Pseudo-holomorphic curves

Transversality and a Gromov-Witten invariant
of simple curves

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Vorwort

Dies ist eine geringfuegig revidierte Version meiner Diplomarbeit, welche ich im Mai 2002 eingereicht habe. Es geht dabei um das Studium pseudo-holomorpher Kurven in symplektischen Mannigfaltigkeiten (M, ω). Zu vorgegebener fast-komplexer Struktur J betrachtet man also die Familie der Loesungen $\overline{\partial}_J u = 0$ des elliptischen Differentialoperators $\overline{\partial}_J$ – der Cauchy-Riemann Operator. Hierbei werden die beteiligten geometrischen Objekte mit einer Banach Raum Struktur versehen und der Cauchy-Riemann Operator $\overline{\partial}_J$ wird als Schnitt in einem Banach Raum Buendel aufgefasst. Das hat den entscheidenden Vorteil, dass nun Fredholm und Transversalitaets Theorie anwendbar sind. In dieser Studie ist man einerseits sehr flexibel, da sich das J aus einer grossen Menge von ω-zahmen, generischen, fast komplexen Strukturen waehlen laesst; andererseits ist der Raum der Loesungen $\mathcal{M}(A, J)$, das sind einfache, pseudo-holomorphe Kurven, welche eine fixe Klasse $A \in H_2(M)$ representieren, klein genug sodass sinnvolle Aussagen moeglich sind. In der Tat, fuer generisches J ist $\mathcal{M}(A, J)$ eine in natuerlicher Weise orientierte, endlich dimensionale Mannigfaltigkeit. Sind nun J_0, J_1 zwei ω-zahme generische Strukturen, so sieht man dass $\mathcal{M}(A, J_0)$ und $\mathcal{M}(A, J_1)$ orientiert bordant sind. (All das ist Gegenstand von Kapitel 4, wobei der geeignete Bordismus Begriff in Kapitel 5 entwickelt wird.) Eines der Hauptprobleme in der Theorie der pseudo-holomorphen Kurven ist es nun eine gute Kompaktifizierung der Moduli Raeume $\mathcal{M}(A, J)$ zu finden. Unter zusaetzlichen Bedingungen an (M, ω) laesst sich zeigen (Kapitel 5) dass eine Kompaktifizierung durch Hinzugabe von Objekten von Kodimension 2 erzielt werden kann. Dies reicht um eine nicht triviale Bordismen Invariante des Tripels (M, ω, A) zu definieren.

Ein wesentlicher Grund, weswegen pseudo-holomorphe Kurven so interessant sind, ist dass sie viele Probleme der symplektischen Geometrie ueberhaupt erst angreifbar gemacht haben. Im Gegensatz zur Riemann'schen Geometrie sehen ja lokal alle symplektischen Mannigfaltigkeiten von selber Dimension gleich aus. Ein Verdienst der Gromov'schen Theorie ist es nun, Zugang zu globalen Invarianten zu verschaffen; wie etwa jene, oben erwaehnte Bordismen Invariante. Vielschichtige Anwendungen zu diesem Thema finden sich zum Beispiel in den Arbeiten von Gromov [6], McDuff [13], dem Buch von Audin-Lafontaine [3] sowie den darin enthaltenen Verweisen. Ein typisches solches Problem der symplektischen Topologie, das Non-Squeezing Theorem, wird in Abschnitt 5.E. behandelt.

Es gibt auch viele Querverbindungen zur mathematischen Physik. Dies begann vor allem mit der Arbeit von Ruan [23], der die bereits erwaehnte Bordismen Invariante einfuehrte; es handelt sich um die Gromov-Witten Invarianten Φ und Ψ. Es ist letztere Invariante, welche von besonderer physikalische Relevanz ist. Wie dem auch sei, da sich erstere, i.e. Φ, auf einfache Kurven beschraenkt ist deren Definition einfacher und daher wird im Folgenden nur auf diese eingegangen.

Die vorliegende Arbeit folgt in ihrer Gestalt in erster Linie dem Werk von McDuff, Salamon [16]. Weitere Hauptreferenzwerke sind Aebischer et al. [1], Gromov [6], McDuff [13, 14], sowie Ruan [23]. Die Transversalitaets Theorie pseudo-holomorpher Kurven wurde im Wesentlichen in McDuff [13] entwickelt.

Danke. Bei Peter Michor bedanke ich mich fuer die von Wissen und Erfahrung gepraegte Betreuung. Ausserdem moechte ich mich bei den Teilnehmern des Seminars bedanken; ganz besonders bei Stefan Haller, der sich die Muehe machte weite Teile eines noch unfertigen Manuskriptes durchzulesen.

Ich habe das Glueck von meinen Eltern immer unterstuetzt worden zu sein. Die letzten Jahre waren eine schoene Zeit, die ich mit Veronika geteilt habe.

Preface

This is a slightly revised version of my diploma thesis as submitted in May 2002. This paper is concerned with the study of pseudo-holomorphic curves in symplectic manifolds (M, ω) as introduced by Gromov [6] in 1985. Chapter 1 provides some basic facts to be used later on. Chapter 2 states Gromov's theorem about weak convergence of pseudo-holomorphic curves and introduces cusp curves. There are no proofs, however, as this could well be the subject of a (diploma) thesis in its own right. Chapter 3 establishes the necessary background on smooth spaces of mappings. It is shown that the Sobolev space of mappings from a closed Riemannian surface Σ to a manifold M is a smooth manifold modelled on certain Banach spaces. Finally the determinant bundle associated to a family of Fredholm operators is treated. Chapters 4 and 5 follow mainly the book of McDuff, Salamon [16]. Here the theory takes its starting point with the study of the elliptic partial differential equation

$$\overline{\partial}_J u = \tfrac{1}{2}(du + J \circ du \circ j) = 0$$

where $u \colon \Sigma \to M$ is a curve from the Riemann surface (Σ, j) to the almost complex manifold (M, J), and J is assumed to be ω-tame. By introducing a Banach space structure on the various geometric objects one can apply transversality and Fredholm theory to the study of the Banach space section $\overline{\partial}$ and its zeroes, the moduli space

$$\mathcal{M}(A, J) = \{u : \overline{\partial}_J u = 0, u_*[\Sigma] = A, u \text{ is somewhere injective } \}$$

of J-holomorphic curves in a fixed homology class $A \in H_2(M)$. In the generic case $\mathcal{M}(A, J)$ is a finite-dimensional, naturally oriented manifold, and different choices of generic structures J_0, J_1 give rise to oriented bordant moduli spaces. Now one has to do quite some work to show that this bordism is a compact one in some sense. Indeed, under additional assumptions on (M, ω, A), it can be shown that the obstruction against the moduli spaces' being compact is only of codimension 2. This is part of the structure theorem in Section 5.C. Once this is established one has the main ingredients necessary to define the Gromov-Witten invariant Φ as it was first introduced by Ruan [23]. Finally section 5.E. derives the non-squeezing theorem as a first application of the Gromov-Witten invariant.

Besides McDuff, Salamon [16] the most used references are Aebischer et al. [1], Gromov [6], McDuff [13, 14], and Ruan [23]. Most of the transversality theory for pseudo-holomorphic curves is due to McDuff [13].

Vienna, October 2002.

Lebenslauf

- Geboren in Wien, am 12. Oktober 1979.
- 1990 – 1998. Besuch des Gymnasiums auf der Schmelz, Wien.
- 1995/96. In diesem akademischen Jahr habe ich im Rahmen eines AFS-Programmes eine High School in Denver, USA, besucht.
- 1998/99. Beginn des Physik Studiums an der Universitaet Wien; der erste Studienabschnitt ist abgeschlossen, dabei wird es jedoch bleiben.
- 1999/00. Beginn des Mathematik Studiums an der Universitaet Wien.
- Das Wintersemester 2001/02 habe ich, unterstuetzt von einem Erasmus Stipendium, an der Universidad de Santiago de Compostela, Spanien, verbracht.

Contents

Chapter 1

Basic notions

1.A. Almost complex structures

If M is a smooth (real) manifold then a TM valued 1-form $J \in \Omega^1(M;TM)$ is called an *almost complex* structure on M if $J(p) \circ J(p) = -\mathrm{id}_{T_pM}$ for all $p \in M$, and (M,J) is said to be an *almost complex* manifold. The set of all almost complex structures on M will be denoted by $\mathcal{J}(M)$.
A smooth manifold M modelled on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ is a *complex manifold* if its chart changing maps are holomorphic maps on open domains. If this is the case, one can equip the tangent space at every point with a complex structure such that it becomes a complex vector space. The thus obtained almost complex structure on M is called the *induced* almost complex structure. In particular, every complex manifold is an almost complex manifold.

1. If $M = \mathbb{R}^{2n}$ then $\mathcal{J}(\mathbb{R}^{2n}) = \mathcal{J}(2n,\mathbb{R}) = \{J \in \mathrm{End}(\mathbb{R}^{2n}) : J^2 = -\mathrm{id}\}$, and *the action $GL(2n,\mathbb{R}) \times \mathcal{J}(2n,\mathbb{R}) \to \mathcal{J}(2n,\mathbb{R})$, $(g,J) \mapsto g \cdot J = g \circ J \circ g^{-1}$ is transitive, and there is a natural identification of homogeneous spaces*

$$\frac{\mathrm{GL}(2n,\mathbb{R})}{\mathrm{GL}(n,\mathbb{C})} \cong \mathcal{J}(2n,\mathbb{R}).$$

Indeed, let $J, J' \in \mathcal{J}(2n,\mathbb{R})$ and consider a transformation $g \in \mathrm{GL}(2n,\mathbb{R})$, $g \times \cdots \times g : (x^1, Jx^1, \ldots, x^n, Jx^n) \mapsto (y^1, J'y^1, \ldots, y^n, J'y^n)$ of J- and J'-complex bases. Then

$$\begin{aligned} g^{-1}J'(y^i + J'y^k) &= Jx^i - x^k \\ &= J(x^i + Jx^k) \\ &= Jg^{-1}(y^i + J'y^k) \end{aligned}$$

for all $i,k \in \{1,\ldots,n\}$ yields $J' = g \cdot J$. The stabilizer of $J \in \mathcal{J}(2n,\mathbb{R})$ in the group is $\mathrm{GL}(2n,\mathbb{R})_J = \{g \in \mathrm{GL}(2n,\mathbb{R}) : g \cdot J = J \Leftrightarrow g \circ J = J \circ g\}$.

Thus also the second statement follows since $\mathrm{GL}(2n,\mathbb{R})_{J_0} = \mathrm{GL}(n,\mathbb{C})$ with the standard complex structure J_0 on $\mathbb{C}^n$. $\square$

2. Let (V,ω) be a real, finite dimensional symplectic vector space, i.e. $\omega \in \Lambda^2 V^*$, and $\breve{\omega} : V \to V^*$, $v \mapsto \omega(v, \_)$ is an isomorphism. A complex structure $J \in \mathcal{J}(V)$ that satisfies

$$\omega(x, Jx) > 0 \ \forall x \neq 0 \tag{T}$$

is called ω-*tame*, and $\mathcal{J}_{\mathrm{t}}(\omega)$ is the set of all ω-tame structures on V. The pair (ω, J) induces a scalar product

$$g_{\mathrm{t}}(x,y) = \frac{1}{2}(\omega(x,Jy) - \omega(Jx,y)),$$

and J is a skew-symmetric isometry with respect to g_{t}. $\square$

3. The set $\mathcal{J}_{\mathrm{c}}(\omega)$ of complex structures satisfying

$$\omega(x, Jx) > 0 \ \forall x \neq 0 \tag{T}$$

$$\omega(x,y) = \omega(Jx, Jy) \ \forall x,y \iff J^*\omega = \omega \tag{C}$$

is the set of ω-*compatible* structures on V. A pair (ω, J) is compatible if and only if

$$g_{\mathrm{c}}(x,y) = \omega(x, Jy)$$

defines a scalar product on V: $g_{\mathrm{c}}(x,y) = \omega(x,Jy) = \omega(Jx, JJy) = \omega(y, Jx) = g_{\mathrm{c}}(y,x)$, and $\omega(x,y) = g_{\mathrm{c}}(x,-Jy) = -g_{\mathrm{c}}(Jy,x) = \omega(Jx,Jy)$. Again J is a skew-symmetric isometry, and if the tame pair (ω, J) is compatible then g_{t} and g_{c} coincide.
Moreover, on a metric and symplectic vector space (V, g, ω) there also is an automorphism $J := \breve{g}^{-1} \circ \breve{\omega}$, $g(Jx,y) = \omega(x,y)$, and the following are equivalent: $J^*\omega = \omega \iff J^2 = -\mathrm{id}_V \iff J^*g = g$. Indeed,

$$\begin{aligned}
\omega(Jx,Jy) = \omega(x,y) \iff g(J^2x, Jy) &= \omega(Jx, Jy) \\
&= \omega(y, -x) \\
&= g(Jy, -x) \\
&= g(-x, Jy) \\
\iff J^2 &= -\mathrm{id} \\
\iff J^{-1} &= -J \\
\iff g(Jx, Jy) &= \omega(x, Jy) \\
&= \omega(-Jy, x)
\end{aligned}$$

$$= g(y,x).$$

If one of these equivalent conditions is fulfilled then J is an ω-compatible complex structure; $\omega(x,Jx) = g(x,x) > 0$ for all $x \neq 0$. □

4. Example: consider $\mathbb{R}^{2n}$ with its standard dual basis $(e^1, \ldots, e^{2n})$, then the canonical structures

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \& \quad g_0 = \sum_{i=1}^{2n} e^i \otimes e^i \quad \& \quad \omega_0 = \sum_{i=1}^{n} e^i \wedge e^{i+n}$$

are compatible: $\omega_0(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, J_0 \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}) = (e^1 \wedge e^2)(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \begin{pmatrix} -y^2 \\ y^1 \end{pmatrix}) = x^1y^1 + y^2x^2 = g_0(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \begin{pmatrix} y^1 \\ y^2 \end{pmatrix})$, and the general case follows by using more indices.□

5. Let W be a finite dimensional vector space with dual W^*. Then $(W \times W^*, \overline{\omega})$ is a symplectic vector space with symplectic form $\overline{\omega}((x,x^*),(y,y^*)) := \langle y^*, x\rangle - \langle x^*, y\rangle$, where $\langle \_, \_\rangle$ denotes the duality pairing. If $(x_1, \ldots, x_n)$ is a basis of $W = W^{**}$, $(x_1^*, \ldots, x_n^*)$ its dual basis then, for any $\{a,b,c,d\} \subseteq \{1, \ldots, n\}$,

$$\overline{\omega}((x_a, x_b^*), (x_c, x_d^*)) = \langle x_d^*, x_a\rangle - \langle x_b^*, x_c\rangle = (\sum_{k=1}^{n} l_k^* \wedge l_k)((x_a, x_b^*), (x_c, x_d^*))$$

implies $\overline{\omega} = \sum_{k=1}^n l_k^* \wedge l_k$, where $l_k := 0_W \oplus x_k$, and $l_k^* := x_k^* \oplus 0_{W^*}$. In particular the transversal subspaces $W \times \{0\}$ and $\{0\} \times W^*$ are Lagrangian. □

6. Have (V, ω, J, g) carry compatible structures as above, and define the ω-*orthogonal* of a linear subspace $L \subseteq V$ to be $L^\circ := \{x \in V : \omega(L,x) = \breve{\omega}(L)(x) = \langle \breve{\omega}(L), x\rangle = \{0\}\}$. L is called *Lagrangian* if $L = L^\circ$, and since $\dim L + \dim L^\circ = \dim V$ it follows that $\dim L = \frac{1}{2}\dim V$. Observe furthermore that

$$JL = J(L^\circ) = \{Jx : \omega(x,L) = \omega(Jx, JL) = \{0\}\} = (JL)^\circ, \text{ and}$$
$$JL = L^\perp := \{x \in V : g(L,x) = \{0\}\} \text{ (the } g\text{-}orthogonal\text{)}:$$

$x \in L \iff g(Jx,y) = \omega(x,y) = 0\ \forall y \in L \iff Jx \in L^\perp$. Thus L and JL are transversal Lagrangian subspaces. With the two-form from above the mapping

$$(V, \omega) = (L \oplus JL, \omega) \xrightarrow{\psi} (L \oplus L^*, \overline{\omega})$$

$$x \oplus Jy \longmapsto x \oplus \breve{\omega}(-Jy)$$

becomes a symplectomorphism:

$$\begin{aligned}\omega(x_1 \oplus Jy_1, x_2 \oplus Jy_2) &= \omega(x_1, Jy_2) + \omega(Jy_1, x_2)\\ (\psi^*\overline{\omega})(x_1 \oplus Jy_1, x_2 \oplus Jy_2) &= \langle \breve{\omega}(-Jy_2), x_1\rangle - \langle \breve{\omega}(-Jy_1), x_2\rangle\\ &= \omega(x_1, Jy_2) + \omega(Jy_1, x_2)\end{aligned}$$

for all $x_i, y_i \in L$. □

7. *The action* $a : \mathrm{Sp}(V,\omega) \times \mathcal{J}_c(\omega) \to \mathcal{J}_c(\omega)$, $(g, J) \mapsto g \cdot J := gJg^{-1}$ *is transitive.* First note that $\operatorname{im} a \subseteq \mathcal{J}_c(\omega)$ since $\omega(gJg^{-1}x, gJg^{-1}y) = \omega(x, y)$ by the assumptions. Let $J, J_1 \in \mathcal{J}_c(\omega)$, and fix a lagrangian subspace $L \subseteq V$. There exist complex bases $(x_1, \ldots, x_n, Jx_1, \ldots, Jx_n)$ and $(y_1, \ldots, y_n, J_1y_1, \ldots, J_1y_n)$ of $L \times JL$ and $L \times J_1L$, respectively, and a transformation $\varphi \in \mathrm{GL}(V)$ such that $\varphi(x_i) = y_i$ and $\varphi(Jx_i) = J_1y_i$. It follows as above that $J_1 = \varphi J \varphi^{-1}$. Now $(L \times L^*, \overline{\omega})$ is a symplectic vector space, and $\varphi_1 := \varphi|_L \oplus (\varphi|_L^{-1})^*$ satisfies

$$\varphi_1^*\overline{\omega} = \varphi_1^* \sum_{i=1}^n y_i^* \wedge y_i = \sum_{i=1}^n y_i^* \circ \varphi|_L \wedge y_i^{**} \circ (\varphi|_L^{-1})^* = \sum_{i=1}^n x_i^* \wedge x_i^{**} = \overline{\omega}.$$

By the previous example the map $\psi \colon (V, \omega) = (L \oplus JL, \omega) \to (L \oplus L^*, \overline{\omega})$, $(x, Jy) \mapsto (x, \breve{\omega}(-Jy))$ is a symplectomorphism. Thus also the composition $\psi^{-1} \circ \varphi_1 \circ \psi : (V, \omega) \to (L \oplus L^*, \overline{\omega}) \to (L \oplus L^*, \overline{\omega}) \to (V, \omega)$ is symplectomorphic, and furthermore has the property that

$$\begin{aligned}(\psi^{-1} \circ \varphi_1 \circ \psi \circ J)(x_i, Jx_k) &= (\psi^{-1} \circ \varphi_1 \circ \psi)(-x_k, Jx_i)\\ &= (\psi^{-1} \circ \varphi_1)(-x_k, \breve{\omega}(-Jx_i))\\ &= \psi^{-1}(-y_k, \breve{\omega}(-J_1y_i))\\ &= (-y_k, J_1y_i)\\ &= J_1(y_i, J_1y_k)\\ &= (J_1 \circ \psi_0)(x_i \oplus Jx_k)\end{aligned}$$

where $\psi_0 := \psi^{-1} \circ \varphi_1 \circ \psi \in \mathrm{Sp}(V, \omega)$, and this makes the action transitive, for $\psi_0 \cdot J = \psi_0 J \psi_0^{-1}$. *Therefore, fixing a Lagrangian subspace of V and a compatible pair (ω, J) yields*

$$\mathcal{J}_c(\omega) = \frac{\mathrm{Sp}(V,\omega)}{\mathrm{U}(V, g_c)}.$$

Note that $gJ = Jg \iff g_c(gx, gy) = \omega(gx, Jgy) = \omega(gx, gJy) = \omega(x, Jy) = g_c(x, y) \iff g_c(gx, y) = g_c(x, g^{-1}y)$ which makes the isotropy subgroup at J assume the form $\mathrm{Sp}(V, \omega)_J = \mathrm{U}(V, g_c)$. □

Proposition 1.1. *Let V be a real, finite dimensional vector space equipped with compatible structures (ω, J_0). Then the transformation $\mu_i : \mathcal{J}_i(\omega) \to E_1(\mathcal{S}_i)$, $J \mapsto (J+J_0)^{-1} \circ (J - J_0)$ is a diffeomorphism for $i \in \{t, c\}$.*

This proposition is a version of the Cayley transformation. The statement is not empty since example (4) shows that it is nonempty in one coordinate system, and thus in any, for the argument may be carried out independently of all such choices.

$$\begin{aligned}
&\mathcal{S}_t := \{S \in \mathrm{L}(V) : J_0 S + S J_0 = 0\}\\
&\mathcal{S}_c := \{S \in \mathrm{L}(V) : J_0 S + S J_0 = 0 \ \ \& \ \ g_c(x, Sy) = g_c(Sx, y)\}\\
&E_1(\mathcal{S}_i) := \{S \in \mathcal{S}_i : \|S\| < 1\} \text{ for } i \in \{t, c\}
\end{aligned}$$

Proof. The map is well-defined: $x \neq 0 \Rightarrow \omega(x, (J+J_0)x) = \omega(x, Jx) + \omega(x, J_0 x) > 0 \Rightarrow J + J_0 \in \mathrm{GL}(V)$. Consider first the tame case $\mu : \mathcal{J}_t(\omega) \to E_1(\mathcal{S}_t)$, and let

$$S := \mu(J) = (J+J_0)^{-1}(J - J_0) = (A + \mathrm{id})^{-1}(A - id) \text{ where } A = J_0^{-1} J.$$

Now there is the formula

$$\begin{aligned}
\|Ax + x\|_c^2 - \|Ax - x\|_c^2 &= g_c(Ax + x, Ax + x) - g_c(Ax - x, Ax - x)\\
&= 4 g_c(J_0^{-1} J x, x)\\
&= 4 g_c(J_0 x, Jx)\\
&= 4\omega(x, Jx) > 0 \text{ for all } x \neq 0,
\end{aligned}$$

since (ω, J_0) are assumed compatible. Hence $\|S\| < 1$, and $\mathrm{im}\,\mu \subseteq E_1(\mathcal{S}_t)$ because anti-commutativity follows by

$$\begin{aligned}
J_0\mu(J) + \mu(J)J_0 &= J_0(J+J_0)^{-1}(J-J_0) + (J+J_0)^{-1}(J-J_0)J_0\\
&= (-J_0 - J)^{-1} - (J_0 J J_0 - J)^{-1}\\
&\quad + (-J_0 + J_0 J J_0)^{-1} + (J + J_0)^{-1}\\
&= 0.
\end{aligned}$$

Via $(J+J_0)^{-1}(J-J_0) = S \Leftrightarrow J - J_0 = JS + J_0 S \Leftrightarrow J = J_0(\mathrm{id}+S)(\mathrm{id}-S)^{-1}$ the inverse of μ is computed to be the map

$$\begin{aligned}
\mu^{-1} : E_1(\mathcal{S}_t) &\longrightarrow \mathcal{J}_t(\omega)\\
S &\longmapsto J_0(\mathrm{id} + S)(\mathrm{id} - S)^{-1},
\end{aligned}$$

which is well-defined since $\|S\| < 1$ makes $\mathrm{id} - S$ invertible. Thus $\mu_t : \mathcal{J}_t(\omega) \to E_1(\mathcal{S}_t)$ is a diffeomorphism, and so is $\mu_c : \mathcal{J}_c(\omega) \to E_1(\mathcal{S}_c)$ since J is compatible if and only if $\mu(J) = S$ is symmetric with respect to g_c:

$$\omega(Jx, Jy) = \omega(x, y) \text{ for all } x, y \in V$$

$$\begin{aligned}
&\iff \omega(J_0(\mathrm{id}+S)(\mathrm{id}-S)^{-1}x, J_0(\mathrm{id}+S)(\mathrm{id}-S)^{-1}y) = \omega(x,y) \\
&\overset{(*)}{\iff} \omega((\mathrm{id}+S)x, (\mathrm{id}+S)y) = \omega((\mathrm{id}-S)x, (\mathrm{id}-S)y) \\
&\iff \omega(x,Sy) + \omega(Sx,y) = -\omega(x,Sy) - \omega(Sx,y) \\
&\iff g_c(Sx,y) = \omega(Sx,J_0y) = -\omega(x,SJ_0y) = \omega(x,J_0Sy) = g(x,Sy).
\end{aligned}$$

The directions $(*)$ follow since J_0 is ω-compatible, and by substituting $x' = (\mathrm{id}-S)x$ and $y' = (\mathrm{id}-S)y$. $\square$

Corollary 1.2. *The spaces $\mathcal{J}_t(\omega)$ and $\mathcal{J}_c(\omega)$ are contractible.* $\square$

Proposition 1.3. *Let (M,ω) be a symplectic manifold. Then there exist nonempty fiber bundles*

$$\begin{aligned}
&\mathcal{J}_t(M,\omega) \longrightarrow M && \text{with fibers } \mathcal{J}_t(M,\omega)_x = \mathcal{J}_t(T_xM,\omega_x), \\
&\mathcal{J}_c(M,\omega) \longrightarrow M && \text{with fibers } \mathcal{J}_c(M,\omega)_x = \mathcal{J}_c(T_xM,\omega_x).
\end{aligned}$$

Moreover, the spaces $\Gamma(\mathcal{J}_t(M,\omega) \to M)$ and $\Gamma(\mathcal{J}_c(M,\omega) \to M)$ are contractible.

Proof. The above arguments could all be carried out independently of a choice of basis. $\square$

$\mathcal{J}(\omega)$. We now introduce some notation which will be used in later chapters.

$$\mathcal{J}(\omega) := \Gamma(\mathcal{J}_t(M,\omega) \to M)$$

is the space of ω-tame almost complex structures on M, that is the set of all $J \in \Omega^1(M;TM)$ such that $J_x^2 = -\mathrm{id}_{T_xM}$ and $\omega_x(X,J_xX) > 0$ for all $X \in T_xM \setminus \{0\}$ and all $x \in M$. Compatible structures will be only of minor significance for later development. $\square$

Lemma 1.4. *Let $(U_i,\psi_i)_i$ be a fiber bundle atlas for $\mathcal{J}_t(M,\omega) \to M$, and $(\rho_i)_i$ a partition of unity subordinate to the open cover $(U_i)_i$. If $J_i : U_i \to \mathcal{J}_t(\mathbb{R}^n,\omega_0)$ is a smooth mapping into the standard fiber of $\mathcal{J}_t(M,\omega)$ then $J\colon x \mapsto \sum_i \rho_i(x)\psi_i(x,J_i(x))$ is a globally well defined element $J \in \mathcal{J}(\omega)$.*

A fiber bundle chart of a given fiber bundle $E \xrightarrow{p} M$ with standard fiber S consists of a pair (U,ψ) such that $U \subseteq M$ is open and $\psi\colon E|_U = p^{-1}(U) \to U \times S$ is a diffeomorphism with the property that $p = \mathrm{pr}_1 \circ \psi$.

Proof. By proposition 1.1 the standard fiber $\mathcal{J}_t(\mathbb{R}^n,\omega_0)$ can be identified with an open unit ball in a vector space. Therefore, the sum in the definition of J makes sense for a suitable partition of unity $(\rho_i)_i$. $\square$

8. Let (M, J) be an almost complex manifold then the *Nijenhuis-tensor* is defined by

$$N_J(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] \ \text{ for } X,Y \in \mathfrak{X}(M).$$

Note that $N_J(JX,Y) = -JN_J(X,Y)$. Now assume that (M, J) is a complex manifold, i.e. existence of holomorphic charts $\psi_i : U_i \to \mathbb{C}^n$, $d\psi_i \circ J = i \circ d\psi_i$. *Then the Nijenhuis-tensor vanishes identically.* This is a local question. So assume $(M, J) = (\mathbb{R}^{2n}, J)$, and there is a biholomorphic map $f = (f_1, \ldots, f_n) : \mathbb{R}^{2n} \to \mathbb{C}^n$, i.e. $(df_k)_{k=1}^n$ is a basis of $\Omega^1(\mathbb{R}^{2n})$, and $df_k \circ J = i \circ df_k$. Then

$$\begin{aligned} df_k N_J(X,Y) &= [JX,JY](f_k) - J[X,JY](f_k) - J[JX,Y](f_k) - [X,Y](f_k) \\ &= -[X,Y](f_k) + [X,Y](f_k) + [X,Y](f_k) - [X,Y](f_k) \\ &= 0, \end{aligned}$$

since also $X(if_k) = iX(f_k)$. This holds for all $X, Y \in \mathfrak{X}(\mathbb{R}^{2n})$, and $k \in \{1, \ldots, n\}$, thus $N_J \equiv 0$. □

Theorem (Newlander-Nirenberg). *If (M, J) is an almost complex manifold then J is integrable if and only if the Nijenhuis-tensor vanishes.*

J is called *integrable* if M carries a complex structure that induces J.

