FINAL REPORT FOR THE PROJECT 'LIE THEORY AND APPLICATIONS II', `P17108-N04`

PETER W. MICHOR

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN
NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA.

UND

ERWIN SCHRÖDINGER INSTITUT FÜR MATHEMATISCHE PHYSIK
BOLZMANNGASSE 9, A-1090 WIEN, AUSTRIA.

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2. Brief project report, english

Description of the results obtained

The results obtained in this project may be grouped as follows.

Shape space analysis. Shape space, for this project, is the orbit space of the action of the group of diffeomorphisms of the circle (the reparametrization group) on the space of immersions of the circle into the plane. The aim is to find good Riemannian metrics which allow applications in pattern recognition and vision. Here in the paper [M107] via the Hamiltonian approach many metrics were investigated, together with their conserved quantities (one of them is the reparametrization momentum) and their sectional curvatures. Recall from a result of the predecessor of this project, that the L^2-metric on the space of immersions has zero geodesic distance on the orbit space under the reparametrization group. These metrics come in 3 flavors: Some are derived from the the L^2-metric by multiplying it with a function of the length of the curve (a conformal change) or by multiplying the L^2-integrand with a function of length and curvature (this is called almost local). the second flavor are the metrics which come from the Sobolev H^n-inner product on the space of immersions. The last flavor comes from using a suitable Sobolov metric on the diffeomorphism group of the plane and treating shape space as a homogeneous space. This is the metric that has found already many applications, particularly in medicine. These applications are done at the 'Center for imaging sciences' at the Johns Hopkins University.

A version of one of the metrics in [M107] turns out to be isometric to a quotient of (an open subset in) an infinite dimensional Grassmannian of 2-planes in Hilbert space. Here recent formulas of Neretin allow explicit geodesics and explicit formulas for the geodesic distance. This is done in the paper [M111].

In [M108] the homotopy type of the rotation degree 0 immersed plane curves is determined: $\pi_1 = \mathbb{Z}$ and $\pi_2 = \mathbb{Z}$, all others vanish.

[M109] is a review article on Riemannian metrics on infinite dimensional regular Lie groups, containing detailed presentations of the Hamiltonian approach. In the end two PDE's (following papers by Constantin, Kolev, Kappeler and others) from the Burgers' and KdV hierachy are treated in detail. This are the notes for lecture which P. Michor gave in order to prepare for the existence results in [M107].

Other results on Diffeomorphism groups. In [M101], for a symplectic manifold (M, ω) with exact symplectic form, a 2-cocycle on the group of symplectomorphisms is constructed. In some cases this cocycle is not trivial.

In [M106] this is generalized to the following situation: Let M be a G-manifold and ω a G-invariant exact m-form on M. We indicate when these data allow us to constract a cocycle on a group G with values in the trivial G-module $\mathbb{R}$ and when this cocycle is nontrivial.

[V] For a perfect ideal $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ with $\mathfrak{g}$-invariant symmetric bilinear form b one considers the the continuous cohomology class in $H^2_c(\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ defined by the 2-cocycles of the form $b([X, -], -)$ for $X \in \mathfrak{g}$. There is an obstruction to extend this class to $\mathfrak{g}$. If it vanishes, invariant symmetric bilinear forms the corresponding abelian extensions of $\mathfrak{g}$ by $(\mathfrak{g}/\mathfrak{h})^*$ are constructed. This is applied to Lie algebras of symplectic vector fields and of divergence free vector fields.

Choosings roots alias lifting of mappings over orbit mappings, and invariant theory. In the topic choosing roots of a parameterized family of polynomials as smoothly (in the parameters) as possible, we had several new results: The roots of a smooth curve of hyperbolic (i.e., all roots real) polynomials may not in general be parameterized smoothly, even not $C^{1,\alpha}$ for any $\alpha > 0$. A sufficient condition for the existence of a smooth parameterization is that no two of the increasingly ordered continuous roots meet of infinite order.

In [R5] refined sufficient conditions for smooth solvability are given in the case that polynomials have certain symmetries. In general a C^{3n} curve of hyperbolic polynomials of degree n admits twice differentiable parameterizations of its roots. If the polynomials have symmetries the bound $3n$ can be lowered.

