3$. Define:

$$H_F(x) = \int_{\mathbb{R}^n} \min\left(\tfrac{1}{|x|^{n-2}}, \tfrac{1}{|y|^{n-2}}\right) F(y)dy$$
$$= \frac{1}{|x|^{n-2}} \int_{|y|\leq|x|} F(y)dy + \int_{|y|\geq|x|} \frac{F(y)}{|y|^{n-2}} dy$$

Then H_F is the convolution of F with $\frac{1}{|x|^{n-2}}$, is in C^4 and:

$$\partial_i(H_F)(x) = -(n-2)\frac{x_i}{|x|^n} \int_{|y|\leq|x|} F(y)dy$$
$$\partial_i\partial_j(H_F)(x) = (n-2)\left(\frac{nx_ix_j - \delta_{ij}|x|^2}{|x|^{n+2}} \int_{|y|\leq|x|} F(y)dy - V_n \frac{x_ix_j}{|x|^2} F(x)\right)$$

If $n = 2$, the same holds if you replace $1/|x|^{n-2}$ by $\log(1/|x|)$ and omit the factors $(n-2)$ in the derivatives.

no L	$K_{0,0} = P_{\mathrm{div}=0} = \delta_{ij}\delta_0 + (\partial_i\partial_j H)_{\mathrm{distr}}$
no L	$K_{0,\eta} = G_\eta^{(p)} * P_{\mathrm{div}=0}$ – see above
$L_{\varepsilon,0} = I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}$	$K_{\varepsilon,0} = \delta_{ij}\delta_0 + \partial_i\partial_j H_\varepsilon$
$L_{\varepsilon,\eta} = \left(I - \frac{\eta^2}{p}\triangle\right)^p \circ$ $\circ \left(I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}\right)$	$K_{\varepsilon,\eta} = \delta_{ij} G_\eta^{(p)} + \partial_i\partial_j (G_\eta^{(p)} * H_\varepsilon)$

Theorem: *Let $\varepsilon \geq 0, \eta > 0, p \geq (n+3)/2$ and $K = K_{\varepsilon,\eta}$ be the corresponding kernel. For any vector-valued distribution m_0 whose components are finite signed measures, consider the Lagrangian equation for a time varying C^1-diffeomorphism $\varphi(\cdot, t)$ with $\varphi(x,0) \equiv x$:*

$$\partial_t \varphi(x,t) = \int K(\varphi(x,t) - \varphi(y,t))(D\varphi(y,t))^{-1,\top} m_0(y) dy.$$

Here $D\varphi$ is the spatial derivative of φ. This equation has a unique solution for all time t.

Proof: The Eulerian velocity at φ is:

$$V_\varphi(x) = \int K(x - \varphi(y))(D\varphi(y))^{-1,\top} m_0(y)dy$$

and $W_\varphi(x) = V_\varphi(\varphi(x))$ is the velocity in 'material' coordinates. Note that because of our assumption on m_0, if φ is a C^1-diffeomorphism, then V_φ and W_φ are C^1 vector fields on $\mathbb{R}^n$; in fact, they are as differentiable as K is, for suitably decaying m. The equation can be viewed as a the flow equation for the vector field $\varphi \mapsto W_\varphi$ on the union of the open sets

$$U_c = \left\{\varphi \in C^1(\mathbb{R}^n)^n : \|\operatorname{Id} - \varphi\|_{C^1} < 1/c,\ \det(D\varphi) > c\right\},$$

where $c > 0$. The union of all U_c is the group $\operatorname{Diff}_{C_b^1}(\mathbb{R}^n)$ of all C^1-diffeomorphisms which, together with their inverses, differ from the identity by a function in $C^1(\mathbb{R}^n)^n$ with bounded C^1-norm. We claim this vector field is locally Lipschitz on each U_c:

$$\|W_{\varphi_1} - W_{\varphi_2}\|_{C^1} \leq C.\|\varphi_1 - \varphi_2\|_{C^1}$$

where C depends only on c: Use that K is uniformly continuous and use $\|D\varphi^{-1}\| \leq \|D\varphi\|^{n-1}/|\det(D\varphi)|$.

