R-TRANSFORMS FOR SOBOLEV H^2-METRICS ON SPACES OF PLANE CURVES

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Dedicated to David Mumford on the occasion of his 76th birthday

Abstract. We consider spaces of smooth immersed plane curves (modulo translations and/or rotations), equipped with reparameterization invariant weak Riemannian metrics involving second derivatives. This includes the full H^2-metric without zero order terms. We find isometries (called R-transforms) from some of these spaces into function spaces with simpler weak Riemannian metrics, and we use this to give explicit formulas for geodesics, geodesic distances, and sectional curvatures. We also show how to utilise the isometries to compute geodesics numerically.

Contents

1. Introduction

In this article we will study four different Sobolev H^2-type metrics on the infinite dimensional manifold of parametrized curves in the plane, $\operatorname{Imm}(S^1, \mathbb{R}^2)$. This space is of interest due to its connections to the field of mathematical shape analysis. Riemannian metrics are used in shape analysis, since they equip the space with a distance function that can be used for comparison or classification of objects;

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they also allow to locally linearize the space via the exponential map and thus to generalize linear statistical methods to these — in general – highly nonlinear spaces.

In applications to shape analysis one is mostly interested not in the curve itself, but only in the shape that it represents. Two curves represent the same shape, if they differ by a reparameterization or relabelling of the points. For this reason we will only be interested in metrics that are invariant under the action of the reparameterization group $\operatorname{Diff}(S^1)$.

The arguably simplest reparametrization invariant metric on $\operatorname{Imm}(S^1,\mathbb{R}^2)$ is the L^2-metric

$$G_c(h,k) = \int_{S^1} \langle h, k\rangle \,\mathrm{d}s\,.$$

Here $h,k \in T_c \operatorname{Imm}(S^1,\mathbb{R}^2)$ are tangent vectors with foot point $c \in \operatorname{Imm}(S^1,\mathbb{R}^2)$ and $\mathrm{d}s$ denotes arc-length integration, i.e., $\mathrm{d}s = |c'(\theta)|\,\mathrm{d}\theta$. Unfortunately this metric is unsuitable for shape analysis, because the induced geodesic distance vanishes, i.e., any two curves can be joined by paths of arbitrary short length. This surprising result was proven first for the quotient space $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ in [25]; later it was generalized in [24] to the space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ of type M submanifolds of N where M is compact and $\dim M \leq \dim N$, as well as the diffeomorphism group $\operatorname{Diff}(M)$; using a combination of both results it was shown in [3] that the distance also vanishes on $\operatorname{Imm}(M,N)$. Note that this is a purely infinite dimensional phenomenon – in finite dimensions the geodesic distance is always positive, due to the local invertibility of the exponential map.

The vanishing of the geodesic distance for the L^2-metric led to the search for stronger metrics, that would be suitable for shape analysis. Candidates, that have been considered, include the L^2-metric weighted by curvature [25]:

$$G^A_c(h,k) = \int_{S^1} (1 + A\kappa_c^2)\langle h, k\rangle \,\mathrm{d}s\,,$$

or the length of the curve [34, 29]:

$$G^\Phi_c(h,k) = \Phi(\ell_c) \int_{S^1} \langle h, k\rangle \,\mathrm{d}s\,.$$

Here κ_c denotes the curvature of the curve, ℓ_c its length and $\Phi : \mathbb{R} \to \mathbb{R}_{>0}$ is a suitable positive function. These metrics have been generalized to higher dimensional immersions in [8, 9].

A different approach to strengthen the metric and the one, that we will use in this article is to add terms involving higher derivatives of the tangent vectors to the metric, leading to metrics of the form

$$G_c(h,k) = \int_{S^1} \sum_{j=0}^{k} a_j \langle D_s^j h, D_s^j k\rangle \, ds\,,$$

where $D_s h = \frac{1}{|c'|} h'$ denotes the arc-length derivative of h and a_j are weights, possibly depending on the curve c. More generally one can consider metrics that are defined via a field of symmetric pseudo- differential operators $L_c : T_c \operatorname{Imm}(S^1,\mathbb{R}^2) \to T_c \operatorname{Imm}(S^1,\mathbb{R}^2)$ by

$$G_c(h,k) = \int_{S^1} \langle L_c h, k\rangle \, ds = \int_{S^1} \langle h, L_c k\rangle \, ds\,.$$

This approach leads to the class of Sobolev-type metrics, which were independently introduced in [11, 26, 32] and studied further in [7, 10, 22]. Often the operator field L will have a kernel and thus G^L will be a metric only a certain quotient of $\operatorname{Imm}(S^1, \mathbb{R}^2)$, e.g., if all constant vector fields are in its kernel, then one has to pass to the quotient $\operatorname{Imm} / \operatorname{Tra}$ of plane parametrized curves modulo translations. An overview of the various metrics on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ can be found in [6].

While Sobolev-type metrics are a natural generalization of the L^2-metric, their numerical treatment is unfortunately rather involved. This stems mainly from the fact that the geodesic equation of a Sobolev-type metric of order k is generally a highly nonlinear PDE of order $2k$. There are exceptions. For the family of first order metrics on $\operatorname{Imm}(S^1, \mathbb{R}^2)/ \operatorname{Tra}$ given by

$$G_c^{a,b}(h,h) = \int_{S^1} a^2 \langle D_s h, n\rangle^2 + b^2 \langle D_s h, v\rangle^2 \, ds \,,$$

with $a, b > 0$ there exists an isometric transformation of the space $\operatorname{Imm}(S^1, \mathbb{R}^2)/ \operatorname{Tra}$, called the R-transform, which tremendously simplifies the computation of geodesics; see [35, 31, 19, 4]. Apart from simplifying the computations, the representation via the R-transform also permits us to compute the curvature and in some special cases to obtain explicit formulas for geodesics.

There have been some attempts to solve the geodesic equation directly for order one metrics on curves [27] and surfaces [2]. Metrics of higher order on the other hand are still practically untouched. The only exception is [30], discussing the homogenous H^2-metric on the space of plane curves modulo similitudes. It is therefore of interest to develop representations of higher order metrics, that have the potential to simplify computations of geodesics.

In this article we continue the investigation started in [4] and use similar methods to study four different H^2-type metrics, namely:

((7) in Sect. 3) $$G_c(h,k) = \int_{S^1} \kappa^{-3/2} \langle D_s^2 h, n\rangle \langle D_s^2 k, n\rangle + \langle D_s h, v\rangle\langle D_s k, v\rangle \, \mathrm{d}s \,,$$

((13) in Sect. 4) $$G_c(h,k) = \int_{S^1} \langle D_s h, v\rangle\langle D_s k, v\rangle + \langle D_s^2 h, n\rangle \langle D_s^2 k, n\rangle \, \mathrm{d}s \,,$$

((19) in Sect. 5) $$G_c(h,k) = \int_{S^1} \langle D_s h, D_s k\rangle + \langle D_s^2 h, n\rangle \langle D_s^2 k, n\rangle \, \mathrm{d}s \,,$$

((26) in Sect. 6) $$G_c(h,k) = \int_{S^1} \langle D_s h, D_s k\rangle + \langle D_s^2 h, D_s^2 k\rangle \, \mathrm{d}s \,.$$

Despite its seemingly complicated nature, metric (7) is completely amenable to the R-tranform treatment: For open curves it is flat and we get explicit formulas for geodesics and the geodesic distance. The image of the R-transform of the space of closed curves is a codimension 2 splitting submanifold of an open (with respect to a finer topology) set in a pre-Hilbert space. See Sect. 3.3 for an explanation of the form of this metric.

For the metric (13) the image of the corresponding R-transform for open curves is $C^\infty([0,2\pi], (\mathbb{R}_{>0} \times \mathbb{R}, g))$ with a weak L^2-type metric; here g is a curved metric on $\mathbb{R}_{>0} \times \mathbb{R}$, for which we manage to derive (somewhat) explicit formulas for geodesics. The image of the space of closed curves is again a codimension 2 splitting submanifold.

For metric (19) the image of the space of open curves under the R-transform is splitting submanifold of infinite codimension in $C^\infty([0,2\pi],(\mathbb{R}_{>0}\times S^1\times\mathbb{R},g))$ described by a system of ODEs.

The picture for metric (26) is again more complicated but managable; we do not include full results in this paper.

2. Background material and notation

In this paper we use convenient analysis in infinite dimensions as described in [20].

2.1. Notation. Let M denote either S^1 or $[0,2\pi]$ and let $c: M\to\mathbb{R}^2$ be a regular curve, i.e., $c'(\theta)\neq 0$. We denote the curve parameter by $\theta\in M$ and differentiation by $'$, i.e., $c'=\partial_\theta c$. Since c is an immersion, the unit-length tangent vector $v=c'/|c'|$ is well-defined. Denote by J the rotation by $\frac{\pi}{2}$. Rotating v we obtain the unit-length normal vector

$$n=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} v = Jv\,.$$

We will denote by $D_s=\frac{1}{|c'|}\partial_\theta$ the derivative with respect to arc-length and by $\mathrm{d}s=|c'|\,\mathrm{d}\theta$ the integration with respect to arclength. To summarize, we have

$$v=D_s c\,,\qquad n=Jv\,,\qquad D_s=\frac{1}{|c'|}\partial_\theta\,,\qquad \mathrm{d}s=|c'|\,\mathrm{d}\theta\,.$$

The curvature can be defined as

$$\kappa=\langle D_s v,n\rangle\,,$$

where $\langle\,,\,\rangle$ denotes the Euclidean inner product. The orthonormal frame (v,n) satisfies the Frenet equations,

$$\begin{aligned} D_s v &= \kappa n\\ D_s n &= -\kappa v\,. \end{aligned}$$

We define the *turning angle* $\alpha: M\to\mathbb{R}/2\pi\mathbb{Z}$ of a curve c by $v(\theta)=\exp(i\alpha(\theta))=(\cos\alpha,\sin\alpha)$; we shall often treat S^1, $\mathbb{R}/2\pi\mathbb{Z}$, and the intervall $[0,2\pi]$ with endpoints identified, as the same space.

2.2. The manifold of plane curves. The space of *closed immersed curves*,

$$\operatorname{Imm}(S^1,\mathbb{R}^2)=\left\{c\in C^\infty(S^1,\mathbb{R}^2)\,:\,c'(\theta)\neq 0\right\}\,,$$

is an open set in the manifold $C^\infty(S^1,\mathbb{R}^2)$ with respect to the C^∞-topology and thus itself a smooth manifold. The tangent space of $\operatorname{Imm}(S^1,\mathbb{R}^2)$ at the point c consists of all vector fields along the curve c. It can be described as the space of sections of the pullback bundle $c^*T\mathbb{R}^2$,

$$T_c\operatorname{Imm}(S^1,\mathbb{R}^2)=\Gamma(c^*T\mathbb{R}^2)=\left\{h:\quad \begin{array}{ccc} & & T\mathbb{R}^2\\ & \overset{h}{\nearrow} & \downarrow\pi\\ S^1 & \xrightarrow{\ c\ } & \mathbb{R}^2\end{array}\right\}.$$

Since the tangent bundle $T\mathbb{R}^2$ is trivial, we can identify $T_c\operatorname{Imm}(S^1,\mathbb{R}^2)$ with the space of $\mathbb{R}^2$-valued functions on S^1,

$$T_c\operatorname{Imm}(S^1,\mathbb{R}^2)\cong C^\infty(S^1,\mathbb{R}^2)\,.$$

If we drop the periodicity condition we obtain the manifold of *open immersed curves*,

$$\operatorname{Imm}([0,2\pi],\mathbb{R}^2) = \left\{c \in C^\infty([0,2\pi],\mathbb{R}^2) \,:\, c'(\theta) \neq 0\right\} .$$

The tangent space of $\operatorname{Imm}([0,2\pi],\mathbb{R}^2)$ is similar to that for closed curves, only with S^1 replaced by $[0,2\pi]$. Whenever we describe results that work for both open and closed curves we will write $\operatorname{Imm}(M,\mathbb{R}^2)$, with M standing for either S^1 or $[0,2\pi]$.

All metrics in this article will be degenerate on $\operatorname{Imm}(M,\mathbb{R}^2)$ – their kernel will consist of translations or Euclidean motions. This leads us to consider the spaces $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}$ and $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$. Here Tra denotes the translation group on $\mathbb{R}^2$, i.e., $\operatorname{Tra} \cong \mathbb{R}^2$ and Mot denotes the group $\mathbb{R}^2 \ltimes SO(2)$ of Euclidean motions. We will identify the quotients with subspaces of $\operatorname{Imm}(M,\mathbb{R}^2)$ (sections for the left action of the translation group or the motion group) in the following way

$$\begin{aligned}\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra} &\cong \{c \in \operatorname{Imm}(M,\mathbb{R}^2) : c(0) = 0\}\\ \operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot} &\cong \{c \in \operatorname{Imm}(M,\mathbb{R}^2) : c(0) = 0 \text{ and } \alpha(0) = 0\}\,.\end{aligned}$$

The tangent spaces are then given by

$$\begin{aligned}T_c\left(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}\right) &\cong \{h \in C^\infty(M,\mathbb{R}^2) : h(0) = 0\}\\ T_c\left(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}\right) &\cong \{h \in C^\infty(M,\mathbb{R}^2) : h(0) = 0, \langle D_s h(0), n(0)\rangle = 0\}\,.\end{aligned}$$

If we want sections that are invariant under the reparameterization group $\operatorname{Diff}(S^1)$, we shall consider instead

$$\begin{aligned}\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra} &\cong \{c \in \operatorname{Imm}(M,\mathbb{R}^2) : \int_M c \,\mathrm{d}s = 0\}\,,\\ \operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot} &\cong \{c \in \operatorname{Imm}(M,\mathbb{R}^2) : \int_M c \,\mathrm{d}s = 0 \text{ and } \int_M \alpha \,\mathrm{d}s = 0\}\,.\end{aligned}$$

If we mean any of these sections, we shall write $\mathcal{C}(M,\mathbb{R}^2)$: in all cases it is the intersection of $\operatorname{Imm}(M,\mathbb{R}^2)$ with a closed linear subspace of $C^\infty(M,\mathbb{R}^2)$. We will also need the space of positively oriented convex curves

$$\operatorname{Imm}_{\mathrm{conv}}(M,\mathbb{R}^2) := \{c \in \operatorname{Imm}(M,\mathbb{R}^2) : \kappa(c) > 0\}\,,$$

which is an open set in $\operatorname{Imm}(M,\mathbb{R}^2)$ and thus itself a smooth manifold.

2.3. **Variational formulae.** We will need formulae that express, how the quantities that have been introduced in the previous sections change, if we vary the underlying curve c. For a smooth map F from $\operatorname{Imm}(M,\mathbb{R}^2)$ to any convenient vector space we denote by

$$dF(c).h = D_{c,h}F = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(c+th) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(\widetilde{c}(t, \quad))$$

the variation in the direction h, where $\widetilde{c} : \mathbb{R}\times M \to \mathbb{R}^2$ is any smooth variation with $\widetilde{c}(0,\theta) = c(\theta)$ and $\partial_t|_0 \widetilde{c}(t,\theta) = h(\theta)$ for all θ. Examples of maps F include v, n, α, $|c'|$, κ. In the following lemma we collect the basic variational formulae that we will use throughout the article.

Lemma 2.3.1. *The first variations of the the turning angle α, the unit tangent vector v, the normal vector n, the length element $|c'|$ and the curvature κ are given by*

$$d\alpha(c).h = \langle D_s h, n\rangle \tag{1}$$

$$dv(c).h = \langle D_s h, n\rangle n \tag{2}$$

$$dn(c).h = -\langle D_s h, n\rangle v \tag{3}$$

$$d(|c'|).h = \langle D_s h, v\rangle |c'| \tag{4}$$

$$d\kappa(c).h = \langle D_s^2 h, n\rangle - 2\kappa\langle D_s h, v\rangle\,. \tag{5}$$

Proof. The proof of these formulae can be found for example in [26]. □

2.4. **Riemannian metrics on spaces of curves.** A Riemannian metric on the manifold of curves is a smooth family of positive definite inner products $G_c(.,.)$ with $c \in \operatorname{Imm}(M,\mathbb{R}^2)$, i.e.,

$$G_c : T_c \operatorname{Imm}(M,\mathbb{R}^2) \times T_c \operatorname{Imm}(M,\mathbb{R}^2) \to \mathbb{R}\,.$$

Each metric is weak in the sense that G_c, viewed as linear map

$$G_c : T_c \operatorname{Imm}(M,\mathbb{R}^2) \to \big(T_c \operatorname{Imm}(M,\mathbb{R}^2)\big)'$$

from $T_c \operatorname{Imm}(M,\mathbb{R}^2)$ into its dual, which consists of $\mathbb{R}^2$-valued distributions on M, is injective but not surjective. In this article, we will study metrics G that are induced by an operator field L via

$$G_c(h,k) = \int_M \langle L_c h, k\rangle \,\mathrm{d}s = \int_M \langle h, L_c k\rangle \,\mathrm{d}s\,,$$

with c a curve and $h,k \in T_c \operatorname{Imm}(M,\mathbb{R}^2)$. The *momentum* $p = L_c h \otimes \mathrm{d}s \in (T_c \operatorname{Imm}(M,\mathbb{R}^2))'$ allows us to represent the metric as

$$G_c(h,k) = \langle p, k\rangle_{T\mathrm{Imm}}\,.$$

Furthermore, we will be interested only in metrics that are invariant under the action of the reparameterization group $\operatorname{Diff}(M)$, that is, for each $\varphi \in \operatorname{Diff}(M)$ the metric has to satisfy

$$G_{c\circ\varphi}(h\circ\varphi, k\circ\varphi) = G_c(h,k)\,.$$

In this case the operator field L inducing the metric is invariant under reparameterizations.

Remark. The reason for this restriction is that in applications to shape analysis one is mostly interested not in the curve c itself, but only in the shape that the curve represents. Two curves c and e represent the same shape, if they differ by a reparameterization or relabelling of the points, i.e., $c = e \circ \varphi$. Thus one passes to the quotient

$$\operatorname{Imm}(M,\mathbb{R}^2) \to B_i(M,\mathbb{R}^2) := \operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Diff}(M)\,,$$

of shapes modulo reparameterizations. The quotient $B_i(M,\mathbb{R}^2)$ is an orbifold; see [25]. Up to technicalities, equivalence classes $[c] \in B_i(M,\mathbb{R}^2)$ correspond to the image $c(M) \subset \mathbb{R}^2$ of the curve. Given a reparameterization invariant metric on $\operatorname{Imm}(M,\mathbb{R}^2)$, it induces a metric on $B_i(M,\mathbb{R}^2)$, such that the projection map is a Riemannian submersion. See [8] for details.