In [M105] the following is proved: Any sufficiently often differentiable curve in the orbit space V/G of a real finite dimensional orthogonal representation $G \to O(V)$ of a finite group G admits a differentiable lift into the representation space V with locally bounded derivative. As a consequence any sufficiently often differentiable curve in the orbit space V/G can be lifted twice differentiably which is in general best possible. These results are then generalized generalized to arbitrary polar representations.

In [R7] the the regularity of the roots of complex monic polynomials $P(t)$ of degree n depending smoothly on a real parameter t is studied. If $P(t)$ is C^∞ and no two of the continuously chosen roots meet of infinite order of flatness, then there exists a locally absolutely continuous parameterization of the roots. Simple examples show that the conclusion is best possible. This result will follow from the proposition that for any t_0 there exists a positive integer N such that $t \mapsto P(\pm(t-t_0)^N)$ admits smooth parameterizations of its roots near t_0. Provided that $P(t)$ is C^n, the roots may be parameterized differentiably if and only if whenever roots meet they meet of order at least 1. Applications to the perturbation theory of normal matrices and unbounded normal operators with compact resolvents and common domain of definition are given. The eigenvalues and eigenvectors of a C^∞ curve of such operators can be arranged locally in an absolutely continuous way, provided that no two of the eigenvalues meet of infinite order of flatness.

[M110] shows the following: If $u \mapsto A(u)$ is a $C^{1,\alpha}$-mapping having as values unbounded self-adjoint operators with compact resolvents and common domain of definition, parametrized by u in an (even infinite dimensional) space then any continuous arrangement of the eigenvalues $u \mapsto \lambda_i(u)$ is $C^{0,1}$ in u. If $u \mapsto A(u)$ is

$C^{0,1}$, then the eigenvalues may be chosen $C^{0,1/N}$ (even $C^{0,1}$ if $N = 2$), locally in u, where N is locally the maximal multiplicity of the eigenvalues.

Symplectic and Poisson geometry. [Ho2] considers the Poisson reduced space $(T^*Q)/K$, where the action of the compact Lie group K on the configuration manifold Q is of single isotropy type and is cotangent lifted to T^*Q. Realizing $(T^*Q)/K$ as a Weinstein space one determines the induced Poisson structure and its symplectic leaves. We thus extend the Weinstein construction for principal fiber bundles to the case of surjective Riemannian submersions $Q \to Q/K$.

[Ho3]: Suppose that a Lie group G acts properly on a configuration manifold Q. We study the symplectic quotient of T^*Q with respect to the cotangent lifted G-action at an arbitrary coadjoint orbit level $\mathcal{O}$. In particular, if $Q = Q_{(H)}$ is of single orbit type we show that the symplectic quotient of T^*Q at $\mathcal{O}$ can be constructed through a minimal coupling procedure involving the smaller cotangent bundle T^*Q_H, the symplectic quotient of $\mathcal{O}$ at 0 with respect to the H-action, and the diagonal Hamiltonian $N(H)/H$-action on these symplectic spaces. A prescribed connection on $Q_H \to Q_H/(N(H)/H)$ then yields a computationally effective way of explicitly realizing the symplectic structure on each stratum of the symplectic quotient of T^*Q. In an example this result is combined with the projection method to produce a stratified Hamiltonian system with very well hidden symmetries.

[M100]: For a Lie algebroid, divergences chosen in a classical way lead to a uniquely defined homology theory. They define also, in a natural way, modular classes of certain Lie algebroid morphisms. This approach, applied for the anchor map, recovers the concept of modular class due to S. Evens, J.-H. Lu, and A. Weinstein.

Comparison with the plan in the application

In the application the following was written. I comment on the results.

1) *In previous papers we studied the problem of choosing the roots of one parameter families of polynomials as differentiable as possible. This problem generalizes to an invariant theoretic question about orthogonal representations of Lie groups. We plan to continue our investigation of this problem. Also we would like to consider polynomials which depend on more than one parameter, and study the relation to entanglement in quantum mechanics.* This succeeded better than I could hope. No application to quantum informatics were given.