Proof. Can be found in Newlander, Nirenberg [18]. □

1.B. Local properties of J-curves

Throughout this section let (Σ, j) be a closed Riemann surface, and (M, J) a compact almost complex manifold. The relevant notation for Sobolev spaces as used in this section is introduced in section 3.C..

Definition. A smooth map $f\colon (M_1, J_1) \to (M_2, J_2)$ between almost complex manifolds is called *pseudo-holomorphic* or (J_1, J_2)*-holomorphic* if its differential is a (J_1, J_2)-linear map, i.e. $T_x f \circ J_1(x) = J_2(f(x)) \circ T_x f$ for all $x \in M_1$. If (Σ, j) is a Riemann surface, and $u : (\Sigma, j) \to (M, J)$ is a (j, J)-holomorphic map then u is said to be a *pseudo-holomorphic curve* or *J-holomorphic curve* or simply a *J-curve*. In particular pseudo-holomorphic curves always are parametrized.

Let $\Sigma = \mathbb{C}P^1 = S^2 = \mathbb{C}_\infty$, then the 6-dimensional automorphism group

$$G := \mathrm{PSL}(2,\mathbb{C}) = \{\phi_g\colon \mathbb{C}P^1 \to \mathbb{C}P^1, z \mapsto \frac{az+b}{cz+d} : g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})\}$$

acts on the space $\mathcal{M}(J)$ of all simple J-curves that represent some fixed homology class from the right by composition, $\mathcal{M}(J)\times G \to \mathcal{M}(J)$, $(u,\phi)\mapsto u\circ\phi$; cf. chapter 4.

We shall be somewhat loose in the use of the word curve. As a rule a curve is an equivalence class of maps under the equivalence relation of reparametrization. However, coming to J-holomorphic maps $u\colon \Sigma\to M$ we mostly follow Gromov's original terminology and speak of J-curves. Thus a parametrized curve is just a map, an unparametrized J-curve is a curve which has a J-holomorphic parametrization, and a parametrized J-curve is a tautology. Admittedly, this is rather unfortunate terminology but should not cause any confusion. Were the author to write this manuscript anew he would settle for the term J-holomorphic map from the beginning.

By definition two curves c^1, c^2 are *equal* if there are parametizations u^1, u^2 of c^1, c^2 and $\phi\in G$ such that $u^2=u^1\circ\phi$, and they are *distinct* if they are not equal. Analogously, two parametrized curves (i.e. maps) u^1 u^2 are said to be *distinct* if there is no $\phi\in G$ such that $u^1\cdot\phi=u^1\circ\phi=u^2$.

A J-holomorphic map $u\colon(\Sigma,j)\to(M,J)$ is called *simple* if it is not multiply covered. It is *multiply covered* if there is a Riemannian surface (Σ',j'), and a holomorphic branched covering $p\colon(\Sigma,j)\to(\Sigma',j')$ of degree greater than 1 such that $u=u'\circ p$ for a J-curve $u'\colon(\Sigma',j')\to(M,J)$. The following proposition will be useful when working with parametrizations of simple cusp curves as introduced in 2.B.

Proposition 1.5. *Let $(u^1,\dots,u^a)\in(C^\infty(\mathbb{C}P^1,M))^a$ be a tuple of simple distinct J-curves, in the sense that $u^i\in u^j\cdot G \iff i=j$. Then there exist points $(z^1,\dots,z^a)\in(\mathbb{C}P^1)^a$ such that*

(i) *$T_{z^i}u^i\neq 0$ and $(u^i)^{-1}(u^i(z^i))=\{z^i\}$,*

(ii) *$u^i(z^i)\in u^j(\mathbb{C}P^1)\iff i=j$.*

Moreover, the set of a-tuples satisfying (i) and (ii) is open and dense in $(\mathbb{C}P^1)^a$, and its complement has codimension at least two.

Proof. See McDuff, Salamon [16, 2.3.2]. □

A map $u\in C^1(\Sigma,M)$ is said to be *injective at z*, and z is called an *injective point* if

(i) $T_zu\colon T_z\Sigma\to T_{u(z)}M$ has maximal rank, and

(ii) $u^{-1}(u(z))=\{z\}$.

This is an open condition. Indeed, e.g. by employing exponential mappings, u can be perturbed to a map $\tilde{u} \in C^1(\Sigma, M)$ such that $\tilde{u}|_U : U \to M$ is an injective immersion for an open neighborhood U of z. A map $u \in C^1(\Sigma, M)$ for which there exist injective points is said to be *somewhere injective.* If $u \in C^1(\Sigma, M)$ is J-holomorphic then condition (i) is equivalent to prescribing $T_z u \neq 0$, and the above proposition may be partially rephrased in saying that simple J-curves have injective points.
A map $u\colon \Sigma \to M$ is J-holomorphic if and only if it verifies the equation

$$\overline{\partial}_J u = \frac{1}{2}(Tu + J \circ Tu \circ j) = 0.$$

$\overline{\partial}_J$ is the *Cauchy-Riemann operator* on (M, J) and its properties will be further explored in later chapters. For now consider the case $(\Sigma, j) = (D, i)$ where $D = \{z \in \mathbb{C} : |z| < 1\}$. The above equation may then be rewritten as

$$\overline{\partial}_J u = \tfrac{1}{2}(\tfrac{\partial u}{\partial x} + (J \circ u)\tfrac{\partial u}{\partial y})dx + \tfrac{1}{2}(\tfrac{\partial u}{\partial y} - (J \circ u)\tfrac{\partial u}{\partial x})dy.$$

Proposition 1.6 (Elliptic regularity). *Let $p > 2$, $l > 0$, and consider an almost complex structure J on M which is of class C^l. If $u \in W^{1,p}(\Sigma, M)$ is J-holomorphic then u is of class C^{l+1}.*

Proof. The strategy of this proof is to show that $\overline{\partial}_J u = 0$ implies $\Delta u = g$ with g of class C^{l-1} so that we may apply elliptic regularity results which are known for the Laplace operator Δ.
This is a local question and we assume that $(\Sigma, j) = (D, i)$, and $M = U \subseteq \mathbb{R}^n$ is open. Furthermore, we specialize to the case where $J \in C^\infty(U, \operatorname{End}\mathbb{R}^n)$ with $J^2 = -\mathrm{id}$. Let $f \in C^\infty(D, U)$ and define

$$\begin{aligned} &\tfrac{\partial f}{\partial x} + (J \circ f)\tfrac{\partial f}{\partial y} = f_x + (J \circ f) f_y =: h^1(f), \\ &\tfrac{\partial f}{\partial y} - (J \circ f)\tfrac{\partial f}{\partial x} =: h^2(f). \end{aligned}$$

Because f is smooth it follows that

$$f_{xx} + f_{yy} + (J \circ f)_x f_y - (J \circ f)_y f_x = h^1(f)_x + h^2(f)_y.$$

If $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is the Laplace operator, and $\langle \_ , \_ \rangle$ denotes the L^2 scalar product on square integrable maps $D \to U$ then this translates to

$$\begin{aligned} &\langle \Delta f, \varphi \rangle - \langle (J \circ f)_x f_y - (J \circ f)_y f_x, \varphi \rangle = -\langle h^1(f)_x + h^2(f)_y, \varphi \rangle \\ \Longleftrightarrow\ &\langle df, d\varphi \rangle = \langle (J \circ f)_x f_y - (J \circ f)_y f_x, \varphi \rangle + \langle h^1(f), \varphi_x \rangle + \langle h^2(f), \varphi_y \rangle \end{aligned}$$

for $\varphi \in C^\infty_{\mathrm{cp}}(D,U)$. By definition $C^\infty_{\mathrm{cp}}(D,U) \subseteq W^{1,p}(D,U)$ is dense. Thus, given $u \in W^{1,p}(D,U)$ with $\overline{\partial}_J u = 0$, we may find a sequence $(f_n)_n$ in $C^\infty_{\mathrm{cp}}(D,U)$ such that $f_n \to u$ in the $W^{1,p}$-norm. By lemma 3.8 the assumption $p > 2$ implies that $J_*\colon W^{1,p}(D,U) \to W^{1,p}(D,U)$, $u \mapsto J \circ u$ is continuous, and this yields

$$\begin{aligned} &h^i(f_n) \longrightarrow h^i(u) = 0, \qquad \text{for } i \in \{1,2\}, \\ &(J \circ f_n)_x (f_n)_y - (J \circ f_n)_y (f_n)_x \longrightarrow (J \circ u)_x u_y - (J \circ u)_y u_x \end{aligned}$$

both with respect to the L^p-norm. Hence, for all $\varphi \in C^\infty_{\mathrm{cp}}(D,U)$,

$$\begin{aligned} \langle du, d\varphi\rangle &= \langle (J \circ u)_x u_y - (J \circ u)_y u_x, \varphi\rangle \\ \iff \Delta u &= (J \circ u)_x u_y - (J \circ u)_y u_x \in L^p(D, \mathbb{R}^n) \end{aligned}$$

where the latter equation is to be read in the distributional sense. In other words u is the weak solution to an elliptic equation, and the corresponding regularity theorems imply that $u \in W^{2,p/2}(D,U)$; see [8]. If we can show that the right hand side of this last equation is as regular as the differential of J then we are finished. A first step is to show that $J \circ u \in C^1(D,U)$, then one may proceed iteratively.
Assume that $p = 3$. The Sobolev lemma 3.5 then implies that $u \in W^{1,6}(D,U)$, and one more application of elliptic regularity yields $u \in W^{2,3}(D,U)$. By lemma 3.8 we conclude that $J \circ u \in W^{2,3}(D,U) \hookrightarrow C^1(D,U)$ and the inclusion is continuous. Repeating this process we find that $J \circ u \in C^l(D,U)$, i.e. $\Delta u \in C^{l-1}(D,U)$, and elliptic regularity implies $u \in C^{l+1}(D,U)$. □

Elliptic regularity thus established for pseudo-holomorphic curves will turn out to be a powerful tool for further development. It is the basic reason why an implicit function theorem on Banach spaces can be employed, and gives an inroad to transversality theorems. The condition $p > 2$ which will be assumed throughout the paper corresponds to $\dim \Sigma = 2$.

Lemma 1.7. *If Σ is connected and the non-constant J-holomorphic curve $u\colon \Sigma \to M$ has vanishing infinity jet $j^\infty_z u = 0$ at a point $z \in \Sigma$ then $u \equiv 0$.*

Proof. This is a local question and we may assume that $(\Sigma, j) = (D, i)$, $M = U \subseteq \mathbb{R}^n$ is open, and $z = 0$. As shown above u solves the equation

$$\Delta u = (J \circ u)_x u_y - (J \circ u)_y u_x.$$

Because J and its derivatives are bounded it follows that u satisfies the second order elliptic differential inequality

$$|\Delta u(z)| \le K(|u(z)| + |u_x(z)| + |u_y(z)|)$$

for some constant $K \in \mathbb{N}$ and all $z \in D$. This puts us in a position to apply Aronszajns theorem. $\square$

Theorem 1.8 (Aronszajn). *Suppose $u \in W^{2,2}(D, \mathbb{R}^n)$ satisfies the pointwise estimate*

$$|\Delta u(z)| \leq K(|u(z)| + |u_x(z)| + |u_y(z)|)$$

for some constant K and almost all $z \in D$, and vanishes to infinite order at 0 in the sense that

$$\int_{|z|\leq r} |u(z)| = O(r^k)$$

for all $k \in \mathbb{N}$. Then $u \equiv 0$.

Proof. This is proved in Aronszajn [2]. $\square$

Lemma 1.9. *If Σ is connected and the J-holomorphic curve $u : \Sigma \to M$ is not constant then the set of critical points of u, namely $CP := \{z \in \Sigma : T_z u = 0\}$, is finite.*

Proof. As Σ is supposed compact we need to show that CP is discrete. Again we can resort to the local model $(\Sigma, j) = (D, i)$ and $M = U \subseteq \mathbb{R}^n$ is open. Furthermore, without loss, we assume that $0 \in CP$, $u(0) = 0$, and that $J(0) = J_0$ is standard. Because u is non-constant we can employ the previous lemma to find $l \geq 1$ such that $\mathrm{Tay}_0^l u = 0$ while $\mathrm{Tay}_0^{l+1} u \neq 0$; here $\mathrm{Tay}_0^l u$ denotes the Taylor expansion at 0, without constant term, of u up to order l, i.e.

$$u(z) = 0 + \sum_{k=1}^{l} \tfrac{1}{k!} d^k u(0) \cdot z^k + o(|z|^l) =: \mathrm{Tay}_0^l u \cdot z + o(|z|^l)$$

for small $z \in D$, and $z^k = (z, \ldots, z)$, k-times. By Taylor's theorem it follows that $J(u(z)) = J_0 + o(|z|^l)$ for small z. Thus

$$0 = \mathrm{Tay}_0^{l-1}(du + (J \circ u) du\, i) = \mathrm{Tay}_0^{l-1} du + J_0 \mathrm{Tay}_0^{l-1} du\, i$$

and hence $d\mathrm{Tay}_0^l u + J_0 d\mathrm{Tay}_0^l u\, i = 0$. Therefore $\mathrm{Tay}_0^l u : \mathbb{C} \to \mathbb{C}^n$ is a holomorphic polynomial of degree l, which implies

$$u(z) = az^l + o(|z|^l), \qquad du(z) = laz^{l-1} + o(|z|^{l-1})$$

where $a \in \mathbb{C}^n \setminus \{0\}$. Thus $du(z) \neq 0$ for $z \neq 0$. $\square$

With the final part of this chapter we follow Nijenhuis, Woolf [19]. Let $B = B_1 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $0 < \alpha < 1$. For $f \in C(B,\mathbb{C})$ we consider the norm

$$\begin{aligned} ||f||_{\infty,\alpha} &:= ||f||_\infty + ||f||_\alpha \\ &= \sup_{z\in B} |f(z)| + \sup_{z\neq w} \tfrac{|f(z)-f(w)|}{|z-w|} \end{aligned}$$

and the Banach space

$$C^\alpha(B,\mathbb{C}) := \{f \in C(B,\mathbb{C}) : ||f||_{\infty,\alpha} < \infty\}.$$

More generally, let $k \geq 0$, $n \geq 1$ and define

$$\begin{aligned} &C^{k+\alpha}(B,\mathbb{C}^n) \\ &\quad := \{f = (f^i)_i \in C^k_\circ(B,\mathbb{C}^n) : \max\nolimits_{1\leq i\leq n} ||d^k f^i||_\infty < \infty \text{ for } 1 \leq i \leq n\} \end{aligned}$$

where $C^k_\circ(B,\mathbb{C}^n)$ denotes the set of all functions $f \in C^k(B^\circ,\mathbb{C}^n)$ such that each $d^j f^i$ is extendable to a continuous function on all of B for $0 \leq j \leq k$ and $1 \leq i \leq n$. The extension will be tacitly carried out and denoted by the same symbol.
Let $B_\delta := \{z : |z| <\leq 1\}$ and $D_\delta = B_\delta^\circ$ its interior. For $f \in C(B_\delta,\mathbb{C})$ the operator T is defined by

$$Tf(z_0) = -\tfrac{1}{2\pi i} \int_{B_\delta} \frac{f(z)}{f(z) - f(z_0)} d\overline{z} \wedge dz.$$

This is well defined and its properties are given in the following lemma.

Lemma 1.10. *Let $\delta \in (0,1]$. Then $T\colon C(B_\delta,\mathbb{C}) \to C(B_\delta,\mathbb{C})$ is well-defined and has the following properties.*

(i) $||Tf||_\infty \leq 4\delta ||f||_\infty$.

(ii) $||Tf||_\alpha \leq \frac{216}{1-\alpha}\delta^{1-\alpha}||f||_\infty$.

(iii) *If $f \in C^\alpha(B_\delta,\mathbb{C})$ then $Tf \in C^{1+\alpha}(B_\delta,\mathbb{C})$, and furthermore $\overline{\partial}Tf = f$.*

Proof. This is proved in all detail in Section 6 of Nijenhuis, Woolf [19]. □

Proposition 1.11. *Let $A = (a_{ik})_{ik} \in C^\infty(D, \operatorname{End}\mathbb{C}^n)$, with $||A||_\infty < \infty$ and $v = (v^i)_i \in \mathbb{C}^n$ arbitrary. Then there is a number $\delta_0 \in (0,1]$ which depends on $||A||_\infty$ such that the following holds. For arbitrary $\delta \in (0,\delta_0]$ there exists $f = (f^i)_i \in C^\infty(D_\delta, \operatorname{End}\mathbb{C}^n)$ which solves*

$$\overline{\partial} f^i + \sum_{k=1}^n a_{ik} f^k = 0, \qquad f^i(0) = v^i, \qquad 1 \leq i \leq n.$$

Note that we mean $\operatorname{End}\mathbb{C}^n = \operatorname{End}_{\mathbb{R}}\mathbb{C}^n$.

Proof. The plan is to apply a contraction principle. We claim that there is a $\delta_0 > 0$ such that the mapping

$$\begin{aligned} K\colon C^\alpha(B_\delta, \mathbb{C}^n) &\longrightarrow C^\alpha(B_\delta, \mathbb{C}^n), \\ f = (f^i)_i &\longmapsto -(Tf^i)_i + (Tf^i(0))_i + v \end{aligned}$$

is a contraction for all $\delta \in (0, \delta_0]$ and arbitrarily fixed $\alpha \in (0,1)$. Thus let $f, g \in C^\alpha(B_\delta, \mathbb{C}^n)$. By the above lemma it is true that

$$\begin{aligned} ||Kf - Kg||_{\infty,\alpha} &\leq \max_{1\leq i\leq n} ||T \textstyle\sum_{k=1}^n a_{ik} f^k - T \sum_{k=1}^n a_{ik} g^k||_{\infty,\alpha} \\ &\quad + \max_{1\leq i\leq n} ||T(\textstyle\sum_{k=1}^n a_{ik} f^k)(0) - T(\sum_{k=1}^n a_{ik} g^k)(0)||_{\infty,\alpha} \\ &\leq 8\delta \max_{1\leq i\leq n} || \textstyle\sum_{k=1}^n a_{ik} f^k - \sum_{k=1}^n a_{ik} g^k||_\infty \\ &\quad + \tfrac{216}{1-\alpha}\delta^{1-\alpha} \max_{1\leq i\leq n} || \textstyle\sum_{k=1}^n a_{ik} f^k - \sum_{k=1}^n a_{ik} g^k||_\infty \\ &\leq (8\delta + \tfrac{216}{1-\alpha}\delta^{1-\alpha}) ||A||_\infty ||f-g||_\infty. \end{aligned}$$

As $1-\alpha > 0$ there clearly exists $\delta_0 > 0$ such that $(8\delta_0 + \frac{216}{1-\alpha}\delta_0^{1-\alpha})||A||_\infty = \frac{1}{2}$, say. Let $\delta \in (0, \min\{1, \delta_0\}]$. By the Banach contraction theorem there is a unique $f \in C^\alpha(B_\delta, \mathbb{C}^n)$ satisfying $Kf = f$. Another application of the above lemma now yields $f = Kf \in C^{1+\alpha}(B_\delta, \mathbb{C}^n)$ as well as

$$\overline{\partial} f = (\overline{\partial} f^i)_i = -(\sum_{k=1}^n a_{ik} f^k)_i + 0 = -Af$$

and

$$f(0) = Kf(0) = 0 + v.$$

Because A is smooth the proof is now comleted by the the following sublemma, which asserts that, in fact, $f|_{D_\delta} \in C^\infty(D_\delta, \mathbb{C}^n)$. $\square$

Sublemma 1.12. *Let $p > 2$, $f \in W^{1,p}(D, \mathbb{C}^n)$, and $A \in C^\infty(D, \operatorname{End}\mathbb{C}^n)$ such that*

$$(\overline{\partial} + A) f = 0.$$

Then $f \in C^\infty(D, \mathbb{C}^n)$.

Proof. It is convenient to identify $\mathbb{C}^n = (\mathbb{R}^{2n}, J_0)$. According to the proof of proposition 1.6 we have to show that $\Delta f = g \in L^p(D, \mathbb{R}^{2n})$. Let $(f_n)_n$

be a sequence in $C^\infty_{\mathrm{cp}}(D,\mathbb{R}^{2n})$ such that $f_n \to f$ in $W^{1,p}(D,\mathbb{R}^{2n})$. Introduce $h^i(f)$ defined as in 1.6 with $i \in \{1,2\}$. Then

$$h^i(f_n) \to h^i(f) \qquad \text{in } L^p(D,\mathbb{R}^{2n}) \text{ for } i \in \{1,2\},$$

and thus there is a weak equation $\Delta f = g \in L^p(D,\mathbb{R}^{2n})$. Indeed,

$$\begin{aligned}
\langle df, d\varphi\rangle &= \lim \langle df_n, d\varphi\rangle \\
&= \lim \langle \Delta f_n, \varphi\rangle \\
&= \lim \langle -\partial_x^2 f_n - \partial_y^2 f_n, \varphi\rangle \\
&= \lim \langle -\partial_x h^1(f_n) - \partial_y h^2(f_n), \varphi\rangle \\
&= \langle h^1(f), \partial_x\varphi\rangle + \langle h^2(f), \partial_y\varphi\rangle \\
&= \langle -Af, \partial_x\varphi\rangle + \langle -Af, \partial_y\varphi\rangle \\
&= \langle \partial_x A\cdot f + A\cdot\partial_x f + \partial_y A\cdot f + A\cdot\partial_y f, \varphi\rangle
\end{aligned}$$

for all $\varphi \in C^\infty_{\mathrm{cp}}(D,\mathbb{R}^{2n})$. Because A is smooth the iterative argument to show $f \in C^\infty(D,\mathbb{R}^{2n})$ is now the same as in 1.6. $\square$

Chapter 2

Gromov compactness

This chapter is only intended as a convenient reference for further developments, and almost all of the results are stated without proof. A thorough treatment of Gromov's compactness theorem would go far beyond the intentions of this paper. There are, however, many excellent references for this compactness theorem. Gromov's original proof [6] used isoperimetric inequalities, and has been explained by Pansu in [3], which has been further elaborated by Hummel [9]. McDuff and Salamon [16] give a different proof based on a result proved by elliptic bootstrapping. This list of references is by no means complete.

$G = \mathrm{PSL}(2, \mathbb{C})$ shall again denote the group of fractional reparametrizarions acting on $C^\infty(S^2, M)$ by composition from the right. The WO^k-topology which will be used below is defined in section 3.A..

2.A. Energy

1. Fixing a metric μ on the Riemann surface (Σ, j) induces the *Hodge star operator* $\star : \Lambda^k T^*\Sigma \to \Lambda^{2-k} T^*\Sigma$ which is defined by the equation $\alpha \wedge \star\beta = \mu(\alpha, \beta)\mathrm{vol}_\mu$, and satisfies $\star df = e_1(f)e^2 - e_2(f)e^1$ for all $f \in C^\infty(\Sigma, \mathbb{R})$:

$$\begin{aligned} e^i \wedge (e_1(f)e^2 - e_2(f)e^1) &= \delta^i_1 e_1(f) - 0 - 0 + \delta^i_2 e_2(f), \\ \mu(e^i, df)\underbrace{\mathrm{vol}_\mu(e_1, e_2)}_{=1} &= \mu(e^i, e_1(f)e^1 + e_2(f)e^2) \\ &= \delta^{i1} e_1(f) + \delta^{i2} e_2(f) \text{ for } i \in \{1, 2\}. \end{aligned}$$

Here (e_1, e_2) was a positively oriented μ-orthonormal frame, and (e^1, e^2) its dual. Similarly one checks that $\star\alpha = -\alpha \circ j$. $\square$

2. If (M, g) is a Riemann manifold and $u \in C^\infty(\Sigma, M)$ then we can define the norm of its derivative in a point-wise manner by

$$\|du\|^2 := \mathrm{tr}((du)^* \circ du) = \sum_{i=1}^{2} \langle ((du)^* \circ du)(e_i), e_i \rangle = \sum_{i=1}^{2} g(du(e_i), du(e_i)),$$

again with a μ-orthonormal frame (e_1, e_2). The $\star$ generalizes to an operation on vector valued forms $\star \otimes \mathrm{id} : \Lambda^k T^*\Sigma \otimes f^*TM \to \Lambda^{2-k} T^*\Sigma \otimes f^*TM$, and g induces an operation $\overline{g}$ on forms $\alpha, \beta \in \Omega(\Sigma; f^*TM)$ defined by

$$\overline{g}(\alpha, \beta)(v, w) = g(\alpha(v), \beta(w)) - g(\alpha(w), \beta(v))$$

where $v, w \in T\Sigma$. This definition yields

$$\begin{aligned} \overline{g}(du, \star du)(e_1, e_2) &= g(du(e_1), du(e_1)) - g(du(e_2), -du(e_2)) \\ &= \mathrm{tr}((du)^* \circ du) \\ &= \|du\|^2 \mathrm{vol}_\mu(e_1, e_2) \end{aligned}$$

If now $u \in C^\infty(\Sigma, M)$ is a J-holomorphic curve, and $g = \omega \circ (\mathrm{id} \times J)$ is a compatible metric then this implies that

$$\begin{aligned} \|du\|^2 \mathrm{vol}_\mu(v, w) &= \overline{g}(du, -du \circ j)(v, w) \\ &= -g(du(v), Jdu(w)) + g(du(w), Jdu(v)) \\ &= 2g(Jdu(v), du(w)) \\ &= 2\omega(du(v), du(w)) \\ &= 2(u^*\omega)(v, w). \end{aligned}$$

□

Lemma 2.1. *Let (M, ω, J) be a manifold with compatible structures, (Σ, j, μ) a closed Riemann surface, and $u : (\Sigma, j) \to (M, J)$ a pseudo holomorphic curve.*

(i) *The* energy identity

$$E(u) := \frac{1}{2} \int_\Sigma \|du\|^2 \mathrm{vol}_\mu = \int_\Sigma u^*\omega = \langle [\omega], u_*[\Sigma] \rangle$$

holds, and $E(u) = E(u, \Sigma)$ is called the energy *of u.*

(ii) *If $u \colon \Sigma \to M$ is furthermore assumed to be an immersion then the pull back metric u^*g induces a volume form which is $\mathrm{vol}_{u^*g} = u^*\omega$.*

If $u\colon \Sigma \to M$ represents a fixed homology class A, i.e. $u_*[\Sigma] = A$ we will also use the notation $\omega(A) := \langle [\omega], A \rangle = E(u)$.

Proof. The first part is immediate from the above. For the second part compute

$$u^*\omega(e, je) = \omega(du\cdot e, Jdu\cdot e) = g(du\cdot e, du\cdot e) = u^*g(e,e) = \mathrm{vol}_{u^*g}(e, je)$$

where (e, je) is a basis of $T\Sigma$. □

Let D be the open unit disk in $\mathbb{C}$, and $D^* := D \setminus \{0\}$.

Theorem 2.2 (Removing singularities). *Let (M, ω) be a compact, symplectic manifold equipped with an ω-compatible structure $J \in \mathcal{J}(\omega)$. If $u : D^* \to M$ is a J-holomorphic map with finite energy $E(u) < \infty$ then u extends to a J-holomorphic map $\tilde{u}\colon D \to M$.*

Proof. This is proved in [1], [3], [9], [14], and [16]. □

It is quite interesting to note the following result which is not unrelated to the previous one.

Theorem (Generalized Weierstrass). *Let $(j_n)_n$ be a sequence of complex structures on Σ which WO^∞ converges to complex structure j. If $(u_n : \Sigma \to M)_n$ is a sequence of (j_n, J)-holomorphic maps that converges to a map $u : \Sigma \to M$ in the WO^0-topology then it converges in the WO^∞-topology. Moreover, the limit $u\colon \Sigma \to M$ is (j, J)-holomorphic.*

Proof. This is proved in Hummel [9]. □

2.B. Cusp curves

1. As an example consider the blow up of $\mathbb{C}^2$ at 0, that is

$$M := \{(z, w, [t_0, t_1]) \in \mathbb{C}^2 \times \mathbb{C}P^1 : zt_1 - wt_0 = 0\}$$

with its induced complex structure, and charts

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow M \\ a_1 : (z, w) &\longmapsto (w, zw, [1, z]) \\ a_2 : (z, w) &\longmapsto (zw, w, [z, 1]). \end{aligned}$$

Then the canonical projection $\pi : M \to \mathbb{C}^2$ gives an isomorphism $\pi|_{M\setminus\pi^{-1}(0)} : M \setminus \pi^{-1}(0) \to \mathbb{C}^2 \setminus \{0\}$ when restricted to the complement of the exceptional divisor $\pi^{-1}(0) = \{a_i(z, w) : w = 0, i \in \{1, 2\}\}$.
For $\varepsilon \in \mathbb{C} \setminus \{0\}$ define the holomorphic curves $u_\varepsilon : t \mapsto u_\varepsilon(t) = \pi^{-1}(\varepsilon, t)$, $\mathbb{C} \to M$. Locally these curves assume the form

$$u^1_\varepsilon(t) := (a_1^{-1} \circ u_\varepsilon)(t) = (\frac{t}{\varepsilon}, \varepsilon) \quad \& \quad u^2_\varepsilon(t) := (a_2^{-1} \circ u_\varepsilon)(t) = (\frac{\varepsilon}{t}, t).$$

On compact subsets not containing the origin $t = 0$ one has uniform convergence $u_\varepsilon \longrightarrow (u : t \mapsto \pi^{-1}(0, t))$, $\pi^{-1}(0, t) = (0, t, [0, 1])$. However, the holomorphic curve $u : \mathbb{C} \setminus \{0\} \to M$ has a removable singularity at zero; the formula extends smoothly to $t = 0$.
Rescale the curves u_ε in the first chart via $\varphi_\varepsilon : t \mapsto \frac{t}{\varepsilon} = t'$ to $u'_\varepsilon = u^1_\varepsilon \circ \varphi_\varepsilon^{-1} : t' \mapsto (t', \varepsilon)$. As $\varepsilon \longrightarrow 0$ one has that $u'_\varepsilon \longrightarrow (u' : t' \mapsto (t', 0))$ uniformly, and thus $(a_1 \circ u')(t') = (0, 0, [1, t'])$, which parametrizes $\pi^{-1}(0) \setminus \{(0, 0, [0, 1])\}$, and the same point is missing on $\operatorname{im} u$, too; the curve $a_1 \circ u'$ has the removable singularities 0 and ∞, it thus parametrizes a sphere. Hence gluing at the origin will produce the connected union $\pi^{-1}(\{0\} \times \mathbb{C})$, a line with a (spherical) bubble attached to at the origin. □

Definition (Cusp curve). Let (M, J) be an almost complex manifold. A (smooth, J-holomorphic) *cusp curve* $c : \mathbb{C}P^1 \to M$ consists of (unparametrized) curves $c^i : \mathbb{C}P^1 \to M$, $i \in \{1, \dots, a\}$ such that:

(i) the union $\operatorname{im} c^1 \cup \dots \cup \operatorname{im} c^a$ is connected.