The following lemma provides a useful way to calculate the geodesic equation of such a metric.

Lemma 2.4.1. *Let $\mathcal{C}(M,\mathbb{R}^2) = \operatorname{Imm}(M,\mathbb{R}^2) \cap V \subset \operatorname{Imm}(M,\mathbb{R}^2)$ be any of the sections mentioned in 2.2, where V is the corresponding closed linear subspace of $C^\infty(M,\mathbb{R}^2)$. Let $G = G^L$ be a weak Riemannian metric on $\mathcal{C}(M,\mathbb{R}^2)$ induced by an operator field L via*

$$G_c^L(h,k) = \int_M \langle h, L_c k\rangle \,\mathrm{d}s\,,$$

for $c \in \mathcal{C}(M,\mathbb{R}^2)$ and $h,k \in T_c\mathcal{C}(M,\mathbb{R}^2)$. Let $\bar{L}_c = L_c(\quad) \otimes \mathrm{d}s$ and let $H_c(c_t,c_t) \in (T_c\mathcal{C}(M,\mathbb{R}^2))'$ be defined via $D_{c,m}(G_c(h,h)) = \langle H_c(h,h), m\rangle_V$, where $\langle\ ,\ \rangle_V$ denotes the dual pairing between V' and V.

Then the geodesic equation exists if and only if $\frac{1}{2}H_c(h,h) - (D_{c,h}L_c)(h)$ is in the image of $\bar{L}_c$ and $(c,h) \mapsto \bar{L}_c^{-1}\big(\frac{1}{2}H_c(h,h) - (D_{c,h}L_c)(h)\big)$ is smooth. The geodesic equation can be written as:

$$\boxed{\begin{aligned} p &= L_c c_t \otimes \mathrm{d}s = \bar{L}_c(c_t)\\ p_t &= \frac{1}{2}H_c(c_t,c_t)\end{aligned}} \qquad \text{or} \qquad \boxed{c_{tt} = \frac{1}{2}\bar{L}_c^{-1}(H_c(c_t,c_t)) - \bar{L}_c^{-1}\big(\partial_t(\bar{L}_c)(c_t)\big)}$$

This lemma is an adaptation to the special situation here of the more general result [23, Sect. 2.4]; there a *robust weak Riemannian manifold* is one where the conclusion of this lemma holds together with a compatibility of the chart structure with the G^L-completions of all tangent spaces. Compare also with the version for diffeomorphism groups [5, Sect. 3.2]. In the sequel we shall compute $H_c(h,h)$ in many situations, but often we shall not check that it lies in the image of $\bar{L}$; the latter will follow directly from the representation of the metric via R-transforms.

Proof. To calculate the first variation of the energy we consider a one-parameter family of curves $c : (-\varepsilon,\varepsilon) \times [0,1] \times M \to \mathbb{R}^2$ with fixed endpoints. The variational parameter will be denoted by $\sigma \in (-\varepsilon,\varepsilon)$ and the time-parameter by $t \in [0,1]$. We calculate:

$$\begin{aligned}
\partial_\sigma \frac{1}{2}\int_0^1 G_c(c_t,c_t)\,\mathrm{d}t &= \frac{1}{2}\int_0^1 (\partial_\sigma G_c)(c_t,c_t)\,\mathrm{d}t + \int_0^1 G_c(\partial_\sigma c_t, c_t)\,\mathrm{d}t\\
&= \frac{1}{2}\int_0^1 \langle H_c(c_t,c_t), c_\sigma\rangle_V\,\mathrm{d}t + \int_0^1 G_c(\partial_t c_\sigma, c_t)\,\mathrm{d}t\\
&= \frac{1}{2}\int_0^1 \langle H_c(c_t,c_t), c_\sigma\rangle_V\,\mathrm{d}t + \int_0^1 \langle \bar{L}_c(c_t), \partial_t c_\sigma\rangle_V\,\mathrm{d}t\\
&= \frac{1}{2}\int_0^1 \langle H_c(c_t,c_t), c_\sigma\rangle_V\,\mathrm{d}t + 0 - \int_0^1 \langle \partial_t(\bar{L}_c(c_t)), c_\sigma\rangle_V\,\mathrm{d}t\\
&= \int_0^1 G_c^L\Big(\bar{L}_c^{-1}\big(\frac{1}{2}H_c(c_t,c_t) - \partial_t(\bar{L}_c c_t)\big), c_\sigma\Big)\,\mathrm{d}t\\
&= \int_0^1 G_c^L\Big(\bar{L}_c^{-1}\big(\frac{1}{2}H_c(c_t,c_t) - (\partial_t\bar{L}_c)(c_t)\big) - c_{tt}, c_\sigma\Big)\,\mathrm{d}t\,.
\end{aligned}$$ □

2.5. **The L^2-metric on $C^\infty(M,N)$ and $H^k(M,N)$.** To show well-posedness of the geodesic equation we will need to work with the L^2-metric on Sobolev completions of manifolds of mappings. Here we summarise the necessary results.

Let M be a compact manifold with volume form μ and (N,g) a Riemannian manifold. We assume that both M,N are finite dimensional. Let $k > \dim M/2+1$.

Then the k-th order Sobolev completion $H^k(M,N)$ of order $C^\infty(M,N)$ is a Hilbert manifold, and the tangent space is given by

$$T_q H^k(M,N) = \{h \in H^k(M,TN) : \pi \circ h = q\}\,.$$

All results of this section also hold for $k = \infty$, i.e., for the Fréchet manifold $C^\infty(M,N)$.

We consider the weak Riemannian metric on $TH^k(M,N)$:

$$G_q^{L^2}(h,k) = \int_M g_{q(x)}(h(x),k(x))\,\mathrm{d}\mu(x)\,. \tag{6}$$

The following theorem summarizes the properties of G^{L^2}.

Theorem 2.5.1. *Let $k \in \mathbb{N}$ satisfy $k > \dim M/2+1$ and let G^{L^2} be defined by* (6)*.*

(1) *G^{L^2} defines a smooth weak Riemannian metric on $H^k(M,N)$.*
(2) *Let $\exp^g : TN \to N$ be the exponential map on (N,g), defined on a neighbourhood of the zero-section. Then*
$$\exp^{L^2} : X \mapsto \exp^g \circ X$$
is the exponential map $\exp^{L^2} : TH^k(M,N) \to H^k(M,N)$ of the G^{L^2}-metric. It is a C^∞-mapping defined on a neighbourhood of the zero-section.
(3) *Let $\Xi^g : TN \to TTN$ be the geodesic spray of (N,g). Then*
$$\Xi^{L^2} : X \mapsto \Xi^g \circ X$$
is the geodesic spray $\Xi^{L^2} : TH^k(M,N) \to TTH^k(M,N)$ of the G^{L^2}-metric. It is a C^∞-mapping.
(4) *Let $R^g : TN \times TN \times TN \to TN$ be the curvature tensor of (N,g). Then*
$$R^{L^2} : (X,Y,Z) \mapsto R^g \circ (X,Y,Z)$$
is the curvature tensor $R^{L^2} : TH^k \times TH^k \times TH^k \to TH^k$ of the G^{L^2}-metric. It is a C^∞-mapping.

The proof of this theorem can be found in [12, Thm. 9.1], [12, Cor. 9.3], [1, Prop. 2] and [28, Prop. 3.4].

Given a submanifold of $H^k(M,N)$, the smoothness of the induced geodesic spray can be shown using the following theorem.

Theorem 2.5.2. *Let $k \in \mathbb{N}$ be as above and let $\mathscr{M}$ be a smooth submanifold of $H^k(M,N)$, such that the projection $\operatorname{Proj}^{\mathscr{M}} : TH^k(M,N) \restriction \mathscr{M} \to T\mathscr{M}$ is smooth. Then the geodesic spray of the metric G^{L^2} on $\mathscr{M}$ is given by*

$$\Xi^{\mathscr{M}} = \operatorname{Proj}^{\mathscr{M}} \circ \Xi^{L^2}$$

and it is a smooth map.

This theorem is proven in [12, Thm. 11.1].

3. A flat H^2-type metric

3.1. **The metric and its geodesic equation.** In this section we will study an H^2-type metric on $\operatorname{Imm}_{\text{conv}}(M,\mathbb{R}^2)/\operatorname{Mot}$ – the space of strictly convex curves. This metric has vanishing vanishing curvature for $M=[0,2\pi]$ and for $M=S^1$ the space is isometric to a codimension 2 submanifold of a flat space. The metric is given by

$$G_c(h,k)=\int_M \kappa^{-3/2}\langle D_s^2h,n\rangle\langle D_s^2k,n\rangle+\langle D_sh,v\rangle\langle D_sk,v\rangle\,\mathrm{d}s\,. \tag{7}$$

Note that the metric is only defined for strictly convex curves, i.e., those satisfying $\kappa>0$. It is sometimes more convenient to write G via its associated operator L,

$$G_c(h,k)=\int_M\langle L_ch,k\rangle\,\mathrm{d}s=\int_M\langle h,L_ck\rangle\,\mathrm{d}s\,,$$

where $L_c: T_c\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)\to T_c\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)$ is given by

$$L_ch=D_s^2\big(\kappa^{-3/2}\langle D_s^2h,n\rangle n\big)\big)-D_s\big(\langle D_sh,v\rangle v\big)\,.$$

Lemma 3.1.1. *The null space of the bilinear form $G_c(.,.)$ is spanned by constant vector fields and infinitesimal rotations, i.e.,*

$$\ker(G_c)=\big\{h\in T_c\operatorname{Imm}_{\text{conv}}(M,\mathbb{R}^2): h=a+b.Jc,\ a\in\mathbb{R}^2, b\in\mathbb{R}\big\}\ .$$

Proof. The null space of a symmetric bilinear form A on $V\times V$ is the set

$$\ker(A)=\{v\in V: A(v,w)=0,\forall w\in V\}\,.$$

Thus for all h in the kernel of G_c we have $G_c(h,h)=0$. From this we see that for all $h\in\ker(G_c)$ we have $\langle D_sh,v\rangle=0$ and $\langle D_s^2h,n\rangle=0$. If h satisfies the two conditions then we also have $G_c(h,k)=0$ for all $k\in T_c\operatorname{Imm}_{\text{conv}}(M,\mathbb{R}^2)$ and thus we see that the null space of G_c consists exactly of these h with $G_c(h,h)=0$. The condition $\langle D_sh,v\rangle=0$ yields $D_sh=b.n$, with $b\in C^\infty(M,\mathbb{R})$, and the condition $\langle D_s^2h,n\rangle=0$ implies that b is constant. Taking the antiderivative of D_sh we obtain the desired result. ☐

As an immediate consequence of Lem. 3.1.1 we obtain that G_c is a weak Riemannian metric on $\operatorname{Imm}_{\text{conv}}(M,\mathbb{R}^2)/\text{Mot}$. Its geodesic equation is giben by the following theorem.

Theorem 3.1.2. *On the manifold $\operatorname{Imm}_{\text{conv}}(M,\mathbb{R}^2)/\operatorname{Mot}$ of plane parametrized curves modulo Euclidean motions $G_c(.,.)$ defines a weak Riemannian metric. For $M=S^1$ the geodesic equation is given by*

$$\begin{aligned}
p&=Lc_t\otimes\mathrm{d}s\\
&=D_s^2\big(\kappa^{-3/2}\langle D_s^2c_t,n\rangle n\big)-D_s\big(\langle D_sc_t,v\rangle v\big)\otimes\mathrm{d}s\,,\\
p_t&=D_s\Big(\frac12\langle D_sc_t,v\rangle^2v-\langle D_sc_t,n\rangle\langle D_sc_t,v\rangle n-D_s\big(\kappa^{-3/2}\langle D_sc_t,n\rangle\langle D_s^2c_t,n\rangle\big)v\\
&\qquad+\kappa^{-3/2}\langle D_s^2c_t,v\rangle\langle D_s^2c_t,n\rangle n-\frac34D_s\Big(\kappa^{-5/2}\langle D_s^2c_t,n\rangle^2n\Big)\Big)\otimes\mathrm{d}s\,,
\end{aligned}$$

with the additional constraint

$$c_t\in T_c(\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Mot})\cong\{h\in C^\infty(M,\mathbb{R}^2): h(0)=0,\langle D_sh(0),n(0)\rangle=0\}\,.$$

The first part of the theorem applies to both $M=S^1$ and $M=[0,2\pi]$. The geodesic equation on the space of closed curves would additionaly contain a series of boundary terms arising from integrations by parts.

Remark. Note that $L_c : T_c\big(\operatorname{Imm}(M,\mathbb{R}^2)/\mathrm{Mot}\big) \to T_c\big(\operatorname{Imm}(M,\mathbb{R}^2)/\mathrm{Mot}\big)$ is not an elliptic operator, since the highest derivative appears only in the normal direction. Thus we cannot apply the well-posedness results from [26] or [7]. Nevertheless we will show in Sects. 3.4 and 3.5 that the geodesic equation is locally well-posed both on the space of open curves as well as the space of closed curves.

Proof. To calculate the formula for the geodesic equation we use Lem. 2.4.1. Using the variational formulas from Sect. 2.3 we calculate for a fixed vector field h the variation of the components of the metric G:

$$\begin{aligned} D_{c,m}\left(\langle D_s h, v\rangle\right) &= -\langle D_s m, v\rangle\langle D_s h, v\rangle + \langle D_s m, n\rangle\langle D_s h, n\rangle \\ D_{c,m}\left(\langle D_s^2 h, n\rangle\right) &= -\langle D_s h, n\rangle D_s\left(\langle D_s m, v\rangle\right) - \langle D_s^2 h, v\rangle\langle D_s m, n\rangle \\ &\quad - 2\langle D_s^2 h, n\rangle\langle D_s m, v\rangle\,. \end{aligned}$$

Thus we obtain the following expression for the variation of the metric:

$$\begin{aligned} D_{c,m}(G_c(h,h)) = \int_{S^1} & -\langle D_s m, v\rangle\langle D_s h, v\rangle^2 + 2\langle D_s m, n\rangle\langle D_s h, n\rangle\langle D_s h, v\rangle \\ & - 2\kappa^{-3/2}\langle D_s h, n\rangle\langle D_s^2 h, n\rangle D_s\left(\langle D_s m, v\rangle\right) - 2\kappa^{-3/2}\langle D_s^2 h, v\rangle\langle D_s^2 h, n\rangle\langle D_s m, n\rangle \\ & - \frac{3}{2}\kappa^{-5/2}\langle D_s^2 m, n\rangle\langle D_s^2 h, n\rangle^2 \,\mathrm{d}s\,. \end{aligned}$$

We can now calculate $H_c(h,h)$ using a series of integrations by parts:

$$\begin{aligned} H_c(h,h) = D_s\Big(&\langle D_s h, v\rangle^2 v - 2\langle D_s h, n\rangle\langle D_s h, v\rangle n - 2D_s\big(\kappa^{-3/2}\langle D_s h, n\rangle\langle D_s^2 h, n\rangle\big)v \\ &+ 2\kappa^{-3/2}\langle D_s^2 h, v\rangle\langle D_s^2 h, n\rangle n - \frac{3}{2}D_s\Big(\kappa^{-5/2}\langle D_s^2 h, n\rangle^2 n\Big)\Big) \otimes \mathrm{d}s\,. \end{aligned}$$

The existence of the geodesic equation, i.e., the invertibility of $\bar{L}$ will follow from the representation of the metric via the R-transform. The R-transform is an isometry from $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$ onto its image $\operatorname{im}(R)$ and the image is a smooth submanifold of the space $(C^\infty(M,\mathbb{R}^2), G^{L^2})$ with a smooth orthogonal projection $TC^\infty(M,\mathbb{R}^2) \restriction \operatorname{im}(R) \to T\operatorname{im}(R)$. Theorem 2.5.2 shows that the geodesic spray of the L^2-meric restricted to $\operatorname{im}(R)$ exists and is smooth and thus we can pull it back via R to $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$. Hence the geodesic equation exists on $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$. □

3.2. **The R-transform.** Consider the map

$$R : \left\{ \begin{array}{ccc} \operatorname{Imm}_{\mathrm{conv}}(M,\mathbb{R}^2)/\mathrm{Mot} & \to & C^\infty(M,\mathbb{R}^2) \\ c & \mapsto & \sqrt{|c'|}\,(2, 4\kappa^{1/4}) \end{array} \right. , \tag{8}$$

and equip the space $C^\infty(M,\mathbb{R}^2)$ with the L^2-Riemannian metric,

$$G_q^{L^2}(h,k) = \int_M \langle h(\theta), k(\theta)\rangle\,\mathrm{d}\theta\,. \tag{9}$$

Here $q \in C^\infty(M,\mathbb{R}^2)$ and $h, k \in T_q C^\infty(M,\mathbb{R}^2)$. Note that the Riemannian metric G^{L^2} doesn't depend on the point q. The space $(C^\infty(M,\mathbb{R}^2), G^{L^2})$ is therefore a flat Riemannian manifold.

Theorem 3.2.1. *The map*

$$R\,:\,(\operatorname{Imm}_{\mathrm{conv}}(M,\mathbb{R}^2)/\operatorname{Mot}, G) \to (C^\infty(M,\mathbb{R}^2), G^{L^2})$$

is an injective isometry between weak Riemannian manifolds, i.e.,

$$G_c(h,k) = G^{L^2}_{R(c)}(dR(c).h, dR(c).k) = \int_M \langle dR(c).h(\theta), dR(c).k(\theta)\rangle \,\mathrm{d}\theta$$

for $h, k \in T_c \operatorname{Imm}_{\text{conv}}(M, \mathbb{R}^2)/\operatorname{Mot}$.