2) *We plan to pursue our study of generalized Cayley transforms from Lie groups to their Lie algebras. Particular attention will be paid to fields with nonzero characteristic.* This was not done. No collaborator with expertise in geometry over finite fields was found.

3) *We consider the space of unparameterized simple close curves in the plane. This can be thought of as the space of two dimensional shapes. We started to study a class of (weak) Riemannian metrics on this space, its geodesic equation,* *curvature and the induced geodesic distance. We plan to compare this metric to the Weil–Peterssen metric which is used in Teichmüller theory and string theory.* This also suceeded beyond expectations. Many results and two very important papers for shape space came out of this. But still no progress on the relation to Weil–Peterssen metric.

4) *The project is to continue the investigation of the geometry of orbit spaces of isometric Lie group actions. There are relations to interesting dynamical systems generalizing the Calogero–Moser system.* Orbit spaces were investigated deeply in the activity of shape space. The reduction theory for cotangent bundles was developed by Hochgerner and Rainer. There are aspects of the cotangent bundle case which are not present in the general reduction theory.

5) *We want to continue our study of extending an infinitesimal group action to a group action on an enlarged manifold. We also plan to study flow completions of positive semigroups. As an application this should give a method to investigate viscosity solutions of Burgers' equation.* Not much work was done in this direction. The paper [M104] is still unfinished.

6) *The cohomology of a Poisson manifold inherits a rich structure. Particularly a filtration which generalizes Brylinsky's space of Poisson harmonic forms. Every Poisson mapping has to preserve this structure. This should yield restrictions on the homotopy type of Poisson mappings. So we would like to compute this structure for nice Poisson manifolds such as Hamiltonian fibrations. It looks as this method also can be used to get information about the singularities of a Poisson manifold.*

7) *There is a close connection between spectral geometry and dynamics. For example the incidence numbers in the Morse–Novikov complex and the number of closed trajectories of a closed one form can be recovered from spectral geometry. We plan to extend this to Morse–Bott–Novikov situation.*

8) *The vortex filament equation for circles in three dimensional space generalizes to a Hamiltonian equation on the space of codimension two submanifolds in a Riemannian manifold. This is a non-linear evolution equation. As a first step in the study of this equation we plan to establish short time existence and uniqueness of solutions.*

With respect to items (6)-(8), Stefan Haller has left this project quite early on since he got a position at the University of Vienna. He has continued this work, but outside of this project. Only his paper [H18] still falls into the project.

Personel development

Stefan Haller left this project in 2005 and got an assistent professor position in the Faculty of Mathematics at the University of Vienna. He defended his Habilitation for the field of Mathematics in February 2005. He has spend 6 month in 2005 at the MPI Mathematics in Bonn, and he has FWF project and is part of the IK 1008-N (doctoral college of the University of Vienna)

Armin Rainer (Dr. 2004) had part-time positions at the Unicversity of Vienna. He recently got an FWF Schrödinger scholarship for 2 years which he will spend in Pisa and in Toronto.

Simon Hochgerner defended his thesis (2005) while employed at this project. He now has the position of assistant professor with Tudor Ratiu at the EPFL Lausanne.

Dennis Westra is doing a second Dr. in Mathematics and he is employed at the IK 1008-N (doctoral college).

More information: Home pages.

- `http://www.mat.univie.ac.at/~michor/`
- `http://www.mat.univie.ac.at/~stefan/`
- `http://www.mat.univie.ac.at/~armin/`
- `http://sma.epfl.ch/~hochgern/`
- `http://arxiv.org/`

Collaborations

Collaborations exist with the coauthors of the papers cited. My main collaborators with intensive collaborations outside of Vienna are: Mark Losik (Saratov), David Mumford (Brown), Janusz Grabowski (Warsaw), and Giuseppe Marmo (Napoli). I hope to restart collaborations with Bert Kostant (MIT) and Tudor Ratiu (Lausanne).

Effects of the project outside the field

Lecture course: 'Geometric evolution equations' (3 hours weekly) SS 2005. continued in WS 2005/6 (3 hours weekly). the lecture notes from part of this course were published [M109]. The paper [M107] was also presented in this course.