(ii) each *component* c^i can parametrized by a smooth J-holomorphic curve $u^i : \mathbb{C}P^1 \to M$.

(iii) $a > 1$ or the parametrization u^1 is multiply covered.

With condition (iii) we follow McDuff [13], and distinguish between cusp curves and J-curves (–elements of the moduli space).

2. The cusp curve c is called *reduced* if all its components are distinct and can be parametrized by simple J-holomorphic curves. Any cusp curve $c = (c^1, \dots, c^a)$ can be reduced to a reduced cusp curve $\overline{c} = (\overline{c}^1, \dots, \overline{c}^{\overline{a}})$ by deleting all but one copies of repeated components, and replacing multiply covered curves by their underlying simple ones. This process will change the homology class, but not the image of the cusp curve. However, there still will be integers $(\lambda_1, \dots, \lambda_{\overline{a}})$ such that $A = \sum_{i=1}^{\overline{a}} \lambda_i[\overline{c}^i]$ where $A = \sum_{i=1}^{a}[c^i]$ is the homology class represented by c. □

3. We can visualize the domain of a cusp curve as being a tree of $\mathbb{C}P^1$s stacked upon each other. The tree as a whole is connected, for technical reasons, however, it is convenient to enumerate the components, such that also the union of $\mathbb{C}P^1$s corresponding to numbers lesser equal to k is connected, for any $k \leq a$. *Any cusp curve* $c = (c^1, \ldots, c^a)$ *can be ordered in such a way that* $\operatorname{im} c^1 \cup \cdots \cup \operatorname{im} c^k$ *is connected for all* $k \leq a$. Indeed, assume this holds for $k - 1$ then there necessarily exists a new component, called c^k, such that $\operatorname{im} c^1 \cup \cdots \cup \operatorname{im} c^k$ is connected, but for $k = 1$ nothing is to show. If c is reduced the intersection pattern can be expressed by the existence of numbers $j_2, \ldots, j_a$ and points $\{z_k, w_k \in \mathbb{C}P^1 : 2 \leq k \leq a\}$ such that $1 \leq j_k < k$, and $u^{j_k}(w_k) = u^k(z_k)$. □

4. *Every cusp curve can be parametrized by a single, smooth, but not J-holomorphic map* $\mathbb{C}_\infty \to M$. Assume $c = (c^1, \ldots, c^a)$ is ordered as above, and let v_k parametrize $(c^1, \ldots, c^k)$ for some $k \geq 1$. By reparametrization we can further assume that $v_k(\infty) = u^{k+1}(0)$. Now choose a smooth bump function $\rho \colon \mathbb{R} \to [0, 1]$ with $\rho(x) = 1$ for $x \leq \frac{1}{2}$ & $x \geq 2$ and which vanishes locally around 1. Then define the smooth map

$$v_{k+1}(z) = v_k(\rho(|z|^2)^{-1} z) \text{ for } |z| < 1,$$
$$v_{k+1}(z) = u^{k+1}(\rho(|z|^2) z) \text{ for } |z| \geq 1.$$

This map parametrizes $c = (c^1, \ldots, c^{k+1})$ since it covers v_k on the unit disk, and c^{k+1} on its complement, but the holomorphicity is destroyed by the bump function. The induction assumption $k = 1$ again is trivial. □

Definition (Weak convergence). A sequence of (J-holomorphic) maps $(u_n : \mathbb{C}P^1 \to M)_n$ *converges weakly* to a cusp curve $c = (c^1, \ldots, c^a)$ if the following conditions are satisfied for some parametrization $u = (u^1, \ldots, u^a)$.

(i) For all $k \in \{1, \ldots, a\}$ there exists a sequence $(g_n^k)_n \subseteq G$, and a finite subset $S^k \subseteq \mathbb{C}P^1$ such that $u_n \cdot g_n^k \longrightarrow u^k$ uniformly with all derivatives on compact subsets of $\mathbb{C}P^1 \setminus S^k$.

(ii) There is a sequence of orientation preserving (not holomorphic) diffeomorphisms $(f_n : \mathbb{C}P^1 \to \mathbb{C}P^1)_n$, and a smooth parametrization v of c as in (4) such that $u_n \circ f_n \longrightarrow v$ with respect to the WO^0-topology.

To be precise, this is the *weak C^∞-topology*. For C^k-curves the *weak C^k-topology* is defined analogously with the appropriate change in point (i).

5. By point (ii), for every $x \in \operatorname{im} c$ there is a sequence of points $(x_n \in \operatorname{im} u_n)_n$ such that $x_n \longrightarrow x$. Conversely, any sequence $\operatorname{ev}(u_n, z_n) = u_n(z_n)$ has a subsequence converging to a point on the limiting cusp curve. Weak convergence is a coarser notion than that of WO^∞-convergence. □

6. Moreover, it follows that, for n sufficiently large, u_n is homotopic to the connected union $u^1\#\ldots\#u^a$. This, in turn, implies that $c_1(A_n) = \sum_{k=1}^a c_1(A^k)$ and also that $\omega(A_n) = \sum_{k=1}^a \omega(A^k)$ for large $n \in \mathbb{N}$. Here $(u^1, \ldots, u^a)$ is a parametrization of the cusp curve c and $A^k = u^k_*[\mathbb{C}P^1]$. □

Theorem 2.3. *Let (M, J) be a compact, almost complex manifold, (Σ, j) a Riemann surface without boundary (but not necessarily compact), and $(J_n)_n$ a sequence in $\mathcal{J}(M)$ such that $J_n \to J$ in the WO^∞-topology. Assume $(u_n)_n$ is a sequence of J_n-curves such that $\|du\|_{L^\infty} < K$ for some constant $K \in \mathbb{N}$. Then $(u_n)_n$ has a subsequence that converges uniformly with all derivatives on compact subsets to a J-holomorphic curve u.*

Proof. This theorem is proved in appendix B of [16]. □

Theorem 2.4. *Let (M, ω) be a compact symplectic manifold an assume that $A \in H_2(M)$ is indecomposable Consider a sequence $(J_n)_n$ of ω-tame almost complex structures, and a sequence of maps $(u_u)_n$ where $u_n \in \mathcal{M}(A, J_n)$.*
If $J_n \to J$ in the WO^∞-topology the there exists a reparametrization sequence $(\phi_n)_n$ and a subsequence (U_{n_k}) such that $u_{n_k} \circ \phi_{n_k} \to u \in \mathcal{M}(A, J)$ with respect to the WO^∞-topology.

A homology class $A \in H_2(M)$ is said to be *spherical* if it lies in the image of the Hurewicz homomorphism $h_2 \colon \pi_2(M) \to H_2(M)$, $[u] \mapsto u_*[S^2]$. Furthermore, A is said to be *indecomposable* if there do not exist spherical classes $A^1, \ldots, A^m$ with $\omega(A^i) > 0$ and $m > 1$ such that $A = \sum_{i=1}^m A^i$. Moduli spaces are treated in chapter 4.

Proof. Because the proof shows how bubbles can appear we will sketch it under the following, stronger assumption: there is no spherical homology class B such that $0 < \omega(B) < \omega(A)$, and $J \in \mathcal{J}(\omega)$ shall be fixed. More details may be found in McDuff [14] and McDuff, Salamon [16].
Sketch: It is convenient to identify $\mathbb{C}P^1 = S^2$. By contradiction, start with a sequence $(u_n)_n$ in $\mathcal{M}(A, J)$ that projects to a sequence in $\mathcal{M}(A, J)/G$ without convergent subsequences. Throughout this proof 'to converge' shall be short for 'to converge in all derivatives separately uniformly on compact subsets'. By the energy identity $(u_n)_n$ is uniformly bounded in the $W^{1,2}$-norm by the number $\omega(A)$. (Boundedness in the $W^{1,p}$-norm with $p > 2$ would actually suffice to find a convergent subsequence. The situation $p = 2$, however, is known as the Sobolev borderline case, and compactness is more subtle – bubbling off can occur.) By using theorem 2.3 and passing to a subsequence one finds conformal maps $\psi_n \colon \mathbb{C} \to S^2$ such that $v_n := u_n \circ \psi_n$ fulfill:

$$E(v_n) \leq E(u_n) = \omega(A), \qquad \text{and} \qquad \|T_0 v_n\| = \sup_z \|T_z v_n\| = 1.$$

We can apply 2.3 again to find a subsequence of $(v_n)_n$ which converges to a J-holomorphic map $v\colon \mathbb{C} \to M$ such that

$$E(v) \leq \omega(A), \qquad \text{and} \qquad \|T_0 v\| = 1.$$

By removal of singularities $v\colon \mathbb{C} \to M$ may be extended to a J-holomorphic map –again denoted by– $v : S^2 \to M$. Let $B = v_*[S^2]$. If $\omega(B) = E(v) < \omega(A)$ we have a contradiction, and $(u_n)_n$ must have a subsequence which converges when projected to $\mathcal{M}(A, J)/G$.
Assume $\omega(B) = \omega(A)$, and let $n \in S^2$ denote the north pole, corresponding to ∞ – the removed singularity of v. We find a sequence $(\phi_n)_n$ in G so that the reparametrized sequence $(u_n \circ \phi_n)_n$ has a subsequence converging to v in $W^{1,p}_{\mathrm{loc}}(S^2 \setminus \{n\}, M)$ with $p > 2$. Now, 2.3 implies that there are two possibilities; either $(u_n \circ \phi_n)_n$ has a subsequence converging to v in $W^{1,p}_{\mathrm{loc}}(S^2, M)$; or we may proceed as above to find rescalings $(\phi'_n)_n$ such that $(u_n \circ \phi'_n)_n$ converges – after passage to a subsequence – near n to a non-constant J-holomorphic map $w\colon S^2 \to M$. This new bubble also satisfies $\omega(C) = E(w) \leq \omega(A)$ where $C = w_*[S^2]$. Because v and w are –roughly speaking– limits of disjoint pieces of $(u_n)_n$ it follows that $\omega(B) + \omega(C) \leq \omega(A)$, and hence $\omega(C) = E(w) = 0$ which is absurd since w was constructed to be non-constant. As stated, more details, also of this last step, are available in [16]. □

Theorem 2.5 (Gromov). *Let (M, ω) be a compact symplectic manifold and $(J_n)_n$ be sequence of ω-tame almost complex structures on M converging to J in the WO^∞-topology. Then any sequence of J_n-curves $u_n\colon \mathbb{C}P^1 \to M$ with $\sup E(u_i) \leq K < \infty$ has a subsequence weakly converging either to a cusp curve or a J-curve.*

The limiting almost complex structure need not be tame.

Proof. There are different versions and proofs of this theorem. Discussions, proofs, and further references can be found in [3, Pansu's contribution], [6], [9], and [16]. □

Corollary 2.6. *$K > 0$, $J \in \mathcal{J}_t(\omega)$. Then there is an open neighborhood of J such that* $\{A \in H_2(M)$ has a J'-holomorphic representative with $\omega(A) \leq K\}$ *is finite for all J' in this neighborhood.*

Proof. Assume not, and use the notation of chapter 4. Then there are sequences $(J_n)_n$ of almost complex structures converging to J, and $(u_n)_n$ with $u_n \in \mathcal{M}(A_n, J_n)$ such that the homology classes $(A_n)_n$ all are pairwise distinct and $E(u_n) = \omega(A_n) \leq K$. Thus there is a subsequence $(u_{n_k})_k$ weakly converging to a J-holomorphic (cusp) curve that represents some class A.

But then also $A_{n_k} \to A$, and since $H_2(M)$ is discrete $(A_{n_k})_k$ has to be finally constant, contradicting the subsequence property. $\square$

Chapter 3

Global analysis

3.A. Spaces of mappings

This section follows mostly Kriegl, Michor [11] and Michor [17].

Throughout this section X, Y, Z will denote finite dimensional smooth manifolds.

1 (Pullback bundle). Let $E \xrightarrow{\pi} Y$ be a vector bundle, and $f : X \to Y$ a smooth mapping. We consider the pullback bundle $f^*E \xrightarrow{f^*\pi} X$ with its associated vector bundle homomorphism and the following notation:

$$\begin{array}{ccc} f^*E & \xrightarrow{\pi^* f} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Since the pullback bundle is obtained by pulling back the transition functions it has the same typical fiber as $E \xrightarrow{\pi} Y$, and as manifolds one has $f^*E = X \times_Y E = \{\, (x, v) \in X \times E : f(x) = \pi(v) \in Y \,\}$. For compact X there is a canonical isomorphism

$$\begin{aligned} \Gamma(f^*TY) &\longrightarrow C^\infty(X, TY)_f \\ s &\longmapsto (\pi^* f)_*(s) = (\pi^* f) \circ s \\ (\mathrm{id}_X, h) &\longleftarrow\!\shortmid h; \end{aligned}$$

where $C^\infty(X, TY)_f := \{h \in C^\infty(X, TY) : f = \pi \circ h\}$. Sometimes it will be convenient to identify $\Gamma(f^*TY) = C^\infty(X, TY)_f$. $\square$

2 (CO-topology). For topological Hausdorff spaces X, Y the *compact-open* or *CO-topology* on $C(X, Y)$ is given by the sub-basis $\{N(K, U) := \{f \in$

$C(X,Y) : f(K) \subseteq U\}$ for $K \subseteq X$ compact & $U \subseteq Y$ open$\}$. Since points are compact in Hausdorff spaces this is a finer topology than the pointwise concept and hence Hausdorff itself. If $f \in C(X,Y)$ then also $f_*\colon C(Z,X) \to C(Z,Y)$, $g \mapsto f \circ g$ and $f^*\colon C(Y,Z) \to C(X,Z)$, $g \mapsto g \circ f$ are continuous for the CO-topology on all spaces:

$$(f_*)^{-1}(N(K,U)) = \{g \in C(Z,X)\colon g(K) \subseteq f^{-1}(U)\} = N(K, f^{-1}U),$$
$$(f^*)^{-1}(N(K,U)) = \{g \in C(X,Y))\colon (g \circ f)(K) \subseteq U\} = N(f(K),U)$$

are again members of the sub-basis of the CO-topology. □

3. Let X, Y be smooth manifolds, X compact. *Then $C(X,Y)$ equipped with the compact-open topology can be continuously modelled on spaces $\Gamma(f^*TY \to X)$ with $f \in C^\infty(X,Y)$.* Let $f \in C^\infty(X,Y)$, and (U,∇) *admissible* in the sense that $U \subseteq TY$ is open, ∇ is a connection, and the corresponding exponential mapping $\exp : U \to \exp U =: V \subseteq Y \times Y$ should be a diffeomorphism onto an open neighborhood V of the diagonal. Then

$$\Gamma^0(f^*U) = \{\xi \in \Gamma^0(f^*TY)\colon \xi(X) \subseteq f^*U := (\pi^* f)^{-1}(U)\} \subseteq \Gamma^0(f^*TY) \text{ and}$$
$$U_f := \{g : (f,g)(X) \subseteq V\} \subseteq C(X,Y)$$

are open for the CO-topologies by almost definition. By (2) the bijective mapping

$$\begin{aligned} \varphi_f\colon \Gamma^0(f^*U) &\longrightarrow \operatorname{im}\varphi_f = U_f, \\ \xi &\longmapsto \exp_f \circ (\pi^* f) \circ \xi, \\ (\mathrm{id}_X, \exp_f^{-1} \circ g) &\longleftarrow g \end{aligned}$$

is a homeomorphism: $\operatorname{im}\varphi_f = \{g \in C(X,Y) : \exists h \in C(X,TY)_f : (f,g) = \exp \circ h\} = U_f$; $(\varphi_f^{-1} \circ \varphi_f)(\xi) = (\mathrm{id}, \exp_f^{-1} \circ \exp_f \circ (\pi^* f) \circ \xi) = \xi$, and $(\varphi_f \circ \varphi_f^{-1})(g) = \exp_f \circ (\pi^* f) \circ (\mathrm{id}, \exp_f^{-1} \circ g) = g$. Since to every $g \in C(X,Y)$ there is a triple (f,U,∇) as above such that $g \in U_f = \operatorname{im}\varphi_f$, the collection $\{\varphi_f^{-1}, U_f\}$ defines continuous atlas on $C(X,Y)$. □

4 (Jets). Let $(U_i,\psi_i)_i$, $(V_j,\varphi_j)_j$ be smooth atlases for X, Y respectively. If $f : X \to Y$ is a smooth mapping and $f(U_i) \cap V_j \neq \emptyset$ then we will write $f_{ij} := \varphi_j \circ f \circ \psi_i^{-1}\colon \psi_i(U_i) \to \varphi_j(V_j)$.
The *k-jet extension of f* at $x \in U_i \subseteq X$ is defined to be

$$j^k_x f := \big(\psi_i(x), f_{ij}(\psi_i(x)), df_{ij}(\psi_i(x)), \tfrac{1}{2!}d^2 f_{ij}(\psi_i(x)), \dots, \tfrac{1}{k!}d^k f_{ij}(\psi_i(x))\big),$$

and this definition is independent of the choices: let $f,g \in C^\infty(X,Y)$ then the following are equivalent. (Where we assume without loss that $\psi_i(x) = 0$.)

(i) $j^k_x f = j^k_x g$.

(ii) The local mappings f_{ij} and g_{ij} have the same Taylor developments up to and including order k. That is $f_{ij}(0) + \mathrm{Tay}^k_0 f_{ij} = g_{ij}(0) + \mathrm{Tay}^k_0 g_{ij}$.

(iii) $T^k_x f = T^k_x g$ where T^k is the k-th iterated tangent bundle functor.

Clearly, (i) $\iff$ (ii). By the kinematic definition of the tangent space (i) $\iff$ (iii) is true as well. Now to have the same k-jet at a point $x \in X$ is an equivalence relation on $C^\infty(X, Y)$ and $j^k_x f$ is the resulting equivalence class of f. For $0 \le k \le \infty$ the space of all k-jets of smooth maps from X to Y is

$$J^k(X, Y) := \{j^k_x f : f \in C^\infty(X, Y), x \in X\}.$$

It is natural to define the *source mapping* $\alpha = \alpha_k \colon J^k(X, Y) \to X$, $j^k_x f \mapsto x$, and the *target mapping* $\beta = \beta_k \colon J^k(X, Y) \to Y$, $j^k_x f \mapsto f(x)$. More generally, for $k \ge l$ we can consider the natural projections π^k_l which are given by truncation of the Taylor series at order l:

$$\dots \xrightarrow{\pi^{k+1}_k} J^k(X, Y) \xrightarrow{\pi^k_l} J^l(X, Y) \xrightarrow{\pi^l_0 = (\alpha, \beta)} J^0(X, Y) = X \times Y$$

that is $\pi^k_l : j^k_x f \mapsto j^l_x f$. We shall also use the notation $J^k_x(X, Y) := \alpha^{-1}(x)$, $J^k(X, Y)_y := \beta^{-1}(y)$, $J^k_x(X, Y)_y := J^k_x(X, Y) \cap J^k(X, Y)_y = (\alpha, \beta)^{-1}(x, y)$. $\square$

Theorem 3.1. $0 \le k \le \infty$. *Let X, Y be smooth manifolds of dimension* $\dim X = n$ *and* $\dim Y = m$.

(i) *$J^k(X, Y)$ is a smooth manifold. If $f : X \to Y$ is smooth then the k-jet extension mapping $j^k f : X \to J^k(X, Y)$, $x \mapsto j^k_x f$ is smooth, too. Clearly, $\alpha \circ j^k f = \mathrm{id}_X$ and $\beta \circ j^k f = f$.*

(ii) *$J^k(X, Y) \xrightarrow{(\alpha, \beta)} X \times Y$ is a fiber bundle with standard fiber*

$$\bigoplus_{i=1}^{k} \mathrm{Poly}^i(\mathbb{R}^n, \mathbb{R}^m) := \bigoplus_{i=1}^{k} \mathrm{Hom}^i_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m).$$

(iii) *If $h \colon Y \to Z$ is smooth then $J^k(X, h) \colon J^k(X, Y) \to J^k(X, Z)$, $j^k_x f \mapsto j^k_x(h \circ f)$ is smooth, too.*

(iv) *For $k \ge l$ the smooth projection $\pi^k_l \colon J^k(X, Y) \to J^l(X, Y)$ is a surjective submersion. Moreover, $\pi^k_l \circ J^k(X, h) = J^l(X, h) \circ \pi^k_l$.*

Proof. This is proved in Kriegl, Michor [11]. □

5 (WO^k-topology). Consider the graph mapping which is defined by graph: $C(X,Y) \to C(X, X \times Y)$, $f \mapsto (\mathrm{graph}(f) \colon x \mapsto (x, f(x)))$. On $C(X,Y)$ we define the *wholly open* or WO-topology by the basis $\{ f \in C(X,Y) : f(X) \subseteq U \}$ where U runs through a basis of in Y open sets. This topology is not Hausdorff, since maps with the same image cannot be separated.

For $0 \leq k \leq \infty$ the *Whitney* C^k or simply WO^k-topology on $C^\infty(X,Y)$ is the initial topology with respect to the k-jet extension

$$j^k : C^\infty(X,Y) \longrightarrow C(X, J^k(X,Y))_{WO},$$

where the target -as indicated- carries the WO-topology. Note that the WO^k-topology is also well-defined on $C^l(X,Y)$ if only $k \leq l$. A basis is given by sets of the form $\{ f \in C^\infty(X,Y) : j^k(X) \subseteq U \}$, where $U \subseteq J^k(X,Y)$ is open. Since it is finer than the topology of pointwise convergence this is a Hausdorff topology. Now, any smooth mapping $g : Y \to Z$ induces $J^k(X,g) : J^k(X,Y) \to J^k(X,Z)$, $j^k_x f \mapsto j^k_x(g \circ f)$, another smooth map. Hence $g_* : C^\infty(X,Y) \to C^\infty(X,Z)$, $f \mapsto g_*(f) = g \circ f$ is continuous for the just described topologies.

If X is compact then 'to converge in the WO^k-topology' on $C^\infty(X,Y)$ is just a shorter synonym for 'to converge uniformly on compact subsets in all derivatives of order lesser equal to k'. □

Lemma 3.2. *Let $E \xrightarrow{\pi_E} X$ and $F \xrightarrow{\pi_F} X$ be finite dimensional vector bundles over X. Let $\alpha : E \to F$ be a smooth, fiber respecting mapping. Then also $\alpha_* : \Gamma_c(E) \to \Gamma_c(F)$, $\xi \mapsto \alpha_*(\xi) = \alpha \circ \xi$ is smooth, and moreover its derivative computes to $d(\alpha_*) = (d_v\alpha)_*$, where the right hand side is the* vertical derivative *given by $d_v\alpha(\xi(x), \eta(x)) := \frac{d}{dt}|_0 \alpha(\xi(x) + t\eta(x))$.*

Proof. By point (3) above α_* is continuous. To show smoothness it clearly suffices to show the asserted formula, for then one can conclude recursively. Formally this is obvious. Let $x \in X$, $\xi, \eta \in \Gamma_{\mathrm{cp}}(E)$. *We claim that*

$$\lim_{t \to 0} \frac{\alpha_*(\xi + t\eta) - \alpha_*(\xi)}{t} = (d_v\alpha)_*(\xi, \eta),$$

where the limit is to be taken with respect to the WO^∞-topology. This is a local question. So choose an open, in both bundles trivializing neighborhood $W \ni x$, and write $\xi(x) = (x, \xi_x)$, $\eta(x) = (x, \eta_x)$, and $(\alpha_*(\xi))(x) = (x, \xi_x)$. Now we have to show that for all $K \underset{\mathrm{cp}}{\subseteq} W$ and all $a \in \mathbb{N}_0^n$

$$\sup_{x \in K} |\partial^a (\frac{\alpha_x(\xi_x + t\eta_x) - \alpha_x(\xi_x)}{t}) - \partial^a(d\alpha_x(\xi_x) \cdot \eta_x)| \longrightarrow 0,$$

as $t \to 0$, and where $a \in \mathbb{N}_0^n$ is a multi index with $n = \dim X$. Fix K, restrict to small t, and apply Taylor's theorem to the case $a = 0$ to obtain

$$\frac{\alpha_x(\xi_x + t\eta_x) - \alpha_x(\xi_x)}{t} - \frac{1}{t} d\alpha_x(\xi_x) \cdot t\eta_x = \frac{1}{t} o(|t\eta_x|^2).$$

The above holds uniformly in K, and the Landau symbol is given by

$$o(|t\eta_x|^2) = \int_0^1 (1-\mu)|d^2\alpha_x(\xi_x + \mu t\eta_x) \cdot (t\eta_x, t\eta_x)| \, d\mu.$$

Since, here, derivatives commute with the integral we have for general $a \in \mathbb{N}_0^n$ that

$$|\partial^a(\frac{\alpha_x(\xi_x + t\eta_x) - \alpha_x(\xi_x)}{t}) - \partial^a(d\alpha_x(\xi_x) \cdot \eta_x)| = o(|t\eta_x|^{|a|+1})$$

uniformly on K. □

Theorem 3.3. *Let X, Y be smooth, finite dimensional manifolds. Then $C^\infty(X, Y)$ is a smooth manifold modelled on spaces $\Gamma_c(f^*TY \to X)$.*

Proof. This theorem is discussed and proved in Kriegl, Michor [11]. At first sight the proof is rather similar to the one of theorem 3.11 below. As a matter of fact, however, it is much more involved because it goes beyond the realm of Banach spaces. □

3.B. Ω-lemma

This subsection follows Palais [20], where quite an interesting development is presented. Assuming, roughly speaking, only continuity it is shown that composition from the left by smooth functions is even smooth, and this holds for all fiber bundles over compact base manifolds.

For any smooth manifold M denote the category of smooth vector bundles over M and smooth fiber preserving maps by $\mathrm{FVB}(M)$. In what follows $\mathcal{L}$ shall denote a functor from $\mathrm{FVB}(M)$ to the category of Banach spaces and continuous maps that satisfies the Ω-condition. For example, the functor Γ satisfies the Ω-condition.

Definition (Ω-condition). The functor $\mathcal{L}$ is said to satisfy the *Ω-condition* if for all compact manifolds M the following holds true:

(i) If $E \in \mathrm{ObFVB}(M)$ then the inclusion $\mathcal{L}(E \to M) \hookrightarrow \Gamma^0(E \to M)$ holds and is continuous linear.

(ii) If $E, F \in \mathrm{ObFVB}(M)$ and $f \in \mathrm{Mor}_{(E,F)}\mathrm{FVB}(M)$ then $f_* : \Gamma^0(E \to M) \to \Gamma^0(F \to M)$, $\sigma \mapsto f \circ \sigma$ restricts to a continuous map $\mathcal{L}(f) := f_*|_{\mathcal{L}(E)} \colon \mathcal{L}(E \to M) \to \mathcal{L}(F \to M)$.

(iii) If N is another compact manifold, $\varphi \in C^\infty(M, N)$ a diffeomorphism, and $E \in \mathrm{ObFVB}(N)$ then $\varphi^* \colon \mathcal{L}(E \to N) \to \mathcal{L}(\varphi^* E \to M)$, $\sigma \mapsto \sigma \circ \varphi$ shall be a continuous linear map.