As a result we can integrate the vector field for short times in $\mathrm{Diff}_{C^1_b}(\mathbb{R}^n)$. But since $(D\varphi(y,t))^{-1,\top} m_0(y)$ is then again a signed finite $\mathbb{R}^n$-valued measure,

$$\int V_{\varphi(\cdot,t)}(x)(D\varphi(y,t))^{-1,\top} m_0(y)dx = \|V_{\varphi(\cdot,t)}\|_{L_{\varepsilon,\eta}}$$

is actually finite for each t. Using the fact that in EPDiff the $L_{\varepsilon,\eta}$-energy $\|V_{\varphi(\cdot,t)}\|_{L_{\varepsilon,\eta}}$ of the $L_{\varepsilon,\eta}$-geodesic is constant in t, we get a bound on the norm $\|V_{\varphi(\cdot,t)}\|_{H^p}$, depending of course on η but independent of t, hence a bound on $\|V_{\varphi(\cdot,t)}\|_{C^1}$. Thus $\|\varphi(\cdot,t)\|_{C^0}$ grows at most linearly in t. But $\partial_t D\varphi = DW_\varphi = DV_\varphi \cdot D\varphi$ which shows us that $D\varphi$ grows at most exponentially in t. Hence $\det D\varphi$ can shrink at worst exponentially towards zero, because $\partial_t \det(D\varphi) = \mathrm{Tr}(\mathrm{Adj}(D\varphi).\partial_t D\varphi)$. Thus for all finite t, the solution $\varphi(\cdot,t)$ stays in a bounded subset of our Banach space and the ODE can continue to be solved. QED.

Lemma: *If $\eta \geq 0$ and $\varepsilon > 0$ are bounded above, then the norm*

$$\|v\|^2_{k,\varepsilon,\eta} = \sum_{|\alpha|\leq k} \int \langle D^\alpha L_{\varepsilon,\eta} v, D^\alpha v \rangle dx$$

is bounded above and below by the metric, with constants independent of ε and η:

$$\|v\|^2_{H^k} + \frac{1}{\varepsilon^2}\|\mathrm{div}(v)\|^2_{H^k} + \sum_{k+1\leq|\alpha|\leq k+p} \eta^{2(|\alpha|-k)} \int |D^\alpha v|^2 + \frac{1}{\varepsilon^2}|D^\alpha \mathrm{div}(v)|^2$$

Main estimate: *Assume k is sufficiently large, for instance $k \geq (n+2p+4)$ works, then the velocity field of a solution satisfies:*

$$\boxed{|\partial_t \left(\|v\|^2_{k,\varepsilon,\eta}\right)| \leq C.\|v\|^3_{k,\varepsilon,\eta}}$$

where, so long ε and η are bounded above, the constant C is independent of ε and η.

Theorem: *Fix k, p, n with $p > n/2 + 1, k \geq n + 2p + 4$ and assume $(\varepsilon, \eta) \in [0, M]^2$ for some $M > 0$. Then there are constants t_0, C such that for all initial $v_0 \in H^{k+p+1}$, there is a unique solution $v_{\varepsilon,\eta}(x, t)$ of EPDiff (including the limiting Euler case) for $t \in [0, t_0]$. The solution $v_{\varepsilon,\eta}(\cdot, t) \in H^{k+p+1}$ depends continuously on $\varepsilon, \eta \in [0, M]^2$ and satisfies $\|v_{\varepsilon,\eta}(\cdot, t)\|_{k,\varepsilon,\eta} < C$ for all $t \in [0, t_0]$.*

Theorem: *Take any k and M and any smooth initial velocity $v(\cdot, 0)$. Then there are constants t_0, C such that Euler's equation and $(\varepsilon, 0)$-EPDiff have solutions v_0 and v_ε respectively for $t \in [0, t_0]$ and all $\varepsilon < M$ and these satisfy:*

$$\|v_0(\cdot, t) - v_\varepsilon(\cdot, t)\|_{H^k} \leq C\varepsilon.$$

Theorem: *Let $\varepsilon > 0$. Take any k and M and any smooth initial velocity $v(\cdot, 0)$. Then there are constants t_0, C such that $(\varepsilon, 0)$-EPDiff and (ε, η)-EPDiff have solutions v_0 and v_η respectively for $t \in [0, t_0]$ and all $\varepsilon, \eta < M$ and these satisfy:*