Nearly all papers were presented in various seminars.

P. Michor was member of the organizing committe of the conference 'Symmetry in Geometry and Physics – 65 years of D. V. Alekseevsky' University of Roma 1, September 14th – 17th, 2005.

P.Michor was member of the organizing committee of: Summer 2007 Program on the Geometry and Statistics of Shape Spaces at the Statistical and Applied Mathematical Sciences Institute (SAMSI) in North Carolina. Workshop on Geometry and Statistics of Shape Spaces, including: Tutorials, July 7-8, 2007; Workshop, July 9-11, 2007; Working Groups, July 12-13, 2007.

Publications, peer-refereed

[MH] Peter W. Michor, *Topics in Differential Geometry*, To appear in: Graduate Studies in Mathematics, Vol. 93, American Mathematical Society, Providence, August 2008. 495 pages.

[M94] Andreas Kriegl, Mark Losik, Peter W. Michor, Armin Rainer, *Lifting mappings over invariants of finite groups.* Acta Math. Univ. Comenianae **77** (2008), 93-122. ESI Preprint 1415, arXiv:math.AG/0312030.

[M100] Janusz Grabowski, Giuseppe Marmo, Peter W. Michor, *Homology and modular classes of Lie algebroids.* Annales del'Institut Fourier **56** (2006), 69-83. ESI Preprint 1373, arXiv:math.DG/0310072

[M101] Rais S. Ismagilov, Mark Losik, Peter W. Michor, *A 2-cocycle on a group of symplectomorphisms.* Moscow Math. J. **6** (2006), 307–315. arXiv:math.GR/0410100, ESI preprint 1525.

[M105] Andreas Kriegl, Mark Losik, Peter W. Michor, and Armin Rainer, *Lifting smooth curves over invariants for representations of compact Lie groups, III.* J. Lie Theory **16** (2006), 579–600. arXiv:math.RT/0504101, ESI preprint 1806.

[M106] Mark Losik, Peter W. Michor, *Cohomology for a group of diffeomorphisms of a manifold preserving an exact form.* International Journal of Geometric Methods in Modern Physics **3**, 5-6 (2006), 1117–1130. arXiv:math.DG/0602489, ESI preprint 1799.

[M107] Peter W. Michor, David Mumford, *An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach.* Applied and Computational Harmonic Analysis **23** (2007), 74-113. arXiv:math.DG/0605009, ESI preprint 1798.

[M108] Hiroki Kodama, Peter W. Michor, *The homotopy type of the space of degree 0 immersed curves.* Revista Matemática Complutense **19** (2006), no. 1, 227–234. arXiv:math.DG/0509694, ESI preprint 1703.

[M109] Peter W. Michor, *Some Geometric Evolution Equations Arising as Geodesic Equations on Groups of Diffeomorphism, Including the Hamiltonian Approach.* Phase space analysis of Partial Differential Equations. Series: Progress in Non Linear Differential Equations and Their Applications, Vol. **69**. Bove, Antonio; Colombini, Ferruccio; Santo, Daniele Del (Eds.). Birkhauser Verlag 2006. Pages 133–215.

[M111] Laurent Younes, Peter W. Michor, Jayant Shah, David Mumford, *A Metric on Shape Space with Explicit Geodesics.* Rend. Lincei Mat. Appl. **9** (2008) 25-57. arXiv:0706.4299.

[H18] Dan Burghelea, Stefan Haller, *Torsion, as a function on the space of representations.* In: *C*-Algebras and Elliptic Theory II*, Burghelea, D.; Melrose, R.; Mishchenko, A.S.; Troitsky, E.V. (Eds.). Trends in Mathematics, Birkhäuser Verlag, Basel, 2008, 41–66. arXiv:math.DG/0507587.

[Ho2] Simon Hochgerner, Armin Rainer, *Singular Poisson reduction of cotangent bundles.* Revista Matemática Complutense **19** (2006), 431–466. arXiv:math.SG/0508455, ESI preprint 1681.

[R5] Mark Losik, Armin Rainer, *Choosing roots of polynomials with symmetries smoothly.* Revista Matemática Complutense **20** (2007), 267–291. arXiv:math.CA/0603660, ESI preprint 1792.