1. Let $\{U^i\}_{i=1}^n$ be an open cover of the compact manifold M, and $s^i \in \mathcal{L}(E)$. *If s is any section of $E \to M$ such that $s|_{U^i} = s^i|_{U^i}$ then $s \in \mathcal{L}(E)$.* Choose a smooth partition of unity $\{\psi_i\}_{i=1}^n$ subordinate to $\{U^i\}_{i=1}^n$. Then $s = \sum_{i^1}^n \psi_i s = \sum_{i^1}^n \psi_i s^i \in \mathcal{L}(E)$. This is the *localization property* of $\mathcal{L}$. □

2. *Direct sum property.* The natural isomorphism $\Gamma^0(E \oplus F) \cong \Gamma^0(E) \oplus \Gamma^0(F)$ restricts to an isomorphism $\mathcal{L}(E \oplus F) \cong \mathcal{L}(E) \oplus \mathcal{L}(F)$. □

3. *If $E = \mathbb{R} \times M \to M$ is the trivial line bundle then $\mathcal{L}(E)$ is even a Banach algebra.* First note that fiber wise multiplication $m : E \oplus E \to E$, $(x, t, x, s) \mapsto (x, ts)$ is smooth. Then $m_* : \Gamma^0(E) \oplus \Gamma^0(E) \to \Gamma^0(E)$, $(s_1, s_2) \mapsto s_1 \cdot s_2$ gives the Banach algebra structure on $\Gamma^0(E)$ which by definition restricts to $\mathcal{L}(E)$. □

4. Maps $f \in \mathrm{Mor}_{(E,F)}\mathrm{FVB}(M)$ are by definition smooth and fiber respecting. This means that we can consider the *vertical derivative* $\mathcal{L}(d_v f)(\sigma, \tau)(x) = d_v f(\sigma(x), \tau(x)) := d_v f(\sigma(x))(\tau(x)) := T_{\sigma(x)}(f|_{E_x}) \cdot \tau(x) = \frac{d}{dt}|_0 f(\sigma(x) + \tau(x))$ for sections $\sigma, \tau \in \mathcal{L}(E)$. This is well defined because the restriction of f to the fibers is smooth, making $d_v f \colon E \to \mathrm{Hom}(E, F)$ smooth and fiber respecting, and hence $\mathcal{L}(d_v f) \colon \mathcal{L}(E) \to \mathcal{L}(\mathrm{Hom}(E, F))$ is continuous. Furthermore the r-th vertical derivative is defined by induction, $d_v^r f := d_v(d_v^{r-1} f) : E \to \mathrm{Hom}(E, \mathrm{Hom}^{r-1}_{\mathrm{sym}}(E; F)) = \mathrm{Hom}^r_{\mathrm{sym}}(E; F) := \mathrm{Hom}_{\mathrm{sym}}(E^r, F)$ – vector space of r-linear, symmetric maps $E^r \to F$. Again $d_v^r f$ is smooth and $\mathcal{L}(d_v^r f) \colon \mathcal{L}(E) \to \mathcal{L}(\mathrm{Hom}^r_{\mathrm{sym}}(E; F))$ is continuous. □

Theorem 3.4 (Ω-lemma). *Let M be a smooth, compact manifold, $E \to M$, $F \to M$ vector bundles, and $f \in C^\infty(E, F)$ a fiber respecting map. Then $\mathcal{L}(f) \colon \mathcal{L}(E) \to \mathcal{L}(F)$, $\sigma \mapsto f \circ \sigma$ is smooth, and moreover $d^r \mathcal{L}(f) = \mathcal{L}(d_v^r f)$.*

Proof. First $r = 1$. By localization we can restrict to the case $M = B \subseteq \mathbb{R}^n$ is the closed unit ball, and $E = B \times \mathbb{R}^m$, $F = B \times \mathbb{R}^l$. The $\mathcal{L}$-norms are all denoted by the same symbol $\|\cdot\|$. Fix $\sigma \in \mathcal{L}(E)$. We need to show that $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all $\tau \in \mathcal{L}(E)$ with $\|\tau\| < \delta$:

$$\|\mathcal{L}(f)(\sigma + \tau) + \mathcal{L}(f)(\sigma) - \mathcal{L}(d_v f)(\sigma, \tau)\| \leq \varepsilon \|\tau\|.$$

By the direct sum property and the universal property of products, to answer this question it suffices to consider $F = B \times \mathbb{R}$, which makes the map $\mathrm{ev}_x \circ \mathrm{ev}_\tau\colon \mathcal{L}(\mathrm{Hom}^2_{\mathrm{sym}}(E;F)) \to \mathcal{L}(F) \hookrightarrow \Gamma^0(F) \to \mathbb{R}$, $l \mapsto l(x)(\tau(x), \tau(x))$ a continuous linear functional on $\mathcal{L}(\mathrm{Hom}^2_{\mathrm{sym}}(E;F))$. With $f_x := f|_{E_x}$ and Taylor's theorem this yields

$$
\begin{aligned}
\mathcal{L}(f)(\sigma+\tau)(x) + \mathcal{L}(f)(\sigma)(x) - \mathcal{L}(d_v f)(\sigma,\tau)(x)& \\
&= f_x(\sigma(x)+\tau(x)) + f_x(\sigma(x)) - df_x(\sigma(x)) \cdot \tau(x) \\
&= \int_0^1 (1-t) d^2 f_x(\sigma(x) + t\tau(x))(\tau(x), \tau(x)) dt \\
&= \int_0^1 (1-t) \mathcal{L}(d_v^2 f)(\sigma + t\tau) dt(\tau)(x)
\end{aligned}
$$

for all $x \in M$. Now $\mathcal{L}(d_v^2 f)\colon \mathcal{L}(E) \to \mathcal{L}(\mathrm{Hom}^2_{\mathrm{sym}}(E;F))$ is continuous, and this just means that $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all $\tau \in \mathcal{L}(E)$ with $\|\tau\| < \delta$: $\|\mathcal{L}(d_v^2 f)(\sigma + t\tau)\| < \varepsilon$ for all $t \in [0,1]$.
Thus $\mathcal{L}(f)$ is C^1, and for $r > 1$ we can proceed by induction. Assume $\mathcal{L}(f)$ is C^{r-1} and $d^{r-1}\mathcal{L}(f) = \mathcal{L}(d_v^{r-1} f)$. But then $d_v^{r-1} f\colon E \to \mathrm{Hom}^{r-1}_{\mathrm{sym}}(E;F)$ is smooth and fiber respecting, and hence $d^r \mathcal{L}(f) = d(d^{r-1}\mathcal{L}(f)) = d\mathcal{L}(d_v^{r-1} f) = \mathcal{L}(d_v^r f)$. $\square$

The above theorem constitutes the intermediate case in the development in Palais [20], and it will actually be sufficient for this paper. Intermediate step means that in order to show a property, like the Ω-lemma, in the category of fiber bundles and smooth maps one passes through the category of vector bundles and smooth maps. As mentioned above Γ satisfies the Ω-condition, so, in particular, 3.4 reproves 3.2 for compact manifolds.

3.C. Sobolev spaces

1. Let $O \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}_0$, $1 \le p < \infty$. Then the *Sobolev space* $W^{k,p}(O, \mathbb{R}^m)$ is the completion of $C_c^\infty(O, \mathbb{R}^m)$ with respect to the *Sobolev* $W^{k,p}$*-norm*

$$
\|f\|_{k,p} := \Big(\sum_{|\alpha| \le k} \|\partial^\alpha f\|_p^p \Big)^{\frac{1}{p}} := \Big(\sum_{|\alpha| \le k} \int |\partial^\alpha f(x)|^p dx \Big)^{\frac{1}{p}},
$$

where multi index notation $\alpha \in \mathbb{N}_0^n$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$ is used. Elements $f \in L^p(O, \mathbb{R}^m)$ can be evaluated on test functions $\varphi \in \mathcal{D}(O, \mathbb{R}^m) := C_c^\infty(O, \mathbb{R}^m)$ by putting $\langle f, \varphi \rangle := \int f\varphi$; $\mathcal{D}(O, \mathbb{R}^m)$ is equipped with the topology of uniform convergence in all derivatives on all compact subsets and support contained in one fixed subset. Thus we can consider the linear continuous embedding $L^p(O,\mathbb{R}^m) \hookrightarrow \mathcal{D}'(O,\mathbb{R}^m)$, $f \mapsto \langle f, \_\rangle$. Moreover, integration by parts motivates to define the *distributional derivative* by $\langle \partial^\alpha f, \varphi\rangle := (-1)^{|\alpha|}\langle f, \partial^\alpha \varphi\rangle$, and we can identify $W^{k,p}(O,\mathbb{R}^m)$ with those distributions that have k-th distributional derivative in (the embedded image of) $L^p(O,\mathbb{R}^m)$. In general $W^{k,p}(O,\mathbb{R}^m)$ is only a Banach space, for $k=2$ however it is even a Hilbert space. See [Yosida, 1965]. Since finite sums of $W^{k,p}$-norms again are $W^{k,p}$-norms one has the formula $W^{k,p}(O,\mathbb{R}^m) = \bigoplus_{i=1}^m W^{k,p}(O,\mathbb{R})$. Without explicitly mentioning it we will often make use of the fact that the finite direct sum equals the finite product. □

2. Since embedding of differentiable functions into distributions gives rise to a commutative rectangle with ordinary and distributional derivative as parallel lines, both are denoted by the same symbol ∂. *For $|\alpha| \le k$ the distributional derivative $\partial^\alpha \colon W^{k,p}(O,\mathbb{R}^m) \to W^{k-\alpha,p}(O,\mathbb{R}^m)$ is a linear and continuous operator.* Let $f_n \to f$ in $W^{k,p}(O,\mathbb{R}^m) \subseteq \mathcal{D}'(O,\mathbb{R}^m)$ then $\langle \partial^\alpha f_n, \varphi\rangle = (-1)^{|\alpha|}\langle f_n, \partial^\alpha\varphi\rangle \to (-1)^{|\alpha|}\langle f, \partial^\alpha\varphi\rangle = \langle \partial^\alpha f, \varphi\rangle$ for all $\varphi \in \mathcal{D}(O,\mathbb{R}^m)$, i.e $\partial^\alpha f_n \to \partial^\alpha f$. □

3. For $p_i \ge 1$, $f_i \in L^{p_i}(O,\mathbb{R}) = W^{0,p_i}(O,\mathbb{R})$ the *Hölder inequality* $\|f_1 f_2\|_{p_1} \le \|f_1\|_{p_2}\|f_2\|_{p_3}$ holds whenever $\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}$. By induction this implies

$$\begin{aligned}\|f_1 \cdots f_{r-1} f_r\|_{(\sum_{i=1}^r \frac{1}{p_i})^{-1}} &\le \|f_1 \cdots f_{r-1}\|_{(\sum_{i=1}^{r-1} \frac{1}{p_i})^{-1}} \cdot \|f_r\|_{p_r} \\ &\le \|f_1\|_{p_1} \cdots \|f_r\|_{p_r},\end{aligned}$$

since $f_1 \cdots f_{r-1} \in L^{(\sum_{i=1}^{r-1} \frac{1}{p_i})^{-1}}(O,\mathbb{R})$, also by induction. □

4. Let X be a compact manifold equipped with a strictly positive measure μ, and $E \to X$ shall be a vector bundle with typical fiber $\mathbb{R}^m$. Choose a finite atlas $\{(U_i,\varphi_i) : i = 1,\dots,m\}$ that has in $E \to X$ trivializing patches, and a partition of unity $\{\rho_i\}_{i=1}^m$ subordinate to the open cover $\{U_i\}_{i=1}^m$. Then the *Sobolev $W^{k,p}$-norm* on $\Gamma(E \to X)$ is

$$\|s\|_{k,p}^E := \sum_{i=1}^m \|(\varphi_i^{-1})^*(\rho_i s)\|_{k,p}.$$

This definition is not independent of the choices. Note that this just means that $(\varphi_i^{-1})^* : W^{k,p}(E|_{U_i}) \to W^{k,p}((\varphi_i^{-1})^* E|_{U_i}) = W^{k,p}(\varphi_i(U_i) \times \mathbb{R}^m)$ is a topological isomorphism with respect to the chosen norm. □

Definition. $k \in \mathbb{N}_0$, $1 \le p < \infty$. Let X be a compact manifold equipped with a strictly positive measure μ, and $E \to X$, shall be a vector bundle with typical fiber $\mathbb{R}^m$. Then the *Sobolev space* $W^{k,p}(E \to X)$ is the completion of $\Gamma(E \to X)$ with respect to the Sobolev $W^{k,p}$-norm. *The topology of* $W^{k,p}(E \to X)$ *is well-defined and independent of the choices made. In particular, if* $\{(U_i, \varphi_i) : i = 1, \dots, m\}$ *is an atlas that has in* $E \to X$ *trivializing patches then we can identify* $(\varphi_i^{-1})^* \colon W^{k,p}(E|_{U_i}) \cong W^{k,p}(\varphi_i(U_i) \times \mathbb{R}^m)$. This is the *localization property* of $W^{k,p}$.

Proof. See Palais [20]. □

Theorem 3.5. $k, l \in \mathbb{N}_0$, $1 \le p, q < \infty$. *Let* X *be a compact manifold,* $\dim X = n$, *equipped with a strictly positive measure* μ, *and* $E \to X$ *shall be a vector bundle.*

Sobolev lemma. *If* $k - \frac{n}{p} \ge l - \frac{n}{q}$ *and* $k \ge l$ *then the injection* $W^{k,p}(E) \hookrightarrow W^{l,q}(E)$ *holds and is continuous. If* $k - \frac{n}{p} > l - \frac{n}{q}$ *and* $k > l$ *it is even compact.*

Rellich's theorem. *If* $k - \frac{n}{p} > l$ *then* $W^{k,p}(E) \hookrightarrow \Gamma^l(E)$ *holds and is furthermore continuous and compact.*

Proof. See Palais [20]. □

This embedding theorem implies that $W^{k,p}$ satisfies the first part of the Ω-condition under assumptions which will always be met in this paper. The lemmata below (which, too, rely on the above theorem) will imply that $W^{k,p}$ indeed satisfies the Ω-condition.

Lemma 3.6. *Let* X *be a compact manifold with measure* μ, $\dim X = n$, $1 \le p_i, q < \infty$, *and* $k_i = \frac{n}{p_i} - s_i \ge 0$ *then point wise multiplication*

$$\mu : \bigoplus_{i=1}^{r} W^{k_i,p_i}(X, \mathbb{R}) \longrightarrow W^{0,q}(X, \mathbb{R}) = L^q(X, \mathbb{R}),$$
$$(\sigma_1, \dots, \sigma_r) \longmapsto \sigma_1 \cdot \dots \cdot \sigma_r$$

is continuous under the assumption that $\frac{1}{q} > \frac{1}{n} \sum_{s_i > 0} s_i$.

Proof. Define numbers $q_i \in \mathbb{N}$ such that $s_i > 0 \Rightarrow \frac{1}{q_i} = \frac{s_i}{n}$; if $s_i = 0$ or $s_i < 0$ then the q_i's shall be chosen large enough such that $\frac{1}{q} > \sum_{i=1}^{r} \frac{1}{q_i} = \sum_{s_i \le 0} \frac{1}{q_i} + \sum_{s_i > 0} \frac{1}{q_i}$ holds. Thus, always $k_i - \frac{n}{p_i} = -s_i \ge -\frac{n}{q_i}$, and the assumption $k_i \ge 0$ guarantees continuity of the inclusion $\iota_i : W^{k_i,p_i}(X, \mathbb{R}) \hookrightarrow W^{0,q_i}(X, \mathbb{R}) = L^{q_i}(X, \mathbb{R})$. Since $\mu(X) < \infty$ we can employ the Hölder inequality to conclude $\|\sigma_1 \cdots \sigma_r\|_q \le \|\sigma_1 \cdots \sigma_r\|_{(\sum_{i=1}^{r} \frac{1}{q_i})^{-1}} \le \|\sigma_1\|_{q_1} \cdots \|\sigma_r\|_{q_r}$ for all $\sigma_i \in L^{q_i}(X, \mathbb{R})$. Whence $\mu \circ \oplus_{i=1}^{r} \iota_i \colon \bigoplus_{i=1}^{r} W^{k_i,p_i}(X, \mathbb{R}) \to L^q(X, \mathbb{R})$ is continuous. □

Lemma 3.7. *$1 \le p < \infty$, $k > \frac{n}{p}$. Let $B \subseteq \mathbb{R}^n$ be the open unit ball, and $\alpha_1, \dots, \alpha_r \in \mathbb{N}_0^n$ multi indices with $\sum_{i=1}^r |\alpha_i| \le k$ and $|\alpha_i| > 0$ for all i. Then*

$$\bigoplus_{i=1}^r W^{k,p}(B, \mathbb{R}) \longrightarrow L^p(B, \mathbb{R}), \quad (f_1, \dots, f_r) \longmapsto \partial^{\alpha_1} f_1 \cdots \partial^{\alpha_r} f_r$$

is a continuous r-linear map.

Proof. By (2) the map $\partial^\alpha \colon W^{k,p}(B, \mathbb{R}) \to W^{k-\alpha,p}(B, \mathbb{R})$ is continuous linear. Put $t := |\{i \in \{1, \dots, r\} : |\alpha_i| > k - \frac{n}{p}\}|$ and $s_i := |\alpha_i| + \frac{n}{p} - k$. By the above lemma we need to check that $\frac{n}{p} > \sum_{s_i > 0} s_i$ which is equivalent to $\frac{n}{p} + t(k - \frac{n}{p}) > \sum_{|\alpha_i| > k - \frac{n}{p}} |\alpha_i|$. We consider three cases:

($t = 0$) $\frac{n}{p} > 0$, clear.

($t = 1$) $\sum_{|\alpha_i| > k - \frac{n}{p}} |\alpha_i| = |\alpha_{i_0}| < \sum_{i=1}^r |\alpha_i| \le k$, since all $|\alpha_i| > 0$.

($t > 1$) $\sum_{i=1}^r |\alpha_i| \le k \Longrightarrow \sum_{|\alpha_i| > k - \frac{n}{p}} |\alpha_i| < \sum_{|\alpha_i| > k - \frac{n}{p}} |\alpha_i| + (t-1)(k - \frac{n}{p}) \le k + (t-1)(k - \frac{n}{p}) = \frac{n}{p} + t(k - \frac{n}{p})$.

The case ($t = 1$) assumes $r > 1$; if $r = 1$ the lemma follows from the Sobolev lemma 3.5. □

Lemma 3.8. *$1 \le p < \infty$, $k > \frac{n}{p}$. Let $B \subseteq \mathbb{R}^n$ be the open unit ball, and $f \in C^\infty(B \times \mathbb{R}^r, \mathbb{R}^s)$. Then the mapping*

$$C(B, \mathbb{R}^r) = \bigoplus^r C(B, \mathbb{R}) \longrightarrow C(B, \mathbb{R}^s),$$
$$\sigma = (\sigma_1, \dots, \sigma_r) \longmapsto f(\_, \sigma_1(\_), \dots \sigma_r(\_)) = f_*(\mathrm{id}_B, \sigma)$$

restricts to a continuous mapping

$$W^{k,p}(B, \mathbb{R}^r) = \bigoplus^r W^{k,p}(B, \mathbb{R}) \longrightarrow W^{k,p}(B, \mathbb{R}^s),$$
$$\sigma = (\sigma_1, \dots, \sigma_r) \longmapsto f(\_, \sigma_1(\_), \dots \sigma_r(\_)) = f_*(\mathrm{id}_B, \sigma).$$

Proof. By the universal property of the product

$$W^{k,p}(B, \mathbb{R}^s) = \bigoplus^s W^{k,p}(B, \mathbb{R}) = \prod^s W^{k,p}(B, \mathbb{R})$$

the problem reduces to the case $r = 1$. Let $\sigma = (\sigma_1, \dots, \sigma_r) \in (C^\infty(B, \mathbb{R}))^r$ then the chain rule yields a continuous mapping

$$\sigma \longmapsto \partial^\alpha (f \circ (\mathrm{id}_B, \sigma)) = \sum_{|\beta_1| + \ldots + |\beta_r| \le |\alpha|} \varphi_{\beta_1, \dots, \beta_r}(\mathrm{id}_B, \sigma) \cdot \partial^{\beta_1} \sigma_1 \dots \partial^{\beta_r} \sigma_r;$$

since the $\varphi_{\beta_1,\ldots,\beta_r}$ are (more than) continuous functions. We need to show that this extends to a continuous function $\bigoplus^r W^{k,p}(B,\mathbb{R}) \to L^p(B,\mathbb{R})$ for all $|\alpha| \leq k$. Indeed, the mapping

$$\bigoplus^r W^{k,p}(B,\mathbb{R}) \hookrightarrow \bigoplus^r C(B,\mathbb{R}) \longrightarrow C(B,\mathbb{R}),$$
$$\sigma \longmapsto (\varphi_{\beta_1,\ldots,\beta_r})_*(\mathrm{id}_B,\sigma)$$

is continuous for the compact-open topology on $C(B,\mathbb{R})$; $\bigoplus^r W^{k,p}(B,\mathbb{R}) \to L^p(B,\mathbb{R})$, $\sigma \mapsto \partial^{\beta_1}\sigma_1 \ldots \partial^{\beta_r}\sigma_r$ is continuous for $|\beta_1| + \ldots + |\beta_r| \leq |\alpha| \leq k$; and multiplication $C(B,\mathbb{R}) \times L^p(B,\mathbb{R}) \to L^p(B,\mathbb{R})$, $(g,h) \mapsto g \cdot h$ is continuous. □

Proposition 3.9. *Let* $1 \leq p < \infty$, $k > \frac{n}{p}$, X *a compact manifold,* $\dim X = n$, $E \to X$, $F \to X$ *vector bundles with standard fibers* $\mathbb{R}^r$, $\mathbb{R}^s$ *respectively, and* $f \in C^\infty(E,F)$ *fiber preserving. Then*

$$f_* \colon W^{k,p}(E) \to W^{k,p}(F), \quad \sigma \mapsto f \circ \sigma$$

is continuous.

Proof. This is a local question, and by localization it suffices to consider $M = B \subseteq \mathbb{R}^n$ open unit ball, $E = B \times \mathbb{R}^r$, and $F = B \times R^s$. But then the assertion is just a reformulation of the previous lemma. □

Corollary 3.10. *If* $1 \leq p < \infty$, $k > \frac{n}{p}$ *then* $W^{k,p}$ *satisfies the* Ω*-condition.*

Proof. The first part is theorem 3.5 and the second the above proposition. Third part: X, Y compact manifolds, $E \to Y$ a vector bundle, and $f : X \to Y$ a diffeomorphism. Then $f^* : W^{k,p}(E) \to W^{k,p}(f^*E)$ is a linear bijection. If (U,ψ) is chart on Y with trivializing patch in $E \to Y$ then $(f^*\psi, f^{-1}U)$ is a chart on X with in $f^*E \to X$ trivializing patch. Then $f^*|_V = (f^*\psi)^* \circ (\psi^{-1})^* : W^{k,p}(E|_V) \to W^{k,p}(f^*E|_{f^{-1}(V)})$ is a topological isomorphism by the localization property. Thus f^* is an isomorphism which is topological when restricted to trivializing patches, and thereby a topological isomorphism as expected. □

Definition. Let $k,p \in \mathbb{N}$, $kp > \dim X$, and X, Y smooth manifolds, and compact then the Sobolev mappings are those which are deformations of smooth mappings, that is

$$W^{k,p}(X,Y) := \{g \in C(X,Y) :$$

$$\exists f \in C^\infty(X,Y), \exists(U,\nabla), \exists \xi \in W^{k,p}(f^*U \to X), \text{ with } g = \exp_f^\nabla \xi\}.$$

This works with the following notation: the data (U,∇) consists of an open neighborhood of the zero section $U \subseteq TY$, and a connection ∇ such that the corresponding exponential map $\exp^\nabla : U \to Y \times Y$, $\zeta \mapsto (\pi(\zeta), \exp^\nabla_{\pi(\zeta)}\zeta)$ is a diffeomorphism onto its open image. For this subsection only, call such a pair (U,∇) *admissible*. Furthermore $g(x) := \exp^\nabla_{f(x)}\xi(x) := (\exp^\nabla_f \xi)(x) = (\mathrm{pr}_2 \circ \exp^\nabla \circ \xi)(x)$. For the pullback bundle see 3.A.(1). Tacitly, we always identify $W^{k,p}(f^*TY \to X) = W^{k,p}(X,TY)_f = \{h \in W^{k,p}(X,TY) : \pi_Y \circ h = f\}$.

Theorem 3.11. *Let X, Y smooth manifolds, X compact, $k,p \in \mathbb{N}$ and $kp > n = \dim X$ then the Sobolev completion $W^{k,p}(X,Y)$ is a smooth Banach manifold modelled on spaces $W^{k,p}(f^*TY \to X)$ where $f \in C^\infty(X,Y)$. Moreover, $W^{k,p}(X,Y)$ is completely metrizable.*

Proof. Let $f \in C^\infty(X,Y)$ and (U,∇) admissible such that $\exp : U \to \exp U =: V \subseteq Y \times Y$ is a diffeomorphism onto an open neighborhood of the diagonal. Then

$$\begin{aligned} \psi_f \colon W^{k,p}(f^*U) &\longrightarrow \operatorname{im}\psi_f = U_f, \\ \xi &\longmapsto \exp_f \circ (\pi^* f) \circ \xi, \\ (\mathrm{id}_X, \exp_f^{-1} \circ g) &\longleftarrow g \end{aligned}$$

is a bijection onto $U_f := \{g \in W^{k,p}(X,Y) \colon (f,g)(X) \subseteq V\} = \operatorname{im}\psi_f$; $(\psi_f^{-1} \circ \psi_f)(\xi) = (\mathrm{id}, \exp_f^{-1} \circ \exp_f \circ (\pi^* f) \circ \xi) = \xi$, and $(\psi_f \circ \psi_f^{-1})(g) = \exp_f \circ (\pi^* f) \circ (\mathrm{id}, \exp_f^{-1} \circ g) = g$, cf. section 3.A. We define the topology on $W^{k,p}(X,Y)$ to be the final one with respect to the sink

$$\psi_f \colon W^{k,p}(f^*U) \longrightarrow W^{k,p}(X,Y) \qquad f \in C^\infty(X,Y),\ (U,\nabla) \text{ admissible.}$$

By theorem 3.5 the embedding $\iota : W^{k,p}(f^*TY) \hookrightarrow \Gamma^0(f^*TY)$ is continuous. By 3.A.(3) $\Gamma^0(f^*U)$ is open, implying the same for $W^{k,p}(f^*U) = \iota^{-1}(\Gamma^0(f^*U)) \subseteq W^{k,p}(f^*TY)$. If the triples (f_1, U_1, ∇_1), (f_2, U_2, ∇_2) produce charts $(\psi_{f_1}^{-1}, U_{f_1})$, $(\psi_{f_2}^{-1}, U_{f_2})$ respectively such that $U_{f_1} \cap U_{f_2} \neq \emptyset$ then $\psi_{f_2}^{-1} \circ \psi_{f_1} : W^{k,p}(f_1^* W_1) \to W^{k,p}(f_2^* W_2)$ is a smooth map between open subsets: $U_{f_1} \cap U_{f_2}$ is open by definition of topology, and hence $\psi_{f_i}^{-1}(U_{f_1} \cap U_{f_2}) = W^{k,p}(f_i^* W_i)$ is open for some open subset $W_i \subseteq U_i$ and $i \in \{1,2\}$. Now we can apply the Ω-lemma 3.4 to conclude that the chart changing

$$\begin{aligned} \psi_{f_2}^{-1} \circ \psi_{f_1} \colon W^{k,p}(f_1^* W_1) &\longrightarrow W^{k,p}(f_2^* W_2) \\ \xi &\longmapsto \exp_{f_1} \circ (\pi^* f_1) \circ \xi \end{aligned}$$

$$
\begin{aligned}
&\longmapsto (\mathrm{id}, \exp_{f_2}^{-1} \circ \exp_{f_1} \circ (\pi^* f_1) \circ \xi) \\
&\quad = (\mathrm{id} \times \exp_{f_2})^{-1} \circ ((\mathrm{id} \times \exp_{f_1}) \circ (\mathrm{id}, (\pi^* f_1 \circ \xi) \\
&\quad = ((\mathrm{id} \times \exp_{f_2})^{-1} \circ (\mathrm{id} \times \exp_{f_1}))_*(\xi)
\end{aligned}
$$

is smooth: $(\mathrm{id} \times \exp_{f_2})^{-1} \circ (\mathrm{id} \times \exp_{f_1}) : f_1^* W_1 \to f_2^* W_2$ is smooth and fiber respecting; it is well defined since $W_i = \exp_{f_i}^{-1}(\exp(U_1) \cap \exp(U_2))$ – where $\exp^{\nabla_1}$ and $\exp^{\nabla_1}$ are only implicitly distinguished. The above defined topology on $W^{k,p}(X, Y)$ is Hausdorff: indeed, the canonical inclusion $W^{k,p}(X, Y) \hookrightarrow C(X, Y)$ is a continuous embedding. This follows by applying Rellich's theorem 3.5 to the bottom line in

$$
\begin{array}{ccc}
W^{k,p}(X, Y) & \xrightarrow{\subseteq} & C(X, Y) \\
{\scriptstyle \psi_f}\uparrow & & \downarrow{\scriptstyle \varphi_f^{-1}} \\
W^{k,p}(f^*TY) & \xrightarrow{\subseteq} & \Gamma^0(f^*TY)
\end{array}
$$

The mappings here are only defined locally, and $\varphi_f \colon \Gamma^0(f^*TY) \to C(X, Y)$ is the chart mapping from 3.A.(3). Thus $W^{k,p}(X, Y)$ is continuously embedded in a topological Hausdorff space; this embedding is not initial.