$$\|v_0(\cdot, t) - v_\eta(\cdot, t)\|_{H^k} \leq C\eta^2.$$

Vortons: Soliton-like solutions via landmark theory

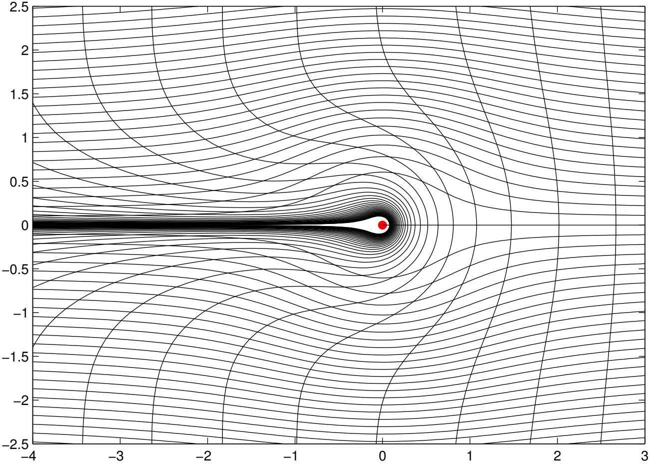
We have a C^1 kernel, so we can consider solutions in which momentum m is supported in a finite set $\{P_1, \cdots, P_N\}$, so that the components of the momentum field are given by $m^i(x) = \sum_a m_{ai}\delta(x - P_a)$. The support is called the set of landmark points and in this case, EPDiff reduces to a set of Hamiltonian ODE's based on the kernel $K = K_{\varepsilon,\eta}, \varepsilon \geq 0, \eta > 0$:

$$\boxed{\begin{aligned} \text{Energy } E &= \sum_{a,b} m_{ai} K_{ij}(P_a - P_b) m_{jb} \\ \frac{dP_{ai}}{dt} &= \sum_{b,j} K_{ij}(P_a - P_b) m_{bj} \\ \frac{dm_{ai}}{dt} &= -\sum_{b,j,k} \partial_{x_i} K_{jk}(P_a - P_b) m_{aj} m_{bk} \end{aligned}}$$

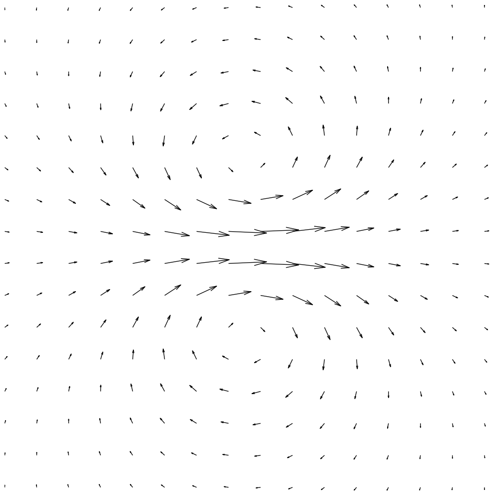
where a, b enumerate the points and i, j, k the dimensions in $\mathbb{R}^n$. These are essentially Roberts' equations from 1972.

One landmark point

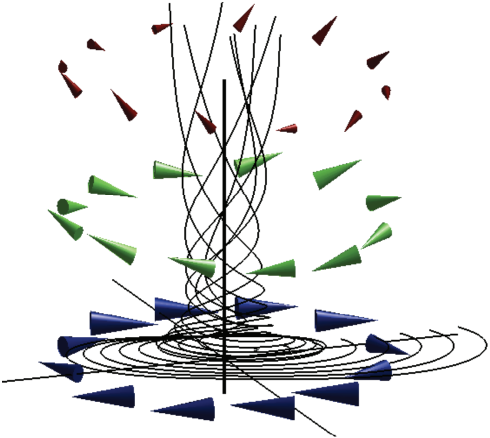
Its momentum must be constant hence so is its velocity. Therefore the momentum moves uniformly in a straight line ℓ from $-\infty$ to $+\infty$.



Momentum is tranformed to vortex-like velocity field by kernel $K_{0,\varepsilon}$

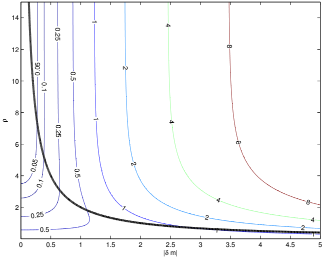


The dipole given by the kernel $K_{0,\eta}$ in dimension 2.



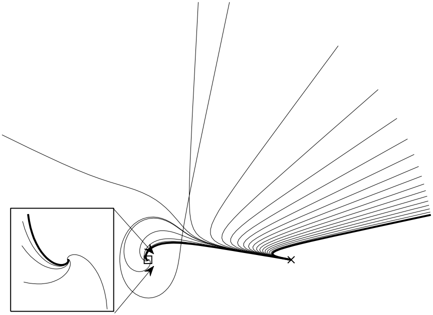
Streamlines and MatLab's 'coneplot' to visualize the vector field given by the x_1-derivative of the kernel $K_{0,1}$ times the vector $(1, 2, 0)$.