[R6] Armin Rainer, *Orbit projections as fibrations.* To appear in Czechoslovak Math. J. arXiv:math.DG/0610513, ESI preprint 1854.

[R9] Armin Rainer, *Orbit projections of proper Lie groupoids as fibrations.* to appear in Czechoslovak Math. J. arXiv:0712.0706

[V] Cornelia Vizman, *Invariant forms on Lie algebra extensions.* Comm. in Algebra. **35** (2007), 1791–1806.

Publications, not yet peer-refereed

[M104] Franz W. Kamber, Peter W. Michor, *Orbifold-like and proper $\mathfrak{g}$-manifolds.* Preliminary version.

[M110] Andreas Kriegl, Peter W. Michor, Armin Rainer, *Many parameter Lipschitz perturbation of unbounded operators.* arXiv:math.FA/0611506.

[Ho3] Simon Hochgerner, *Spinning particles in a Yang-Mills field*, arXiv:math.SG/0602062, ESI preprint 1773.

[Ho4] L. Feher, S. Hochgerner, *The mechanical connection and dynamical r-matrices.* In preparation.

[R7] Armin Rainer, *Perturbation of complex polynomials and normal operators.* arXiv:math.CA/0611633.

[R8] Armin Rainer, *Invariant functions in Denjoy-Carleman classes.* arXiv:0711.3163

All papers are available via the homepages listed above.

Some lectures

Lectures of P. Michor

192. Riemannian geometries on spaces of plane curves and of immersions. Conference: Differential geometry and its Applications, Prague, August 30-September 3, 2004
193. Riemannian geometries on spaces of plane curves. Seminar, Centre interfacultaire Bernoulli, Ecole polytechnique federale de Lausanne, September 15, 2004.
194. Vanishing geodesic distance on spaces of submanifolds and of diffeomorphisms. Seminar, Centre interfacultaire Bernoulli, Ecole polytechnique federale de Lausanne, September 21, 2004.
195. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms Geometry seminar, Univ. Vienna, October 13, 2004.
196. From Riemannian geometries on spaces of plane curves to vanishing geodesic distance on spaces of submanifolds and of diffeomorphisms. Conference 'Progress in Geometry', MPI Leipzig, November 25–27, 2004.
197. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. Winter school on Geometry and Physics, Srni (CZ). January 15–22, 2005.
198. From Riemannian geometries on spaces of plane curves to vanishing geodesic distance on spaces of submanifolds and of diffeomorphisms. Seminar, Dipartimento di Matematica, Università di Pisa, February 15, 2005.
199. Geodesic equations on diffeomorphism groups, sometimes leading to integrable systems. Seminar, Dipartimento di Matematica, Università di Pisa, February 15, 2005.
200. Curvature of shape space. ARCC Workshop on Statistical Inferences on Shape Manifolds, American Institute of Mathemaics, Palo Alto, May 6-9, 2005.