Proof. Using formulas (4) and (5) we obtain

$$D_{c,h}\left(\sqrt{|c'|}\right) = \tfrac{1}{2}\langle D_s h, v\rangle \sqrt{|c'|}$$
$$D_{c,h}\left(\kappa^{1/4}\sqrt{|c'|}\right) = \tfrac{1}{4}\kappa^{-3/4}\langle D_s^2 h, n\rangle \sqrt{|c'|}\,,$$

and thus the derivative of the R-transform is

$$dR(c).h = \left(\langle D_s h, v\rangle \sqrt{|c'|},\ \kappa^{-3/4}\langle D_s^2 h, n\rangle \sqrt{|c'|}\right)\,.$$

Hence

$$\langle dR(c).h, dR(c).h\rangle = \left(\langle D_c h, v\rangle^2 + \kappa^{-3/2}\langle D_s^2 h, n\rangle^2\right)|c'|\,,$$

and the first statement of the theorem follows.

To show that R is injective we recall that one can recover the turning angle up to a constant from the curvature function κ and the arclength $|c'|$ via $D_s\alpha = \kappa$, or equivalently

$$\alpha(\theta) - \alpha(0) = \int_0^\theta \kappa |c'| \,\mathrm{d}\theta\,.$$

Choosing a different value for $\alpha(0)$ results in a rotation of the curve. From the turning angle α and the arclength $|c'|$ one can reconstruct the curve c up to a translation by integration: $c(\theta) - c(0) = \int_0^\theta e^{i\alpha}|c'|\,\mathrm{d}\theta$. □

3.3. **A motivation for this metric.** The choice of the factor $\kappa^{-3/2}$ in front of the term $\langle D_s^2 h, n\rangle^2$ in (7) appears to be arbitrary. One possibility to construct a second order Sobolev type metric on $\operatorname{Imm}(M, \mathbb{R}^2)$ as the pullback of the flat G^{L^2}-metric, is to use an R-transform, that has a component of the form $R^j(c) = \sqrt{|c'|}f(\kappa)$, where $f : \mathbb{R} \to \mathbb{R}$ is some smooth function. The derivative of R^j with respect to c is

$$dR^j(c)h = \sqrt{|c'|}\left(f'(\kappa)\langle D_c^2 h, n\rangle - 2\kappa f'(\kappa)\langle D_c h, v\rangle + \tfrac{1}{2}f(\kappa)\langle D_c h, v\rangle\right)\,.$$

The pullback metric would then contain a term

$$\int_{S^1}\left(f'(\kappa)\langle D_c^2 h, n\rangle - 2\kappa f'(\kappa)\langle D_c h, v\rangle + \tfrac{1}{2}f(\kappa)\langle D_c h, v\rangle\right)^2 \mathrm{d}s\,.$$

In order to avoid cross-derivatives in the metric, the function $f(\kappa)$ needs to satisfy

$$\frac{1}{2}f(\kappa) = 2\kappa f'(\kappa)\,.$$

Solutions to this ODE are given by

$$f(\kappa) = \begin{cases} C\kappa^{1/4}, & \kappa > 0 \\ -C(-\kappa)^{1/4}, & \kappa < 0 \end{cases},$$

with $C \in \mathbb{R}$ and the corresponding R-transform with $R^j(c) = 4\sqrt{|c'|}\kappa^{1/4}$ induces the following term in the metric

$$\int_{S^1} \kappa^{-3/2}\langle D_c^2 h, n\rangle^2 \,\mathrm{d}s\,.$$

Thus the factor $\kappa^{-3/2}$ is the unique choice to obtain a second order Sobolev type metric, which is flat on the space $\operatorname{Imm}_{\text{conv}}(M,\mathbb{R}^2)/\operatorname{Mot}$.

3.4. **The space of open curves.** The image of the R-transform on the space $\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)$ of open convex curves is the set of all $\mathbb{R}^2_{>0}$-valued functions, which is an open set in $C^\infty([0,2\pi],\mathbb{R}^2)$.

Theorem 3.4.1. *The R-transform*

$$R : \operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot} \to C^\infty([0,2\pi],\mathbb{R}^2_{>0}) \underset{\text{open}}{\subset} C^\infty([0,2\pi],\mathbb{R}^2)$$

is a diffeomorphism and its inverse is given by

$$R^{-1} : \begin{cases} \operatorname{im}(R)_{\text{op}} & \to \quad \operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot} \\ q & \mapsto \quad 2^{-2}\int_0^\theta q_1^2 \exp\left(i\int_0^\sigma 2^{-6}q_1^{-2}q_2^4\,\mathrm{d}\tau\right)\mathrm{d}\theta \end{cases} . \tag{10}$$

The space $(\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, G)$ *is a flat and geodesically convex Riemannian manifold.*

Proof. It is shown in Thm. 3.2.1 that the R-transform is injective. The surjectivity will follow directly from the inversion formula. For the inversion formula note that we can reconstruct

$$|c'| = 2^{-2}q_1^2 \qquad \kappa = (\tfrac{1}{2}q_1^{-1}q_2)^4 \qquad \alpha' = 2^{-6}q_1^{-2}q_2^4$$

and that

$$c(\theta) = c(0) + \int_0^\theta \exp\left(i\alpha(0)\right)\exp\left(i\int_0^\sigma \alpha(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\sigma\,.$$

Since the initial conditions $c(0)$ and $\alpha(0)$ are unspecified, this determines the curve up to translations and rotations. The flatness follows from Thm. 2.5.1, since $\mathbb{R}^2_{>0}$ is flat. Given two curves c_0 and c_1, the minimizing geodesic connecting them is given by

$$c(t,\theta) = R^{-1}\left(tR(c_1) + (1-t)R(c_0)\right)(\theta)\,,$$

thus showing that $\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$ is geodesically convex. □

Remark. Since $\mathbb{R}^2_{>0}$ is geodesically incomplete, the same is true for the space $\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$. A geodesic will leave the space, when it fails to be an immersion, i.e., $|c'(t,\theta)| = 0$ for some (t,θ), or when it stops being convex, i.e., $\kappa(t,\theta) = 0$. While the term $\kappa^{-3/2}\langle D_s^2 h, n\rangle^2$ in the metric penalizes a curve from straightening out, it is not strong enough to prevent it.

The R-transform allows us to give explicit formulas for geodesics and for the geodesic distance.

Theorem 3.4.2. *Given two curves* $c_0, c_1 \in \operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$ *the unique geodesic connecting them is given by*

$$c(t,\theta) = R^{-1}\left(tR(c_1) + (1-t)R(c_0)\right)(\theta)\,,$$

and their geodesic distance is

$$\operatorname{dist}^G_{\text{op}}(c_0,c_1)^2 = \int_0^{2\pi} 16\left(\sqrt{|c_1'|}\kappa_1^{1/4} - \sqrt{|c_0'|}\kappa_0^{1/4}\right)^2 + 4\left(\sqrt{|c_1'|} - \sqrt{|c_0'|}\right)^2 \mathrm{d}\theta\,.$$

Proof. This follows from Thm. 3.4.1. □

Remark. The formula for the geodesic distance implies in particular that the following functions are Lipschitz continuous

$$\sqrt{|c'|} : \Big(\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, \operatorname{dist}^G_{\text{op}}\Big) \to L^2([0,2\pi],\mathbb{R})$$
$$\kappa^{1/4}\sqrt{|c'|} : \Big(\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, \operatorname{dist}^G_{\text{op}}\Big) \to L^2([0,2\pi],\mathbb{R})\,.$$

From the Lipschitz continuity of $\sqrt{|c'|}$ and the identity $\sqrt{\ell_c} = \|\sqrt{|c'|}\|_{L^2}$ we can then conclude that

$$\sqrt{\ell_c} : \Big(\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, \operatorname{dist}^G_{\text{op}}\Big) \to \mathbb{R}$$

is also Lipschitz continuous. An immediate consequence of this is the following lower bound for the geodesic distance:

$$\operatorname{dist}^G_{\text{op}}(c_0,c_1) \geq 2\left|\sqrt{\ell_{c_1}} - \sqrt{\ell_{c_0}}\right|\,.$$

3.5. **The space of closed curves.** When we restrict our attention to the space $\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ of closed, strictly convex curves, the image of the R-transform is no longer an open subset of $C^\infty(S^1,\mathbb{R}^2)$.

Define the following function $H_{\text{cl}} : \operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)/\operatorname{Mot} \to \mathbb{R}^2$, which measures how far away the preimage of q is from being a closed curve:

$$H_{\text{cl}}(q) = \int_0^{2\pi} q_1(\theta)^2 \exp(i\alpha(q)(\theta))\,\mathrm{d}\theta\,,\quad \alpha(q)(\theta) = 2^{-6}\int_0^\theta q_1(\sigma)^{-2}q_2(\sigma)^4\,\mathrm{d}\sigma\,.$$

The gradients of the components of H_{cl} with respect to the G^{L^2}-metric are

$$\operatorname{grad} H^1_{\text{cl}} = \begin{pmatrix} 2q_1\cos\alpha(q) + 2^{-5}q_2^4q_1^{-3}\int_\theta^{2\pi} q_1^2\sin\alpha(q)\,\mathrm{d}\sigma \\ -2^{-4}q_2^3q_1^{-2}\int_\theta^{2\pi} q_1^2\sin\alpha(q)\,\mathrm{d}\sigma \end{pmatrix} \tag{11a}$$

$$\operatorname{grad} H^2_{\text{cl}} = \begin{pmatrix} 2q_1\sin\alpha(q) - 2^{-5}q_2^4q_1^{-3}\int_\theta^{2\pi} q_1^2\cos\alpha(q)\,\mathrm{d}\sigma \\ 2^{-4}q_2^3q_1^{-2}\int_\theta^{2\pi} q_1^2\cos\alpha(q)\,\mathrm{d}\sigma \end{pmatrix}. \tag{11b}$$

The function H_{cl} characterizes the image of the R-transform.

Lemma 3.5.1. *The image of* $\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ *under the R-transform is*

$$\operatorname{im}(R)_{\text{cl}} = \left\{q \in C^\infty(S^1,\mathbb{R}^2_{>0}) : H_{\text{cl}}(q) = 0\right\}\,.$$

It is a splitting submanifold of $C^\infty(S^1,\mathbb{R}^2_{>0})$ of codimension 2.

For $q \in \operatorname{im}(R)_{\text{cl}}$ the orthogonal complement of $T_q\operatorname{im}(R)_{\text{cl}}$ with respect to the G^{L^2}-metric is spanned by $\operatorname{grad} H^1_{\text{cl}}(q), \operatorname{grad} H^2_{\text{cl}}(q)$*, given in* (11).

Proof. To characterize the image $\operatorname{im}(R)_{\text{cl}}$ we recall the inversion formula (10) from Thm. 3.4.1. For $q \in C^\infty(\mathbb{R}_{>0}\times\mathbb{R})$ it follows immediately that $R^{-1}(q)$ is a closed curve if and only if $H_{\text{cl}}(q) = 0$.

We now show that $\operatorname{im}(R)_{\text{cl}}$ is a splitting submanifold. Fix $q_0 \in \operatorname{im}(R)_{\text{cl}}$ and define the codimension 2 closed linear subspace

$$T_{q_0}(\operatorname{im}(R)_{\text{cl}}) := \big(\mathbb{R}.\operatorname{grad} H^1_{\text{cl}}(q_0) + \mathbb{R}.\operatorname{grad} H^2_{\text{cl}}(q_0)\big)^{\perp, G^{L^2}} \subset C^\infty(S^1,\mathbb{R}^2).$$

We consider the affine isomorphism

$$A_{q_0} : \begin{cases} T_{q_0}(\operatorname{im}(R)_{\text{cl}}) \times \mathbb{R}^2 & \to \quad C^\infty(S^1,\mathbb{R}^2) \\ (h,x,y) & \mapsto \quad q_0 + x.\operatorname{grad} H^1_{\text{cl}}(q_0) + y.\operatorname{grad} H^2_{\text{cl}}(q_0) \end{cases}.$$

Then the derivative $D_2(H_{\text{cl}} \circ A_{q_0})(h,x,y)$ of the smooth mapping

$$H_{\text{cl}} \circ A_{q_0} : T_{q_0}(\operatorname{im}(R)_{\text{cl}}) \times \mathbb{R}^2 \to \mathbb{R}^2$$

is bounded and invertible for all (h,x,y) near $(0,0,0)$, and thus $\operatorname{im}(R)_{\text{cl}} = H_{\text{cl}}^{-1}(0)$ is a smooth splitting submanifold by the implicit function theorem with parameters in a convenient vector space; see [33, Lem. 1], [17] or [15].

Since $\operatorname{grad} H^1_{\text{cl}}(q)$ and $\operatorname{grad} H^2_{\text{cl}}(q)$ are linearly independent in $C^\infty(S^1,\mathbb{R}^2)$ for any $q \in C^\infty(S^1, {\mathbb{R}_{>0}}^2)$, they form a basis of $(T_q \operatorname{im}(R)_{\text{cl}})^\perp$. □

Remark. Since the tangent space $T_q \operatorname{im}(R)_{\text{cl}}$ has codimension 2, the orthogonal projection $\operatorname{Proj}^{\text{im}} : TC^\infty(S^1,\mathbb{R}^2) \restriction \operatorname{im}(R)_{\text{cl}} \to T\operatorname{im}(R)_{\text{cl}}$ is given by

$$\operatorname{Proj}^{\text{im}}(q).h = h - \langle h, w^1(q)\rangle w^1(q) - \langle h, w^2(q)\rangle w^2(q)\,, \tag{12}$$

where $w^1(q), w^2(q)$ is an orthonormal basis of $(T_q \operatorname{im}(R)_{\text{cl}})^\perp$. In particular $\operatorname{Proj}^{\text{im}}$ is smooth.

Using the orthonormal basis $w^1(q), w^2(q)$ of $\operatorname{im}(R)^\perp_{\text{cl}}$ and the Gauß equation one can calculate the curvature of $\operatorname{im}(R)_{\text{cl}}$ and hence also of $\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)/\operatorname{Mot}$. This has been done for a different metric in [4].

The geodesic equation on $\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ is well-posed in appropriate Sobolev completions. We refer to Sect. 7 and in particular to Sect. 7.2 for a detailed discussion and proofs. The spaces $\operatorname{Imm}^{j,k}(S^1,\mathbb{R}^2)$ are defined in (29).

Theorem 3.5.2. *For $k \geq 2$ and initial conditions $(c_0,u_0) \in T\operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^2)$, the geodesic equation has solutions in $\operatorname{Imm}^{j+1,j+2}(S^1,\mathbb{R}^2)$ for each $2 \leq j \leq k$. The solutions depend C^∞ on t and the initial conditions and the domain of existence is independent of j.*

In particular for smooth initial conditions $(c_0,u_0) \in T\operatorname{Imm}(S^1,\mathbb{R}^2)$ the geodesic equation has smooth solutions.

Remark. Since $\operatorname{im}(R)_{\text{cl}} \subset \operatorname{im}(R)_{\text{op}}$, the geodesic distance functions satisfy

$$\operatorname{dist}^G_{\text{op}}(c_0,c_1) \leq \operatorname{dist}^G_{\text{cl}}(c_0,c_1)\,.$$

Thus, the remark after Thm. 3.4.2 also holds for the space $\operatorname{Imm}_{\text{conv}}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ of closed curves, i.e., the functions $\sqrt{|c'|}$, $\kappa^{1/4}\sqrt{|c'|}$ and $\sqrt{\ell_c}$ are Lipschitz continuous with respect to the geodesic distance.

4. The second metric

4.1. **The metric and its geodesic equation.** The metric studied in the previous section is defined only for strictly convex curves. Consider the following related metric

$$G_c(h,k) = \int_M \langle D_s h, v\rangle\langle D_s k, v\rangle + \langle D_s^2 h, n\rangle\langle D_s^2 k, n\rangle \,\mathrm{d}s\,, \tag{13}$$

with $c \in \operatorname{Imm}(M,\mathbb{R}^2)$ and $h,k \in T_c\operatorname{Imm}(M,\mathbb{R}^2)$. This metric is defined for all curves. After integrating the expression of the metric by parts

$$\begin{aligned} G_c(h,k) &= \int_{S^1} \langle D_s h, v\rangle\langle D_s k, v\rangle + \langle D_s^2 h, n\rangle\langle D_s^2 k, n\rangle \,\mathrm{d}s \\ &= \int_{S^1} -\langle k, D_s\left(\langle D_s h, v\rangle v\right)\rangle + \langle k, D_s^2\left(\langle D_s^2 h, n\rangle n\right)\rangle \,\mathrm{d}s\,. \end{aligned}$$

we obtain the associated operator field L of the metric on the space $\operatorname{Imm}(S^1,\mathbb{R}^2)$ of closed curves,

$$L_c h = D_s^2\big(\langle D_s^2 h, n\rangle n\big) - D_s\left(\langle D_s h, v\rangle v\right)\,.$$

The null space of G_c is the same as in Sect. 3.

Lemma 4.1.1. *The null space of the bilinear form $G_c(.,.)$ is spanned by constant vector fields and infinitesimal rotations, i.e.,*

$$\ker(G_c) = \big\{h \in T_c \operatorname{Imm}(M,\mathbb{R}^2) : h = a + b.Jc,\ a \in \mathbb{R}^2, b \in \mathbb{R}\big\}\,.$$

Proof. See the proof of Lem. 3.1.1. □

It follows from this lemma that G_c is a Riemannian metric on $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$. We will use Lem. 2.4.1 to calculate its geodesic equation.

Theorem 4.1.2. *On the manifold $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$ of plane parametrized curves modulo Euclidean motions $G_c(.,.)$ defines a weak Riemannian metric. For $M = S^1$ the geodesic equation is given by*

$$\begin{aligned}
p &= Lc_t \otimes \mathrm{d}s = \big(D_s^2\big(\langle D_s^2 c_t, n\rangle n\big) - D_s\left(\langle D_s c_t, v\rangle v\right)\big) \otimes \mathrm{d}s\\
p_t &= \frac{1}{2} D_s\Big(\langle D_s c_t, v\rangle^2 v - 2\langle D_s c_t, n\rangle\langle D_s c_t, v\rangle n - 2D_s\big(\langle D_s c_t, n\rangle\langle D_s^2 c_t, n\rangle\big)v\\
&\qquad + 2\langle D_s^2 c_t, v\rangle\langle D_s^2 c_t, n\rangle n + 3\langle D_s^2 c_t, n\rangle^2 n\Big) \otimes \mathrm{d}s\,,
\end{aligned}$$

with the additional constraint

$$c_t \in T_c(\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Mot}) \cong \{h \in C^\infty(M,\mathbb{R}^2) : h(0) = 0, \langle D_s h(0), n(0)\rangle = 0\}\,.$$

Remark. Similarily as in Sect. 3, the operator $L_c : T_c\big(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}\big) \to T_c\big(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}\big)$ is not an elliptic operator, since the highest derivative appears only in the normal direction. Again we cannot apply the well-posedness results from [26] or [7]. Instead we will show in Sects. 4.3 and 4.4, that the geodesic equation is locally well-posed both on the space of open and closed curves.

