PROJECT APPLICATION: LIE THEORY AND APPLICATIONS

Peter W. Michor

Institut für Mathematik, Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.
And: Erwin Schrödinger Institut für Mathematische
Physik, Boltzmanngasse 9, A-1090 Wien, Austria.

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The research in this project follows different lines which are explained below. The project should be located at the Erwin Schrödinger Institute and at the Institute for Mathematics of the University of Vienna jointly.

1. Invariant Theory. Collaboration with Dmitri Alekseevky from Moscow, Mark Losik from Saratov (RU) and Andreas Kriegl from Vienna.

In [65] we investigated the following problem: Let

$$P(t) = x^n - \sigma_1(t)x^{n-1} + \dots + (-1)^n\sigma_n(t)$$

be a polynomial with all roots real, smoothly parametrized by t near 0 in $\mathbb{R}$. Can we find n smooth functions $x_1(t), \dots, x_n(t)$ of the parameter t defined near 0, which are the roots of $P(t)$ for each t? We showed that this is possible under quite general conditions: real analyticity or no two roots should meet of infinite order. Some applications to perturbations of unbounded operators in Hilbert space are also given.

This problem can be reformulated in the following way: Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates (the roots), and consider the polynomial mapping $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{R}^n \to \mathbb{R}^n$ whose components are the elementary symmetric polynomials (the coefficients). Given a smooth curve $c : \mathbb{R} \to \sigma(\mathbb{R}^n) \subset \mathbb{R}^n$, is it possible to find a smooth lift $\bar{c} : \mathbb{R} \to \mathbb{R}^n$ with $\sigma \circ \bar{c} = c$?

In [73] we tackled the following generalization of this problem. Consider an orthogonal representation of a compact Lie group G on a real vector space V. Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. Then the mapping

$$\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$$

defines a bijection of the orbit space V/G to the semialgebraic set $\sigma(V) \subseteq \mathbb{R}^n$. A curve

$$c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$$

in the orbit space V/G is called smooth if it is smooth as a curve in $\mathbb{R}^n$. This is well defined, i.e. does not depend on the choice of generators.
Problem. Given a smooth curve $c : \mathbb{R} \to V/G$ in the orbit space, does there exist a smooth lift to V, i.e. a smooth curve $\bar{c} : \mathbb{R} \to V$ with $c = \sigma \circ \bar{c}$?
We gave satisfactiry answers under similar conditions as in the paper [65].

In this project we want to investigate these two questions, but replace $\mathbb{R}$ by higher dimensional manifolds. Easy counterexamples show that liftings will be more rare and that strong conditions will be necessary.

2. Completely integrable systems and double Lie groups. Collaboration with the physicist Guiseppe Marmo from Napoli and with Janusz Grabowski from Warsaw. In this project we want to push further the investigations from the papers [53], [70], [71], and [83] which give some indications about how to build new integrable systems from old ones by the use of Poisson mappings: notably multiplications on Poisson Lie groups. The paper [71] generalizes the notion of a completely integrable system from a torus to an arbitrary Poisson Lie group.

3. Geodesics on infinite dimensional Lie groups and completely integrable systems. Collaboration with Gerard Misiolek (Notre Dame University, US) and Tudor Ratiu (ETH Lausanne).

In [69] a careful presentation is given of the known fact, that the geodesic equation on the Virasoro-Bott group of the right invariant H^0-metric is the Korteveg-De Vries equation. This allows for Jacobi fields, curvature, etc. In [79] this is extended to the H^1-metric, where another infinite dimensional completely integrable system comes up, namely the Camassa-Holmes equation. There are many directions of further research possible, see in particular point 6 of the application.

4. Further investigations of infinite dimensional regular Lie groups. Joint work with Andreas Kriegl, J. Teichmann, and St. Haller. This is a continuation of the work [62], [G], [75].

In particular the following topics should be treated: Up to now the question, whether there exist non-regular convenient or even Fréchet-Lie-groups is unsolved. It is not clear how necessary the hypothesis of regularity is in convenient Lie theory (see [G]). By the methods of Lipschitz metrics it is possible to characterize regularity by some simple conditions involving those metrics and smooth curves, however, there are no good counterexamples (see [T0], [T4]).

Locally arcwise connected topological groups without one-parameter subgroups are rare objects: one starting example is given by the integer valued L_2-functions on $[0, 1]$, which form a closed locally arcwise connected subgroup of the abelian group $L_2([0, 1])$, the Hilbert space of real valued L_2-functions. The aim is to find among these abelian groups some convenient one to prove the conjecture that non-regular convenient Lie groups do exist.