201. From Riemannian geometries on spaces of plane curves to vanishing geodesic distance on spaces of submanifolds and of diffeomorphisms. Seminar, Center for Imaging Science, and Department of Mathematics, John Hopkins University, Baltimore, May 25, 2005.
202. Riemannian geometries on spaces of planar shapes. Plenary lecture, Conference 'Differential Geometry and Physics' Eötvös Loránd University, Budapest, Hungary 29 August - 2 September, 2005.
203. Orbit spaces. Minisymposium 'Geometric Topology', 16. Internationaler Kongress der Österreichischen Mathematischen Gesellschaft — Jahrestagung der Deutschen Mathematiker-Vereinigung, Klagenfurt 18. - 23. September 2005.
204. Tutorial on Sobolev spaces, ILH groups, and EPDiff. Retreat " NSF-Focused Research Group: The geometry, mechanics, and statistics of the infinite dimensional shape manifolds", Center od imaging sciences, Johns Hopkins University, October 17-23, 2005.
205. Hamiltonian approach to Riemannian metrics on spaces of curves. Retreat " NSF-Focused Research Group: The geometry, mechanics, and statistics of the infinite dimensional shape manifolds", Center od imaging sciences, Johns Hopkins University, October 17-23, 2005.
206. The Hamiltonian approach to Riemann metrics on plane curve and shape space. Berufungsvortrag Universitaet Wien, 25. Oktober 2005.
207. Hamiltonian approach to Riemannian metrics on spaces of curves. Conference "Phace space analysis of PDE's", Pienza, November 8-13, 2005.
208. The Homotopy type of the space of immersions $S^1 \to \mathbb{R}^2$ mod reparametrizations. Winter school geometry and physics, Srni/CZ, January 14-21, 2006.
209. The geometry of the space of planar shapes - geodesics and curvature. Colloquium, University of Bordeaux, February 9, 2006.
210. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. Differential geometry seminar, University of Bordeaux, February 10, 2006.
211. The geometry of the space of planar shapes - geodesics and curvature. Differential geometry seminar. Berlin, Humboldt-Universität, 27. Februar 2006.
212. Tutorial on ODE's. Mesopotamia Seminar, Program on evolutionaly dynamics, Harvard University, March 29, 2006.
213. Geometries on the space of planar shapes - geodesics and curvatures. IMA workshop "Shape Spaces", Minnesota, April 3-7, 2006.
214. The symplectic approach to the Euler Poincare equation on $\mathrm{Diff}(S^1)$. Lecture in the lecture course of David Mumford at Brown University. April 12, 2006.
215. Curvature on a certain infinite dimensional Stiefel manifold related to the H^1-metric on shape space. "EPDiff and the Manifolds of Shape" conference, Santa Fe, NM, July 24-28, 2006. Sponsored by Los Alamos National Lab's Center for Nonlinear Studies and Johns Hopkins University.

216. Neretin Geodesics leading to computable geodesic distance in vision. Winter school in Geometry and Physics, Srni, January 13-19, 2007.
217. The proof of the Poincare conjecture (following Hamilton and Perelman). Kolloquium der Fakultät für Mathematik, Universität Wien, January 24, 2007
218. Grassmann. Lecture in the seminar on History of Mathematics of David Mumford, Brown University, February 27, 2007.
219. Neretin Geodesics leading to computable geodesic distance in vision. Central European Seminar in Differential Geometry, Brünn, March 20, 2007
220. Nicolas Bourbaki. Vortrag im Institut Wiener Kreis, Universitaet Wien, 14. Mai 2007.
221. A specific Geometry on shape space with explicit geodesics. Seminaire, Université Paul Verlaine – Metz, June 27, 2007
222. Hamiltonian approach to Riemannian metrics on spaces of curves. Conference "Phase space analysis of PDE's", Certosa di Pontigliano (near Siena), October 10-13, 2007.
223. Hamiltonian approach to Riemannian structures on spaces of planar shapes. University of Vienna Science Day at Peking University, October 19, 2007.
224. Hamiltonian methods in the geometry of shape space. Conference Renormalization Group Flow and Ricci Flow, ESI, October 22-26, 2007
225. Hamiltonian approach to Riemannian structures on spaces of planar shapes. Geometric Mechanics, CIRM Luminy, November 19-23, 2007
226. A metric on shape space with compatable geodesics. Winter school in Geometry and Physics, Srni, January 12-18, 2008.

Lectures by Armin Rainer

January 2007: 27th Winter School Geometry and Physics in Srní, Czech Republic. Talk: Orbit projections as fibrations.

November 2006: Seminar Sophus Lie XXXII at the Erwin Schrödinger Institute (ESI) in Vienna, Austria.

January 2006: 26th Winter School Geometry and Physics in Srní, Czech Republic. Talk: Choosing roots of polynomials with symmetries smoothly.

July 2005: Summer School and Conference on Poisson Geometry at the ICTP in Trieste, Italy.

February 2005: Visitor at the Department of Mathematics at the University of Pisa from February 13th to February 25th. Talk: Choosing roots of polynomials smoothly and lifting smooth curves over invariants.

January 2005: 25th Winter School Geometry and Physics in Srní, Czech Republic. Talk: Lifting smooth curves over invariants.

Lectures by Simon Hochgerner See his homepage.