Proof. To apply Lem. 2.4.1 we need to compute the H_c-gradient of the metric. Using the variational formulae from Sect. 2.3 we first calculate the variation of the metric

$$\begin{aligned}
D_{c,m}(G_c(h,h)) &= \int_{S^1} -\langle D_s m, v\rangle\langle D_s h, v\rangle^2 + 2\langle D_s m, n\rangle\langle D_s h, n\rangle\langle D_s h, v\rangle\\
&\quad - 2\langle D_s h, n\rangle\langle D_s^2 h, n\rangle D_s\left(\langle D_s m, v\rangle\right) - 2\langle D_s^2 h, v\rangle\langle D_s^2 h, n\rangle\langle D_s m, n\rangle\\
&\quad - 3\langle D_s^2 h, n\rangle^2\langle D_s m, v\rangle\,\mathrm{d}s\,,
\end{aligned}$$

and we integrate to obtain

$$\begin{aligned}
H_c(h,h) &= D_s\Big(\langle D_s h, v\rangle^2 v - 2\langle D_s h, n\rangle\langle D_s h, v\rangle n - 2D_s\big(\langle D_s h, n\rangle\langle D_s^2 h, n\rangle\big)v\\
&\qquad + 2\langle D_s^2 h, v\rangle\langle D_s^2 h, n\rangle n + 3\langle D_s^2 h, n\rangle^2 n\Big) \otimes \mathrm{d}s\,.
\end{aligned}$$

Regarding the existence of the geodesic equation, see the proof of Thm. 3.1.2. □

4.2. **The R-transform.** Consider the map

$$R : \left\{ \begin{array}{ccc} \operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot} & \to & C^\infty(M,\mathbb{R}^2) \\ c & \mapsto & (\sqrt{|c'|}, \kappa|c'|^2) \end{array} \right. \tag{14}$$

On $\mathbb{R}^2$ we define the following Riemannian metric

$$g_{(q_1,q_2)} = \begin{pmatrix} 4 & 0 \\ 0 & q_1^{-6} \end{pmatrix},$$

and we equip the space $C^\infty(M,(\mathbb{R}^2,g))$ with the L^2-metric,

$$G_q^{L^2}(h,k) = \int_M g_{q(\theta)}(h(\theta),k(\theta))\,\mathrm{d}\theta\,. \tag{15}$$

Here $q \in C^\infty(M,\mathbb{R}^2)$ and $h,k \in T_qC^\infty(M,\mathbb{R}^2)$. Note that as opposed to (9) the metric (15) does depend on the footpoint q. We will sometimes write $G_q^{L^2,g}$ to emphasize this dependence on the metric g. Since $(\mathbb{R}^2,g)$ is not a flat Riemannian manifold, neither is $(C^\infty(M,(\mathbb{R}^2,g)),G^{L^2})$.

Theorem 4.2.1. *With the metrics g and G^{L^2} defined as above, the map*

$$R\,:\,(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot},G) \to (C^\infty(M,(\mathbb{R}^2,g)),G^{L^2})$$

is an injective isometry between weak Riemannian manifolds, i.e.,

$$G_c(h,k) = G^{L^2}_{R(c)}(dR(c).h,dR(c).k) = \int_M g_{R(c)(\theta)}(dR(c).h(\theta),dR(c).k(\theta))\,\mathrm{d}\theta\,,$$

for $h,k \in T_c\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Mot}$.

Proof. Using the formulas (4) and (5) we obtain

$$\begin{aligned} D_{c,h}\left(\sqrt{|c'|}\right) &= \tfrac{1}{2}\langle D_sh,v\rangle\sqrt{|c'|} \\ D_{c,h}\left(\kappa|c'|^2\right) &= \langle D_s^2h,n\rangle|c'|^2 - 2\kappa\langle D_sh,v\rangle|c'|^2 + 2\kappa\langle D_sh,v\rangle|c'|^2 \\ &= \langle D_s^2h,n\rangle|c'|^2\,, \end{aligned}$$

and thus the derivative of the R-transform is

$$dR(c).h = \left(\tfrac{1}{2}\langle D_sh,v\rangle\sqrt{|c'|},\langle D_s^2h,n\rangle|c'|^2\right).$$

Hence

$$g_{R(c)}(dR(c).h,dR(c).h) = \left(\langle D_ch,v\rangle^2 + \langle D_s^2h,n\rangle^2\right)|c'|$$

and the first statement of the theorem follows. Injectivity has already been shown in the proof of Thm. 3.2.1. □

4.3. **The space of open curves.** The image of the R-transform on the space $\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$ of open curves is the set of all $\mathbb{R}_{>0}\times\mathbb{R}$-valued functions, which is an open subset of $C^\infty([0,2\pi],\mathbb{R}^2)$.

Theorem 4.3.1. *With the metrics g and G^{L^2} defined as above, the R-transform*

$$R\,:\,\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot} \to C^\infty([0,2\pi],\mathbb{R}_{>0}\times\mathbb{R}) \underset{\text{open}}{\subset} C^\infty([0,2\pi],\mathbb{R}^2)$$

is a diffeomorphism. Its inverse is given by

$$R^{-1} : \left\{ \begin{array}{ccc} C^\infty([0,2\pi],\mathbb{R}_{>0}\times\mathbb{R}) & \to & \operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot} \\ q & \mapsto & \int_0^\theta q_1^2\exp\left(i\int_0^\sigma q_2q_1^{-2}\,\mathrm{d}\tau\right)\,\mathrm{d}\sigma \end{array} \right. .$$

The space $(\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, G^{L^2,g})$ *is a geodesically convex Riemannian manifold.*

Proof. The characterization of the image and the inversion formula can be proven as in Thm. 3.4.1 using

$$|c'| = q_1^2 \qquad \kappa = q_1^{-4} q_2 \qquad \alpha' = q_1^{-2} q_2\,.$$

It is shown in Sect. 4.5 that $(\mathbb{R}_{>0}\times\mathbb{R}, g)$ is geodesically convex and that the geodesic connecting two points is unique. Given two curves c_0 and c_1, the minimizing geodesic connecting them is described in (16). Thus $\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$ is geodesically convex. □

Remark. Since $\mathbb{R}_{>0}\times\mathbb{R}$ is geodesically incomplete (see Sect. 4.5), so is the space $\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$. A geodesic will leave the space, when it fails to be an immersion, i.e., when $c'(t,\theta)=0$ for some (t,θ).

The R-transform allows us to give formulas for geodesics and a lower bound for the geodesic distance.

Theorem 4.3.2. *Given two curves* $c_0, c_1 \in \operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$ *the unique geodesic connecting them is given by*

$$c(t,\theta) = R^{-1}(q(t))(\theta)\,, \tag{16}$$

where for each $\theta\in[0,2\pi]$ *the curve* $t\mapsto q(\cdot,\theta)$ *is the geodesic connecting* $R(c_0)(\theta)$ *and* $R(c_1)(\theta)$. *The geodesic distance is bounded from below by*

$$\begin{aligned}&\operatorname{dist}^G_{\text{op}}(c_0,c_1)^2\\ &\geq \int_0^{2\pi} \frac{2^{15/8}\left(|c_1'|^2\kappa_1 - |c_0'|^2\kappa_0\right)^2}{\left(|c_0'|^4 + |c_1'|^4 + \frac{1}{2A}\left|\,|c_1'|^2\,\kappa_1 - |c_0'|^2\kappa_0\right|\right)^{3/2}} + 16\left(\sqrt{|c_1'|} - \sqrt{|c_0'|}\right)^2 \mathrm{d}\theta\,,\end{aligned}$$

where A *is the constant*

$$A = \int_0^1 \frac{z^6\,\mathrm{d}x}{\sqrt{1-z^6}} = 0.30358\ldots\,.$$

Proof. The proof of this theorem follows directly from the analysis of the finite dimensional metric g, see Sect. 4.5. □

Remark. The formula for the geodesic distance implies in particular that the function

$$\sqrt{|c'|} : \left(\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, \operatorname{dist}^G_{\text{op}}\right) \to L^2([0,2\pi],\mathbb{R})$$

is Lipschitz continuous, Since $\sqrt{\ell_c} = \|\sqrt{|c'|}\|_{L^2}$ we can then conclude that

$$\sqrt{\ell_c} : \left(\operatorname{Imm}_{\text{conv}}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, \operatorname{dist}^G_{\text{op}}\right) \to \mathbb{R}$$

is also Lipschitz continuous. This implies the following lower bound for the geodesic distance:

$$\operatorname{dist}^G_{\text{op}}(c_0,c_1) \geq 4\left|\sqrt{\ell_{c_1}} - \sqrt{\ell_{c_0}}\right|\,.$$

The space $(\operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}, G^{L^2})$ is not flat. The representation via R-transform allows us to calculate its curvature.

Theorem 4.3.3. *The sectional curvature of the plane spanned by h, k is given by*

$$\mathrm{k}_c(P(h,k)) = \frac{-3\int_0^{2\pi}\left(\langle D_s h, v\rangle\langle D_s^2 k, n\rangle - \langle D_s k, v\rangle\langle D_s^2 h, n\rangle\right)^2 \,\mathrm{d}s}{G_c(h,h)G_c(k,k) - G_c(h,k)^2}\,,$$

with $h, k \in T_c \operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{Mot}$. In particular the sectional curvature is non-positive.

Proof. The sectional curvature of $(C^\infty([0,2\pi],\mathbb{R}^2), G^{L^2})$ is the integral over the pointwise sectional curvatures, see Thm. 2.5.1. The statement now follows from the curvature formulas for g given in Sect. 4.5. □

4.4. **The space of closed curves.** The image of the space $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ of closed curves under the R-transform is not an open subset of $C^\infty(S^1,\mathbb{R}^2)$.

Consider the following function $H_{\mathrm{cl}} : \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Mot} \to \mathbb{R}^2$, which measures how far away the preimage of q is from being closed:

$$H_{\mathrm{cl}}(q) = \int_0^{2\pi} q_1(\theta)^2 \exp(i\alpha(q)(\theta))\,\mathrm{d}\theta\,, \qquad \alpha(q)(\theta) = \int_0^\theta q_2(\sigma)q_1(\sigma)^{-2}\,\mathrm{d}\sigma\,.$$

The gradients of the components of H_{cl} with respect to the G^{L^2}-metric are

$$\operatorname{grad}(H^1_{\mathrm{cl}}) = \begin{pmatrix} \frac{1}{2}q_1\cos\alpha(q) + \frac{1}{2}q_2q_1^{-3}\int_\theta^{2\pi} q_1^2 \sin\alpha(q)\,\mathrm{d}\sigma \\ -q_1^4\int_\theta^{2\pi} q_1^2\sin\alpha(q)\,\mathrm{d}\sigma \end{pmatrix} \tag{17a}$$

$$\operatorname{grad}(H^2_{\mathrm{cl}}) = \begin{pmatrix} \frac{1}{2}q_1\sin\alpha(q) - \frac{1}{2}q_2q_1^{-3}\int_\theta^{2\pi} q_1^2 \cos\alpha(q)\,\mathrm{d}\sigma \\ +q_1^4\int_\theta^{2\pi} q_1^2\cos\alpha(q)\,\mathrm{d}\sigma \end{pmatrix}. \tag{17b}$$

The function H_{cl} permits us to characterize the image of the R-transform.

Lemma 4.4.1. *The image of* $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ *under the R-transform is given by*

$$\operatorname{im}(R)_{\mathrm{cl}} = \left\{q \in C^\infty(S^1, \mathbb{R}_{>0}\times\mathbb{R}) : H_{\mathrm{cl}}(q) = 0\right\}\,.$$

It is a splitting submanifold of $C^\infty(\mathbb{R}_{>0}\times\mathbb{R})$ of codimension 2.

For $q \in \operatorname{im}(R)_{\mathrm{cl}}$, the orthogonal complement of $T_q\operatorname{im}(R)_{\mathrm{cl}}$ with respect to the G^{L^2}-metric is spanned by $\operatorname{grad} H^1_{\mathrm{cl}}(q), \operatorname{grad} H^2_{\mathrm{cl}}(q)$, *given in* (17).

Proof. Mutatis mutandis, we can reuse the proof of Lem. 3.5.1. □

Remark. The orthogonal projection $\operatorname{Proj}^{\mathrm{im}} : TC^\infty(S^1,\mathbb{R}^2) \restriction \operatorname{im}(R)_{\mathrm{cl}} \to T\operatorname{im}(R)_{\mathrm{cl}}$ is again given by (12) and it is a smooth map.

The geodesic equation on $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Mot}$ is well-posed in appropriate Sobolev completions. We refer to Sect. 7 and in particular to Sect. 7.2 for a detailed discussion and proofs. The spaces $\operatorname{Imm}^{j,k}(S^1,\mathbb{R}^2)$ are defined in (29).

Theorem 4.4.2. *For $k \geq 2$ and initial conditions $(c_0, u_0) \in T\operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^2)$, the geodesic equation has solutions in $\operatorname{Imm}^{j+1,j+2}(S^1,\mathbb{R}^2)$ for each $2 \leq j \leq k$. The solutions depend C^∞ on t and the initial conditions and the domain of existence is independent of j.*

In particular for smooth initial conditions $(c_0,u_0) \in T\operatorname{Imm}(S^1,\mathbb{R}^2)$ the geodesic equation has smooth solutions.

Remark. Since $\operatorname{im}(R)_{\text{cl}} \subset \operatorname{im}(R)_{\text{op}}$, the geodesic distance functions satisfy

$$\operatorname{dist}^G_{\text{op}}(c_0, c_1) \le \operatorname{dist}^G_{\text{cl}}(c_0, c_1)\,.$$

Therefore the remark after Thm. 4.3.2 also holds for the space $\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Mot}$ of closed curves, i.e., the functions $\sqrt{|c'|}$ and $\sqrt{\ell_c}$ are Lipschitz continuous with respect to the geodesic distance.

4.5. **Analysis of the Riemannian manifold** $(\mathbb{R}_{>0} \times \mathbb{R}, g)$**.** The metric is $g = 4dx^2 + x^{-6}dy^2 = \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2$, where $\sigma^1 = 2dx$, $\sigma^2 = x^{-3}dy$ is an orthonormal coframe.

First we note that the Levi-Civita connection has a 2-dimensional symmetry group; its transformations map geodesics to geodesics:

$$T_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y+a \end{pmatrix}, \quad a \in \mathbb{R}\,; \qquad H_r \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ r^4 y \end{pmatrix}, \quad r \in \mathbb{R}_{>0}\,.$$

Namely, each T_a and each reflection $y \mapsto -y$ is an isometry. On the other hand, $H_r^* g = 4d(rx)^2 + (rx)^{-6} d(r^4 y)^2 = r^2 g$, and a constant multiple of a Riemannian metric has the same geodesics. Since $H_r\, T_a\, H_r^{-1} = T_{r^4 a}$, this is the semidirect product $\mathbb{R}_{>0} \ltimes \mathbb{R}$ with multiplication $(T_{a_1}, H_{r_1})(T_{r_2}, H_{r_2}) = (T_{a_1 + r_1^4 a_2} H_{r_1 r_2})$.

We have $d\sigma^1 = 0$ and $d\sigma^2 = -3x^{-4}\, dx \wedge dy = -(3/2)x^{-1}\sigma^1 \wedge \sigma^2$. Using Cartan's structure equation we compute the connection form $\omega \in \Omega^1(\mathbb{R}_{>0} \times \mathbb{R}; \mathfrak{o}(2))$,

$$\begin{aligned} -d\sigma^1 &= 0 + \omega^1_2 \wedge \sigma^2 = 0 \\ -d\sigma^2 &= \omega^2_1 \wedge \sigma^1 + 0 = \tfrac{3}{2x}\sigma^1 \wedge \sigma^2 \end{aligned} \implies \omega = \begin{pmatrix} 0 & \frac{3}{2x^4}dy \\ -\frac{3}{2x^4}dy & 0 \end{pmatrix}$$

The *curvature matrix* and the *Gauss curvature* are then:

$$\Omega = d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & -\frac{3}{x^2}\sigma^1 \wedge \sigma^2 \\ \frac{3}{x^2}\sigma^1 \wedge \sigma^2 & 0 \end{pmatrix}, \qquad \operatorname{scal}(g) = -\frac{3}{x^2}\,.$$

This already implies that there are no conjugate points along any geodesic and that the Riemannian exponential mapping centered at each point is a diffeomorphism onto its image.

We use the dual orthonormal frame $s_1 = \frac{1}{2}\partial_x$, $s_2 = x^3\partial_y$. Then the velocity of a curve $q(t)$ is $\dot q = \dot x \partial_x + \dot y \partial_y = 2\dot x(s_1 \circ q) + \dot y x^{-3}(s_2 \circ q)$, and the *geodesic equation* is as follows:

$$\begin{aligned} 0 = \nabla_{\partial_t} \dot q &= \nabla_{\partial_t}\big(2\dot x(s_1 \circ q) + \dot y x^{-3}(s_2 \circ q)\big) \\ &= (\ddot x + \tfrac{3}{4}x^{-7}\dot y^2)\partial_x + \big(\ddot y - 6x^{-1}\dot x \dot y\big)\partial_y \end{aligned} \qquad \text{or} \qquad \boxed{\begin{aligned} \ddot x &= -\tfrac{3}{4}x^{-7}\dot y^2 \\ \ddot y &= 6x^{-1}\dot x \dot y \end{aligned}}$$

Note that $(x_0 + t\dot x_0, y_0)$ are incomplete geodesics. Hence, if $\dot y(t) = 0$ for some t, then for all t. If $\dot y > 0$, it never can change sign, always $\ddot x < 0$ and the geodesic, which is curving always to the left, leaves the space for $t \to \infty$. The case $\dot y < 0$ is mapped to $\dot y > 0$ by the reflection $(x, y) \mapsto (x, -y)$.