$W^{k,p}(X, Y)$ *is completely metrizable.* Embed $Y \hookrightarrow T \subseteq \mathbb{R}^r$ with large $r \in \mathbb{N}$, and let $T \to Y$ be a tubular neighborhood of Y in $\mathbb{R}^r$. Now, $W^{k,p}(X, T)$ is completely metrizable and it only remains to show that $W^{k,p}(X, Y) \subseteq W^{k,p}(X, T)$ is closed. This is true since point evaluations are continuous. Let $(f_n)_n$ be a sequence in $W^{k,p}(X, Y)$, $f_n \to f$. Assume there were a point $x \in X$ such that $f(x) \notin Y$. By the above we can embed $W^{k,p}(X, Y) \hookrightarrow C(X, Y)$ continuously, thus $f_n(x) \to f(x)$. But $Y \subseteq T$ is the zero section, hence closed, and $f_n(x) \in Y$ for all $n \in \mathbb{N}$ implies $f(x) \in Y$. $\square$

Lemma 3.12. *Let X, Y be smooth manifolds, X compact, $p > \dim X = n$, $k \in \mathbb{N}$ and $\dim Y = m$. Choose smooth atlases $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$, $(V_\beta, \psi_\beta)_{\beta \in B}$ for X, Y respectively, and assume that $g \in C(X, Y)$.*

(i) *Let $(\alpha, \beta) \in A \times B$. If $g \in W^{k,p}(X, Y)$ and $g(U_\alpha) \subseteq V_\beta$ then it follows that $\psi_\beta \circ g \circ \varphi_\alpha^{-1} \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$.*

(ii) *Conversely, if $\psi_\beta \circ g \circ \varphi_\alpha^{-1} \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ for all $(\alpha, \beta) \in A \times B$ then it follows that $g \in W^{k,p}(X, Y)$.*

Proof. *(i)* Fix $(\alpha, \beta) \in A \times B$ such that $g(U_\alpha) \subseteq V_\beta$. Let $f \in C^\infty(X, Y)$ and $\xi \in W^{k,p}(f^*TY)$ such that $g = \exp_f \circ (\pi^* f) \circ \xi$. In this proof the upper star will denote the morphisms belonging to the pullback bundle as well as pullback of vector fields, e.g. $(f^*\pi)^*(\varphi_\alpha^{-1}) \circ (\varphi_\alpha^{-1})^*\xi = \xi \circ \varphi_\alpha^{-1}$. Notation is also introduced by the following diagram.

$$\begin{array}{ccccc}
(\varphi_\alpha^{-1})^* f^*TY & \xrightarrow{(f^*\pi)^*(\varphi_\alpha^{-1})} & f^*TY & \xrightarrow{\pi^* f} & TY \\
{\scriptstyle (\varphi_\alpha^{-1})^*(f^*\pi)}\downarrow & & {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\varphi_\alpha(U_\alpha) & \xrightarrow{\varphi_\alpha^{-1}} & X & \xrightarrow{f} & Y
\end{array}$$

The pullback bundle is constructed by pulling back the transition functions, hence $(\varphi_\alpha^{-1})^* f^*TY = (f \circ \varphi_\alpha^{-1})^*TY$, and $\pi^* f \circ (f^*\pi)^*\varphi_\alpha^{-1} = \pi^*(f \circ \varphi_\alpha^{-1})$, as well as $(\varphi_\alpha^{-1})^*(f^*\pi) = (f \circ \varphi_\alpha^{-1})^*\pi$. This yields

$$\begin{aligned}
\psi_\beta \circ g \circ \varphi_\alpha^{-1} &= \psi_\beta \circ \exp_f \circ \pi^* f \circ \xi \circ \varphi_\alpha^{-1} \\
&= \psi_\beta \circ \exp_f \circ \pi^* f \circ (f^*\pi)^*(\varphi_\alpha^{-1}) \circ (\varphi_\alpha^{-1})^*\xi \\
&= \psi_\beta \circ \exp_f \circ \pi^*(f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha^{-1})^*\xi \\
&=: h \circ (\varphi_\alpha^{-1})^*\xi.
\end{aligned}$$

Now $(\varphi_\alpha^{-1})^* : W^{k,p}(f^*TY|_{U_\alpha}) \to W^{k,p}(\varphi_\alpha(U_\alpha) \times \mathbb{R}^m) \cong W^{k,p}(\varphi_\alpha(U_\alpha), \mathbb{R}^m)$, $\xi \mapsto (\varphi_\alpha^{-1})^*\xi = (\mathrm{id}, \sigma) \mapsto \sigma$ is continuous almost by definition. The assertion follows since $h : \varphi_\alpha(U_\alpha) \times \mathbb{R}^m \to \psi_\beta(V_\beta) \hookrightarrow \mathbb{R}^m$ is smooth, thus making $W^{k,p}(\varphi_\alpha(U_\alpha), \mathbb{R}^m) \to W^{k,p}(\varphi_\alpha(U_\alpha), \mathbb{R}^m)$, $\sigma \mapsto h \circ (\mathrm{id}, \sigma) = h \circ (\varphi_\alpha^{-1})^*\xi$ continuous by lemma 3.8. In particular $\psi_\beta \circ g \circ \varphi_\alpha^{-1} \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$.
(ii) Let $g_{\alpha\beta} := \psi_\beta \circ g \circ \varphi_\alpha^{-1} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ for all $(\alpha, \beta) \in A \times B$, and let $f \in C^\infty(X, Y)$ such that $g = \exp_f \circ (\pi^* f) \circ \xi$ for some $\xi \in \Gamma^0(f^*TY)$, cf. 3.A.(3). Continuing the notation from *(i)* we find that $(\varphi_\alpha^{-1})^*\xi = h^{-1} \circ g_{\alpha\beta} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ by 3.8. This is equivalent to $\xi \in W^{k,p}(f^*TY)$, and hence $g \in W^{k,p}(X, Y)$. □

3.D. Fredholm operators

This section outlines some basic properties of differential operators on manifolds. A more detailed description of the following may be found in Palais et al. [21]. Another reference is Wells [26].

For the notation on jets and spaces thereof, as used in the sequel, see section 3.A. Let $p\colon E \to X$ be a smooth vector bundle with standard fiber $\mathbb{R}^s$ over the n-dimensional base manifold X. Then

$$J^k E := J^k(E \to X) := \coprod_{x \in X} \{ j^k_x s \in J^k_x(X, E) : s \in \Gamma(E) \}$$

is a closed sub-manifold of $J^k(X,E)$. Considering the source projection α we obtain the smooth vector bundle $\alpha\colon J^kE \to X$ with standard fiber

$$\bigoplus_{l=1}^{k} \mathrm{Poly}^l(\mathbb{R}^n, \mathbb{R}^s) = \bigoplus_{l=1}^{k} \mathrm{Hom}^l_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^s) \cong \bigoplus_{|\alpha|\le k} \mathbb{R}^s;$$

here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi index and $|\alpha| := \alpha_1 + \ldots + \alpha_n$. For later use we also introduce the notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}, \ldots, \partial x_n^{\alpha_n}}.$$

Lemma 3.13 (Jet bundle exact sequence). *Let $k \in \mathbb{N}$. The following sequence of vector bundles and vector bundle homomorphisms is exact.*

$$0 \longrightarrow \mathrm{Hom}^k_{\mathrm{sym}}(TX; E) \xrightarrow{\ \iota\ } J^kE \xrightarrow{\pi^k_{k-1}} J^{k-1}E \longrightarrow 0.$$

For $\eta_x \in T^*_xX$ and $v_x \in E_x$ the suspected injection is defined by

$$\iota\colon S^k(\eta_x) \otimes v_x \longmapsto j^k_x\big(\tfrac{1}{k!}(g - g(x))^k s\big)$$

where $g \in C^\infty(X, \mathbb{R})$, $s \in \Gamma(E)$ are chosen such that $dg(x) = \eta_x$ and $s(x) = v_x$.

$S^k(T^*X)$ denotes the sub-bundle of $\bigotimes^k T^*X$ which is given by symmetrization. With this definition we may identify

$$\mathrm{Hom}^k_{\mathrm{sym}}(T_xX; E_x) = S^k(T^*_xX) \otimes E_x.$$

Proof. For each $x \in X$ there clearly exists an exact sequence of the form

$$\begin{aligned} 0 \longrightarrow S^k(T^*_xX) \otimes E_x &= \mathrm{Hom}^k_{\mathrm{sym}}(T_xX; E_x) \\ &\longrightarrow \bigoplus_{l=1}^{k} \mathrm{Hom}^l_{\mathrm{sym}}(T_xX; E_x) = J^k_xE \xrightarrow{\pi^k_{k-1}} J^{k-1}E \longrightarrow 0 \end{aligned}$$

where the injection is the natural one. Thus we need to find a (local) section of $E \to X$ whose only nonzero Taylor coefficient is the one of order k. Let $\eta_x \in T^*_xX$ and $v_x \in E_x$. Then there exist infinitely many $g \in C^\infty(X, \mathbb{R})$ and $s \in \Gamma(E)$ such that $g(x) = 0$, $dg(x) = \eta_x$, and $s(x) = v_x$. Now we may compute locally

$$\iota(S^k(\eta_x) \otimes v_x) = j^k_x\big(\tfrac{1}{k!}(g - g(x))^k s\big) = (x, 0, \ldots, 0, S^k(dg(x)) \otimes s(x))$$

to see that ι really is the hoped for, well-defined injection. $\square$

Definition. Let $E \to X$, $F \to X$ be vector bundles. A *k-th order differential operator* from E to F is linear map $D\colon \Gamma(E) \to \Gamma(F)$ such that, for $x \in X$, $j^k_x s = 0$ implies that $Ds(x) = 0$.
The $C^\infty(X, \mathbb{R})$-module of all k-th order differential operators will be denoted by $\mathrm{DOP}_k(E, F)$.

The k-jet extension mapping is a *universal k-th order differential operator* in the sense that it is a k-th order differential operator, and, for all $D \in \mathrm{DOP}_k(E, F)$, there exists a unique vector bundle homomorphism $T \in \Gamma(\mathrm{Hom}(J^k E, F))$ such that $D = T \circ j^k$.
Indeed, by tautology, $j^k_x s = 0$ implies $j^k_x s = 0$. Let $D \in \mathrm{DOP}_k(E, F)$. For each $x \in X$ the linear mapping $j^k_x : \Gamma(E) \to J^k_x E$ is surjective, and, since $\ker j^k_x \subseteq \ker(\mathrm{ev}_x \circ D)$ is a linear subspace, we can consider the following commutative diagram of linear maps.

$$\begin{tikzcd}
\Gamma(E) \arrow[r, "\pi_x", "\text{onto}"'] \arrow[d, "\mathrm{ev}_x \circ D"'] & \frac{\Gamma(E)}{\ker j^k_x} \arrow[r, "\cong"] \arrow[d, "\text{onto}"] & J^k_x E \\
F_x & \frac{\Gamma(E)}{\ker(\mathrm{ev}_x \circ D)} \arrow[l, "\exists!"'] &
\end{tikzcd}$$

where the surjective maps are the natural projections. Identifying $\pi_x(s) = j^k_x s$, this yields $\mathrm{ev}_x \circ D = T_x \circ j^k_x$ for a uniquely determined linear map $T_x\colon J^k_x E \to F_x$. As this construction is independent of the base point x, we obtain the asserted homomorphism $T : \Gamma(J^k E) \to \Gamma(F)$ which is uniquely determined, and by definition verifies $Ds(x) = (T \circ j^k)(x)(s) = (T(x) \circ j^k_x)(s)$. This proves the following lemma.

Lemma 3.14. *The mapping $(j^k)^* : \Gamma(\mathrm{Hom}(J^k E, F)) \to \mathrm{DOP}_k(E, F)$, $T \mapsto T \circ j^k$ is an isomorphism of $C^\infty(X, \mathbb{R})$-modules.* $\square$

Lemma 3.15. *Let $\dim X = n$ and $p\colon E \to X$, $q : F \to X$ vector bundles with standard fiber $\mathbb{R}^s$, $\mathbb{R}^t$ respectively. A linear map $D : \Gamma(E) \to \Gamma(F)$ is a k-th order differential operator from E to F if and only if every local representation of D is of the form*

$$\sum_{|\alpha| \le k} A_\alpha D^\alpha$$

for smooth maps $A^\alpha\colon \mathbb{R}^n \to L(\mathbb{R}^s, \mathbb{R}^t)$.

Proof. Let $D \in \mathrm{DOP}_k(E, F)$. We choose an open patch $U \subseteq X$ which trivializes E and F, i.e. $U \xrightarrow{\cong} \mathbb{R}^n$, and $E|_U \xrightarrow{\cong} \mathbb{R}^n \times \mathbb{R}^s$ as well as $F|_U \xrightarrow{\cong}$

$\mathbb{R}^n \times \mathbb{R}^t$. By the above lemma there exists a unique homomorphism $T : \Gamma(J^kE) \to \Gamma(F)$ such that $D = T \circ j^k : \Gamma(E) \to \Gamma(F)$. If $\underline{s} : \mathbb{R}^n \to \mathbb{R}^s$ denotes a local representation of $s \in \Gamma(E)$ then a local representation of $j^k s$ is given by $(D^\alpha \underline{s})_{|\alpha|\le k} \in \bigoplus_{\alpha \le k} C^\infty(\mathbb{R}^n, \mathbb{R}^s)$. In the same spirit, a local representation of $T \in \Gamma(\mathrm{Hom}(J^kE, F))$ is of the form $(A_\alpha)_{|\alpha|\le k} \in \bigoplus_{|\alpha|\le k} C^\infty(\mathbb{R}^n, L(\mathbb{R}^s, \mathbb{R}^t))$. Point-wise matrix multiplication of these two yields the local representative of D, namely $\sum_{|\alpha|\le k} A_\alpha D^\alpha$.

For the converse assume that, locally, D looks like $\sum_{|\alpha|\le k} A_\alpha D^\alpha$. By definition, for a smooth map $\underline{s} : \mathbb{R}^n \to \mathbb{R}^s$ we have, for $x \in \mathbb{R}^n$, $j^k_x \underline{s} = 0$ if and only if $D^\alpha \underline{s}(x) = 0$ for all $\alpha \le k$. Thus we find that $D|_U \in \mathrm{DOP}_k(E|_U, F|_U)$, and therefore we may write $D|_U(s) = T^U \circ j^k s$ for all $s \in \Gamma(E|_U)$ and a unique homomorphism $T^U \in \Gamma(\mathrm{Hom}(J^kE|_U, F_U))$. It follows that $D = T \circ j^k \in \mathrm{DOP}_k(E, F)$. □

Let 0 be the (image of the) zero section in $T^*X \to X$. We write $\pi_0 : T^*X \backslash 0 \to X$ for the restricted foot point projection.

Definition. Let $k \in \mathbb{Z}$, and $E \to X$, $F \to X$ vector bundles. We define the set of *symbols* to be

$$\begin{aligned}\mathrm{Smbl}_k(E, F) := \{&\sigma \in \Gamma(\mathrm{Hom}(\pi_0^*E, \pi_0^*F) \to T^*X \setminus 0) : \\ &\sigma(x, \lambda\eta) = \lambda^k \sigma(x, \eta) \text{ for all } (x, \eta) \in T^*X \setminus 0 \text{ and } \lambda > 0\},\end{aligned}$$

and this is a linear subspace of $\Gamma(\mathrm{Hom}(\pi_0^*E, \pi_0^*F))$. The subspace of *symbols of differential operators* or *polynomial symbols* is

$$\mathrm{Smbl}_k^{\mathrm{pol}}(E, F) := \Gamma\big(\mathrm{Poly}^k(T^*X, \mathrm{Hom}(E, F)) \to X\big).$$

(The choice of name for the latter space will be justified below.) The *symbol* $\sigma_k(D)$ of a differential operator $D \in \mathrm{DOP}_k(E, F)$ is defined by, for $(x, \eta) \in T^*X \setminus 0$ and $v \in E_x$,

$$\sigma_k(D)(x, \eta)v = D(\tfrac{1}{k!}(g - g(x))^k s)(x)$$

where $g \in C^\infty(X, \mathbb{R})$, $s \in \Gamma(E)$ are chosen such that $dg(x) = \eta$ and $s(x) = v$. It is immediate from the above that the thus defined symbol homomorphism $\sigma_k(D)(x, \eta) : E_x \to F_x$ is well-defined.

Proposition 3.16 (Symbol exact sequence). *The following is an exact sequence of vector spaces and linear maps.*

$$0 \longrightarrow \mathrm{DOP}_{k-1}(E, F) \xrightarrow{\subseteq} \mathrm{DOP}_k(E, F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k^{\mathrm{pol}}(E, F) \longrightarrow 0.$$

Proof. Let $\iota : \mathrm{Hom}^k_{\mathrm{sym}}(TX; E) \cong S^k(T^*X) \otimes E \hookrightarrow J^kE$ be given by the point-wise formula, for $x \in X$, $\eta \in T^*_xX$, $v \in E_x$,

$$\iota(s^k(\eta) \otimes v) = j^k_x(\tfrac{1}{k!}(g - g(x))^k s)$$

where $g \in C^\infty(X, \mathbb{R})$, $s \in \Gamma(E)$ are chosen such that $dg(x) = \eta$ and $s(x) = v$. Consider the exact sequence from lemma 3.13

$$0 \longrightarrow S^k(T^*X) \otimes E \xrightarrow{\iota} J^kE \longrightarrow J^{k-1}E \longrightarrow 0.$$

As $\mathrm{Hom}(\_, F)$ is a contravariant functor which is exact, i.e. preserves exact sequences, we obtain the exact sequence of vector bundles

$$0 \longrightarrow \mathrm{Hom}(J^{k-1}E, F) \longrightarrow \mathrm{Hom}(J^kE, F) \xrightarrow{\iota^*} \mathrm{Hom}(S^k(T^*X) \otimes E, F) \longrightarrow 0.$$

By associativity of the tensor product it follows that $\mathrm{Hom}(S^k(T^*X) \otimes E, F) = S^k(TX) \otimes E^* \otimes F = S^k(TX) \otimes \mathrm{Hom}(E, F) = \mathrm{Poly}_k(T^*X, \mathrm{Hom}(E, F))$; indeed, the identification can be described as $\lambda\colon \mathrm{Hom}(S^k(T^*X) \otimes E, F) \ni (T \circ \iota) \mapsto \lambda(T \circ \iota)$ which is given by $\lambda(T \circ \iota)(x, \eta) \cdot v = (T \circ \iota)(S^k(\eta) \otimes v)$.
Thus applying the covariant exact functor $\Gamma(\_)$ yields exactness of

$$\begin{aligned} 0 \longrightarrow \mathrm{DOP}_{k-1}(E, F) \longrightarrow \mathrm{DOP}_k(E, F) &\cong \Gamma(\mathrm{Hom}(J^kE, F)) \\ &\xrightarrow{\lambda_* \circ (\iota^*)_*} \Gamma\big(\mathrm{Poly}^k(T^*X, \mathrm{Hom}(E, F))\big) = \mathrm{Smbl}^{\mathrm{pol}}_k(E, F) \longrightarrow 0 \end{aligned}$$

which is a sequence of vector spaces and linear maps. So it only remains to check that $(\lambda \circ \iota^*)_*(T) = \sigma_k(T \circ j^k) = \sigma_k(D)$; where $T \in \Gamma(\mathrm{Hom}(J^kE, F))$ and $T \circ j^k = D \in \mathrm{DOP}_k(E, F)$. Let $x \in X$, $\eta \in T^*_xX$, and $v \in E_x$. Then

$$\begin{aligned} (\lambda \circ \iota^*)_*(T)(x, \eta) \cdot v &= (T \circ \iota)(S^k(\eta) \otimes v) \\ &= T(j^k_x(\tfrac{1}{k!}(g - g(x))^k s)) \\ &= D(\tfrac{1}{k!}(g - g(x))^k s)(x) \\ &= \sigma_k(D)(x, \eta) \cdot v \end{aligned}$$

where $g \in C^\infty(X, \mathbb{R})$, $s \in \Gamma(E)$ are chosen such that $dg(x) = \eta$ and $s(x) = v$. $\square$

Thus every symbol $\sigma \in \mathrm{Smbl}^{\mathrm{pol}}_k(E, F)$ corresponds to some differential operator $D \in \mathrm{DOP}_k(E, F)$, and D is unique up to addition of an element in $\mathrm{DOP}_{k-1}(E, F)$.
For completeness sake we remark that we can use the local form of $Ds(x)$, namely $\sum_{|\alpha| \le k} A_\alpha(0) D^\alpha \underline{s}(0)$ to obtain the local formula for $\sigma_k(D)(x, \eta)v$, namely

$$\tfrac{1}{k!} \sum_{|\alpha| \le k} A_\alpha(0) D^\alpha \underline{g}^k(0) \cdot v = \sum_{|\alpha| = k} \underline{\eta}^\alpha A_\alpha(0) \cdot \underline{v}$$

where underlined letters denote local representatives – in a chart centered at x and we assumed $\underline{g}(0) = 0$; indeed, $D^\alpha \underline{g}^k(0) = 0$ for $|\alpha| < 0$ and $D^\alpha \underline{g}^k(0) = k!\underline{\eta}^\alpha$ for $|\alpha| = k$.

Definition. A symbol of a differential operator $\sigma \in \mathrm{Smbl}_k^{\mathrm{pol}}(E, F)$ is called *elliptic* if $\sigma(x, \eta)\colon E_x \to F_x$ is an isomorphism for all $\eta \in T_x^* X \setminus 0$ and $x \in X$. A differential operator $D \in \mathrm{DOP}_k(E, F)$ is said to be *elliptic* if its symbol $\sigma_k(D)$ is elliptic.

By the symbol exact sequence we see that a k-th order differential operator can never be $k+1$-st order elliptic operator. On the other hand, if D is an elliptic k-th order differential operator then so is $D + D_{k-1}$ for all $D_{k-1} \in \mathrm{DOP}_{k-1}(E, F)$.
If $D\colon \Gamma(E) \to \Gamma(F)$ is a k-th rder differential operator we can consider its extension $D_{s,p}$ to the Banach spaces of Sobolev sections, i.e. $D_{s,p}\colon W^{s,p}(E) \to W^{s-k,p}(F)$.

Theorem 3.17. *Let $D \in \mathrm{DOP}_k(E, F)$ be a differential operator which is elliptic. Then the following hold.*

(i) $\ker D_{s,p} = \ker D$.

(ii) $\dim\ker D_{s,p} < \infty$ *and* $\mathrm{dimcoker}\, D_{s,p} = \dim \frac{W^{s-k,p}(F)}{\mathrm{im}\, D_{s,p}} < \infty$.

Proof. The first point is a consequence of elliptic regularity. A proof may be found in Wells [26]. □

Definition. Let V, W be Banach spaces. A linear map $L\colon V \to W$ is called *Fredholm* if $\dim\ker L < \infty$ and $\dim\mathrm{coker}\, L < \infty$.

Thus the above theorem may be paraphrased in saying that the extension of an elliptic differential operator to the Sobolev completions is Fredholm.

Theorem 3.18 (Sard-Smale). *Let X, Y, Z be smooth Banach manifolds, Z finite dimensional, and $f\colon X \to Y$ a C^k-Fredholm mapping.*

(i) *If $k > \max\{\mathrm{index}\, f, 0\}$ then, except for a subset of first category, all points of Y are regular values of f.*

(ii) *Let $g : Z \to Y$ be a C^l-embedding, and $l \geq k > \max\{\mathrm{index}\, f + \dim Z, 0\}$ then there exists a C^l-embedding $g_1 : Z \to Y$, which is arbitrarily WO^l-close to g, such that f and g_1 are transversal. Moreover, if there is a subset $A \subseteq Z$ such that f is transversal to $g|_A$ then g_1 may be chosen such that $g|_A = g_1|_A$.*

Proof. See Smale [25]. □

For Banach manifolds X, Y a map $f \in C^k(X,Y)$ is called *Fredholm* if its differential $T_x f\colon T_x X \to T_{f(x)} Y$ is a Fredholm operator for all points $x \in X$, and $\operatorname{index} f := \operatorname{index} T_x f$. Since X is assumed to be connected this does not depend on the particular choice of $x \in X$. For any map $f \in C^k(X,Y)$ a point $x \in X$ is called a *regular point* of f if $T_x f\, T_x X \to T_{f(x)} Y$ is surjective. Points that are not regular are called *critical points*, and elements in their image are *critical values*. The complement of the set of critical values is the set of *regular values*. In particular, if $x \notin \operatorname{im} f$ then x is regular.

3.E. Determinant bundle

References for this subsection have been Donaldson, Kronheimer [4], and Floer, Hofer [5]. Quillen [22] takes a slightly different approach towards this subject: assuming that the relevant vector spaces are of Hilbert type he constructs a holomorphic line bundle $\coprod_T \det T \to \operatorname{Fred}(E,F)$. However, in the present context this assumption will not always be readily available – we will be mostly concerned with spaces of the type $W^{1,p}(\Sigma, M)$ where $p > 2 = \dim \Sigma$. This latter prescription is the basic ingredient which makes such tools as the Ω-lemma or elliptic regularity accessible.

Lemma 3.19. *Let*

$$0 \longrightarrow E_1 \xrightarrow{l_1} E_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} E_n \longrightarrow 0$$

be an exact sequence of finite dimensional vector spaces and linear maps. Then there exists a canonical isomorphism

$$\Phi : \bigotimes_i \Lambda^{\max} E_{2i} \longrightarrow \bigotimes_i \Lambda^{\max} E_{2i-1}.$$

Proof. For $n = 4$. The general case only requires more typing. Consider

$$\begin{aligned} 0 \longrightarrow (E_1, (a_1^i)) \xrightarrow{l_1} (E_2, (l_1 \cdot a_1^i, a_2^j)) &\xrightarrow{l_2} (E_3, (l_2 \cdot a_2^j, a_3^k)) \\ &\xrightarrow{l_3} (E_4, (l_3 \cdot a_3^k)) \longrightarrow 0, \end{aligned}$$

where, e.g., $(a_1^i)_{i=1}^{\dim E_1}$ is a basis of E_1, and $(a_2^j)_{j=1}^{\dim E_2 - \dim E_1}$ are chosen such that $(l_1 \cdot a_1^i, a_2^j)_{i,j}$ is a basis of E_2. This works because the sequence is exact, and the vector spaces in question are finite dimensional. Now define

$$\Phi : \Lambda^{\max} E_2 \otimes \Lambda^{\max} E_4 \longrightarrow \Lambda^{\max} E_1 \otimes \Lambda^{\max} E_3$$

which is given by

$$l_1 a_1^1 \wedge \cdots \wedge l_1 a_1^{\max} \wedge a_2^1 \wedge \cdots \wedge a_2^{\max} \otimes l_3 a_3^1 \wedge \cdots \wedge l_3 a_3^{\max}$$
$$\longmapsto a_1^1 \wedge \cdots \wedge a_1^{\max} \otimes l_2 a_2^1 \wedge \cdots \wedge l_2 a_2^{\max} \wedge a_3^1 \wedge \cdots \wedge a_3^{\max},$$

where, e.g., $a_1^{\max} = a_1^{\dim E_1}$ but $a_3^{\max} \neq a_3^{\dim E_3 - \dim E_2}$, in general. The so defined Φ is independent of the choices, and constitutes the asserted isomorphism. $\square$

Let E, F be Banach spaces, X a manifold, and $f : X \to \mathrm{Fred}(E, F)$ smooth. Since $\mathrm{Fred}(E, F) \subseteq L(E, F)$ is open with respect to the norm topology it can be considered as a submanifold. Assume that the map $X \ni x \mapsto \dim \ker f(x) \in \mathbb{N}$ is locally constant then the same is true for $x \mapsto \dim \ker f(x) - \mathrm{index} f(x) = \dim \mathrm{coker}\, f(x)$. Hence $\coprod_{x \in X} \ker f(x) \to X$ is a sub-bundle of the trivial bundle $X \times E \to X$, and $\coprod_{x\in X} \mathrm{coker}\, f(x) \to X$ is a quotient-bundle of $X \times F \to X$. Thus the set theoretical line bundle $\coprod_{x\in X} \Lambda^{\max} \ker f(x) \otimes (\Lambda^{\max} \mathrm{coker}\, f(x))^* \to X$ is smooth.