Two landmark points



Level sets of energy for the collision of two vortons with $\overline{m} = 0$, $\eta = 1$, $\omega = 1$. The coordinates are $\rho = |\delta P|$ and $|\delta m|$, and the state space is the double cover of the area above and right of the heavy black line, the two sheets being distinguished by the sign of $\langle \delta m, \delta P \rangle$. The heavy black line which is the curve $\rho \cdot |\delta m| = \omega$ where $\langle \delta m, \delta P \rangle = 0$. Each level set is a geodesic. If they hit the black line, they flip to the other sheet and retrace their path. Otherwise ρ goes to zero at one end of the geodesic.

Geodesics in the δP plane all starting at the point marked by an X but with $\overline{m} = m_1 + m_2 =$const. along the y-axis varying from 0 to 10. Here $\eta = 1$, the initial point is $(5, 0)$ and the initial momentum is $(-3, .5)$. Note how the two vortons repel each other on some geodesics and attract on others. A blow up shows the spiraling behavior as they collapse towards each other.



The generalized Euler flow on the space immersions

Let now $(N, \bar{g})$ be a Riemannian manifold (of bounded geometry), and $\operatorname{Imm}(M, N)$ the space of all immersions $M \to N$. For $f \in \operatorname{Imm}$ we have:

- $T_f \operatorname{Imm} = \Gamma(f^* TN) = \{h : M \to TN : \pi_N \circ h = f\}$.
- $T_f \operatorname{Imm} \ni h = Tf.h^\top + h^\perp \in Tf.\mathfrak{X}(M) \oplus \Gamma(\operatorname{Nor}(f))$
- $g = f^* \bar{g}$ be the induced metric on M.
- $\operatorname{vol}(g) = \operatorname{vol}(f^* \bar{g})$ the induced volume form.
- ∇^g, $\nabla^{\bar{g}}$, $S = S^f \in \Gamma(S^2 T^* M \otimes Nor(f))$ second fund. form.
- $\operatorname{Tr}^g(S) \in \Gamma(\operatorname{Nor}(f))$ mean curvature.

The differential of the pullback metric

$$\begin{aligned} \mathrm{Imm} \to \Gamma(S^2_{>0} T^*M), &\qquad f \mapsto g = f^*\bar{g}, \qquad \text{is given by} \\ D_{(f,h)} g = 2 \operatorname{Sym} \bar{g}(\nabla h, Tf) &= -2\bar{g}(h^\perp, S) + 2 \operatorname{Sym} \nabla (h^\top)^\flat \\ &= -2\bar{g}(h^\perp, S) + \mathcal{L}_{h^\top} g. \end{aligned}$$

The differential of the volume density

$$\begin{aligned} \mathrm{Imm} \to \mathrm{Vol}(M), &\qquad f \mapsto \mathrm{vol}(g) = \mathrm{vol}(f^*\bar{g}) \qquad \text{is given by} \\ D_{(f,h)} \mathrm{vol}(g) &= \mathrm{Tr}^g \big(\bar{g}(\nabla h, Tf)\big) \mathrm{vol}(g) \\ &= \Big(\mathrm{div}^g(h^\top) - \bar{g}\big(h^\perp, \mathrm{Tr}^g(S)\big) \Big) \mathrm{vol}(g). \end{aligned}$$

Let us fix a volume density on M.

Theorem. [Mathieu Molitor 2012, for embeddings] *The space*

$$\begin{aligned}&\mathrm{Imm}_{\mu,\mathrm{Tr}(S)}(M,N) = \\ &\qquad = \{f \in \mathrm{Emb}(M,N) : \mathrm{vol}(f^*\bar{g}) = \mu, \mathrm{Tr}(S^f) \textit{ nowhere } 0\}\end{aligned}$$

of volume preserving immersions with nowhere vanishing mean curvature is a tame Fréchet submanifold of $\mathrm{Imm}(M,N)$.

The proof uses the Hamilton-Nash-Moser implicit function theorem.

Some weak Riemannian metrics on spaces of immersions

For $f \in \mathrm{Imm}(M,N)$ and $h,k,\dots \in T_f\mathrm{Imm}(M,N)$;

$$G^0_f(h,k) = \int_M \bar g(h,k)\mathrm{vol}(f^*\bar g), \qquad \text{the } L^2-\text{metric.}$$

Geodesic distance vanishes for G^0.