Now we eliminate time from the geodesic equation. The second equation can be rewritten as

$$\begin{aligned} \frac{\ddot y}{\dot y} = 6\frac{\dot x}{x} \quad &\iff \quad \log(|\dot y|) = 6\log(x) + \log(|\dot y_0|) - 6\log(x_0) \\ &\iff \quad \dot y = C_1 x^6 \quad \text{where} \quad C_1 = \frac{\dot y_0}{x_0^6}. \end{aligned}$$

Inserting this, the first equation becomes

$$\ddot{x} = -\tfrac{3}{4}C_1^2 x^5 \iff 2\ddot{x}\dot{x} = -\tfrac{3}{2}C_1^2 x^5 \dot{x} \iff \dot{x}^2 = -\tfrac{1}{4}C_1^2 x^6 + \dot{x}_0^2$$
$$\iff \dot{x} = \operatorname{sign}(\dot{x}_0)\big(C_2^2 - \tfrac{1}{4}C_1^2 x^6\big)^{1/2} \quad \text{where} \quad C_2 = \dot{x}_0.$$

Therefore,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\operatorname{sign}(\dot{x}_0)C_1 x^6}{(C_2^2 - \frac{1}{4}C_1^2 x^6)^{1/2}} = \frac{C_1 x^6}{C_2(1 - |\frac{C_1}{2C_2}|^2 x^6)^{1/2}} = \frac{2C^3 x^6}{(1 - (|C|x)^6)^{1/2}}\,,$$
$$C = \Big(\frac{C_1}{2C_2}\Big)^{1/3} = \Big(\frac{\dot{y}_0}{2x_0^6 \dot{x}_0}\Big)^{1/3}.$$

Note that $\operatorname{sign}(C) = \operatorname{sign}\Big(\frac{\dot{y}_0}{\dot{x}_0}\Big)$. We get y as a function of x, namely

$$y(x) = y_0 + \int_{x_0}^{x} \frac{2C^3 z^6\,\mathrm{d}z}{(1 - (|C|z)^6)^{1/2}} = y_0 + \frac{2\operatorname{sign}(C)}{|C|^4}\int_{|C|x_0}^{|C|x} \frac{z^6\,\mathrm{d}z}{\sqrt{1 - z^6}}$$

for $0 < x_0, x \le 1/|C|$. Thus the trajectory of a geodesic is given by:

$$\boxed{\begin{aligned} &y(x) = y_0 + \operatorname{sign}\Big(\frac{\dot{y}_0}{\dot{x}_0}\Big)\frac{2}{|C|^4}\big(F(|C|x) - F(|C|x_0)\big) \\ &\qquad \text{for } 0 \le x_0 \le x \le 1/|C|\,, \quad \text{where} \\ &F(u) = \int_0^u \frac{z^6\,\mathrm{d}z}{\sqrt{1 - z^6}} \qquad \text{with} \qquad A := F(1) = 0.30358\ldots. \end{aligned}} \tag{18}$$

Let us now assume that $\dot{y}_0 > 0$ and $\dot{x}_0 > 0$. Then

$$y(0) = y_0 - \frac{2}{|C|^4}F(|C|x_0)\,, \qquad y(1/|C|) = y(0) + \frac{2F(1)}{|C|^4}\,,$$
$$\frac{dy}{dx}(0) = 0\,, \qquad \frac{dy}{dx}(1/|C|) = \infty\,.$$

The reflection $y \mapsto -y + 2y(1/|C|)$ maps this geodesic to its other half which returns to $x = 0$.

Given two points $(x_0 > 0, y_0)$ and $(x_1 > 0, y_1)$, without loss satisfying $x_0 < x_1$ and $y_0 < y_1$, we shall now *determine the unique connecting geodesic trajectory.* By our assumption we have $\dot{y}_0 > 0$ and $\dot{x}_0 > 0$, thus $\operatorname{sign}(C) = \operatorname{sign}(\dot{y}_0/\dot{x}_0) = 1$.

Case 1. The two points lie on the right travelling branch of the geodesic arc if we can find $C > 0$ such that

$$y_1 = y_0 + \frac{2}{C^4}\big(F(Cx_1) - F(Cx_0)\big), \qquad 0 < x_0 < x_1 \le \frac{1}{C}.$$

The function $f(C) = f_{x_0,x_1}(C) := \frac{2}{C^4}\big(F(Cx_1) - F(Cx_0)\big)$ is monotone increasing in $0 < C \le \frac{1}{x_1}$, since (note that $F(u)$ is flat for $u \searrow 0$)

$$f'(C) = -\frac{8}{C^5}\big(F(Cx_1) - F(Cx_0)\big) + 2C^2\Big(\frac{x_1^7}{\sqrt{1 - (Cx_1)^6}} - \frac{x_0^7}{\sqrt{1 - (Cx_0)^6}}\Big) > 0.$$

This is the case if and only if

$$0 < y_1 - y_0 \le f_{x_0,x_1}(1/x_1) = 2x_1^4(F(1) - F(x_0/x_1)) = 2x_1^4 \int_{x_0/x_1}^{1} \frac{z^6\,\mathrm{d}z}{\sqrt{1 - z^6}}\,.$$

Case 2. The point (x_0, y_0) lies on the right travelling branch, but (x_1, y_1) is reached only after passing the apex on the left travelling branch. We know that $y_1 - y_0 > f_{x_0,x_1}(1/x_1)$. We have to find the apex $(\bar{x}, \bar{y})$ with $x_0 < x_1 < \bar{x}$, $y_0 < \bar{y} < y_1$, and $C = 1/\bar{x}$. The apex is given by

$$\bar{y} := y_0 + 2\bar{x}^4\big(F(1) - F(x_0/\bar{x})\big) = y_1 - 2\bar{x}^4\big(F(1) - F(x_1/\bar{x})\big).$$

So we have to solve

$$y_1 - y_0 = 2\bar{x}^4\big(2F(1) - F(x_0/\bar{x}) - F(x_1/\bar{x})\big)$$

for $\bar{x} = 1/C > x_1$. Since the right hand side is monotone increasing in $C = 1/\bar{x}$ there is a unique solution.

This implies that $(\mathbb{R}_{>0} \times \mathbb{R}, g)$ is geodesically convex, and there is a unique geodesic connecting any two points.

Since we need it later, we describe the *Riemann curvature tensor*, for $q = (q_1, q_2) \in \mathbb{R}_{>0} \times \mathbb{R}$ and $h, k, \in \mathbb{R}^2$:

$$\begin{aligned}
(\sigma^1 \wedge \sigma^2)_q(h,k) &= \frac{2}{q_1^3}(h_1k_2 - k_1h_2) = \frac{2}{q_1^3}\det(h,k) = \frac{2}{q_1^3}\langle h, ik\rangle \\
\mathcal{R}^g_q(h,k)\ell &= s_1\Omega^1_2(h,k)\sigma^2(\ell) + s_2\Omega^2_1(h,k)\sigma^1(\ell) \\
&= \begin{pmatrix} 0 & -\frac{3}{q_1^8}\langle h, Jk\rangle \\ \frac{12}{q_1^3}\langle h, Jk\rangle & 0 \end{pmatrix}\begin{pmatrix}\ell_1 \\ \ell_2\end{pmatrix} \\
g_q(\mathcal{R}^g_q(h,k)k, h) &= \operatorname{scal}(g)(q).\big(\|h\|^2_{g_q}\|k\|^2_{g_q} - g_q(h,k)^2\big) \\
&= \frac{-3}{q_1^2}\Big(\big(4h_1^2 + \frac{h_2^2}{q_1^6}\big)\big(4k_1^2 + \frac{k_2^2}{q_1^6}\big) - \big(4h_1k_1 + \frac{h_2k_2}{q_1^6}\big)^2\Big) \\
&= \frac{-12}{q_1^8}\,(h_1k_2 - h_2k_1)^2
\end{aligned}$$

Lemma 4.5.1. *The geodesic distance admits the following lower bound*

$$\operatorname{dist}((x_0,y_0),(x_1,y_1)) \geq 2\sqrt{|x_0 - x_1|^2 + \frac{|y_0 - y_1|^2}{2^{1/8}\left(x_0^4 + x_1^4 + \frac{1}{2A}|y_0 - y_1|\right)^{3/2}}}\,.$$

Proof. Let $\gamma(t)$ be the geodesic connecting (x_0, y_0) and (x_1, y_1). The distance between the points, where the continuation of γ intersects the y-axis is less than

$$2A\left(x_0^4 + x_1^4\right) + |y_0 - y_1|\,.$$

Thus

$$\gamma^1(t) \leq 2^{-1/4}\left(x_0^4 + x_1^4 + \tfrac{1}{2A}|y_0 - y_1|\right)^{1/4}\,.$$

Define

$$r = 2^{-1/12}\left(x_0^4 + x_1^4 + \tfrac{1}{2A}|y_0 - y_1|\right)^{1/4}\,,$$

and rescale the point by $(\overline{x}_i, \overline{y}_i) = (rx_i, r^4y_i)$. From the symmetries of the Levi-Civita connection we see that the geodesic connecting $(\overline{x}_0, \overline{y}_0)$ and $(\overline{x}_1, \overline{y}_1)$ is given by $(\overline{\gamma}^1, \overline{\gamma}^2) = (r\gamma^1, r^4\gamma^2)$. Hence

$$\overline{\gamma}^1(t) = r\gamma^1(t) \leq 2^{-1/3} = 4^{-1/6}\,.$$

On the set $\{(x,y) : x \leq 4^{-1/6}\}$ we have

$$g(x,y) = 4dx^2 + x^{-1/6}dy^2 \geq 4dx^2 + 4dy^2$$

and thus

$$\begin{aligned}\operatorname{dist}((x_0,y_0),(x_1,y_1)) &= \operatorname{Len}^g(\gamma) = r^{-1}\operatorname{Len}^g(\overline{\gamma}) \geq r^{-1}\operatorname{Len}^{\mathrm{Eucl}}(\overline{\gamma})\\ &\geq r^{-1}\sqrt{|\overline{x}_0-\overline{x}_1|^2+|\overline{y}_0-\overline{y}_1|^2}\\ &= \sqrt{|x_0-x_1|^2+r^6|y_0-y_1|^2}\,.\end{aligned}$$

This concludes the proof. □

5. The third metric

5.1. **The metric and its geodesic equation.** The metrics studied in the previous two sections both had Euclidean motions in their kernel. In this section we will study the following metric

$$G_c(h,k) = \int_M \langle D_s h, D_s k\rangle + \langle D_s^2 h, n\rangle\langle D_s^2 k, n\rangle \,\mathrm{d}s\,, \tag{19}$$

whose kernel consists only of translations. We can obtain it from (13) by adding the term $\langle D_s h, n\rangle\langle D_s k, n\rangle$. We shall study this metric only on the space of closed curves because the R-transform is not better behaved on the space of open curves.

The associated operator field L is given by

$$L_c h = D_s^2\big(\langle D_s^2 h, n\rangle n\big) - D_s^2 h\,.$$

Lemma 5.1.1. *The null space of the bilinear form $G_c(.,.)$ is spanned by constant vector fields, i.e.,*

$$\ker(G_c) = \left\{h \in T_c\operatorname{Imm}(S^1,\mathbb{R}^2) : h = a,\, a \in \mathbb{R}^2\right\}\,.$$

Proof. We have $h \in \ker(G_c)$ if and only if $D_s h = 0$. □

As an immediate consequence of Lem. 5.1.1 we obtain that G_c is a weak Riemannian metric on $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}$. We will use Lem. 2.4.1 to calculate its geodesic equation.

Theorem 5.1.2. *On the manifold $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}$ of plane curves modulo translations $G_c(.,.)$ defines a weak Riemannian metric. The geodesic equation is given by*

$$\begin{aligned} p &= Lc_t \otimes \mathrm{d}s = \ \left(-D_s^2 c_t + D_s^2\left(\langle D_s^2 c_t, n\rangle n\right)\right) \otimes \mathrm{d}s\,,\\ p_t &= D_s\Big(\frac{1}{2}|D_s h|^2 v - D_s\big(\langle D_s c_t, n\rangle\langle D_s^2 c_t, n\rangle\big)v\\ &\qquad + \langle D_s^2 c_t, v\rangle\langle D_s^2 c_t, n\rangle n + \frac{3}{2}\langle D_s^2 c_t, n\rangle^2 v\Big) \otimes \mathrm{d}s, \end{aligned}$$

with the additional constraint:

$$c_t \in T_c\left(\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}\right) \cong \{h \in C^\infty(S^1,\mathbb{R}^2) : h(0) = 0\}\,.$$

Remark. Similarily, as in Sects. 3 and 4 the operators

$$L_c : T_c\left(\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}\right) \to T_c\left(\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}\right)$$

are not elliptic operators and thus we again cannot apply the well-posedness results from [26] or [7].

Proof. Using the variational formulas from Sect. 2.3 we calculate the variation of the metric:

$$\begin{aligned} D_{c,m}(G_c(h,h)) = \int_{S^1} &-\langle D_s m, v\rangle\langle D_s h, D_s h\rangle - 2\langle D_s^2 h, n\rangle\langle D_s h, n\rangle D_s\left(\langle D_s m, v\rangle\right) \\ &- 2\langle D_s^2 h, n\rangle\langle D_s^2 h, v\rangle\langle D_s m, n\rangle - \langle D_s^2 h, n\rangle^2\langle D_s m, v\rangle \,\mathrm{d}s\,. \end{aligned}$$

We can now calculate $H_c(h,h)$ using a series of integrations by parts,

$$\begin{aligned} H_c(h,h) = D_s\Big(&|D_s h|^2 v - 2D_s\big(\langle D_s h, n\rangle\langle D_s^2 h, n\rangle\big)v \\ &+ 2\langle D_s^2 h, v\rangle\langle D_s^2 h, n\rangle n + 3\langle D_s^2 h, n\rangle^2 v\Big) \otimes \mathrm{d}s\,. \end{aligned}$$

Regarding the existence of the geodesic equation, see the proof of Thm. 3.1.2. □

5.2. **The R-transform.** Consider the map

$$R : \left\{ \begin{array}{ccc} \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra} & \to & C^\infty(S^1, \mathbb{R}_{>0}\times S^1\times\mathbb{R})\,, \\ c & \mapsto & (\sqrt{|c'|}, \alpha, \kappa|c'|^2) \end{array}\right. .$$

In order to simplify the notation we shall write, $\mathbb{R}^3_*$ for $\mathbb{R}_{>0}\times S^1\times\mathbb{R}$. On $\mathbb{R}^3_*$ we define the following Riemannian metric

$$g_{(q_1,q_2,q_3)} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & q_1^2 & 0 \\ 0 & 0 & q_1^{-6} \end{pmatrix}\,, \tag{20}$$

and we equip the space $C^\infty(M,(\mathbb{R}^3_*,g))$ with the L^2-Riemannian metric,

$$G_q^{L^2}(h,k) = \int_M g_{q(\theta)}(h(\theta),k(\theta))\,\mathrm{d}\theta\,. \tag{21}$$

Here $q\in C^\infty(M,(\mathbb{R}^3_*,g))$ and $h,k\in T_q C^\infty(M,(\mathbb{R}^3_*,g))$.

The reason for introducing these objects lies in the following theorem.

Theorem 5.2.1. *With the metrics g and G^{L^2} defined as above, the R-transform*

$$R\,:\,(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}, G) \to (C^\infty(M,(\mathbb{R}^3_*,g)), G^{L^2})$$

is an injective isometry between weak Riemannian manifolds, i.e.,

$$G_c(h,k) = G^{L^2}_{R(c)}(dR(c).h, dR(c).k) = \int_M g_{R(c)(\theta)}(dR(c).h(\theta), dR(c).k(\theta))\,\mathrm{d}\theta\,,$$

for $h,k\in T_c\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}$.

Proof. Using the formulas from Sect. 2.3 we calculate the derivative of the R-transform:

$$dR(c)h = \left(\tfrac{1}{2}\langle D_s h, v\rangle\sqrt{|c_\theta|}, \langle D_s h, n\rangle, \langle D_s^2 h, n\rangle|c_\theta|^2\right)\,.$$

Hence

$$g_{R(c)}(dR(c)h, dR(c)h) = \left(\langle D_c h, v\rangle^2 + \langle D_s h, n\rangle^2 + \langle D_s^2 h, n\rangle^2\right)|c_\theta|\,,$$

and the first statement of the theorem follows. The map R is injective on $\operatorname{Imm}/\operatorname{Tra}$ since one can reconstruct the curve c up to translations from $|c'|$ and α. □

Remark. A key difference between this R-transform and the ones used in Sects. 3.2 and 4.2 is that the image has infinite codimension, both for open as well as closed curves.

5.3. **The space of closed curves.**

Theorem 5.3.1. *The image of* $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}$ *under the R-transform is given by*

$$\operatorname{im}(R) = \left\{ (q_1,q_2,q_3)\ :\ \begin{array}{l} (1)\ q_2' = q_1^{-2} q_3 \\ (2)\ \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta = 0 \end{array} \right\}.$$

It is a splitting submanifold of $C^\infty(S^1,\mathbb{R}^3_*)$*. The inverse of the R-transform is given by*

$$R^{-1} : \left\{ \begin{array}{ccc} \operatorname{im}(R) & \to & \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra} \\ q & \mapsto & \int_0^\theta q_1^2 \exp(iq_2)\,\mathrm{d}\sigma \end{array} \right. .$$

We introduce the maps

(22)

$$H_{\text{diff}} : \left\{ \begin{array}{ccc} C^\infty(S^1,\mathbb{R}^3_*) & \to & C^\infty(S^1,\mathbb{R}) \\ q & \mapsto & q_3 - q_1^2 q_2' \end{array} \right. \qquad H_{\text{cl}} : \left\{ \begin{array}{ccc} C^\infty(S^1,\mathbb{R}^3_*) & \to & \mathbb{R}^2 \\ q & \mapsto & \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta \end{array} \right. ,$$

which allow us to write $\operatorname{im}(R) = H_{\text{diff}}^{-1}(0) \cap H_{\text{cl}}^{-1}(0)$.

Proof. The constraint $H_{\text{diff}}(q) = 0$, rewritten in terms of c, is the definition of curvature $\kappa = D_s\alpha$, written as

$$\kappa|c'|^2 = \sqrt{|c'|}^2 \alpha' .$$

The constraint $H_{\text{cl}}(0)$ results from $\int_{S^1} c'\,\mathrm{d}\theta = 0$ and $c' = q_1^2 \exp(iq_2)$. The latter identity implies

$$c(\theta) = c(0) + \int_0^\theta q_1^2 \exp(iq_2)\,\mathrm{d}\sigma .$$

Since $c(0)$ is unspecified, this determines the curve up to translations. Thus we have identified $\operatorname{im}(R)$ and proven the inversion formula.