Definition. E, F Banach spaces, X a manifold, and $f : X \to \mathrm{Fred}(E, F)$ smooth. The *determinant* of a Fredholm operator $T \in \mathrm{Fred}(E, F)$ is the line

$$\det T := \Lambda^{\max} \ker T \otimes (\Lambda^{\max} \mathrm{coker}\, T)^*.$$

The *determinant bundle* associated to the parametrized family of Fredholm operators $(f(x))_x$ is the real line bundle

$$\det f := \coprod_{x\in X} \det f(x) = \coprod_{x \in X} \Lambda^{\max} \ker f(x) \otimes (\Lambda^{\max} \mathrm{coker}\, f(x))^* \longrightarrow X.$$

For general $f : X \to \mathrm{Fred}(E, F)$ this definition is justified by the following

Proposition 3.20. *Let E, F be Banach spaces, X a manifold, and $f : X \to \mathrm{Fred}(E, F)$ smooth. Then $\det f \to X$ is a smooth real line bundle.*

Proof. We want to refine the argument from above. For any $x \in X$ there is a linear map $\alpha : \mathbb{R}^n \to F$ such that $\alpha \oplus f(x) : \mathbb{R}^n \times E \to F$, $(v, a) \mapsto \alpha \cdot v + f(x) \cdot a$ is surjective. But since $f : X \to \mathrm{Fred}(E, F)$ is continuous we can assume that $\alpha \oplus f(y)$ is even surjective for all y in an open neighborhood U_x of x. For the moment, call such a pair (U_x, α) *admissible*. For an admissible pair (U_x, α) consider the continuous map $f_\alpha := (0, \alpha \oplus f) : U_x \to \mathrm{Fred}(\mathbb{R}^n \times E, \mathbb{R}^n \times F)$, i.e. $f_\alpha(y) = (0, \alpha \oplus f(y)) : \mathbb{R}^n \times E \to \mathbb{R}^n \times F$, $(v, a) \mapsto (0, \alpha \cdot v + f(y) \cdot a)$. This map has the property that $y \mapsto \dim \mathrm{coker}\, f_\alpha(y) = \dim \frac{\mathbb{R}^n \times F}{\mathrm{im}\, f_\alpha(y)} = \dim \frac{\mathbb{R}^n \times F}{\{0\} \times F} = \dim \mathbb{R}^n$, and consequently also $y \mapsto \dim \ker f_\alpha(y) = \dim \mathrm{index} f_\alpha(y) + n$ are locally constant. Thus $\det f_\alpha \to U_x$ is a line bundle. For $y \in U_x$ consider now the exact sequence defined as follows:

$$0 \longrightarrow \ker f(y) \xrightarrow{l_1} \ker f_\alpha(y) \xrightarrow{l_2} \mathbb{R}^n \xrightarrow{l_3} \operatorname{coker} f(y) \longrightarrow 0$$

$$a \overset{l_1}{\longmapsto} (0,a),\ (v,a) \overset{l_2}{\longmapsto} v,\ v \overset{l_3}{\longmapsto} [\alpha v],$$

where $[\alpha v] = \alpha v + \operatorname{im} f(y) \in \frac{F}{\operatorname{im} f(y)}$. By the lemma above there is a canonical isomorphism

$$\Lambda^{\max} \ker f(y) \otimes \Lambda^{\max} \mathbb{R}^n \xrightarrow{\simeq} \Lambda^{\max} \ker f_\alpha(y) \otimes \Lambda^{\max} \operatorname{coker} f(y),$$

and multiplying both sides by $(\Lambda^{\max} \mathbb{R}^n)^* \otimes (\Lambda^{\max} \operatorname{coker} f(y))^*$ leads to the natural isomorphism

$$\begin{aligned}
\det f(y) &= \Lambda^{\max} \ker f(y) \otimes (\Lambda^{\max} \operatorname{coker} f(y))^* \\
&\cong \Lambda^{\max} \ker f_\alpha(y) \otimes \Lambda^{\max} \operatorname{coker} f(y) \otimes (\Lambda^{\max} \mathbb{R}^n)^* \otimes (\Lambda^{\max} \operatorname{coker} f(y))^* \\
&\cong \Lambda^{\max} \ker f_\alpha(y) \otimes (\Lambda^{\max} \mathbb{R}^n)^* \\
&= \det f_\alpha(y).
\end{aligned}$$

Thus we have a bijection

$$\begin{tikzcd}
\det f \mid_{U_x} \arrow[r, "\simeq"] \arrow[d] & \det f_\alpha \arrow[d] \\
U_x \arrow[r, equal] & U_x
\end{tikzcd}$$

which fiber wise is an isomorphism. *If (U_x, α), (U_z, β) are admissible data such that $U_x \cap U_z = V \neq \emptyset$ then the transition map $\det f_\alpha \mid_V \to \det f_\beta \mid_V$ is a vector bundle isomorphism.* So $\alpha : \mathbb{R}^n \to F$, and $\beta : \mathbb{R}^m \to F$, and assume without loss that $\mathbb{R}^n \subseteq \mathbb{R}^m$. Consider the special case that $\beta(v,w) = \alpha(v) + \alpha_1(w) = (\alpha \oplus \alpha_1)(v,w)$, where $(v,w) \in \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$, and $\alpha_1 : \mathbb{R}^{m-n} \to F$ is linear. As above, for $y \in V$, there is an exact sequence

$$0 \longrightarrow \ker f_\alpha(y) \xrightarrow{l_1} \ker f_{\alpha\oplus\alpha_1}(y) \xrightarrow{l_2} \mathbb{R}^n \times \mathbb{R}^{m-n} \xrightarrow{l_3} \operatorname{coker} f_\alpha(y) \longrightarrow 0,$$

given by

$$(v,a) \overset{l_1}{\longmapsto} (v,0,a),\ (v,w,a) \overset{l_2}{\longmapsto} (0,v),\ (v,w) \overset{l_3}{\longmapsto} [\alpha v]$$

with its corresponding isomorphism

$$\det f_\alpha(y) = \Lambda^{\max} \ker f_\alpha(y) \otimes \Lambda^{\max} \mathbb{R}^m \otimes (\Lambda^{\max} \mathbb{R}^m)^* \otimes (\Lambda^{\max} \operatorname{coker} f_\alpha(y))^*$$

$$\begin{aligned}
&\cong \Lambda^{\max} \ker f_{\alpha\oplus\alpha_1}(y) \otimes \Lambda^{\max} \operatorname{coker} f_\alpha(y) \\
&\quad \otimes (\Lambda^{\max} \mathbb{R}^m)^* \otimes (\Lambda^{\max} \operatorname{coker} f_\alpha(y))^* \\
&= \det f_{\alpha\oplus\alpha_1}(y),
\end{aligned}$$

since $\mathbb{R}^m = \operatorname{coker} f_{\alpha\oplus\alpha_1}(y)$. Since $\det f_\alpha \to U_x$, and $\det f_\beta \to U_z$ already are vector bundles we thus have a vector bundle isomorphism $\det f_\alpha\ |_V \to \det f_{\alpha\oplus\alpha_1}\ |_V = \det f_\beta\ |_V$.
For general $\beta : \mathbb{R}^m \to F$ we define $\beta_1 := \alpha \oplus \beta : \mathbb{R}^n \times \mathbb{R}^m \to F$, and note that the following is a commutative diagram of vector bundle isomorphisms over V:

$$\begin{array}{ccc}
\det f_\alpha\ |_V & \xrightarrow{\simeq} & \det f_\beta\ |_V \\
\simeq\big\downarrow & & \big\downarrow\simeq \\
\det f_{\alpha\oplus\beta}\ |_V & \xrightarrow{\simeq} & \det f_{\alpha\oplus\beta}\ |_V
\end{array}$$

Indeed, the vertical lines have just been argued to be isomorphisms, and the bottom line is an isomorphism because $f_{\alpha\oplus\beta}(v,w,a) = f_{\beta\oplus\alpha}(w,v,a)$. $\square$

Chapter 4

Transversality

Throughout this chapter (Σ, j) will denote a closed Riemann surface, (M, ω) a compact symplectic manifold of dimension $\dim M = 2n$, and $A \in H_2(M)$ shall be a fixed homology class. The space of all ω-tame almost complex structures on M is $\mathcal{J}(\omega) = \{J \in \Omega^1(M; TM) : J^2 = -\mathrm{id}_{TM},\ \text{and } \omega(v, Jv) > 0 \text{ for all } v \neq 0\}$; this space is contractible, nonempty, and carries the structure of a smooth Fréchet manifold.

4.A. Cauchy-Riemann operator

Recall from section 1.B. that the injectivity condition is open in $C^\infty(\Sigma, M)$. Hence the space

$$\mathcal{N}(A) := \{u \in C^\infty(\Sigma, M) : u_*[\Sigma] = A,\ \text{u is somewhere injective}\}$$

is a smooth manifold modelled on spaces $\Gamma(u^*TM \to \Sigma)$;[1] see also 3.3. As long as A is kept fixed we shall simply write $\mathcal{N}(A) = \mathcal{N}$.

1. Because the condition of being ω-tame is open it follows that $T_J\mathcal{J}(\omega) = \Omega^{0,1}_J(M; TM) = \Gamma(\mathrm{End}^{0,1}_J(TM) \to M)$. The map $(\_)^{0,1} : \mathrm{End}(TM) \to \mathrm{End}^{0,1}_J(TM)$, $l \mapsto \frac{1}{2}(l + JlJ)$ denotes projection onto the sub-vector bundle of conjugate linear endomorphisms – with respect to the (almost) complex structure J. □

2 (Cauchy-Riemann operator). Consider the infinite dimensional vector bundle $\mathcal{E} \to \mathcal{N} \times \mathcal{J}(\omega)$, with total space

$$\mathcal{E} := \coprod_{(u,J) \in \mathcal{N} \times \mathcal{J}(\omega)} \mathcal{E}_{(u,J)}, \qquad \text{and fibers}$$

[1] Since Σ is closed orientable it is a fundamental cycle itself, and has a fundamental class $[\Sigma]$. This class is taken to $u_*[\Sigma] \in H_2(M)$ via the homomorphism $H_2(u) = u_*$. The equation $u_*[\Sigma] = A$ describes a union of connected components in $C^\infty(\Sigma, M)$.

$$\mathcal{E}_{(u,J)} := \Omega^{0,1}_J(\Sigma; u^*TM) = \Gamma(\Lambda^{0,1}T^*\Sigma \otimes_J u^*TM).$$

This bundle has a section $F \in \Gamma(\mathcal{E} \to \mathcal{N} \times \mathcal{J}(\omega))$ which is given by $(u, J) \mapsto F(u, J) = (u, J, \overline{\partial}_J u)$, where

$$\overline{\partial}_J u = \tfrac{1}{2}(du + J \circ du \circ j) = du^{0,1}.$$

The aim of this chapter is to describe the zeroes of this equation. *At a zero* $(u, J) \in \mathcal{N} \times \mathcal{J}(\omega)$ *the vertical derivative of the section* F *computes to*

$$\begin{aligned} T^v_{(u,J)}F \colon T_u\mathcal{N} \times T_J\mathcal{J}(\omega) &\longrightarrow \mathcal{E}_{u,J}, \\ (\xi, Y) &\longmapsto (\nabla\xi)^{0,1} + \tfrac{1}{8}N_J(\xi, \partial_J u) + \tfrac{1}{2}Y du \circ j. \end{aligned}$$

Here N_J is the Nijenhuis tensor of the almost complex structure J. It is given by $N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$ for vector fields X,Y on M. By theorem 3.4. in [10, chapter IX] every almost complex manifold admits a Hermitian connection such that the torsion Tor is proportional to N_J, or $\mathrm{Tor} = \frac{1}{8}N_J$, to be precise. So let TM be equipped with such a Hermitian connection and let ∇ denote its covariant derivative; then J is parallel with respect to this connection, i.e. commutes with the induced parallel transport.

We have already fixed a $(u, J) \in \mathcal{N} \times \mathcal{J}(\omega)$ such that $\bar{\partial}_J u = 0$. Let $\xi \in T_u\mathcal{N} = \Gamma(u^*TM)$ and $(-\varepsilon, \varepsilon) \ni t \mapsto \exp_u t\xi =: u_t$; $u_0 = u$ and $\frac{\partial}{\partial t}|_{t=0} u_t = \xi$. For a point $z \in \Sigma$ denote parallel transport along the reverse of the curve $t \mapsto \exp_{u(z)} t\xi(z) = u_t(z)$ by

$$\mathrm{Pt}^{\exp_{u(z)}(t+\_)\xi(z)}(-t) =: \mathrm{Pt}^{\hat{u}(z)}(t) : T_{u_t(z)}M \to T_{u(z)}M$$

We will also need $\alpha, \beta(v) \in \mathfrak{X}_{\mathrm{loc}}(M)$, that are defined on an open neighborhood of $\bigcup_t \operatorname{im} u_t$ and on this union fulfill

$$\begin{aligned} \alpha(u_t(z)) &= \tfrac{\partial}{\partial s}|_{s=t} u_s(z), \\ \beta(v)(u_t(z)) &= T_z u_t \cdot v(z) \end{aligned}$$

for all $v \in \mathfrak{X}(\Sigma)$ and $z \in \Sigma$. On this union we also have that

$$\begin{aligned} [\alpha, \beta(v)] &= \frac{\partial}{\partial t}\,|_0\, (T\mathrm{Fl}^\alpha_{-t} \circ \beta(v) \circ \mathrm{Fl}^\alpha_t)(u_{t_0}(z)) \\ &= \frac{\partial}{\partial t}\,|_0\, (T_{u_{t_0}(z)}\mathrm{Fl}^\alpha_t)^{-1} \cdot T_{u_{t_0}(z)}\mathrm{Fl}^\alpha_t \cdot T_z u_{t_0} \cdot v(z) \\ &= 0 = [\alpha, \beta(jv)], \end{aligned} \tag{$*$}$$

by the chain rule. $\mathcal{N} \times \mathcal{J}(\omega)$ is a product, and thus each partial may be computed separately. For the first partial we obtain the following.

$$T^v_{(u,J)}F(\xi, 0) \cdot v_z = \frac{\partial}{\partial t}_{|t=0} (\mathrm{Pt}^{\exp(t+\_)\xi}(-t) \circ \overline{\partial}_J \circ \exp t\xi)(u) \cdot v_z$$

$$\begin{aligned}
&= \tfrac{1}{2}\tfrac{\partial}{\partial t}_{|t=0} \mathrm{Pt}^{\hat{u}(z)}(t)(du_t(z)\cdot v_z + J(u_t(z))\cdot du_t(z)\cdot jv_z)\\
&= \tfrac{1}{2}\tfrac{\partial}{\partial t}_{|t=0} \mathrm{Pt}^{\hat{u}(z)}(t)(\beta(v)(u_t(z)))\\
&\quad + J(u(z))\tfrac{1}{2}\tfrac{\partial}{\partial t}_{|t=0} \mathrm{Pt}^{\hat{u}(z)}(t)(\beta(jv)(u_t(z)))\\
&= \tfrac{1}{2}(\nabla_\alpha\beta(v) + J\nabla_\alpha\beta(jv))(u(z))\\
&\overset{(*)}{=} \tfrac{1}{2}(\nabla_{T_z u\cdot v_z}\alpha + J_{u(z)}\nabla_{T_z u\cdot jv_z}\alpha)\\
&\quad + \tfrac{1}{2}(\mathrm{Tor}(\alpha(u(z)),\beta(v)(u(z)))\\
&\quad + J_{u(z)}\mathrm{Tor}(\alpha(u(z)),\beta(jv)(u(z))))\\
&= (\nabla(\alpha\circ u))^{0,1}(v_z)\\
&\quad + \tfrac{1}{8}N_J(\xi_z, \tfrac{1}{2}(T_z u\cdot v_z - J_{u(z)}T_z u\cdot v_z)),
\end{aligned}$$

where we continued to denote covariant differentiation on u^*TM by ∇. The second partial comes out as $T^v_{(u,J)}F(0,Y) = \frac{1}{2}Y(u)\circ du\circ j$ which, however, is obvious. □

3. Fix $J \in \mathcal{J}(\omega)$ and consider $D_u := T^v_{(u,J)}F(\_,0)$ which we shall henceforth regard as the *lineariztion* of $\overline{\partial}_J$ at $u \in \mathcal{N}$. *For every J-holomorphic curve u the linearization $D_u : T_u\mathcal{N} \to \mathcal{E}_{(u,J)}$ is a first order elliptic differential operator.* First note that $D_u(f\xi) = (df\otimes\xi)^{0,1} + fD_u(\xi)$ for all $f \in C^\infty(\Sigma,\mathbb{R})$; a differential operator with this property is called a *$\overline{\partial}$-operator.* Clearly, D_u is a first order differential operator. We have to show that the symbol homomorphism

$$\sigma(D_u)(z,v) : u^*TM_z \to \Lambda^{0,1}T^*_z\Sigma \otimes_J u^*TM_z$$

is an isomorphism for every $z \in \Sigma$ and every nonzero $v \in T^*_z\Sigma\backslash\{0\}$. So choose a function $fC^\infty(\Sigma,\mathbb{R})$ with $f(z) = 0$ and $df(z) = v$. Let $\xi \in \Gamma(u^*TM \to \Sigma)$. Then

$$\begin{aligned}
\sigma(D_u)(z,v)\cdot\xi_z &= D_u(f\xi)(z)\\
&= (df\otimes\xi)^{0,1}(z) + 0 \in \mathrm{Hom}^{0,1}(T_z\Sigma, u^*TM_z)
\end{aligned}$$

vanishes if and only if $0 = \xi_z \in u^*TM_z$. Thus, by reason of dimension, the symbol homomorphism is an isomorphism, and by section 3.D. D_u is an elliptic operator as its symbol is elliptic. In particular all $\overline{\partial}$-operators have the same symbol, and hence are elliptic. For integrable J ellipticity of the Cauchy-Riemann equations is of course well known. Now this paragraph just exploits the obvious fact that the Nijenhuis-tensor N_J –which measures the non-integrability of J– is of lower (zeroth) order and so non-integrability is not an obstruction for the symbol $\sigma(D_u)$ to be elliptic.

If we choose to extend D_u to the appropriate Sobolev-completions then we may say that $D_u : T_u\mathcal{N}^{k,p} \to \mathcal{E}^{k-1,p}_{(u,J)}$ is a Fredholm operator; the just used notation will be introduced in the next section. $\square$

4. If we want to express the vertical tangent mapping to $F_J = F(\_, J)\colon \mathcal{N} \to \mathcal{E}_J$ in terms of a covariant derivative at an arbitrary point $u \in \mathcal{N}$ the result will depend on a choice of a connection on M. To make matters worse, for $F_J(u) = \overline{\partial}_J u \neq 0$, there is not even a canonical splitting of $T_{F_J(u)}\mathcal{E}_J$ into horizontal and vertical part. What we need to show for further development is the following. *The operator D_u is a $\overline{\partial}$-operator; in particular, it is elliptic.* As above, choose a Hermitian connection on (M, J) such that $\mathrm{Tor} = \frac{1}{8}N_J$. Locally we then have

$$T^v_u F_J(\xi)(v) = (\nabla\xi)^{0,1}(v) + \tfrac{1}{8}N_J(\xi, \partial_J u \cdot v) - \Gamma_u(\overline{\partial}_J u \cdot v, \xi)$$

where $v \in T\Sigma$, $\xi \in T_u\mathcal{N}$, and Γ denotes the Christoffel symbols.[2] The equation $D_u(f\xi) = (df \otimes \xi)^{0,1} + fD_u(\xi)$ for $f \in C^\infty(\Sigma, \mathbb{R})$ now yields the claim. $\square$

5. Let $(u, J) \in \mathcal{N} \times \mathcal{J}(\omega))$ such that $\overline{\partial}_J u = 0$. We want to find a specific local formula for $T^v_{(u,J)}F$ in terms of holomorphic coordinates on Σ. So we work with the local model $(D, i) \hookrightarrow (\Sigma, j)$. As noted just above the formula for $T^v_{(u,J)}F$ at a zero (u, J) will not depend upon a choice of connection on $TM \to M$. Let ∇ denote covariant differentiation with respect to some chosen connection. Let $(\xi, Y) \in T_{(u,J)}(\mathcal{N} \times \mathcal{J}(\omega))$, $v \in TD = \mathbb{C} \times \mathbb{C}$, and introduce $(-\varepsilon, \varepsilon) \ni t \mapsto u_t \in \mathcal{N}$, $\alpha, \beta(v) \in \mathfrak{X}_{\mathrm{loc}}(M)$, $\mathrm{Pt}^{\hat{u}(z)}$ defined just like in (2). Then $[\alpha, \beta(v)] = [\alpha, \beta(jv] = 0$, and

$$\begin{aligned}
D_u\xi \cdot v_z &= \tfrac{\partial}{\partial t}|_0\big(\mathrm{Pt}^{\exp(t+\,)\xi}(-t) \circ \overline{\partial}_J \circ \exp t\xi\big)(u)v_z \\
&= \tfrac{1}{2}\tfrac{\partial}{\partial t}|_0\mathrm{Pt}^{\hat{u}(z)}(t)du_t(z)v_z + \tfrac{1}{2}\tfrac{\partial}{\partial t}|_0\mathrm{Pt}^{\hat{u}(z)}(t)(J(u_t(z))du_t(z)iv_z) \\
&= \tfrac{1}{2}(\nabla_\alpha\beta(v))(u(z)) + \tfrac{1}{2}(\nabla_\alpha J)(u(z))\beta(iv)(u(z)) \\
&\quad + \tfrac{1}{2}J(u(z))(\nabla_\alpha\beta(iv))(u(z)) \\
&= \tfrac{1}{2}\nabla_{\beta(v)u(z)}\alpha + 0 + \tfrac{1}{2}J(u(z))\nabla_{\beta(iv)(u(z))}\alpha + 0 + \tfrac{1}{2}(\nabla_\xi J)(z)du(z)iv_z \\
&= \big((\nabla\xi)^{0,1} + \tfrac{1}{2}(\nabla_\xi J)du \cdot i\big)v_z
\end{aligned}$$

with the last equality using $\nabla_{du\cdot v}\alpha = \nabla_v(\alpha \circ u) = \nabla_v\xi$ as in (2). Working locally we have

$$\begin{aligned}
(\nabla\xi)^{0,1} &= \overline{\partial}_J\xi \\
&= \tfrac{1}{2}(\partial_s\xi + \partial_t\xi)ds + \tfrac{1}{2}(\partial_t\xi - \partial_s\xi)dt
\end{aligned}$$

[2] In more classical literature this is $-\Gamma$.

and

$$\begin{aligned}(\nabla_\xi J)du\cdot i &= (\partial_\xi J)(J\circ u)\partial_s u ds + (\partial_\xi J)(J\circ u)\partial_t u dt\\ &= (\partial_\xi J)\partial_t u ds - (J\circ u)(\partial_\xi J)\partial_t u dt.\end{aligned}$$

Furthermore,

$$(Y\circ u)du\cdot i = (Y\circ u)\partial_t u ds - (J\circ u)(Y\circ u)\partial_t u dt.$$

Putting this together the desired formula reads

$$D_u\xi + \tfrac{1}{2}(Y\circ u)du\cdot i = \eta ds - (J\circ u)\eta dt$$

where

$$\eta := \tfrac{1}{2}\big(\partial_s\xi + (J\circ u)\partial_t\xi + (\partial_\xi J)\partial_t u + (Y\circ u)\partial_t u\big).$$

□

4.B. Implicit function theorem

Definition. Let $J\in\mathcal{J}(\omega)$ and $A\in H_2(M)$. Then we call $\mathcal{M}(A,J) := \{u\in\mathcal{N}(A) : \overline{\partial}_J u = 0\}$ the *moduli space* of A-representing curves of the Cauchy-Riemann operator on M at J.

A fact which lies at the core of this whole construction is that proposition 1.6 implies that J-holomorphic curves are actually smooth. Since $\mathcal{M}(A,J) = F(\_,J)^{-1}(0)$, where 0 denotes the image of the zero section in $\mathcal{E}^p$, a sufficient condition for $\mathcal{M}(A,J)$ to be a smooth sub-manifold of $\mathcal{N}^{1,p}(A)$ is that $F(\_,J)$ be transversal to the zero section. Assuming this we will show below that $\mathcal{M}(A,J)$ is furthermore finite dimensional and carries a natural orientation.

$\boldsymbol{\mathcal{N}^{k,p}(A)}$**.** Let $kp > \dim\Sigma = 2$. From section 3.C. recall the Sobolev $W^{k,p}$-completion $W^{k,p}(\Sigma,M)$ of the space of smooth mappings $C^\infty(\Sigma,M)$. Now we consider the space of all A-representing, somewhere injective maps $\mathcal{N}(A)$ which is an open submanifold of $C^\infty(\Sigma,M)$. The Sobolev $W^{k,p}$ completion of this space shall be denoted by $\mathcal{N}^{k,p}(A)$; this is a smooth manifold, and a chart construction is given below. If A is kept fixed we will write $\mathcal{N}^{k,p}(A) = \mathcal{N}^{k,p}$. If $u\in\mathcal{N}^{k,p}$ satisfies $\overline{\partial}_J u = 0$ and J is smooth then lemma 1.6 implies that $u\in C^\infty(\Sigma,M)$, therefore it will be sufficient to work with the space $\mathcal{N}^{1,p}$ (where $p>2$). □

$\boldsymbol{\mathcal{J}^l(\omega)}$**.** Let $l\geq 1$. The space of all almost complex, ω-tame structures on M is defined by

$$\begin{aligned}\mathcal{J}^l(\omega) := \{ & J \in \Gamma^l(T^*M \otimes TM \to M) : \\ & J_x^2 = -\mathrm{id}_{T_xM}, \omega_x(X, JX) > 0 \text{ for all } x \in M \text{ and } X \in T_xM \setminus \{0\}\}.\end{aligned}$$

In view of the above remark we should develop a criterium for the operator $D_u \colon T_u\mathcal{N}^{k,p} \to \mathcal{E}^{k-1,p}_{(u,J)}$ to be surjective. This is done by adjoining an additional parameter space – $\mathcal{J}^l(\omega)$. □

$\boldsymbol{\mathcal{E}^{k-1,p}}$**.** Let $kp > 2$ and $k \leq l$. One more object to introduce is the Banach space vector bundle

$$\begin{aligned}\mathcal{E}^{k-1,p} &:= \coprod_{(u,J)\in\mathcal{N}^{k,p}\times\mathcal{J}^l(\omega)} \mathcal{E}^{k-1,p}_{(u,J)}, \qquad \text{with fibers} \\ \mathcal{E}^{k-1,p}_{(u,J)} &:= W^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes_J u^*TM).\end{aligned}$$

It will be show below that this space carries a smooth structure. We shall write $\mathcal{E}^{0,p} = \mathcal{E}^p$. □

1. Let $f \in \mathcal{N}$ and (U, ∇) an admissible pair in the sense of 3.11. Then charts for $\mathcal{N}^{1,p}$ are given by

$$\begin{aligned}\psi_f : W^{1,p}(f^*U \to \Sigma) &\longrightarrow \mathcal{N}^{1,p}(A), \\ \xi &\longmapsto \exp_f^\nabla \xi\end{aligned}$$

where $\exp_f^\nabla \xi : z \mapsto \exp_{f(z)}^\nabla \xi(z)$. An admissible datum (U, ∇) was defined to consist of an open neighborhood of the zero section $U \subseteq TM$, and a connection ∇ on TM such that the corresponding exponential map $\exp^\nabla :$ $U \to M \times M$, $X \mapsto (\pi(X), \exp_{\pi(X)}^\nabla X)$ is a diffeomorphism onto an open neighborhood of the diagonal. □

2. *The vector bundle* $\mathcal{E}^p \longrightarrow \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)$ *carries a smooth structure.* Let $\Omega^1_p(\Sigma; u^*TM) := L^p(T^*\Sigma \otimes u^*TM \to \Sigma)$, and define the bundle

$$\mathcal{F}^p := \coprod_{u\in\mathcal{N}^{1,p}(A)} \Omega^1_p(\Sigma; u^*TM) \longrightarrow \mathcal{N}^{1,p}(A)$$

For $\mathcal{N}^{1,p}(A)$ and (U, ∇) an admissible pair as above, point (1), consider the mapping

$$\begin{aligned}\varphi_f : W^{1,p}(f^*U \to \Sigma) \times \Omega^1_p(\Sigma; f^*TM) &\longrightarrow \bigcup_{u\in\mathrm{im}\psi_f} \{u\} \times \Omega^1_p(\Sigma; u^*TM) \\ (\xi, s) &\longmapsto (\exp_f^\nabla \cdot\xi, \mathrm{Pt}_f^{\gamma(\xi)}(1) \cdot s).\end{aligned}$$

Explanation of notation: again, it is convenient to identify $W^{1,p}(f^*U \to \Sigma) = W^{1,p}(\Sigma, U)_f = \{h \in W^{1,p}(\Sigma, U) : \pi_M \circ h = f\}$. $\mathrm{Pt}_f^{\gamma(\xi)}(1)_z = \mathrm{Pt}_{f(z)}^{\gamma(\xi)_z}(1)\colon T_{f(z)}M \to T_{u(z)}M$ is parallel transport up to time one along the curve $\gamma(\xi)_z\colon t \mapsto \exp^{\nabla}_{f(z)}(t\xi(z))$, and $\psi_f : W^{1,p}(f^*U) \to \mathcal{N}^{1,p}(A)$, $\xi \mapsto \exp^{\nabla}_f(\xi) = u$ is the chart mapping from point (1). By the universal property of the pullback bundle $\varphi_f\colon W^{1,p}(f^*U) \times \Omega^1_p(\Sigma; f^*TM) \to \coprod_{u\in \mathrm{im}\psi_f} \Omega^1_p(\Sigma; u^*TM)$ is a bijection of set theoretical vector bundles:

$$\begin{array}{ccccc}
T\Sigma & \xrightarrow{s} & f^*TM & \xrightarrow{\pi_M^* f} & TM \\
\| & & & & \downarrow{\scriptstyle \mathrm{Pt}_f^{\gamma(\xi)}(1)\cdot s} \\
T\Sigma & \xrightarrow{\exists !} & u^*TM & \xrightarrow{\pi_M^* u} & TM \\
{\scriptstyle \pi_\Sigma}\downarrow & & {\scriptstyle u^*\pi_M}\downarrow & & \downarrow{\scriptstyle \pi_M} \\
\Sigma & = & \Sigma & \xrightarrow{u} & M
\end{array}$$

Now let (U_i, ∇_i) be admissible data, and $f \in C^\infty(\Sigma, M)$ for $i \in \{1,2\}$, and $\mathrm{im}\,\psi_{f_1} \cap \mathrm{im}\,\psi_{f_2} \neq \emptyset$. Then the gluing maps are given by

$$\begin{aligned}
\varphi_{f_2}^{-1} \circ \varphi_{f1}\colon W^{1,p}(f_1^*U_1 \to \Sigma) \times \Omega^1_p(\Sigma; f_1^*TM) &\longrightarrow \coprod_u \Omega^1_p(\Sigma; u^*TM) \\
&\longrightarrow W^{1,p}(f_2^*U_1 \to \Sigma) \times \Omega^1_p(\Sigma; f_2^*TM) \\
(\xi, s) &\longmapsto (\exp^{\nabla_1}_{f_1}(\xi), \mathrm{Pt}_{f_1}^{\gamma_1(\xi)}(1)\cdot s) \\
&\longmapsto (\underbrace{(\exp^{\nabla_2}_{f_2})^{-1} \circ \exp^{\nabla_1}_{f_1} \circ \xi}_{=:\eta}, (\mathrm{Pt}_{f_2}^{\gamma_2(\eta)}(1))^{-1} \cdot \mathrm{Pt}_{f_1}^{\gamma_1(\xi)}(1)\cdot s).
\end{aligned}$$