5. Attempts for a structure theory of the Lie algebra of vector fields on a finite dimensional Lie group. Collaboration with St. Haller and J. Teichmann.

Our aim is to develop some representation theory for Lie algebras of vector fields on compact Lie groups and the associated infinite dimensional Lie groups. The aim is to develop first some 'algebraic structure theory' of vector fields:

Let G be a connected, compact Lie group and let D be an element of the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ which acts a differential operator $D : \mathfrak{X}(G) \otimes \mathbb{C} \to \mathfrak{X}(G)^{\mathbb{C}}$ on the space of complex vector fields on G, where the left factor acts by left invariant vector fields and the right factor by right invariant ones. We assume furthermore that D is invariant under the action of $G \times G$ such that the eigenspaces are finite dimensional and the sum of the eigenspaces $E(D)$ is dense in $\mathfrak{X}(G)^{\mathbb{C}}$ with respect to the biivariant L_2-norm induced by the negative Cartan-Killing-form on $\mathfrak{g}$. Examples of such elements are the Casimir operators of the left or right G-action or the $G \times G$-action on $\mathfrak{X}(G)^{\mathbb{C}}$, see below. Obviously the sum of the eigenspaces $E(D)$ can be completely reduced into finite dimensional irreducible $G \times G$-subrepresentations by the invariance of D and the finite dimensionality of the eigenspaces. Given any finite dimensional irreducible $G \times G$-subrepresentation of $\mathfrak{X}(G) \otimes \mathbb{C}$, D restricts to this subrepresentation and is consequently scalar by Schur's lemma, so this subrepresentation lies in $E(D)$.

The Lie bracket $[.,.]$ on $\mathfrak{X}(G) \otimes \mathbb{C}$ restricts to $E(D)$: Given two vector fields $v, w \in \mathfrak{X}(G) \otimes \mathbb{C}$ in irreducible $G \times G$-subrepresentations V and W, then $[V, W]$ is a finite dimensional $G \times G$-subrepresentation and can be completely reduced into irreducible $G \times G$-subrepresentations, consequently $[v, w]$ lies in $E(D)$. Our aim is to understand deeply the decomposition into irreducible subrepresentations and to understand the restricted Lie bracket $[.,.] : E(D) \times E(D) \to E(D)$ in algebraic terms.

The Casimir operators of the left or right G-action or the $G \times G$-action on $\mathfrak{X}(G) \otimes \mathbb{C}$ are examples for $G \times G$-invariant differential operators in the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$, which admit a finite dimensional eigenspace decomposition due to ellipticity. The main symbol of these Casimir operators is given in the left trivialization by $diag(D) : \mathcal{F}(G)^n \to \mathcal{F}(G)^n$, where $D : \mathcal{F}(G) \to \mathcal{F}(G)$ denotes a self-adjoint elliptic operator on smooth complex valued functions. The Casimir operator Δ_L of the left G-action and the Casimir operator Δ_R of the right G-action commute and in the right respectively left trivialization they act like the Laplace-Beltrami operator Δ on smooth functions. More precisely, let X be a vector field, then $X = \sum_{i=1}^{n} f^i X_i$, where $\{X_i\}_{1 \leq i \leq n}$ denotes the frame of left invariant vector fields and $\{f^i\}_{1 \leq i \leq n}$ are smooth functions on G. $\Delta_R X_i = 0$, so $\Delta_R X = \sum_{i=1}^{n} (\Delta f^i) X_i$.

By means of harmonic analysis the decomposition in irreducible subrepresentations of the $G \times G$-action on the complex valued smooth functions $\mathcal{F}(G)$ is well understood. It should therefore be easy to understand the irreducible components of the $G \times G$-action on $\mathfrak{X}(G) \otimes \mathbb{C}$. This will be the basis for a further algebraic analysis of the Lie bracket, since for two vector fields X, Y we can search the result $[X, Y]$ in a finite sum of irreducible components.

To obtain more concrete ideas about $G \times G$-subrepresentations we investigated the differential geometry of G, too. The Levi-Civita-connection is given by $\nabla_X Y = \frac{1}{2}[X, Y]$ for left invariant vector fields X, Y or for right invariant vector fields X, Y. Additionally the formula $[X, \nabla_Y Z] = \nabla_{[X,Y]} Z + \nabla_Y [X, Z]$ holds for X left invariant or right invariant and $Y, Z \in \mathfrak{X}(G) \otimes \mathbb{C}$. Furthermore $[X, Y] = 0$ for X any left and

Y any right invariant vector field by commutation of left and right action of G on G. Consequently we can construct non-trivial $G \times G$-subrepresentation by iterated covariant derivation.