To show that $\operatorname{im}(R)$ is a splitting submanifold define

$$\Phi : \left\{ \begin{array}{ccc} C^\infty(S^1,\mathbb{R}^3_*) & \to & C^\infty(S^1,\mathbb{R}^3_*) \\ (q_1,q_2,q_3) & \mapsto & (q_1,q_2,q_3 + q_1^2 q_2') \end{array} \right. .$$

The map Φ is a smooth diffeomorphism and $H_{\text{diff}}^{-1}(0) = \Phi(\{q_3 = 0\})$. Thus $H_{\text{diff}}^{-1}(0)$ is a splitting submanifold of $C^\infty(S^1,\mathbb{R}^3_*)$. Now we will pull back $\operatorname{im}(R)$ by Φ and show that $\Phi^{-1}(\operatorname{im}(R))$ is a splitting submanifold of $\{q_3 = 0\}$. First note that

$$\Phi^{-1}(\operatorname{im}(R)) = (H_{\text{cl}} \circ \Phi)^{-1}(0) \cap \{q_3 = 0\} = H_{\text{cl}}^{-1}(0) \cap \{q_3 = 0\} .$$

The last equality holds because H_{cl} doesn't depend on q_3. That $H_{\text{cl}}^{-1}(0)$ is a splitting submanifold of $\{q_3 = 0\}$ can be shown via the implicit function theorem with convenient parameters like in Lem. 3.5.1. This concludes the proof. □

Next we want to compute the orthogonal projection $\operatorname{Proj}^{\text{im}} : TC^\infty(S^1,\mathbb{R}^3_*) \restriction \operatorname{im}(R) \to T\operatorname{im}(R)$. We do this in two steps. First define the space

$$\operatorname{im}(R)_{\text{op}} = H_{\text{diff}}^{-1}(0) = \{q \in C^\infty(S^1, \mathbb{R}_{>0} \times S^1 \times \mathbb{R}) : q_3 = q_1^2 q_2'\} ,$$

and compute the orthogonal projection $\operatorname{Proj}^{\text{im}}_{op} : TC^\infty(S^1,\mathbb{R}^3_*) \to T\operatorname{im}(R)_{op}$. The space $\operatorname{im}(R)_{\text{op}}$ corresponds to the image of the R-transform on the space of open curves, if S^1 were replaced by $[0,2\pi]$. We can compute the tangent space

$$T_q \operatorname{im}(R)_{\text{op}} = \left\{ \left(h_1, h_2, 2q_1^{-1} q_3 h_1 + q_1^2 h_2'\right)\ :\ h_1,h_2 \in C^\infty(S^1,\mathbb{R}) \right\}$$

for $q \in \operatorname{im}(R)_{\text{op}}$.

Lemma 5.3.2. *Let $q \in \operatorname{im}(R)_{\mathrm{op}}$ and $h \in T_q C^\infty(S^1, \mathbb{R}^3_*)$. Then the orthogonal projection $k = \operatorname{Proj}^{\mathrm{im}}_{\mathrm{op}}(q).h$ of h to the tangent space to $\operatorname{im}(R)_{\mathrm{op}}$ is given by*

$$q_1^2 k_2 - (A k_2')' = B \tag{23a}$$

$$k_1 = \frac{2q_1^8 h_1 + q_1 q_3 h_3 - q_1^3 q_3 k_2'}{2(q_1^8 + q_3^2)} \tag{23b}$$

$$k_3 = 2 q_1 q_2' k_1 + q_1^2 k_3' \tag{23c}$$

with the functions

$$A = \frac{q_1^6}{q_1^8 + q_3^2}\,, \qquad B = q_1^2 h_2 + \left(\frac{2 q_1^3 q_3 h_1 + q_1^{-4} q_3^2 h_3}{q_1^8 + q_3^2} \right)'\,,$$

Furthermore the map $\operatorname{Proj}^{\mathrm{im}}_{\mathrm{op}}$ is smooth.

Proof. The orthogonal projection $k \in T_q \operatorname{im}(R)_{\mathrm{op}}$ of h is defined by the equation

$$G_q^{L^2}(h, a) = G_q^{L^2}(k, a)\,,$$

which has to hold for all $a \in T_q \operatorname{im}(R)_{\mathrm{op}}$. Any such a is of the form

$$a = (a_1, a_2, 2 q_1^{-1} q_3 a_1 + q_1^2 a_2')\,.$$

Thus the above equation reads

$$\begin{aligned} &\int_{S^1} 4 h_1 a_1 + q_1^2 h_2 a_2 + q_1^{-6} h_3 (2 q_1^{-1} q_3 a_1 + q_1^2 a_2')\, \mathrm{d}\theta = \\ &\qquad = \int_{S^1} 4 k_1 a_1 + q_1^2 k_2 a_2 + q_1^{-6} (2 q_1^{-1} q_3 k_1 + q_1^2 k_2')(2 q_1^{-1} q_3 a_1 + q_1^2 a_2')\, \mathrm{d}\theta\,. \end{aligned}$$

which is equivalent to the system

$$\begin{aligned} 4 h_1 + 2 q_1^{-7} q_3 h_3 &= 4 k_1 + 4 q_1^{-8} q_3^2 k_1 + 2 q_1^{-5} q_3 k_2' \\ q_1^2 h_2 - (q_1^{-4} h_3)' &= q_1^2 k_2 - \left(2 q_1^{-5} q_3 k_1 + q_1^{-2} k_2' \right)'\,. \end{aligned}$$

The first equation allows us to express k_1 in terms of k_2',

$$k_1 = \frac{2q_1^8 h_1 + q_1 q_3 h_3 - q_1^3 q_3 k_2'}{2(q_1^8 + q_3^2)}\,,$$

which we then insert into the second equation and obtain

$$q_1^2 k_2 - (A k_2')' = B\,,$$

with A and B defined as above. The map $(q, h) \mapsto k$ is smooth, because the solution of an elliptic equation depends smoothly on the coefficients. □

Now we compute $\operatorname{Proj}^{\mathrm{im}}$. The components of the constraint function H_{cl} are

$$H_{\mathrm{cl}} = \left(\int_{S^1} q_1^2 \cos q_2 \,\mathrm{d}\theta, \int_{S^1} q_1^2 \sin q_2 \,\mathrm{d}\theta \right)\,,$$

and their gradients with respect to the G^{L^2}-metric are given by

$$\operatorname{grad}^{L^2} H^1_{\mathrm{cl}}(q) = (\tfrac{1}{2} q_1 \cos q_2, -\sin q_2, 0) \tag{24a}$$

$$\operatorname{grad}^{L^2} H^2_{\mathrm{cl}}(q) = (\tfrac{1}{2} q_1 \sin q_2, \cos q_2, 0)\,. \tag{24b}$$

Theorem 5.3.3. *Let $q \in \operatorname{im}(R)$. Define $v^i = \operatorname{Proj}^{\mathrm{im}}_{\mathrm{op}}(q).\operatorname{grad}^{L^2} H^i_{\mathrm{cl}}(q)$ for $i = 1, 2$ and let $T(q) : T_q C^\infty(S^1, \mathbb{R}^3_*) \to T_q C^\infty(S^1, \mathbb{R}^3_*)$ be the projection to the orthogonal complement of $\operatorname{span}\{v^1, v^2\}$. Then*

$$\operatorname{Proj}^{\mathrm{im}}(q) = T(q) \circ \operatorname{Proj}^{\mathrm{im}}_{\mathrm{op}}(q)\,, \tag{25}$$

and $\operatorname{Proj}^{\mathrm{im}}$ is smooth.

Proof. Since $\operatorname{span}\{v^1, v^2\}$ is the orthogonal complement to $\operatorname{im}(R)$ within $\operatorname{im}(R)_{op}$, the map $T(q)$ projects $T_q \operatorname{im}(R)_{op}$ down to $T_q \operatorname{im}(R)$. In order for T to be smooth we need that $\{v^1, v^2\}$ are linearly independent. This is equivalent to the condition that $\operatorname{grad}^{L^2} H^1_{\mathrm{cl}}(q)$ and $\operatorname{grad}^{L^2} H^2_{\mathrm{cl}}(q)$ don't differ by an element of $(T_q \operatorname{im}(R)_{op})^\perp$. This is clear from Lem. 5.3.2 and the formulas (24) for the gradients of H^i_{cl}. □

The geodesic equation on $\operatorname{Imm}_{\mathrm{conv}}(S^1, \mathbb{R}^2)/\operatorname{Tra}$ is well-posed in appropriate Sobolev completions. We refer to Sect. 7 and in particular to Sect. 7.2 for a detailed discussion and proofs. The spaces $\operatorname{Imm}^{j,k}(S^1, \mathbb{R}^2)$ are defined in (29).

Theorem 5.3.4. *For $k \geq 2$ and initial conditions $(c_0, u_0) \in T \operatorname{Imm}^{k+1,k+2}(S^1, \mathbb{R}^2)$, the geodesic equation has solutions in $\operatorname{Imm}^{j+1,j+2}(S^1, \mathbb{R}^2)$ for each $2 \leq j \leq k$. The solutions depend C^∞ on t and the initial conditions and the domain of existence is independent of j.*

In particular, for smooth initial conditions $(c_0, u_0) \in T \operatorname{Imm}(S^1, \mathbb{R}^2)$ the geodesic equation has smooth solutions.

6. The full H^2-metric

6.1. **The metric and its geodesic equation.** In this section we will study the full H^2-metric that has translations in the kernel. In comparison to the metric of the previous section we add the missing H^2-term $\langle D_s^2 h, v\rangle$ to the definition of the metric. This yields the bilinear form

$$G_c(h, k) = \int_M \langle D_s h, D_s k\rangle + \langle D_s^2 h, D_s^2 k\rangle \,\mathrm{d}s\,. \tag{26}$$

On closed curves the associated operator field of this pseudo metric is given by

$$L_c h = D_s^4 h - D_s^2 h\,.$$

Lemma 6.1.1. *The null space of the bilinear form $G_c(.,.)$ is spanned by constant vector fields, i.e.,*

$$\ker(G_c) = \left\{h \in T_c \operatorname{Imm}(M, \mathbb{R}^2) : h = a,\, a \in \mathbb{R}^2\right\}\,.$$

Proof. The proof of this lemma is obvious, since the bilinear form G_c measures the full first derivative. □

Remark. In contrast to the other metrics studied in this article the operator L is elliptic and invertible on $T_c \operatorname{Imm}/\operatorname{Tra}$. Therefore we will be able to apply the well-posedness result of [7, 35].

As an immediate consequence of Lem. 3.1.1 we obtain that G_c is a weak Riemannian metric on $\operatorname{Imm}(M, \mathbb{R}^2)/\operatorname{Tra}$. Similarly. as in Sects. 3 and 4 we will use Lem. 2.4.1 to calculate its geodesic equation.

Theorem 6.1.2. *On the manifold* $\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}$ *of plane curves modulo translations, the pseudo metric* $G_c(.,.)$ *is a weak Riemannian metric. The geodesic equation on the manifold of closed curves modulo Euclidean motions* $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}$ *is given by:*

$$p = Lc_t \otimes \mathrm{d}s = D_s^4 c_t - D_s^2 c_t \otimes \mathrm{d}s$$
$$p_t = D_s\Big(3|D_s^2 c_t|^2 v - 2D_s\big(\langle D_s c_t, D_s^2 c_t\rangle v\big) + |D_s c_t|^2 v\Big) \otimes \mathrm{d}s$$

with the additional constraint:

$$c_t \in T_c\left(\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}\right) \cong \{h \in C^\infty(M,\mathbb{R}^2) : h(0) = 0\}\,.$$

For each $k > 11/2$ *and initial conditions* $(c_0,u_0) \in T\operatorname{Imm}^k(S^1,\mathbb{R}^2)$*, the geodesic equation has solutions in* $\operatorname{Imm}^j(S^1,\mathbb{R}^2)$ *for each* $11/2 < j \le k$*. The solutions depend* C^∞ *on* t *and the initial conditions and the domain of existence is independent of* j*. In particular, for smooth initial conditions* $(c_0,u_0) \in T\operatorname{Imm}(S^1,\mathbb{R}^2)$*, the geodesic equation has smooth solutions.*

Proof. Using the variational formulas from Sect. 2.3 and

$$d(D_s^2 h)(c).m = -2\langle D_s m, v\rangle D_s^2 h - D_s(\langle D_s m, v\rangle) D_s h$$

we obtain the variation of the metric:

$$\begin{aligned} D_{c,m}(G_c(h,h)) = \int_{S^1} & -3\langle D_s m, v\rangle\langle D_s^2 h, D_s^2 h\rangle - 2D_s(\langle D_s m, v\rangle)\langle D_s h, D_s^2 h\rangle \\ & - \langle D_s m, v\rangle\langle D_s h, D_s h\rangle \,\mathrm{d}s \end{aligned}$$

We can now calculate $H_c(h,h)$ using a series of integration by parts:

$$H_c(h,h) = D_s\Big(3|D_s^2 h|^2 v - 2D_s\big(\langle D_s h, D_s^2 h\rangle v\big) + |D_s h|^2 v\Big) \otimes \mathrm{d}s\,. \qquad \square$$

The well-posedness result can be proven similar as in [35], see also [7].

6.2. **The R-transform.** We introduce the following transformation

$$R : \left\{ \begin{array}{ccc} \operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra} & \to & C^\infty(M, \mathbb{R}_{>0} \times S^1 \times \mathbb{R}^2)\,, \\ c & \mapsto & (\sqrt{|c'|}, \alpha, D_s|c'|, \kappa|c'|^2) \end{array} \right. .$$

which assigns to each curve c the 4-tuple of functions $(\sqrt{|c'|}, \alpha, D_s|c'|, \kappa|c'|^2)$. In order to simplify the notation we shall write, $\mathbb{R}^4_*$ for $\mathbb{R}_{>0} \times S^1 \times \mathbb{R}^2$. On $\mathbb{R}^4_*$ we define the following Riemannian metric,

$$g_{(q_1,q_2,q_3,q_4)} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & q_1^2 + q_4^2 q_1^{-6} & -q_4 q_1^{-4} & 0 \\ 0 & -q_4 q_1^{-4} & q_1^{-2} & 0 \\ 0 & 0 & 0 & q_1^{-6} \end{pmatrix}, \tag{27}$$

and we equip the space $C^\infty(M,(\mathbb{R}^4_*,g))$ with the L^2-Riemannian metric,

$$G_q^{L^2}(h,k) = \int_M g_{q(\theta)}(h(\theta),k(\theta))\,\mathrm{d}\theta\,. \tag{28}$$

Here $q \in C^\infty(M,(\mathbb{R}^4_*,g))$ and $h,k \in T_q C^\infty(M,(\mathbb{R}^4_*,g))$.

The reason for introducing these objects lies in the following theorem:

Theorem 6.2.1. *With the metrics g and G^{L^2} defined as above, the map*

$$R : (\operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}, G) \to (C^\infty(M,(\mathbb{R}^4_*,g)), G^{L^2})$$

is an injective isometry between weak Riemannian manifolds, i.e.,

$$G_c(h,k) = G^{L^2}_{R(c)}(dR(c).h, dR(c).k) = \int_M g_{R(c)(\theta)}(dR(c).h(\theta), dR(c).k(\theta))\,\mathrm{d}\theta\,,$$

for $h,k \in T_c \operatorname{Imm}(M,\mathbb{R}^2)/\operatorname{Tra}$.

Proof. Using the identity

$$D_{c,h}\left(D_s \circ R\right) = D_s\left(D_{c,h}R\right) - \langle D_s h, v\rangle D_s(R(c))\,,$$

we calculate:

$$\begin{aligned} D_{c,h}\left(D_s|c'|\right) &= D_s\left(|c'|\langle D_s h, v\rangle\right) - \langle D_s h, v\rangle D_s|c'| \\ &= |c'|\langle D_s^2 h, v\rangle + |c'|\langle D_s^2 h, D_s v\rangle \\ &= |c'|\langle D_s^2 h, v\rangle + |c'|\kappa\langle D_s h, n\rangle\,. \end{aligned}$$

Thus we obtain the derivative of the R-map:

$$dR(c).h = \left(\tfrac{1}{2}\langle D_s h, v\rangle\sqrt{|c'|}, \langle D_s h, n\rangle, |c'|\langle D_s^2 h, v\rangle + |c'|\kappa\langle D_s h, n\rangle, \langle D_s^2 h, n\rangle|c'|^2\right)\,.$$

Hence

$$\begin{aligned} g_{R(c)}(dR(c).h, dR(c).h) &= \Big(\langle D_s h, v\rangle^2 + \langle D_s h, n\rangle^2(1+\kappa^2) \\ &\quad + \langle D_s^2 h, v\rangle^2 + 2\langle D_s^2 h, v\rangle\langle D_s h, n\rangle\kappa + \langle D_s h, n\rangle^2\kappa^2 \\ &\quad - 2\langle D_s^2 h, v\rangle\langle D_s h, n\rangle\kappa - 2\langle D_s h, n\rangle^2\kappa^2 + \langle D_s^2 h, n\rangle^2\Big)|c'| \\ &= \left(\langle D_s h, D_s h\rangle + \langle D_s^2 h, D_s^2 h\rangle\right)|c'|\,, \end{aligned}$$

and the first statement of the theorem follows. The map R is injective on $\operatorname{Imm}/\operatorname{Tra}$ since one can reconstruct the curve c up to translations from $|c'|$ and α. □

6.3. **The metric on the space of closed curves.**

Theorem 6.3.1. *The image of* $\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra}$ *under the R-transform is given by*

$$\operatorname{im}(R) = \left\{ (q_1,q_2,q_3,q_4) : \begin{array}{l} (1)\, 2q_1' = q_1 q_3 \\ (2)\, q_2' = q_1^{-2} q_4 \\ (3)\, \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta = 0 \end{array} \right\}\,.$$

It is a splitting submanifold of $C^\infty(S^1,\mathbb{R}^4_*)$. The inverse of the R-transform is given by

$$R^{-1} : \left\{ \begin{array}{ccc} \operatorname{im}(R) & \to & \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Tra} \\ q & \mapsto & \int_0^\theta q_1^2 \exp(iq_2)\,\mathrm{d}\sigma \end{array} \right.\,.$$

Introduce the maps

$$H_{\text{diff}} : \left\{ \begin{array}{ccc} C^\infty(S^1,\mathbb{R}^4_*) & \to & C^\infty(S^1,\mathbb{R}^2) \\ q & \mapsto & \left(q_3 - 2q_1^{-1}q_1', q_4 - q_1^2 q_2'\right) \end{array} \right.$$

$$H_{\text{cl}} : \left\{ \begin{array}{ccc} C^\infty(S^1,\mathbb{R}^3_*) & \to & \mathbb{R}^2 \\ q & \mapsto & \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta \end{array} \right.\,,$$

which allow us to write $\operatorname{im}(R) = H_{\text{diff}}^{-1}(0) \cap H_{\text{cl}}^{-1}(0)$.