This map is smooth by definition of the smooth structure on $\mathcal{N}^{1,p}(A)$, and since parallel transport also depends smoothly on the curve, see Kriegl and Michor [11]. Moreover, $\varphi_{f_2}^{-1} \circ \varphi_{f_1}$ is fiberwise linear, and hence an isomorphism of smooth vector bundles. Via the projection $p^{0,1} : \mathcal{F}^p \times \mathcal{J}^l(\omega) \to \mathcal{E}^p$, $(l, J) \mapsto p^{0,1}(l, J) = p^{0,1}_J(l) = \frac{1}{2}(l + J \circ l \circ j)$, $\mathcal{E}^p$ becomes a smooth sub-vector bundle of the bundle $\mathcal{F}^p \times \mathcal{J}^l(\omega)$ – considering the total spaces pars pro toto. $\square$

3. *$F\colon \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega) \to \mathcal{E}^p$ is a smooth section.* Consider the differential operator $d\colon \mathcal{N}^{1,p}(A) \to \mathcal{F}^p$, $u \mapsto du$. In a trivialization with (U, ∇) admissible, $f \in C^\infty(\Sigma, M)$, $\exp^{\nabla} = \exp$, and continuing the notation from above this appears as

$$\varphi_f^{-1} \circ d \circ \psi_f : W^{1,p}(f^*U \to \Sigma) \longrightarrow \mathcal{N}^{1,p}(A)$$

$$
\begin{aligned}
&\longrightarrow \coprod_{u \in \operatorname{im} \psi_f} \Omega^1_p(\Sigma; u^*TM) \\
&\longrightarrow W^{1,p}(f^*U \to \Sigma) \times \Omega^1_p(\Sigma; f^*TM), \\
\xi &\longmapsto \exp_f \xi \\
&\longmapsto d(\exp_f \xi) \\
&\longmapsto (\xi, (\mathrm{Pt}_f^{\gamma(\xi)}(1))^{-1} \cdot d(\exp_f \xi)).
\end{aligned}
$$

Indeed, $d(\exp_f \xi) = d(\mathrm{pr}_2 \circ \exp \circ \xi) = \mathrm{pr}_2 \circ d(\exp \circ \xi) = (\mathrm{pr}_2 \circ d\exp)_*(d\xi)$. Now $\mathrm{pr}_2 \circ d(\exp)\colon T^2M \to TM \times TM \to TM$ is a smooth fiber respecting map, and so we can apply theorem 3.4 to conclude smoothness of $d\xi \mapsto (\mathrm{pr}_2 \circ d\exp)_*(d\xi)$; differentiation $W^{1,p}(f^*U) \to L^p(f^*U)$, $\xi \mapsto d\xi$ is smooth as well since this is a continuous linear map on Banach spaces. The composition by φ_f^{-1} is somewhat superfluous – it is only needed to obtain the exact definition of differentiability in manifolds. The assertion now follows since $\overline{\partial}_J u = (p^{0,1} \circ (d \times \mathrm{id}_{\mathcal{J}^l(\omega)}))(u, J)$. One could also draw a diagram – similar to the one omitted at the end of (2). $\square$

When D_u is a Fredholm operator then its kernel is finite dimensional with basis $\{l_1, \ldots, l_m\}$ and dual basis $\{l_1^*, \ldots, l_m^*\}$ given by $l_i^*(l_k) = \delta_{ik}$. By Hahn-Banach there exist $\{k_1^*, \ldots, k_m^*\} \subseteq T_u^*\mathcal{N}^{1,p}(A)$ such that $k_i^*|_{\ker D_u} = l_i^*$, and we obtain a continuous linear map $p : T_u\mathcal{N}^{1,p}(A) \to \ker D_u$, $\xi \mapsto \sum_{i=1}^m k_i^*(\xi) l_i$ satisfying $p|_{\ker D_u} = \mathrm{id}$. Thus there is a splitting $T_u\mathcal{N}^{1,p}(A) = \ker D_u \oplus \ker p =: \ker D_u \oplus V$.

If D_u is furthermore surjective then the following theorem constructs J-holomorphic curves in the vicinity of an approximate J-curve u.

Theorem 4.1. *Assume $p > 2$. Let Σ be a closed Riemann surface, (M, ω) a compact symplectic manifold, and continue the notation from above. Assume $u \in \mathcal{N}^{1,p}(A)$ and $J \in \mathcal{J}(\omega)$ are such that the Fredholm operator $D_u\colon T_u\mathcal{N}^{1,p}(A) \to \mathcal{E}^p_{(u,J)}$ is surjective.*

Then there is an open zero neighborhood $N \subseteq \mathcal{E}^p_{(u,J)}$ (dependent on $\|du\|_p$) such that the following holds. If $\overline{\partial}_J u \in N$ then there exist open neighborhoods of zero U, W in $\ker D_u$, V respectively, and a smooth function $f\colon U \to W$, such that $\overline{\partial}_J \exp_u(\xi + f(\xi)) = 0$ for all $\xi \in U$.

Proof. It suffices to assume that u is smooth, so that we can really speak of the tangent space to $\mathcal{N}^{1,p}(A)$ at u, cf. theorem 3.11. On (M, J) choose a Hermitian connection as in 4.A.(4). Since D_u is Fredholm we can decompose $T_u\mathcal{N}^{1,p}(A)$ and consider the smooth mapping

$$\mathcal{F}\colon T_u\mathcal{N}^{1,p}(A) = \ker D_u \oplus V \longrightarrow \mathcal{E}^p_u,$$

$$\xi = \xi_1 \oplus \xi_2 \longmapsto (\mathrm{Pt}^{\exp(1+\cdot)\xi}(-1) \circ \overline{\partial}_J \circ \exp \xi)(u),$$

which is globally well defined. Observe that $T_0\mathcal{F} = D_u$, and, by the open mapping theorem, $D_u|_V = \partial_2\mathcal{F}(0) \colon V \to \mathcal{E}^p_u$ is an isomorphism. As in the proof of the usual implicit function theorem, we want to apply a uniform contraction principle to a map constructed from the above one. Consider

$$G \colon \ker D_u \oplus V \longrightarrow V, \qquad (\xi, \eta) \longmapsto \eta - (\partial_2\mathcal{F}(0)^{-1} \circ \mathcal{F})(\xi + \eta).$$

Since $\partial_2 G(0) \cdot \eta = \eta - \mathrm{id}_V \cdot \eta = 0$, there exists an open, convex zero neighborhood $U \oplus W \subseteq \ker D_u \oplus V$ such that $\|\partial_2 G(\xi,\eta)\| \leq \lambda < 1$ for all $(\xi,\eta) \in U \times \overline{W}$. Assume N is an open zero neighborhood in $\mathcal{E}^p_{(u,J)}$ such that $N \subseteq D_u|_V(W)$, and let $\overline{\partial}_J u \in N$. By the mean value theorem we have $\|G(\xi,\eta_2) - G(\xi,\eta_1)\|_{1,p} \leq M\|\eta_2 - \eta_1\|_{1,p}$ with $M = \max_{0\leq t\leq 1}\|\partial_2 G(\xi,(1-t)\eta_2 + t\eta_1)\| \leq \lambda < 1$.
By assumption there is $\eta \in W$ such that $D_u|_V\eta = \overline{\partial}_J u$, and hence $G(0,\eta) = \eta - (\partial_2\mathcal{F}(0)^{-1} \circ \mathcal{F})(\eta) = 0$. Thus there is $U_1 \subseteq U$ open with the property that $G(U_1, \overline{W}) \subseteq W$. But this means precisely that $G \colon U_1 \times \overline{W} \to W$ is a uniform contraction, and by the corresponding principle we can find a smooth map $f \colon U_1 \to W$ such that $f(\xi) = G(\xi, f(\xi)) = f(\xi) - \partial_2\mathcal{F}(0)^{-1} \cdot \mathcal{F}(\xi + f(\xi))$ for all $\xi \in U_1$. In particular, for any $\xi \in U_1$, one can find the solution $f(\xi)$ by iterating $\eta \mapsto \eta - (\partial_2\mathcal{F}(0)^{-1} \circ \mathcal{F})(\xi + \eta)$ starting with e.g. $\eta = 0$. □

Lemma 4.2. $p > 2$. *Let Σ be a closed Riemann surface, (M,ω) a compact symplectic manifold, and continue the notation from above. Assume that $F : \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega) \to \mathcal{E}^p$ is a submersion whenever $\overline{\partial}_J u = 0$, i.e. $\overline{\partial}_J u = 0$ implies that $T_{(u,J)}F : T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)) \to \mathcal{E}^p$ is surjective.*
Then there exists a smooth mapping $f : \ker T_{(u,J)}F \supseteq U \to W \subseteq E$ which satisfies $F(\exp_{(u,J)}(\xi + f(\xi))) = 0$. E is a topologically complementary vector space to $\ker T_{(u,J)}F$ in $T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega))$.

Proof. $\overline{\partial}_J u = 0$. On (M,J) choose a Hermitian connection as in 4.A.(2). It only remains to show that $\ker T_{(u,J)}F$ splits in $T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega))$. Again, $V \subseteq T_u\mathcal{N}^{1,p}(A)$ denotes a closed linear subspace such that $\ker D_u \oplus V = T_u\mathcal{N}^{1,p}(A)$. Note that

$$\begin{aligned}\ker T_{(u,J)}F &= \{(\xi_1 \oplus \xi_2, Y) : 0 + D_u|_V \cdot \xi_2 + \tfrac{1}{2} Y \cdot du \cdot j = 0\} \\ &= \{(\xi_1 \oplus (D_u|_V)^{-1}(-\tfrac{1}{2} Y \cdot du \cdot j), Y) : \xi_1 \in \ker D_u, Y \in T_J\mathcal{J}^l(\omega)\}.\end{aligned}$$

Consider the continuous map $p : T_u\mathcal{N}^{1,p}(A) \times T_J\mathcal{J}^l(\omega) \to \ker T_{(u,J)}F$, $(\xi_1 \oplus \xi_2, Y) \mapsto (\xi_1 \oplus (D_u|_V)^{-1}(-\tfrac{1}{2} Y \cdot du \cdot j), Y)$. This map satisfies $p|_{\ker T_{(u,J)}F} = \mathrm{id}$ as well as $p \circ p = p$, hence giving rise to the splitting $T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)) =$

$\ker T_{(u,J)}F \oplus \ker p =: \ker T_{(u,J)}F \oplus E$. The rest follows word by word as above by considering the mapping

$$\begin{aligned}\mathcal{F}\colon & T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)) = \ker T_{(u,J)}F \oplus E \longrightarrow \mathcal{E}^p_{(u,J)},\\ &(\xi, Y) = (\xi_1, Y_1) \oplus (\xi_2, Y_2) \longmapsto (\mathrm{Pt}^{\exp(1+\_)(\xi,Y)}(-1) \circ F \circ \exp(\xi, Y))(u, J).\end{aligned}$$

Indeed, observe that $\partial_2\mathcal{F}(0) = T_{(u,J)}F|_E$. □

Lemma 4.3. $p > 2$. *Let Σ be a closed Riemann surface, (M, ω) a compact symplectic manifold, and continue the notation from above. Let $\gamma\colon [0,1] \to \mathcal{J}(\omega)$, $t \mapsto \gamma(t) = J_t$ be a smooth curve, and assume that there exists a point $u_0 \in \mathcal{N}^{1,p}(A)$ such that $\overline{\partial}_{J_0}u_0 = 0$ and $D_{u_0} : T_{u_0}\mathcal{N}^{1,p}(A) \to \mathcal{E}^p_{(u_0,J_0)}$ is surjective.*
Then there is a smooth function $f : [0,1] \times \ker D_{u_0} \to V$ which is defined on open neighborhoods of zero in the respective spaces with the property that $F(\exp_{u_0} t(\xi + f(t,\xi)), J_t) = 0$ for all $(t, \xi) \in \mathrm{domain} f$. V splits $\ker D_{u_0}$.

Proof. On (M, J_0) choose a Hermitian connection as in 4.A.(2). Let $Y_0 \in T_{J_0}\mathcal{J}^l(\omega)$ such that $\exp_{J_0} tY_0 = J_t$ holds for small $\varepsilon > 0$ and all $t \in [0, \varepsilon)$; that is $\exp_{J_0(x)} tY_0(x) = J_t(x)$ for all $x \in M$ and the exponential mapping with respect to the chosen connection. Consider

$$\begin{aligned}\mathcal{F}\colon & [0,1] \times T_{u_0}\mathcal{N}^{1,p}(A) \longrightarrow \mathcal{E}^p_{(u_0,J_0)},\\ &(t, \xi) \longmapsto (\mathrm{Pt}^{\exp(t+\_)(\xi,Y_0)}(-t) \circ F \circ \exp t(\xi, Y_0))(u_0, J_0).\end{aligned}$$

By the open mapping theorem this map has the property that $\partial_3\mathcal{F}(0) = D_{u_0}|_V\colon V \to \mathcal{E}^p_{(u_0,J_0)}$ is an isomorphism. Thus we can consider the map

$$\begin{aligned}G\colon & [0,1] \times \ker D_{u_0} \times V \longrightarrow \mathcal{E}^p_{(u_0,J_0)}\\ &(t, \xi_1, \xi_2) \longmapsto \xi_2 - (\partial_3\mathcal{F}(0)^{-1} \circ \mathcal{F})(t, \xi_1, \xi_2),\end{aligned}$$

and since $\partial_3 G(0) = 0$ it turns out that this is a uniform contraction; cf. proof of 4.1. By the uniform contraction principle there exists a smooth mapping $f : [0,1] \times \ker D_{u_0} \to V$ such that $f(t,\xi) = G(t, \xi, f(t,\xi)) = f(t,\xi) - (\partial_3\mathcal{F}(0)^{-1} \circ \mathcal{F})(t, \xi, f(t,\xi))$, and consequently $0 = F(\exp_{(u_0,J_0)} t(\xi + f(t,\xi), Y_0)) = F(\underbrace{\exp_{u_0} t(\xi + f(t,\xi))}_{=:u_t}, \underbrace{\exp_{J_0} tY_0}_{=J_t}) = \overline{\partial}_{J_t} u_t$ □

In the terminology of subsequent sections this lemma says that a curve $\gamma\colon [0,1] \to \mathcal{J}(\omega)$ that starts at a regular almost complex structure J_0 can be lifted to curve $\tilde{\gamma}\colon [0, \varepsilon) \to \mathcal{M}^l(A)$ in the universal moduli space. By elliptic regularity the points on this curve will be smooth maps.

4.C. Universal moduli spaces

Lemma 1.6 asserts that J-holomorphic curves are as regular as the derivative of J. This reflects the fact that ... *the only meaningful objects are holomorphic curves which are unsensible to a choice of the infinite dimensional phraseology*, Gromov [6]. Thus the space

$$\begin{aligned}\mathcal{M}^l = \mathcal{M}^l(A) &:= \{(u,J) \in \mathcal{N}^{k,p} \times \mathcal{J}^l(\omega) : \bar{\partial}_J u = 0\} \\ &= \{(u,J) \in \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega) : \bar{\partial}_J u = 0\}\end{aligned}$$

is well defined for $p > 2$ and $1 \le k \le l$. It is called the *universal moduli space* of the Cauchy-Riemann operator on M.

Proposition 4.4. $p > 2$. *Then* $\mathcal{M}^l(A)$ *is a smooth Banach manifold for any* $l \in \mathbb{N}$. *It is modelled on spaces of the type* $\ker T_{(u,J)}F$.

Proof. With the notation from above consider the section $F \in \Gamma(\mathcal{E}^p \to \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega))$. *$F$ is transversal to the zero section in $\mathcal{E}^p \to \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)$.* Fix $(u,J) \in \mathcal{M}^l(A)$. Since $\operatorname{im} D_u \subseteq \operatorname{im} T_{(u,J)}F$ and D_u is Fredholm the image $\operatorname{im} T_{(u,J)}F \subseteq \mathcal{E}^p_{(u,J)}$ is closed, and it suffices to check that $T_{(u,J)}F$ has dense range. Assume the contrary, then there exists a non-zero $\eta \in (\operatorname{im} T_{(u,J)}F)^\circ$,[3] that is

$$\int_\Sigma \langle \eta, D_u\xi\rangle(z)\, dz = \int_\Sigma \langle \eta, \tfrac{1}{2}Y(u) \circ Tu \circ j\rangle(z)\, dz = 0$$

for all $(\xi, Y) \in T_u\mathcal{N}^{k,p} \times T_J\mathcal{J}^l(\omega)$. *$\eta$ vanishes on the open and dense set of regular points of u.* Indeed, take such a regular point $z \in \Sigma$ and assume the complement of $\{\eta_z\}^\circ \subseteq \mathcal{E}^p_{(u,J)}|_z = \operatorname{Hom}^{0,1}_J(T_z\Sigma, u^*TM|_z)$ is non-empty. Then $T_z u$ has a left inverse, and hence

$$\begin{aligned} &\exists Z \circ j_z \in \operatorname{Hom}^{0,1}_J(T_z\Sigma, u^*TM|_z) = \operatorname{Hom}^{0,1}_J(T_z\Sigma, T_{u(z)}M) \\ &\text{such that } 0 \neq \langle \eta_z, Z \circ j_z\rangle = \langle \eta_z, Z \circ T_z u^{-1} \circ T_z u \circ j_z\rangle \\ \Longrightarrow \quad &\int_\Sigma \langle \eta, Y(u) \circ Tu \circ j\rangle(z)\, dz = \int_{u^{-1}(U)} \langle \eta, Y(u) \circ Tu \circ j\rangle(z)\, dz \neq 0,\end{aligned}$$

where $Y \in \Gamma^l_{cp}(\operatorname{End}^{0,1}_J(TM) \to M) = T_J\mathcal{J}^l(\omega)$ satisfying $Y(u(z)) \circ T_z u = Z$. Such a section exists locally on an open neighborhood U of $u(z)$, and is then globally well defined by multiplying it with a smooth bump function

[3] For $M \subseteq \mathcal{E}^p_{(u,J)}$ and $\frac{1}{p} + \frac{1}{q} = 1$ we denote the polar of M in $\mathcal{E}^q_{(u,J)}$ by $M^\circ \subseteq \mathcal{E}^q_{(u,J)}$. For subspaces this coincides with the annihilator. If M is absolutely convex then one has $M^{\circ\circ} = \overline{M}$ by the bipolar theorem.

ρ with supp$\rho \subseteq U$ and $\rho(u(z)) = 1$. This contradiction makes $T_{(u,J)}F : T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)) \to \mathcal{E}^p_{(u,J)}$ surjective and lemma 4.2 applicable.

Let $E \subseteq T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega))$ be a closed, linear subspace such that $\ker T_{(u,J)}F \times E = T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega))$, and denote by pr_i the canonical projections, and by $f_{(u,J)} : \ker T_{(u,J)}F \supseteq U \to W \subseteq E$ the smooth mapping satisfying $F(\exp_{(u,J)}((\xi, Y) + f_{(u,J)}(\xi, Y))) = 0$, where U, W are open zero neighborhoods in $\ker T_{(u,J)}F$, E respectively. Furthermore, let $T \subseteq T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega))$ such that $\exp_{(u,J)} : T \to O$ is a diffeomorphism onto an open neighborhood $O \subseteq \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)$ of (u, J). Assume without loss of generality that $\mathrm{pr}_1(T) = U$ and $\mathrm{pr}_2(T) = W$. If now $(u_1, J_1) = \exp_{(u,J)}\zeta \in O \cap \mathcal{M}^l(A)$ then $\mathrm{pr}_1(\zeta) =: (\xi, Y)$, and $\mathrm{pr}_2(\zeta) = f_{(u,J)}(\xi, Y)$ because Banach fixed points are unique. Thus the mapping $\psi_{(u,J)} : \ker T_{(u,J)}F \to \mathcal{M}^l(A)$, $(\xi, Y) \mapsto \exp_{(u,J)}((\xi, Y) + f_{(u,J)}(\xi, Y))$ is a local bijection. *The chart changing maps are smooth.* If $(u', J') \in \mathcal{M}^l(A)$ has the property that $\operatorname{im}\psi_{(u,J)} \cap \operatorname{im}\psi_{(u',J')} \neq \emptyset$, and $(\xi, Y) \in \psi^{-1}_{(u,J)}(\operatorname{im}\psi_{(u',J')})$ then

$$\begin{aligned}\psi^{-1}_{(u',J')} \circ \psi_{(u,J)} \colon \ker T_{(u,J)}F &\longrightarrow \ker T_{(u',J')}F,\\ \xi &\longmapsto \exp_{(u,J)}((\xi, Y) + f_{(u,J)}(\xi, Y)) =: (u_1, J_1)\\ &\longmapsto \exp^{-1}_{(u',J')}(u_1, J_1)\\ &\longmapsto \mathrm{pr}_1(\exp^{-1}_{(u',J')}(u_1, J_1))\end{aligned}$$

is smooth. Thus the collection $(\psi_{(u,J)}, U_{(u,J)})_{(u,J)}$ defines a smooth atlas on $\mathcal{M}^l(A)$. The topology is the identification topology with respect to this atlas. $\square$

For the rest of this section let $\Sigma = \mathbb{C}P^1$, and $G = \mathrm{PSL}(2, \mathbb{C})$ be the reparametrization group acting on $\mathcal{N}^{1,p}(A)$ from the right by composition. In what follows $B_{[r_1,r_2]} := \{z \in \mathbb{C} : r_1 \leq z \leq r_2\}$ denotes the closed annulus centered at 0.

Proposition 4.5. *Let (M, ω) be a compact symplectic manifold, $J \in \mathcal{J}(\omega)$, $u \in \mathcal{M}(A, J)$, and $z_0 \in \mathbb{C}P^1$. Then there exist $\delta > 0$ and (r_1, r_2) with $0 < r_1 < r_2 < \delta$ such that the following is true. For all $X \in T_{u(z_0)}M$ there are tangent vectors $(\xi, Y) \in T_u\mathcal{N}(A) \times T_J\mathcal{J}(\omega)$ satisfying the following. (Where we choose holomorphic coordinates centered at z_0 and thereby identify the coordinate patch with $\mathbb{C}$.)*

(i) $\xi(z_0) = X$.

(ii) $T_{(u,J)}F \cdot (\xi, Y) = 0$, *that is* $(\xi, Y) \in T_{(u,J)}\mathcal{M}(A)$.

(iii) $\operatorname{supp}\xi \subseteq B_{r_2}$ *and* $\operatorname{supp} Y$ *is contained in* $u(V)$ *where* V *is a small neighborhood of the annulus* $B_{[r_1,r_2]}$.

In particular, $\mathrm{ev}\colon \mathcal{M}(A) \to M$, $(u, J) \mapsto u(z_0)$ *is a submersion.*

Proof. Let $X \in T_{u(z_0)}M$ be fixed. Choose a holomorphic chart (U, ψ) centered at z_0. By 1.A. there is a trivialization

$$\begin{array}{ccccc}
\mathbb{C} \times \mathbb{R}^{2n} & \xrightarrow{\Phi} & u^*TM|_U & \xrightarrow{\subseteq} & u^*TM \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C} & \xrightarrow{\psi^{-1}} & U & \xrightarrow{\subseteq} & \mathbb{C}P^1
\end{array}$$

such that $\Phi(z) \circ J_0 = J(u(\psi^{-1}(z))) \circ \Phi(z)$. From now on we will identify $z = \psi^{-1}(z) = s + it$, for simplicity, and $\Phi = \Phi|B_1$. For arbitrary $(\xi, Y) \in T_{(u,J)}(\mathcal{N}(A) \times \mathcal{J}(\omega))$ we define

$$\begin{aligned}
\xi_0 &:= \Phi^{-1} \cdot \xi\colon B \to \mathbb{R}^{2n}, z \mapsto \Phi(z)^{-1}\xi(z), \\
Y_0 &:= \Phi^{-1} \cdot (Y \circ u) \cdot \Phi\colon B \to \operatorname{End}\mathbb{R}^{2n}, \\
\eta_0 &:= \partial_s\xi_0 + J_0\partial_t\xi_0 + A\xi_0 + Y_0\Phi^{-1}\partial_t u
\end{aligned}$$

where $A\colon B \to \operatorname{End}\mathbb{R}^{2n}$ is defined by

$$A\xi_0 = \Phi^{-1}(\partial_s\Phi \cdot \xi_0 + (J \circ u)\partial_t\Phi \cdot \xi_0 + \partial_{\Phi\xi_0}J \cdot \partial_t u).$$

From 4.A.(5) recall the local formula

$$D_u\xi + \tfrac{1}{2}(Y \circ u)du \cdot i = 2\eta ds - 2(J \circ u)\eta dt$$

with

$$\eta := \partial_s\xi + (J \circ u)\partial_t\xi + \partial_\xi J \cdot \partial_t u + (Y \circ u)\partial_t u.$$

Using Leibniz rule now yields

$$\begin{aligned}
\eta - \Phi \cdot \eta_0 &= \partial_s(\Phi\xi_0) - \Phi\partial_s\xi_0 \\
&\quad + (J \circ u)\partial_t(\Phi\xi_0) - (J \circ u)\Phi\partial_t\xi_0 \\
&\quad + (Y \circ u)\partial_t u - (Y \circ u)\partial_t u \\
&\quad + (\partial_\xi J)\partial_t u - \Phi A\xi_0 \\
&= (\partial_s\Phi)\xi_0 + (J \circ u)(\partial_t\Phi)\xi_0 + (\partial_{\Phi\xi_0}J)\partial_t u - \Phi A\xi_0 \\
&= 0.
\end{aligned}$$

Thus we could try to find local solutions ξ_0, Y_0 with $\xi_0(0) = v := \Phi(0)^{-1}(X)$ such that the corresponding η_0 satisfies $\eta_0 = 0$.

By proposition 1.11 there is $\delta_1 \in (0,1)$ such that there exists $\xi_0 \in C^\infty(B_{\delta_1}, \mathbb{R}^{2n})$ solving

$$\partial_s \xi_0 + J_0 \partial_t \xi_0 + A\xi_0 = 0, \qquad \xi_0(0) = v.$$

Claim: There are $\delta_2 > 0$, r_1, r_2 with $0 < r_1 < r_2 < \delta_2$ such that $u|_{B_{[r_1,r_2]}} : B_{[r_1,r_2]} \to M$ is an injective immersion.

If 0 is an injective point then this is obvious. Let 0 be non-injective and choose $\delta_2 > 0$ such that $\{z \in B_{\delta_2} : du(z) = 0\} = \{0\}$. Arguing by contradiction we construct sequences $(z_n)_n$, $(w_n)_n$ in B_{δ_2} such that:

$$z_n \to z \in B_{\delta_2} \setminus \{0\}, w_n \to w \in B_{\delta_2} \setminus \{0\}, z_n \neq z, w_n \neq w,$$
$$u(z_n) = u(w_n), u(z) = u(w).$$

Let $a := \frac{w}{z}$. Without loss we assume that $z \neq w$; otherwise $z_n = w_n$ for large n since $du(z) \neq 0$. Let $v := u \circ a^{-1}$. Then $v(az_n) = u(z_n) = u(w_n)$. Because $du(w) \neq 0$ we can apply lemma 2.2.3. from McDuff, Salamon [16] to find $\varepsilon > 0$ and a holomorphic map $\phi \colon B_\varepsilon(w) \to \mathbb{C}$ such that

$$u|B_\varepsilon = v \circ \phi = u \circ a^{-1} \circ \phi, \qquad \phi(w) = w.$$

Now $a^{-1} \circ \phi \neq \mathrm{id}$; else $w = a^{-1}\phi(w) = z$. Hence $\{x, a^{-1}\phi(x)\} \subseteq u^{-1}(u(x))$ for all $x \in B_\varepsilon(w)$. But this is absurd, for it would mean that the complement of the dense set of injective points contains a ball of nonzero radius.