6. Approximations procedures on regular Fréchet-Lie groups aiming towards solving certain non-linear partial differential equations. This is building on the Thesis [T0] of J. Teichmann and his paper [T4]

Assuming that the model space of a regular Lie group is a Fréchet space given by an inverse limit of Hilbert spaces, so the definition of Lipschitz-metrics is equally a definition of a variational problem, which is easily solved under some condition on the Lie bracket, namely that ad has a continuous transpose with respect to some scalar product. Then the geodesic equation associated to the variational problem is given through $u_t = -ad(u)^{\mathrm{T}} u$ where u denotes the right logarithmic derivative of the geodesic (see [G], section 46.4). Only in the case, where $u \in \ker(ad(u)^{\mathrm{T}})$ for $u \in \mathfrak{g}$ the smooth one-parameter subgroups are the geodesics. With respect to interesting non-linear partial differential equations (for example the Korteweg-De Vries-equation) it is worth studying this situation in concrete cases. The question arises if such naturally appearing differential equations can be solved on the given Lie groups by internal methods, for example by Lipschitz-metrics (see [T0], chapter 3). If this were the case, some interesting geometro-analytic progress in partial differential equations would be possible. To set the program it is first necessary to find some natural approximation procedure for geodesic problems fitting into some successive subtilization of product integrals (see [T0], chapter 3), then to apply the Lipschitz-methods to prove that approximation works well.

7. A non-linear version of Arzelà-Ascoli's theorem on convenient Lie groups. This conjecture is building on the Thesis [T0] of J. Teichmann.

Let E be a Fréchet space such that closed bounded sets are compact and assume that G is a connected regular Lie group modeled on E. By a construction given in the Thesis [T0], there is a sequence of Lipschitz-metrics on G generating the sequential topology on G. Given a subset K of G being bounded with respect to these Lipschitz-metrics, then the conjecture is that K is relatively compact in the topology of G. This would be a non-linear analogue of the Arzelà-Ascoli theorem in the version that bounded subsets of $C^\infty(M)$ of smooth functions on a compact manifold are relatively compact subsets.

8. Actions of finite dimensional Lie groups and structures of orbit spaces. Collaboration with D. Alekseevsky from Moscow and M. Losik from Saratov.

Let G be a Lie group which acts isometrically on a Riemannian manifold M. A section of the Riemannian G-manifold M is a closed submanifold Σ which meets each orbit orthogonally. In this situation the trace on Σ of the G-action is a discrete group action by the generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$, where $N_G(\Sigma) := \{g \in G : g.\Sigma = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : g.s = s \text{ for all } s \in \Sigma\}$. A differential form $\varphi \in \Omega^p(M)$ is called *G-invariant* if $g^*\varphi = \varphi$ for all $g \in G$ and *horizontal* if φ kills each vector tangent to a G-orbit. We denote by $\Omega^p_{\mathrm{hor}}(M)^G$ the space of all horizontal G-invariant p-forms on M which are also called *basic forms*.

In the papers [58] and [64] (this was the result promised for the project P 10037-MAT) it was shown that for a proper isometric action of a Lie group G on a smooth

Riemannian manifold M admitting a section Σ the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G-invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$.

The project is to continue these investigations in the direction of obtaining a better understanding of the geometry of the orbit space of an isometric Lie group action. Paper [80] is in preparation and it contains already some results in this direction.

9. Actions of Lie algebras on manifolds. Collaboration with D. Alekseevsky from Moscow and Franz Kamber from Illinois.

In the paper [56] we stated to investigate the differential geometry of an action of a Lie algebra on a manifold, i.e. only an infinitesimal Lie group action. We want to study how this action can be extended to an enlarged manifold. Some results are already available.

Personnel.

Stefan Haller, 1.3.2001–28.2.2003
Josef Teichmann, 1.7.2001–30.6.2003
N.N. Dissertant, 1.7.2000–30.6.2003
N.N. Dissertant, 1.7.2000–30.6.2003

I apply for the support for 2 postdocs (J. Teichmann and St. Haller) for two years each, 4 x 504.000.-.

Furthermore I apply for support for two PhD-students for 3 years each. 6 x 328.000.-.

Sum: AS 3.984.000,–

If an application for a 'Wissenschaftskolleg Differentialgleichungen' materializes and if I am in this application, I want to shift the support for the PhD-students to the 'Wissenschaftskolleg'.

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Most of the papers including the two theses can be downloaded from my homepage:
http://radon.mat.univie.ac.at/~michor/

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: Peter.Michor@esi.ac.at