Proof. The first component of the constraint $H_{\text{diff}}(q) = 0$ corresponds to the identity $D_s|c'| = \frac{1}{|c'|}\partial_\theta|c'|$. The second component as well as $H_{\text{cl}}(q) = 0$ are the same as in Thm. 5.3.1.

To show that $\operatorname{im}(R)$ is a splitting submanifold define

$$\Phi : \begin{cases} C^\infty(S^1, \mathbb{R}^4_*) & \to & C^\infty(S^1, \mathbb{R}^4_*) \\ (q_1, q_2, q_3, q_4) & \mapsto & (q_1, q_2, q_3 + 2q_1^{-1}q_1', q_4 + q_1^2 q_2') \end{cases} .$$

We note as in the proof of Thm. 5.3.1 that Φ is a smooth diffeomorphism, that $H_{\text{diff}}^{-1}(0) = \Phi(\{q_3 = 0, q_4 = 0\})$ and that

$$\Phi^{-1}(\operatorname{im}(R)) = H_{\text{cl}}^{-1}(0) \cap \{q_3 = 0, q_4 = 0\} .$$

Since H_{cl} doesn't depend on q_3, q_4 we then apply the implicit function theorem with convenient parameters to conclude the proof. □

7. Well-posedness of the geodesic equation

7.1. **Well-posedness for the third metric.** Let G be the Riemannian metric (19) from Sect. 5 on closed curves

$$G_c(h,h) = \int_{S^1} |D_s h|^2 + \langle D_s^2 h, n\rangle^2 \,\mathrm{d}s .$$

For $j \geq 1$, $k \geq 1$ define the spaces

$$\operatorname{Imm}^{j,k}(S^1, \mathbb{R}^2) = \left\{ c \in H^2(S^1, \mathbb{R}^2) \,:\, |c'| \in H^j(S^1, \mathbb{R}), \alpha \in H^k(S^1, S^1) \right\} . \tag{29}$$

Then $\operatorname{Imm}^{j,k}(S^1, \mathbb{R}^2)$ is a Hilbert manifold modelled on $H^j \times H^k$ and a global chart is given by $c \mapsto (|c'|, \alpha)$. For $k \geq 2$ denote by

$$\operatorname{Imm}^k(S^1, \mathbb{R}^2) = \{c \in H^k(S^1, \mathbb{R}^2) \,:\, |c'| > 0\}$$

the space of Sobolev immersions of order k. Note that we have the inclusions

$$\operatorname{Imm}^{\max(j,k)+1}(S^1, \mathbb{R}^2) \subseteq \operatorname{Imm}^{j,k}(S^1, \mathbb{R}^2) \subseteq \operatorname{Imm}^{\min(j,k)+1}(S^1, \mathbb{R}^2) ,$$

and if $j = k$, then

$$\operatorname{Imm}^{j,j}(S^1, \mathbb{R}^2) = \operatorname{Imm}^{j+1}(S^1, \mathbb{R}^2) .$$

The spaces can $\operatorname{Imm}^{j,k}(S^1, \mathbb{R}^2)$ can be seen as a refinement of the Sobolev scale of function spaces.

Theorem 7.1.1. *For $k \geq 2$, the geodesic spray*

$$\Xi^G : T\operatorname{Imm}^{k+1,k+2}(S^1, \mathbb{R}^2)/\operatorname{Tra} \to TT\operatorname{Imm}^{k+1,k+2}(S^1, \mathbb{R}^2)/\operatorname{Tra}$$

of the metric G is smooth.

Combining this theorem with the existence theorem for ODEs, the translation invariance of the geodesic spray from App. A and Thm. A.1, we obtain the following corollary.

Corollary 7.1.2. *For $k \geq 2$ and initial conditions $(c_0, u_0) \in T\operatorname{Imm}^{k+1,k+2}(S^1, \mathbb{R}^2)$, the geodesic equation has solutions in $\operatorname{Imm}^{j+1,j+2}(S^1, \mathbb{R}^2)/\operatorname{Tra}$ for each $2 \leq j \leq k$. The solutions depend C^∞ on t and the initial conditions and the domain of existence is independent of j.*

In particular for smooth initial conditions $(c_0, u_0) \in T\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Tra}$ the geodesic equation has smooth solutions.

Proof of Theorem. The R-transform
$$R(c) = (\sqrt{|c'|}, \alpha, \kappa|c'|^2)$$
extends to a smooth map
$$R : \operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^2)/\operatorname{Tra} \to H^k(S^1,\mathbb{R}^3_*)$$
with $\mathbb{R}^3_* = \mathbb{R}_{>0} \times S^1 \times \mathbb{R}$ and the image of the map is given by
$$\operatorname{im}(R) = \left\{ (q_1,q_2,q_3) : \begin{array}{l} (1)\, q_1 \in H^k, q_2 \in H^{k+1}, q_3 \in H^k \\ (2)\, q_2' = q_1^{-2} q_3 \\ (3)\ \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta = 0 \end{array} \right\}.$$
By Lem. 7.1.3 it is an embedded submanifold of $H^k(S^1,\mathbb{R}^3_*)$. Let g be the Riemannian metric (20) on $\mathbb{R}^3_*$ and G^{L^2} the L^2-metric (15) on $H^k(S^1,\mathbb{R}^3_*)$. The same proof as for Thm. 5.2.1 shows that the R-transform is an isometry between $(\operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^2)/\operatorname{Tra}, G)$ and $(H^k(S^1,\mathbb{R}^3_*), G^{L^2})$.

Denote by
$$\iota : \operatorname{im}(R) \to H^k(S^1,\mathbb{R}^3_*)$$
the inclusion map. The orthogonal projection to $T\operatorname{im}(R)$ is a map
$$\operatorname{Proj}^{\mathrm{im}} : \iota^* TH^k(S^1,\mathbb{R}^3_*) \restriction \operatorname{im}(R) \to T\operatorname{im}(R)\,,$$
and it is given by the same formulas (23) and (25) as in Thm. 5.3.3. It is shown in Lem. 7.1.4 that $\operatorname{Proj}^{\mathrm{im}}$ can be extended to a smooth map
$$\operatorname{Proj}^{\mathrm{im}} : TH^k(S^1,\mathbb{R}^3_*) \to TH^k(S^1,\mathbb{R}^3_*)\,.$$
Denote by
$$\Xi^{L^2} : TH^k(S^1,\mathbb{R}^3_*) \to TTH^k(S^1,\mathbb{R}^3_*)$$
the geodesic spray of the G^{L^2}-metric on $H^k(S^1,\mathbb{R}^3_*)$. It is smooth by Thm. 2.5.1. Theorem 2.5.2 shows that the geodesic spray of the G^{L^2}-metric restricted to $\operatorname{im}(R)$ is given by
$$\Xi^{\mathrm{im}} = T\operatorname{Proj}^{\mathrm{im}} \circ \Xi^{L^2} \circ T\iota : T\operatorname{im}(R) \to TT\operatorname{im}(R)\,,$$
and that this map is smooth as well. The geodesic sprays Ξ^{L^2} and Ξ^{im} are TR-related, i.e., the following diagram commutes.
$$\begin{array}{ccc} TT\operatorname{Imm}^{k+1,k+2}/\operatorname{Tra} & \xrightarrow{TTR} & TT\operatorname{im}(R) \\ \Xi^G \uparrow & & \uparrow \Xi^{\mathrm{im}} \\ T\operatorname{Imm}^{k+1,k+2}/\operatorname{Tra} & \xrightarrow{TR} & T\operatorname{im}(R) \end{array}$$
Since $\operatorname{im}(R)$ is an embedded submanifold of $H^k(S^1,\mathbb{R}^3_*)$ the map
$$R : \operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^3_*)/\operatorname{Tra} \to \operatorname{im}(R)$$
is a diffeomorphism and we can conclude that Ξ^G is smooth. □

Lemma 7.1.3. *Let $k \geq 2$. The image of the R-transform*
$$R : \left\{ \begin{array}{ccc} \operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^2)/\operatorname{Tra} & \to & H^k(S^1,\mathbb{R}^3_*) \\ c & \mapsto & (\sqrt{|c'|}, \alpha, \kappa|c'|^2) \end{array} \right. ,$$
is an embedded submanifold of $H^k(S^1,\mathbb{R}^3_*)$.

Proof. Define the functions

$$\begin{aligned} &C_\alpha : H^k(S^1,\mathbb{R}^3_*) \to \mathbb{R} && C_\alpha(q) = \int_{S^1} q_1^{-2} q_3 \,\mathrm{d}\theta \\ &C_{\mathrm{diff}} : H^k \times H^{k+1} \times H^k \to H^k && C_{\mathrm{diff}} = q_2' - q_1^{-2} q_3 \\ &C_{\mathrm{cl}} : H^k(S^1,\mathbb{R}^3_*) \to \mathbb{R}^2 && C_{\mathrm{cl}}(q) = \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta\,, \end{aligned}$$

which allow us to write the image $\mathrm{im}(R)$ as

$$\mathrm{im}(R) = H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) \cap C_{\mathrm{diff}}^{-1}(0) \cap C_{\mathrm{cl}}^{-1}(0)\,.$$

That $H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z})$ is a submanifold of $H^k(S^1,\mathbb{R}^3_*)$ can be seen via the inverse function theorem on Banach spaces. Let Φ be the map

$$\Phi : \left\{ \begin{array}{ccc} H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) & \to & H^k(S^1,\mathbb{R}^3_*) \\ (u_1,u_2,u_3) & \mapsto & \left(u_1, u_2 + \int u_1^{-2}u_3, u_3\right) \end{array} \right. ,$$

with $\int u_1^{-2} u_3$ denoting the indefinite integral of $u_1^{-2}u_3$. Then Φ is a bijection, when restricted to

$$\Phi : H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) \cap \{u_2 = 0\} \xrightarrow{\cong} H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) \cap H_{\mathrm{diff}}^{-1}(0)\,.$$

This implies that

$$H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) \cap C_{\mathrm{diff}}^{-1}(0) \hookrightarrow H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z})$$

is a submanifold. Finally, that

$$H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) \cap C_{\mathrm{diff}}^{-1}(0) \cap C_{\mathrm{cl}}^{-1}(0) \hookrightarrow H^k(S^1,\mathbb{R}^3_*) \cap C_\alpha^{-1}(2\pi\mathbb{Z}) \cap C_{\mathrm{diff}}^{-1}(0)$$

is a submanifold can again be shown via the implicit function theorem. □

Lemma 7.1.4. *Let $k \geq 2$. The orthogonal projection* $\mathrm{Proj}^{\mathrm{im}}$ *defined in* (25) *can be extended to a smooth map*

$$\mathrm{Proj}^{\mathrm{im}} : TH^k(S^1,\mathbb{R}^3_*) \to TH^k(S^1,\mathbb{R}^3_*)\,.$$

Proof. First we show that $\mathrm{Proj}^{\mathrm{im}}_{\mathrm{op}} : TH^k \to TH^k$ is a smooth map. Using the notation of Lem. 5.3.2 we see that for $(q,h) \in TH^k$ we have $A \in H^k(S^1,\mathbb{R})$ and $B \in H^{k-1}(S^1,\mathbb{R})$. The second component k_2 of $\mathrm{Proj}^{\mathrm{im}}_{\mathrm{op}}$ is given as the solution of the equation

$$q_1^2 k_2 - (Ak_2')' = B$$

and by Lem. B.1 we know that such a $k_2 \in H^{k+1}(S^1,\mathbb{R})$ exists and depends smoothly on (q,h). Then (23) shows that $k_1, k_3 \in H^k(S^1,\mathbb{R})$ thus showing $\mathrm{Proj}^{\mathrm{im}}_{\mathrm{op}}$ to be well-defined and smooth.

The smoothness of the maps $q \mapsto \mathrm{grad}^{L^2} H^i_{\mathrm{cl}}(q)$ for $i = 1,2$ given by (24) is clear and we see by inspection that $\mathrm{grad}^{L^2} H^i_{\mathrm{cl}}(q) \in H^k(S^1,\mathbb{R}^3)$. Therefore $q \mapsto \mathrm{Proj}^{\mathrm{im}}(q).\,\mathrm{grad}^{L^2} H^i_{\mathrm{cl}}(q) =: v^i(q)$ is also smooth. Let $w^1(q), w^2(q)$ be an orthonormal basis of $\mathrm{span}\{v^1(q), v^2(q)\}$, constructed, e.g., via Gram-Schmidt. Then

$$T(q).h = h - \langle h, w^1(q)\rangle w^1(q) - \langle h, w^2(q)\rangle w^2(q)$$

is smooth as well. Thus we conclude that the composition

$$\mathrm{Proj}^{\mathrm{im}}(q).h = T(q).\,\mathrm{Proj}^{\mathrm{im}}_{\mathrm{op}}(q).h$$

is smooth as required. □

7.2. **Well-posedness for the first and second metrics.** The statements of Thm. 7.1.1 and Cor. 7.1.2 also hold for the metrics (7) and (13) from Sect. 3 and Sect. 4 on the space of closed curves. In the proof of Thm. 7.1.1 we need to change the R-transform used to represent the metric and prove the analogues of Lem. 7.1.3 and Lem. 7.1.4, the rest of the proof will remain the same.

Let G be the metric (13) from Sect. 4,

$$G_c(h,k) = \int_{S^1} \langle D_s h, D_s k\rangle + \langle D_s^2 h, n\rangle\langle D_s^2 k, n\rangle \,\mathrm{d}s\,.$$

The image of the R-transform (14) is a submanifold in appropriate Sobolev extensions.

Lemma 7.2.1. *Let $k \geq 2$. The image of the R-transform*

$$R : \begin{cases} \operatorname{Imm}^{k+1,k+2}(S^1,\mathbb{R}^2)/\operatorname{Mot} & \to \quad H^k(S^1,\mathbb{R}_{>0}\times\mathbb{R}) \\ \qquad c & \mapsto \quad (\sqrt{|c'|}, \kappa|c'|^2) \end{cases}$$

is an embedded submanifold of $H^k(S^1,\mathbb{R}_{>0}\times\mathbb{R})$.

Proof. The image is given by

$$\operatorname{im}(R) = \{q \in H^k(S^1,\mathbb{R}_{>0}\times\mathbb{R}) : H_{\mathrm{cl}}(q) = 0\}\,,$$

with the functional H_{cl} given by

$$H_{\mathrm{cl}}(q) = \int_0^{2\pi} q_1^2 \exp\left(i \int_0^\theta q_1(\sigma)^{-2} q_2(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\theta\,.$$

The gradient of H_{cl} was computed in Sect 4.4. Since 0 is a regular point of H_{cl}, the statement of the lemma follows from the implicit function theorem in Banach spaces. □

Since the image is defined by a finite number of constraints, the projection to the orthogonal complement can be written explicitely.

Lemma 7.2.2. *Let $k \geq 2$. The orthogonal projection* $\operatorname{Proj}^{\mathrm{im}}$ *to $T\operatorname{im}(R)$ can be extended to a smooth map*

$$\operatorname{Proj}^{\mathrm{im}} : TH^k(S^1,\mathbb{R}_{>0}\times\mathbb{R}) \to TH^k(S^1,\mathbb{R}_{>0}\times\mathbb{R})\,.$$

Proof. The smoothness of the maps $q \mapsto \operatorname{grad}^{L^2} H^i_{\mathrm{cl}}(q)$ for $i = 1,2$ given by (17) is clear and we see by inspection that $v^i(q) := \operatorname{grad}^{L^2} H^i_{\mathrm{cl}}(q) \in H^k(S^1,\mathbb{R}^2)$. Let $w^1(q), w^2(q)$ be an orthonormal basis of $\operatorname{span}\{v^1(q), v^2(q)\}$, constructed, e.g., via Gram-Schmidt. Then the orthogonal projection is given by

$$\operatorname{Proj}^{\mathrm{im}}(q).h = h - \langle h, w^1(q)\rangle w^1(q) - \langle h, w^2(q)\rangle w^2(q)$$

and is smooth. □

For the metric (7),

$$G_c(h,k) = \int_{S^1} \kappa^{-3/2}\langle D_s^2 h, n\rangle\langle D_s^2 k, n\rangle + \langle D_s h, v\rangle\langle D_s k, v\rangle\,\mathrm{d}s\,,$$

the analogues of the above lemmas can be proven in the same way.

8. Discretization

In this section we describe, how one can use the R-transform to discretize the geodesic equation of second order metrics. To make the exposition more concise we will restrict ourselves to the third metric, described in Sect. 5, although the principles are rather general.

We consider the metric (19),

$$G_c(h,k) = \int_M \langle D_s h, D_s k\rangle + \langle D_s^2 h, n\rangle\langle D_s^2 k, n\rangle \,\mathrm{d}s\,.$$

The space $\operatorname{Imm}/\operatorname{Tra}$ with the metric G is isometric to

$$\operatorname{im}(R) = \left\{ (q_1,q_2,q_3)\;:\; \begin{array}{l} (1)\, q_2' = q_1^{-2} q_3 \\ (2)\, \int_{S^1} q_1^2 \exp(iq_2)\,\mathrm{d}\theta = 0 \end{array} \right\},$$

which is a submanifold of $C^\infty(S^1,\mathbb{R}^3_*)$ equipped with the G^{L^2}-metric. Here $\mathbb{R}^3_* = \mathbb{R}_{>0}\times S^1\times\mathbb{R}$ and we equip it with the non-flat Riemannian metric

$$g = 4\,dq_1\otimes dq_1 + q_1^2\,dq_2\otimes dq_2 + q_1^{-6}\,dq_3\otimes dq_3\,.$$

Instead of discretizing the space $\operatorname{Imm}/\operatorname{Tra}$ and the geodesic equation thereon, we discretize instead $C^\infty(S^1,\mathbb{R}^3_*)$, the metric G^{L^2} and the constraints defining $\operatorname{im}(R)$.