$$G^\varepsilon_f(h,h) = \int_M \Big(\bar g(h,h) + \frac{1}{\varepsilon^2}\big(\mathrm{div}^g(h^\top) - \bar g(h^\perp, \mathrm{Tr}^g S)\big)^2\Big)\mathrm{vol}(f^*\bar g)$$

$$\text{or } = \int_M \Big(\bar g(h,h) + \frac{1}{\varepsilon^2}\bar g(h^\perp, \mathrm{Tr}^g S)^2 + \frac{1}{\varepsilon^2}\mathrm{div}^g(h^\top)^2\Big)\mathrm{vol}(f^*\bar g)$$

$$= \int_M \bar g\Big(L_{\varepsilon,0}h, h\Big)\mathrm{vol}(f^*\bar g), \qquad \text{where}$$

$$L_{\varepsilon,0} = 1 + \frac{1}{\varepsilon^2}\mathrm{Tr}^g(S)\otimes \bar g(\mathrm{Tr}^g S) + \frac{1}{\varepsilon^2}Tf.\mathrm{grad}^g\,\mathrm{div}^g\,(\quad)^\top$$

is a linear differential operator $T_f\mathrm{Imm}(M,N) \to T_f\mathrm{Imm}(M,N)$ of order 2 which depends smoothly on f. It is not elliptic.

Theorem *G^ε on $\mathrm{Imm}(M,N)$ has positive geodesic distance.*

Proof not yet written.

Now we add a regularizing term of order $2p$ to the metric, using a parameter $\eta > 0$,

$$G_f^{\varepsilon,\eta}(h,h) = \int_M \bar{g}\Big(L_{\varepsilon,\eta} h, h\Big) \mathrm{vol}(f^*\bar{g}), \qquad \text{where}$$

$$L_{\varepsilon,\eta} = 1 + \frac{1}{\varepsilon^2} \operatorname{Tr}^g(S) \otimes \bar{g}(\operatorname{Tr}^g S) + \frac{1}{\varepsilon^2} Tf . \operatorname{grad}^g \operatorname{div}^g (\quad)^\top + \frac{\eta^{2p}}{p^p} (\Delta^g)^p$$

For $p \geq 2$ the differential operator $L_{\varepsilon,\eta}$ is elliptic.

Let K_ε and $K_{\varepsilon,\eta}$ be the kernels (inverses) for the operators L_ε and $L_{\varepsilon,\eta}$.

Theorem. [Mathieu Molitor 2012, for embeddings] *For $f \in \mathrm{Imm}(M,N)$ and $g = f^*\bar g$ let us denote by*

$$T_{f,\mathrm{vol}(g)}\mathrm{Imm}(M,N) := \\ = \{h \in T_f\mathrm{Imm}(M,N) : \mathrm{div}^g(h^\top) - \bar g(h^\perp, \mathrm{Tr}(S)) = 0\}.$$

Then for each $h \in T_f\mathrm{Imm}(M,N)$ there exist unique $h^{\mathrm{vol}(g)} \in T_{f,\mathrm{vol}(g)}\mathrm{Imm}(M,N)$ and $p = p^h \in C^\infty(M)$ such that

$$h = h^{\mathrm{vol}(g)} + Tf.\mathrm{grad}^g(p^h) + p^h.\mathrm{Tr}(S).$$

Moreover, the mapping $K_0^f : h \to h^{\mathrm{vol}(g)}$ and $\bar K_0^f h \mapsto p^h$ is bounded linear in h and smooth on f.

If $M = N$ and $f = \mathrm{Id}$ this is the Helmholtz decomposition of vector fields $\mathfrak{X}(M) = \mathfrak{X}_{\mathrm{div}^g=0}(M) \oplus \mathrm{grad}^g(C^\infty(M))$.

Theorem. *Geodesic distance is positive for the weak Riemannian metric* $G^{\varepsilon,0}$ *on each connected component of* $\mathrm{Imm}(M,N)$*.*
Theorem: *Let* $\varepsilon \geq 0, \eta > 0, p \geq (\dim(M)+3)/2$*. Then the geodesic equation for the metric* $G^{\varepsilon,\eta}$ *on* $\mathrm{Imm}(M,N)$ *is globally wellposed:* $\exp : T\mathrm{Imm} \to \mathrm{Imm}$ *is everywhere defined and induces a diffeomorphisms* $(\pi_{\mathrm{Imm}}, \exp) : T\mathrm{Imm} \to \mathrm{Imm} \times \mathrm{Imm}$ *from a neighbourhood of the zero section to a neighbourhood of the diagonal.*
Local wellposedness follows from [BHM2012]. Global wellposedness is NOT yet proved in general, only for the case $\mathrm{Imm}(S^1, \mathbb{R}^2)$, in [BMM2013].

Theorem: *Let* $\varepsilon \geq 0 > 0$*. Then the geodesic equation for the metric* G^{ε} *on* $\mathrm{Imm}(M,N)$ *is locally wellposed:*
Not yet proved.

Thank you for listening.