8.1. **Spatial discretization.** We replace the curve $q\in C^\infty(S^1,\mathbb{R}^3_*)$ by N uniformly sampled points $q^1,\dots,q^N$ with $q^k = q(2\pi k/N)$. Denote by $\Delta\theta = 2\pi/N$ the spatial resolution. The continuous geodesic equation corresponds to a Hamiltonian system with the Hamiltonian

$$E_{\mathrm{cont}}(q,p) = \frac{1}{2}\int_{S^1} g^{-1}_{q(\theta)}(p(\theta),p(\theta))\,\mathrm{d}\theta\,,$$

together with the constraint functions H_{diff}, H_{cl} defined in (22), that define the image of the R-transform. Instead of discretizing the geodesic equations directly, we discretize the Hamiltonian function. The discrete Hamiltonian is

$$E_{\mathrm{discr}}(q^1,\dots,q^N,p^1,\dots,p^N) = \frac{1}{2}\sum_{k=1}^N g^{-1}_{q^k}(p^k,p^k)\Delta\theta\,.$$

To simplify notation we shall denote the discretized curve $(q^1,\dots,q^N)$ again by q and the same for the momentum. The discrete constraint functions are

$$H^k_{\mathrm{diff}}(q,p) = \left(q_1^k\right)^{-2} q_3^k - \left(q_2^{k+1} - q_2^k\right)/\Delta\theta$$
$$H_{\mathrm{cl}}(q,p) = \sum_{k=1}^N \exp(iq_2^k)(q_1^k)^2\Delta\theta\,.$$

Thus we have replaced an infinite-dimensional system by a $3N$-dimensional Hamiltonian system with $N+2$ constraints. The resulting system has $2N-2$ degrees of freedom. This corresponds to discretizing a plane curve by N points and removing translations, again leading to $2N-2$ degrees of freedom.

The advantage of the R-transform is that the Hamiltonian E_{cont} of the continuous system doesn't contain spatial derivatives, since it is related to an L^2-type metric. The spatial derivatives appear in the constraints, in particular in H_{diff}, which enforces that the third component q_3 is a derivative of the second component q_2. However, even though q_3 represents the curvature of the curve and thus a second derivative, the constraint H_{diff} is written in terms of the first derivative only, which can be discretized using first order differences.

Our discrete equations are now Hamilton's equations of a constrained Hamiltonian system. Denote by $H : \mathbb{R}^{3N} \to \mathbb{R}^{N+2}$ the collected constraint functions and let $\lambda \in \mathbb{R}^{N+2}$ be a Lagrange multiplier. Then Hamilton's equations are

$$\begin{aligned} \partial_t q &= \partial_p E_{\mathrm{discr}}(q,p) \\ \partial_t p &= -\partial_q E_{\mathrm{discr}}(q,p) + DH^T(q).\lambda \\ H(q) &= 0\,. \end{aligned} \tag{30}$$

8.2. **Time discretization.** There is a variety of integrators available for the time-discretization of a constrained Hamiltonian system. For example RATTLE is a second-order symplectic method, that preserves constraints exactly. A time-step of RATTLE is given by the following equations. To simplify notation, we denote in this section by $q^j = q(t_j)$ the system at time t_j.

$$\begin{aligned} p^{j+1/2} &= p^j + \frac{\Delta t}{2}\left(-\partial_q E(q^j, p^{j+1/2}) + DH^T(q^j).\lambda_1\right) \\ q^{j+1} &= q^j + \frac{\Delta t}{2}\left(\partial_p E(q^j, p^{j+1/2}) + \partial_p E(q^{j+1}, p^{j+1/2})\right) \\ 0 &= H(q^{j+1}) \\ p^{j+1} &= p^{j+1/2} + \frac{\Delta t}{2}\left(-\partial_q E(q^{j+1}, p^{j+1/2}) + DH(q^{j+1})^T.\lambda_2\right) \\ 0 &= DH(q^{j+1}).\partial_p E(q^{j+1}, p^{j+1})\,. \end{aligned} \tag{31}$$

This method was first proposed in [18]. One first performs a momentum update with half of the timestep using the implicit Euler method and an unknown Lagrange multiplier λ_1. This is followed by a full time-step for the position using the implicit midpoint rule. The Lagrange multiplier λ_1 is determined by the condition $H(q^{j+1})$, which guarantees that the constraints are exactly satisfied in each time-step. Then we perform another half time-step for the momentum with the explicit Euler and determine the Lagrange multiplier λ_2 by requiring the hidden constraint $DH(q^{j+1}).\partial_p E(q^{j+1}, p^{j+1}) = 0$ to be satisfied. See [16, 21] for more details about symplectic integrators.

9. Experiments

In this section we present a series of numerical examples to demonstrate the value of R-transforms for numerical computations. The examples were computed as described in Sect. 8. In all these examples we will only consider the third metric, i.e.,

$$G_c(h,h) = \int_{S^1} \langle D_s h, D_s h\rangle + \langle D_s^2 h, n\rangle^2 ds\,.$$

The curves are discretized with 100 points and since the metric ignores translations we centered all curves such that their center of mass lies at the origin. In Fig. 1 we show two examples of solutions to the geodesic boundary value problem.

The second series of examples (Fig. 2) is concerned with the geodesic initial value problem. It shows two geodesic that starts at the circle with two different initial velocities.

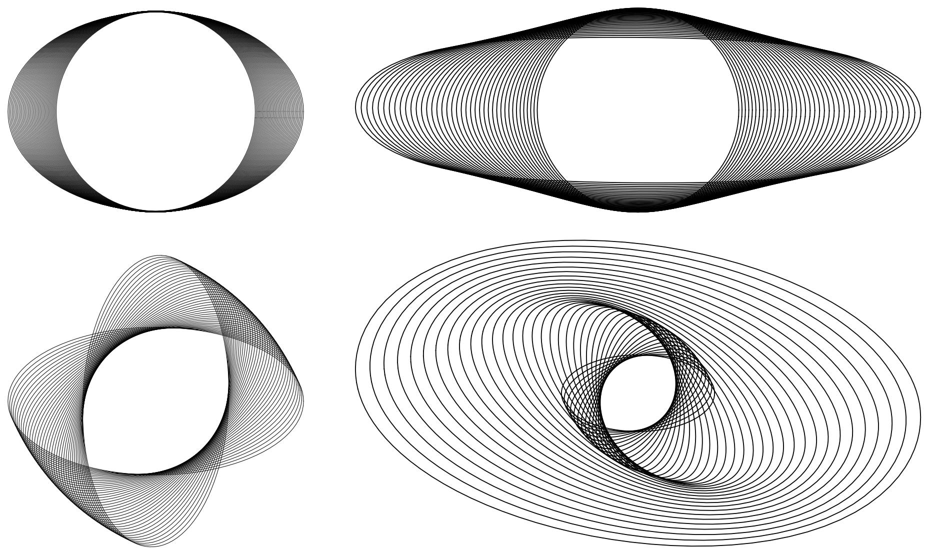

FIGURE 1. First line, left side: Geodesic connecting a circle to an ellipse in time $t = 2$. First line, right side: The geodesic continued until time $t = 6$. Second line, left side: Geodesic connecting a ellipse to a rotated ellipse in time $t = 2$. Second line, right side: The geodesic continued until time $t = 6$.

The last example shows a geodesic in the shape space of umparametrized curves. Following the presentation of [26] we identify this space with the quotient space

$$B_i(S^1, \mathbb{R}^2) = \operatorname{Imm}(S^1, \mathbb{R}^2) / \operatorname{Diff}(S^1).$$

Every reparametrization invariant metric on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ induces a metric on the shape space $B_i(S^1, \mathbb{R}^2)$ such that the projection

$$\pi : \operatorname{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2)$$

is a Riemannian submersion. In this setting geodesics on shape space $B_i(S^1, \mathbb{R}^2)$ correspond to horizontal geodesics on $\operatorname{Imm}(S^1, \mathbb{R}^2)$, i.e., geodesics on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ with horizontal velocity. By the conservation of reparametrization momentum a geodesic with horizontal inital velocity stays horizontal for all time and thus this condition has to be checked at the initial point of the geodesic only. For a detailed description of this construction see [26, 7].

For metrics that are induced by a differential operator field L the horizontality condition can be expressed as

$$h \in \operatorname{Hor}(c) \subset T_c \operatorname{Imm}(S^1, \mathbb{R}^2) \Leftrightarrow L_c h = f.n, \quad f \in C^\infty(S^1)\,,$$

with n denoting the normal field to c. In the following we want to investigate this condition for the third metric,

$$G_c(h, h) = \int_{S^1} \langle D_s h, D_s h \rangle + \langle D_s^2 h, n \rangle^2 ds\,.$$

This metric is induced by the differential operator

$$L_c h = D_s^2\big(\langle D_s^2 h, n \rangle n\big) - D_s^2 h\,.$$

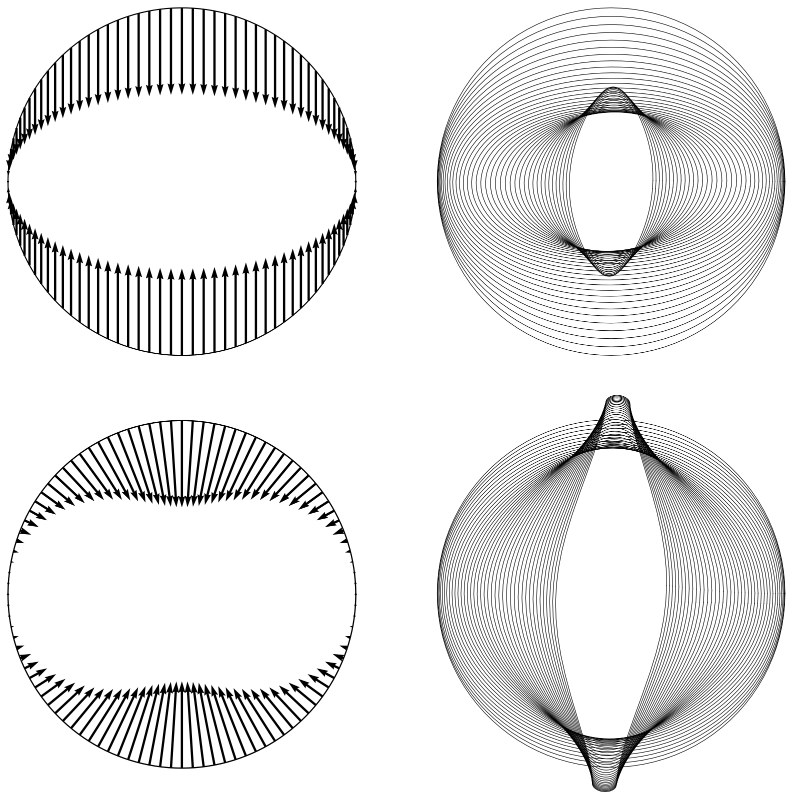

FIGURE 2. Two examples of a geodesic that starts at the circle. On the right hand side the geodesic is printed, whereas on the left hand side we pictured the initial velocity. First line: Initial velocity $h = (0, \sin\theta)$. The geodesic is computed until time 2. Second line: Initial velocity $h = -\sin^2\theta(\cos\theta, \sin\theta)$. The geodesic is computed until time 1.

To simplify the expressions we choose the circle $c(\theta) = (\cos\theta, \sin\theta)$ as the starting point of the geodesic. Then we have

$$|c'| = 1, \quad D_s = \partial_\theta, \quad \partial_\theta n = v \quad \text{and} \quad \partial_\theta v = -n\,.$$

Let $h = an + bv$. Then

$$\begin{aligned} D_s^2 h &= \partial_\theta^2(an + bv) = \partial_\theta(a'n + an' + b'v + bv') \\ &= a''n + 2a'n' + an'' + b''v + 2b'v' + bv'' \\ &= (a'' - a + 2b')n + (-2a' + b'' - b)v \end{aligned}$$

Thus we have

$$\begin{aligned} \partial_\theta^2\big(\langle \partial_\theta^2 h, n\rangle n\big) &= \partial_\theta^2\big((a'' - a - 2b')n\big) \\ &= (a'''' - a'' + 2b''')n + 2(a''' - a' + 2b'')n' + (a'' - a + 2b')n'' \\ &= -2(a''' - a' + 2b'')v + (a'''' - 2a'' + a - 2b' + 2b''')n \end{aligned}$$

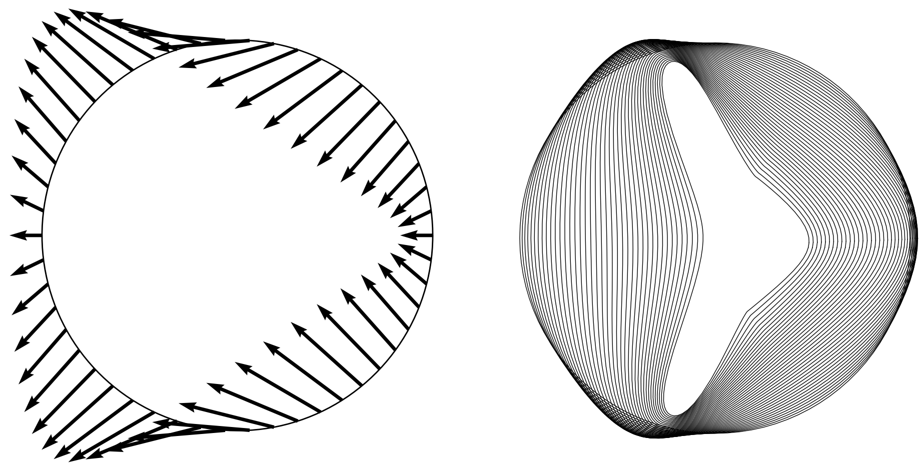

FIGURE 3. A horizontal geodesic on $\mathrm{Imm}(S^1,\mathbb{R}^2)$ with initial velocity $h=-(2-\cos 2\theta, 2\sin 2\theta)$ until time .3.

From this we can read off the tangential and the normal part of Lh:

$$\langle L_c h, v\rangle = -2a''' + 4a' - 5b'' + b$$
$$\langle L_c h, n\rangle = a'''' - 3a'' + 2a + 2b''' - 4b'$$

A horizontal velocity satisfies $\langle L_c h, v\rangle = 0$, which leads to the ODE

$$-2a''' + 4a' - 5b'' + b = 0\,.$$

A solution to this equation is given for example by

$$a = 3\cos\theta, \quad b = \sin\theta\,,$$

This leads the initial velocity

$$h = (-\cos^2\theta - 3\sin^2\theta, -4\cos\theta\sin\theta) = -(2-\cos 2\theta, 2\sin 2\theta)\,.$$

Appendix A. Translation invariant sprays

Let N be a finite dimensional manifold. Consider a spray Ξ on $C^\infty(S^1,N)$ that has smooth extensions to $H^k(S^1,N)$ for $k\geq k_0$. For each initial condition $(q_0,v_0)\in TH^k(S^1,N)$ and each Sobolev order k denote by $J_k(q_0,v_0)$ the maximal domain of existence of its flow. If $(q_0,v_0)\in TH^{k+1}(S^1,N)$ then a-priori we only have the inclusion

$$J_{k+1}(q_0,v_0)\subseteq J_k(q_0,v_0)\,.$$

To obtain that smooth initial conditions have smooth solutions need to know that the length of the maximal existence interval is bounded from below, i.e., $\cap_{k\geq k_0} J_k(q_0,v_0)=\{0\}$ cannot happen. If the spray is invariant under translations, this is ruled out by the following result.

Theorem A.1 (Ebin-Marsden, 1970)**.** *Let the spray Ξ be invariant under translations, i.e.*

$$\Xi(T(\sigma)q, T(\sigma)v) = T(\sigma)\Xi(q,v)\,,$$

with $T(\sigma)q(\theta) = q(\theta+\sigma)$ being the translation group. Then for initial conditions $(q_0, v_0) \in TH^{k+1}(S^1, N)$ we have

$$J_{k+1}(q_0, v_0) = J_k(q_0, v_0)\,.$$

Proof. This result is implicit in [12, Thm. 12.1] and made explicit in [13, Lem. 5.1]. We prove it here only for $N = \mathbb{R}^d$, the general case requiring only slightly more cumbersome notation.

If the spray is translation-invariant, then so is the exponential map,

$$\exp(T(\sigma)q_0, T(\sigma)tv_0) = T(\sigma)\exp(q_0, tv_0)\,.$$

Differentiating at $\sigma = 0$ gives

$$T_{(q_0,tv_0)}\exp.(q_0', tv_0') = \partial_\theta \exp(q_0, tv_0)\,.$$

From our assumption $q_0, v_0 \in H^{k+1}(S^1, \mathbb{R}^d)$ it follows that the left-hand side is an element of $H^k(S^1, \mathbb{R}^d)$. Since $q(t) = \exp(q_0, tv_0)$ we see that $q'(t) \in H^k(S^1, \mathbb{R}^d)$ which implies $q(t) \in H^{k+1}(S^1, \mathbb{R}^d)$. Thus $J_k(q_0, v_0) = J_{k+1}(q_0, v_0)$. □

Appendix B. Regularity for elliptic equations

Lemma B.1. *Let $k \geq 2$ and let L be the operator*

$$Lu = -(au')' + bu$$

with $a \in H^{k-1}(S^1)$, $b \in H^{k-2}(S^1)$ and $a > 0$, $b \geq \varepsilon > 0$ for $\varepsilon \in \mathbb{R}$. The L is a bibounded, invertible operator

$$L : H^k(S^1) \to H^{k-2}(S^1)\,.$$

Furthermore the map $L^{-1} : a, b, f \mapsto L^{-1}_{a,b} f$ is a smooth map

$$L^{-1} : H^{k-1}(S^1) \times H^{k-2}(S^1) \times H^{k-2}(S^1) \to H^s(S^1)\,.$$

Proof. This lemma can be proven in the same way as existence and regularity results are proven for second order elliptic PDEs in, e.g., [14, Chap. 6]. The proofs can be followed line by line, even though we have required less regularity for the coefficient functions. □

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