Let $\delta = \min\{\delta_1, \delta_2\}$, and assume without loss that $u|_V \colon V \to M$ is an injective immersion for some open neighborhood V of $B_{[r_1,r_2]}$. Choose a smooth bump function $\beta \colon B \to [0,1]$ such that $\beta([0,r_1]) = \{1\}$ and $\beta([r_2,1]) = \{0\}$. Consider $\zeta_0 := \beta\xi_0 \in C^\infty(B, \mathbb{R}^{2n})$ and define

$$\sigma := \partial_s \zeta_0 + J_0 \partial_t \zeta_0 + A\zeta_0.$$

By construction $\operatorname{supp} \sigma \subseteq B_{[r_1,r_2]}$. Because the action $(\operatorname{End} \mathbb{R}^{2n})^{0,1} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $(l,v) \mapsto l(v)$ is transitive there is $Y_0 \in C^\infty(B, \operatorname{End} \mathbb{R}^{2n})$ such that

$$Y_0 \cdot \Phi^{-1} \partial_t u = -\sigma \qquad \text{and} \qquad Y_0 \cdot J_0 + J_0 \cdot Y_0 = 0.$$

Choose another smooth bump function $\gamma \colon B \to [0,1]$ with $\gamma(B_{[r_1,r_2]}) = \{1\}$ and $\gamma(B \setminus W) = \{0\}$ where W is another open neighborhood of $B_{[r_1,r_2]}$ such that $\overline{W} \subseteq V$. For $x \in u(V)$ we can define $x \mapsto Y_x = Y_{u(z)} = \Phi(z) Y_0(z) \Phi(z)^{-1} \in \Gamma_{\mathrm{loc}}(\operatorname{End} TM \to M)$ which extends trivially to all of M. Thus we have actually found $\xi := \Phi \cdot \zeta_0$ (also extended trivially), Y that fulfill

$$\xi(0) = \Phi(0)\xi_0(0) = X,$$

$$\begin{aligned}
&\operatorname{supp}\xi \subseteq B_{r_1} \qquad \text{and} \qquad \operatorname{supp} Y \subseteq u(V),\\
&(J\circ u)(Y\circ u) = (J\circ u)\Phi Y_0\Phi^{-1} = -(Y\circ u)(J\circ u),\\
&\eta = \Phi\eta_0 = \Phi(\sigma-\sigma) = 0,
\end{aligned}$$

that is $(\xi, Y) \in T_{(u,J)}\mathcal{M}(A)$ with $\xi(z_0) = X$. □

Recall that $\mathcal{N}^{1,p}$ depends on the homology class $A \in \mathrm{H}_2(M)$, but we omitted to express this dependence since the A was kept fixed. However we might as well have chosen to write $\mathcal{N}^{1,p} = \mathcal{N}^{1,p}(A)$.

1. *The subset* $\mathcal{N}^{1,p}(A^1,\dots,A^n) \subseteq \mathcal{N}^{1,p}(A^1)\times\cdots\times\mathcal{N}^{1,p}(A^n)$*, defined by*

$$\begin{aligned}
&\mathcal{N}^{1,p}(A^1,\dots,A^n)\\
&\quad := \{(u^1,\dots,u^n)\colon u^i \in \mathcal{N}^{1,p}(A^i), \text{there are } (z^1,\dots,z^n) \in (\mathbb{C}P^1)^n\\
&\qquad\qquad \text{such that } u^i(z^i) \in u^j(\mathbb{C}P^1) \Longrightarrow i = j \text{for all } i \in \{1,\dots,n\}\ \}
\end{aligned}$$

is open, and hence a smooth Banach sub-manifold. Indeed, this condition is open since u^j has closed image.

2. By taking first appropriate Whitney sums, and then projection onto complex anti-linear forms, followed by pullback via the canonical inclusion $\iota : \mathcal{N}^{1,p}(A^1,\dots,A^n)\times\mathcal{J}^l(\omega) \hookrightarrow \mathcal{N}^{1,p}(A^1)\times\cdots\times\mathcal{N}^{1,p}(A^n)\times\mathcal{J}^l(\omega)$, the bundle $(\mathcal{E}^p)^n \to \mathcal{N}^{1,p}(A^1,\dots,A^n)\times\mathcal{J}^l(\omega)$ with fibers

$$(\mathcal{E}^p)^n_{(u,J)} = \bigoplus_{i=1}^n \Omega^{0,1}_J(\Sigma;(u^i)^*TM) = \bigoplus_{i=1}^n L^p(\Lambda^{0,1}T^*\Sigma\otimes_J(u^i)^*TM) = \bigoplus_{i=1}^n \mathcal{E}^p_{(u^i,J)}$$

becomes a smooth vector bundle. We will identify $T(\mathcal{N}^{1,p}(A^1,\dots,A^n)\times\mathcal{J}^l(\omega)) = T\iota\cdot T(\mathcal{N}^{1,p}(A^1,\dots,A^n)\times\mathcal{J}^l(\omega))$. □

3. Thus there is again a global version of the Cauchy-Riemann operator given by the smooth section

$$\begin{aligned}
F^n\colon \mathcal{N}^{1,p}(A^1,\dots,A^n)\times\mathcal{J}^l(\omega) &\longrightarrow (\mathcal{E}^p)^n,\\
(u,J) = (u^1,\dots,u^n,J) &\longmapsto (F(u^1,J),\dots,F(u^n,J))
\end{aligned}$$

with differential $T_{(u,J)}F^n(\xi,Y) = (T_{(u^1,J)}F(\xi^1,Y),\dots,T_{(u^1,J)}F(\xi^n,Y))$. As above $F(u^i,J) = (u^i,J,\overline{\partial}_J u^i)$, in a trivialization. □

4. *If* $F^n(u,J) = 0$ *then* $T_{(u,J)}F^n(\_,0) = \partial_1 F^n(u,J)$ *is a Fredholm operator.* It is a $\overline{\partial}$-operator:

$$\partial_1 F^n(u,J)(f\xi) = (D_{u^1}(f\xi^1),\dots,D_{u^n}(f\xi^n))$$

$$= ((df \otimes \xi^1)^{0,1} + f D_{u^1}\xi^1, \dots, (df \otimes \xi^n)^{0,1} + f D_{u^n}\xi^n)$$
$$= (df \otimes \xi)^{0,1} + f\partial_1 F^n(u, J)\xi.$$

Thus $\partial_1 F^n(u,J)$ is a partial differential operator with elliptic symbol by 4.A.(3). $\square$

5. *If* $F^n(u,J) = 0$ *then* $\ker T_{(u,J)}F^n$ *splits in* $T_{(u,J)}(\mathcal{N}^{1,p}(A^1,\dots,A^n)\times\mathcal{J}^l(\omega))$. As above $\ker D_{u^i} \oplus V_i = T_{u^i}\mathcal{N}^{1,p}(A^i)$, and consider the continuous, linear map

$$\begin{aligned} p^n\colon T_{u^1}\mathcal{N}^{1,p}(A^1) \times \cdots \times T_{u^n}\mathcal{N}^{1,p}(A^n) \times T_J\mathcal{J}^l(\omega) \\ \longrightarrow T_{u^1}\mathcal{N}^{1,p}(A^1) \times \cdots \times T_{u^n}\mathcal{N}^{1,p}(A^n) \times T_J\mathcal{J}^l(\omega), \\ ((\xi_1^i \oplus \xi_2^i)_{i=1}^n, Y) \longmapsto ((\xi_1^i \oplus (D_{u^i}|_{V_i})^{-1}(-\tfrac{1}{2}Y \cdot Tu^i \cdot j))_{i=1}^n, Y), \end{aligned}$$

which satisfies $\operatorname{im} p^n = \ker T_{(u^1,J)}F \times \cdots \times \ker T_{(u^n,J)}$ and $p^n \circ p^n = p^n$. Hence $\ker p^n$ gives the asserted topologically complementary subspace when restricted to $T_{(u,J)}(\mathcal{N}^{1,p}(A^1,\dots,A^n) \times \mathcal{J}^l(\omega))$. $\square$

6. Let $F^n(u,J) = 0$, and E a closed, linear subspace complementary to $\ker T_{(u,J)}F^n$, and $T_{(u,J)}F^n$ surjective. Then the smooth mapping

$$\begin{aligned} \mathcal{F}^n\colon T_{(u,J)}(\mathcal{N}^{1,p}(A^1,\dots,A^n) \times \mathcal{J}^l(\omega)) \longrightarrow (\mathcal{E}^p)^n_{(u,J)}, \\ (\xi^1,\dots,\xi^n,Y) \longmapsto \big((\mathrm{Pt}^{\exp(t+\_)(\xi^i,Y)}(-t) \circ F \circ \exp t(\xi^i,Y))(u^i,J)\big)_{i=1}^n \end{aligned}$$

satisfies $T_0\mathcal{F}^n = T_{(u,J)}F^n$, and thereby induces another smooth (locally around zero defined) map $f\colon \ker T_{(u,J)}F^n \to E$ such that the implicit equation $F^n(\exp_{(u,J)}((\xi,Y) + f(\xi,Y))) = 0$ holds. $\square$

$\boldsymbol{\mathcal{M}^l(A^1,\dots,A^n)}$**.** For $p > 2$ the *universal moduli space* of distinct, pseudo-holomorphic curves corresponding to the n-tuple $(A^1,\dots,A^n) \in (H_2(M))^n$ is

$$\begin{aligned} \mathcal{M}^l(A^1,\dots,A^n) := \{(u^1,\dots,u^n,J)\colon \\ J \in \mathcal{J}^l(\omega), u^i \in \mathcal{M}^l(A^i), u^j \in u^i \cdot G \Rightarrow i = j\} \end{aligned}$$

$\subseteq \mathcal{N}^{1,p}(A^1,\dots,A^n) \times \mathcal{J}^l(\omega)$. By elliptic regularity, lemma 1.6, this definition does not depend on the particular $p > 2$.

7. If $(c^1,\dots,c^n)$ is a simple cusp curve then the parametrizing tuple is an element of $\mathcal{M}^l(A^1,\dots,A^n)$. Conversely, if the union $\operatorname{im} u^1 \cup \cdots \cup \operatorname{im} u^n$ of the images of an element $(u^1,\dots,u^n) \in \mathcal{M}^l(A^1,\dots,A^n)$ is nonempty then $(u^1,\dots,u^n)$ is the parametrization of a simple cusp curve. $\square$

8. *The action* $G^n \times \mathcal{M}^l(A^1,\dots,A^n) \to \mathcal{M}^l(A^1,\dots,A^n)$, $(g,u,J) \mapsto (u^1 \cdot g_1^{-1},\dots,u^n \cdot g_n^{-1},J)$ *is free.* This follows from the assumption and the existence of injective points as in section 1.B.

Proposition 4.6. $\mathcal{M}^l(A^1,\dots,A^n)$ *is a smooth Banach manifold. It is modelled on spaces of the type* $\ker T_{(u,J)}F^n$.

Proof. Let $F^n(u^1,\dots,u^n,J) = F^n(u,J) = 0_{(u,J)} \in (\mathcal{E}^p)^n_{(u,J)}$, i.e. $\overline{\partial}_J u^i = 0$ for all $i \in \{1,\dots,n\}$. *The tangent mapping* $T_{(u,J)}F^n \colon T_u\mathcal{N}^{1,p}(A^1,\dots,A^n) \times T_J\mathcal{J}^l(\omega) \to (\mathcal{E}^p)^n_{(u,J)}$ *is surjective.* Because of the inequality

$$\operatorname{codim}\operatorname{im} T_{(u,J)}F^n \le \operatorname{codim}\operatorname{im} \partial_1(u,J)F^n < \infty$$

it again suffices to show that $\operatorname{im} T_{(u,J)}F^n \subseteq (\mathcal{E}^p)^n_{(u,J)}$ is dense or, equivalently, has trivial polar. Let $\eta = (\eta^1,\dots,\eta^n) \in (\operatorname{im}\partial_2 F^n(u,J))^\circ$, and $(z^1,\dots,z^n) \in (\mathbb{C}P^1)^n$ such that

(i) $T_{z^i}u^i \neq 0$, $(u^i)^{-1}(u^i(z^i)) = \{z^i\}$, and

(ii) $u^i(z^i) \in u^j(\mathbb{C}P^1) \iff i = j$.

We show that $(\eta^i(z^i))_{i=1}^n = 0$. By $T_{3\frac{1}{2}}$ (or compactness) there are neighborhoods U_i of z^i such that $u^i(U_i) \cap u^j(\mathbb{C}P^1) \neq \emptyset \iff i = j$, and by adjusting the size of these neighborhoods it follows that

$$\int_{\mathbb{C}P^1} \sum_{i=1}^n \langle \eta^i, \tfrac{1}{2}Y(u^i)\cdot Tu^i \cdot j\rangle(z)dz = \sum_{i=1}^n \int_{u^i(U_i)} \langle \eta^i, \tfrac{1}{2}Y(u^i)\cdot Tu^i \cdot j\rangle(z)dz \neq 0;$$

choose $Y \in T_J\mathcal{J}^l(\omega)$ such that $\langle \eta^i(z), \frac{1}{2}Y(u^i(z)) \cdot T_z u^i \cdot j(z)\rangle \neq 0$ and $\operatorname{supp} Y \subseteq u^1(U_1) \cup \dots \cup u^n(U_n)$. But the set of points $(z^1,\dots,z^n) \in (\mathbb{C}P^1)^n$ satisfying (i) and (ii) is open and dense in $(\mathbb{C}P^1)^n$; see proposition 1.5. Thus $(\operatorname{im} T_{(u,J)}F^n)^\circ \subseteq (\operatorname{im}\partial_2 F^n(u,J))^\circ = \{0\}$.
Let $O \subseteq \mathcal{N}^{1,p}(A^1,\dots,A^n) \times \mathcal{J}^l(\omega)$ be an open neighborhood of $(u,J) \in \mathcal{M}^l(A^1,\dots,A^n)$. Then it follows as above that the locally defined mapping

$$\begin{aligned} \ker T_{(u,J)}F^n &\longrightarrow \mathcal{M}^l(A^1,\dots,A^n) \cap O, \\ (\xi,Y) &\longmapsto \exp_{(u,J)}((\xi,Y) + f_{(u,J)}(\xi,Y)) \end{aligned}$$

is a smooth chart. See proof of proposition 4.4 and point (6) above. $\square$

4.D. Moduli spaces

Consider the natural projections

$$p_1 : \mathcal{N}^{k,p} \times \mathcal{J}^l(\omega) \supseteq \mathcal{M}^l(A) \longrightarrow \mathcal{N}^{k,p},$$
$$p_2 : \mathcal{N}^{k,p} \times \mathcal{J}^l(\omega) \supseteq \mathcal{M}^l(A) \longrightarrow \mathcal{J}^l(\omega),$$

and observe that $T_{(u,J)}p_i = p_i$ for $i \in \{1,2\}$. *The projection $p_2 : \mathcal{M}^l(A) \to \mathcal{J}^l(\omega)$ is a Fredholm map.*[4] Indeed, it is true that

$$\dim \ker T_{(u,J)}p_2 = \dim \{(\xi, 0) : D_u\xi + 0 = 0\} = \dim \ker D_u.$$

But, since $\operatorname{im} T_{(u,J)}p_2 = \{Y \in T_J\mathcal{J}^l(\omega) : Y \circ Tu \circ j \in \operatorname{im} D_u\}$, one also finds $\operatorname{codim}_{T_J\mathcal{J}^l(\omega)} \operatorname{im} T_{u,J}p_2 = \operatorname{codim}_{\mathcal{E}^{k-1,p}_{u,J}} \operatorname{im} D_u$:

$$\begin{tikzcd}
T_{(u,J)}(\mathcal{N}^{k,p} \times \mathcal{J}^l(\omega)) \arrow[r, "T_{(u,J)}F", "\text{onto}"'] \arrow[d, "\text{onto}"'] & \mathcal{E}^{k-1,p}_{(u,J)} \arrow[r, "\text{onto}"'] & \frac{\mathcal{E}^{k-1,p}_{(u,J)}}{\text{im } D_u} \\
\frac{T_{(u,J)}(\mathcal{N}^{k,p} \times \mathcal{J}^l(\omega))}{\ker T_{(u,J)}F} \arrow[r, "\cong"] & \frac{T_{(u,J)}\mathcal{N}^{k,p}}{\operatorname{im} T_{(u,J)}p_1} \oplus \frac{T_J\mathcal{J}^l(\omega)}{\operatorname{im} T_{u,J}p_2} \arrow[u, "\cong"] \arrow[r, "pr_2", "\text{onto}"'] & \frac{T_J\mathcal{J}^l(\omega)}{\operatorname{im} T_{u,J}p_2} \arrow[u, "\cong"]
\end{tikzcd}$$

Surjectivity of $T_{(u,J)}F$ was shown in the proof of 4.4. By definition a point $J \in \mathcal{J}^l(\omega)$ is a regular value of $p_2 : \mathcal{M}^l(A) \to \mathcal{J}^l(\omega)$ if and only if $T_{(u,J)}p_2 : T_{(u,J)}\mathcal{M}^l(A) \to T_J\mathcal{J}^l(\omega)$ is surjective for all $(u, J) \in p_2^{-1}(J) =: \mathcal{M}^l(A, J)$. As a corollary we obtain the following statement: *$J \in \mathcal{J}^l(\omega)$ is a regular value of $p_2 : \mathcal{M}^l(A) \to \mathcal{J}^l(\omega)$ if and only if $D_u : T_u\mathcal{N}^{k,p} \to \mathcal{E}^{k-1,p}_{(u,J)}$ is surjective for all $u \in \mathcal{M}^l(A, J)$.*

Definition. An almost complex, ω-tame structure $J \in \mathcal{J}(\omega)$ is called *regular* if $D_u : T_u\mathcal{N} \to \mathcal{E}_{(u,J)}$ is surjective for all $u \in \mathcal{M}(A, J)$. The set of all almost complex, ω-tame, regular structures on M is denoted by $\mathcal{J}_{\text{reg}}(\omega, A)$. $J \in \mathcal{J}(\omega)$ is said to be *generic* if $\mathcal{M}(A, J)$ is a finite dimensional manifold. By the following theorem the words generic and regular are synonymous.

Theorem 4.7. *Let Σ be a closed Riemann surface, (M, ω) a symplectic, compact manifold, $A \in H_2(M)$, $g = \operatorname{genus} \Sigma$, $2n = \dim M$, and continue the notation from above.*

(i) *If $J \in \mathcal{J}_{\text{reg}}(\omega, A)$ then $\mathcal{M}(A, J)$ is a smooth, finite dimensional manifold. Its dimension is given by $\dim \mathcal{M}(A, J) = n(2 - 2g) + 2c_1(M)(A)$, and $T_u\mathcal{M}(A, J) = \ker D_u$.*

[4] I.e. its derivative at every point is a Fredholm operator.

(ii) *If* $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A)$ *then* $\mathcal{M}(A, J)$ *carries a natural orientation.*

(iii) $\mathcal{J}_{\mathrm{reg}}(\omega, A) \subseteq \mathcal{J}(\omega)$ *is a subset of second category with respect to the* WO^∞*-topology, i.e. its complement is contained in a countable union of closed sets with empty interior.*

In particular, if J is generic and $\dim M \leq 6$ and $c_1(M)(A) < 0$ there cannot be any J-curves $\mathbb{C}P^1 \to M$ representing A – since the reparametrization group $G = \mathrm{PSL}(2, \mathbb{C})$ is 6 dimensional the space of unparametrized A-spheres $\mathcal{M}(A, J)/G$ would then have negative dimension. This will be taken up again in section 5.B.

The class $c_1(M) \in H^2(M)$ is the first Chern class of the bundle $TM \to M$. Its evaluation on classes $A \in H_2(M)$ is denoted by $c_1(M)(A) = \int_A c_1(M)$.

Proof. By elliptic regularity, lemma 1.6, it suffices to exhibit the inverse image $(F(\_, J) \colon \mathcal{N}^{1,p}(A) \to \mathcal{E}^p)^{-1}(0)$ as a smooth manifold, for $(F(\_, J))^{-1}(0) = \mathcal{M}(A, J)$ for smooth structures J.

(i) Consider the smooth section $F : \mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega) \to \mathcal{E}^p$, $(u, J) \mapsto (u, J, \overline{\partial}_J u)$. At a regular point $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A)$ the map $F(\_, J) \colon \mathcal{N}^{1,p}(A) \to \mathcal{E}^p$ is a smooth Fredholm map between smooth Banach manifolds which is transverse to the zero section. Thus $\mathcal{M}(A, J) = (F(\_, J))^{-1}(0) \subseteq \mathcal{N}^{1,p}(A)$ is a smooth sub-manifold of $\mathcal{N}^{1,p}(A)$ of dimension

$$\dim \mathcal{M}(A, J) = \mathrm{index} F(\_, J) = n(2 - 2g) + 2c_1(M)(A)$$

which follows from the index theorem (Hirzebruch-Riemann-Roch or Atiyah-Singer theorem); this theorem can be found in Hirzebruch [7].

The implicit function theorem even provides an explicit chart construction. Let $u \in \mathcal{M}(A, J)$. Decompose $T_u \mathcal{N}^{1,p}(A) = \ker D_u \oplus V$, cf. theorem 4.1. *There is an open zero-neighborhood* $U_u \subseteq \ker D_u$*, and a mapping* $\psi_u \colon U_u \to \mathcal{M}(A, J)$*, which is a diffeomorphism onto its open image.* Let $v \in \mathcal{M}(A, J) \subseteq \mathcal{N}^{1,p}(A)$ be close to u. Then there is a $\zeta \in \Gamma^{1,p}(u^* TM) = T_u \mathcal{N}^{1,p}(A)$ such that $\exp_u(\zeta) = v$, and moreover $\zeta = (\xi_1, \xi_2) \in U \times W$, where $U \times W \subseteq \ker D_u \times V$ is open as in proposition 4.1. By the same proposition there exists a smooth mapping $f_u : U \to W$ such that $\overline{\partial}_J \exp_u(\xi + f_u(\xi)) = 0$ for all $\xi \in U$. But then $\xi_2 = f_u(\xi_1)$, since Banach fixed points are unique.

(ii) Since $D_u = (\nabla_\_)^{0,1} + \frac{1}{8} N_J(\_, \partial_J u) =: L_u + \frac{1}{8} N_J(\_, \partial_J u)$, and $L_u \circ J = J \circ L_u$ while $\frac{1}{8} N_J(\_, \partial_J u) \circ J = -J \circ \frac{1}{8} N_J(\_, \partial_J u)$, the kernel $\ker D_u$ is invariant under J if the Nijenhuis tensor vanishes, that is, if J is integrable.

Consider the smooth function $f : [0, 1] \times \mathcal{M}(A, J) \to \mathrm{Fred}(T\mathcal{N}^{1,p}(A), \mathcal{E}^p)$, $(t, u) \mapsto L_u + t\frac{1}{8} N_J(\_, \partial_J u)$. By 3.E. we can consider the determinant bundle

$$\det f \longrightarrow [0, 1] \times \mathcal{M}(A, J),$$

and this bundle has a canonical, strictly non-zero section. Observe that $\det f(0,u) = \det L_u = \Lambda^{\max} \ker L_u \otimes (\Lambda^{\max} \mathrm{coker}\, L_u)^*$ carries an induced complex structure $\hat{J}$ since $\ker L_u$, and $\operatorname{im} L_u$ are invariant under J. Now there is a vector bundle isomorphism $[0,1] \times \coprod_u \det f(0,u) \cong \det f$ given by the parallel transport $\mathrm{Pt}^{\mathrm{id}_{[0,1]}}(t)_u \colon \{t\} \times \det f(0,u) \to \det f(t,u)$. Thus also

$$\coprod_{u \in \mathcal{M}(A,J)} \det f(1,u) = \Lambda^{\max} T\mathcal{M}(A,J)$$

possesses a strictly non-zero section, namely $\mathrm{Pt}^{\mathrm{id}_{[0,1]}}(t) \circ \hat{J}$, and this induces a natural orientation on $\mathcal{M}(A,J)$.

This orientation does not depend on the Hermitian connection chosen in the computation of D_u, cf. 4.A. (2). Choose any two Hermitian connections ∇_0, ∇_1 in the affine space of all Hermitian connections such that their torsions both equal $\mathrm{Tor}^1 = \mathrm{Tor}^2 = \frac{1}{8} N_J$. Now connect the connections by the path $s \mapsto \nabla_s := (1-s)\nabla_0 + s\nabla_1$ for $s \in [0,1]$. By analogy to above consider the smooth map

$$\begin{aligned} \hat{f} \colon [0,1] \times [0,1] \times \mathcal{M}(A,J) &\longrightarrow \mathrm{Fred}(T\mathcal{N}^{1,p}(A), \mathcal{E}^p), \\ (s,t,u) &\longmapsto L_u^s + t\mathrm{Tor}_u^s \end{aligned}$$

where $L_u^s := (\nabla_{s\_})^{0,1}$, and $\mathrm{Tor}_u^s := \mathrm{Tor}^s(\_, du)^{0,1}$ for the torsion Tor^s of of ∇_s. Note that $\hat{f}(s,t,u)(g\xi) = (dg \otimes \xi)^{0,1} + g\hat{f}(s,t,u)(\xi)$ implies that $\operatorname{im} \hat{f} \subseteq \mathrm{Fred}(T\mathcal{N}^{1,p}(A), \mathcal{E}^p)$ by 4.A.(3). We can again consider the determinant bundle $\det \hat{f} \cong [0,1] \times [0,1] \times \coprod_u \det \hat{f}(s_0, 1, u)$, and by the same argument as above this bundle carries a canonical never vanishing section $\Omega \in \Gamma([0,1] \times [0,1] \times \coprod_u \det \hat{f}(s_0,1,u))$; $\Omega(s,t,u) = (s,t,\Omega_u^{s_0 1})$. For $s_0 = 0$, corresponding to ∇_0, this induces an orientation on $\mathcal{M}(A,J)$ through the volume form $\Omega^{01} \in \Omega^{\max}(\mathcal{M}(A,J))$. Let $(e_1, \ldots, e_{\max})$ be a positively oriented, orthonormal frame on $\mathcal{M}(A,J)$ such that $\Omega^{01}(e_1 \wedge \ldots \wedge e_{\max}) = 1$. Then $\Omega^{s1}(e_1 \wedge \ldots \wedge e_{\max}) \neq 0$ for all $s \in [0,1]$, for otherwise Ω would not be never vanishing. Now the set $S := \{s \in [0,1] : \Omega^{s1}(e_1 \wedge \ldots \wedge e_{\max}) > 0\}$ is given by a continuous inequality and thus it is open, but it is also closed since its complement $\{s \in [0,1] : \Omega^{s1}(e_1 \wedge \ldots \wedge e_{\max}) < 0\}$ too is open for the same reason, and it is non-empty since $0 \in S$. Hence $S = [0,1]$ and the two connections induce volume forms of the same sign.

Here is an even shorter argument why the thus obtained orientation on $\mathcal{M}(A,J)$ should be independent of the choices: Let ∇_0, ∇_1 be two Hermitian connections on (M,J) such that their respective torsions both equal $\frac{1}{8} N_J$. Then $\nabla := \nabla_1 - \nabla_0$ is a Hermitian connection, and computing D_u in terms of the associated covariant derivative ∇ yields $D_u = (\nabla_\_)^{0,1}$. Thus

$\ker D_u$ is invariant under J, whence inducing an orientation on $\mathcal{M}(A, J)$. By the construction in the first step this orientation is compatible with the ones induced by ∇_0 and ∇_1.

(iii) The following argument is due to Taubes and can also be found in [16]. Define the sets

$$\begin{aligned}\mathcal{J}^l_{\text{reg}}(\omega)_K &:= \{J \in \mathcal{J}^l(\omega) : D_u \text{ is onto} \, \forall u \in \mathcal{M}^l(J)_K\}, \text{ and}\\ \mathcal{M}^l(J)_K &:= \{u \in C^l(\Sigma, M) : \overline{\partial}_J u = 0, \| \, du \,\|_\infty \leq K,\\ &\qquad \exists z \in \Sigma \text{ such that } \inf_{w \neq z} \frac{d(u(w)), d(u(z))}{d(w, z)} \geq \frac{1}{K}\}.\end{aligned}$$

In particular, the latter space consists of curves that are somewhere injective. Spaces with no superscripts will consist of elements that are smooth, i.e. of class C^∞. Since $\mathcal{M}^l(J) = \bigcup_{K \in \mathbb{N}} \mathcal{M}^l(J)_K$ it follows that

$$\mathcal{J}^l_{\text{reg}}(\omega) = \bigcap_{K \in \mathbb{N}} \mathcal{J}^l_{\text{reg}}(\omega)_K$$

for all $l \in \mathbb{N}_\infty$. Consider again the projection $p_2 : \mathcal{M}^l(A) \to \mathcal{J}^l(\omega)$ from above. Since p_2 and $\mathcal{M}^l(A)$ are of class C^∞ we can use the Sard-Smale theorem 3.18 to obtain that the set $\{J \in \mathcal{J}^l(\omega) : J \text{ is regular value for } p_2\} = \mathcal{J}^l_{\text{reg}}(\omega) \subseteq \mathcal{J}^l(\omega)$ is of second category with respect to the WO^l-topology. By the argument below applied to $l < \infty$ it follows that $\mathcal{J}^l_{\text{reg}}(\omega)_K \subseteq \mathcal{J}^l(\omega)$ is open. Hence $\mathcal{J}^l_{\text{reg}}(\omega)_K \subseteq \mathcal{J}^l(\omega)$ is also dense, for otherwise $\mathcal{J}^l_{\text{reg}}(\omega)$ had to be meagre.

$\mathcal{J}_{\text{reg}}(\omega)_K \subseteq \mathcal{J}(\omega)$ *is open for all* $K \in \mathbb{N}$. Choose a sequence $(J_n)_n$ in $\mathcal{J}(\omega) \setminus \mathcal{J}_{\text{reg}}(\omega)_K$ converging to J in the WO^∞-topology. Then there are $u_n \in \mathcal{M}(J_n)_K$, and $z_n \in \Sigma$ such that the D_{u_n} are not surjective and furthermore

$$\overline{\partial}_{J_n} u_n = 0, \qquad \|du_n\|_\infty \leq K, \qquad \inf_{w \neq z_n} \frac{d(u_n(w)), d(u_n(z_n))}{d(w, z_n)} \geq \frac{1}{K}. \qquad (*)$$

By the compactness theorem 2.3 there is a diagonal subsequence $(u_{n_k}, z_{n_k}) \to (u, z)$ such that the triple (J, u, z) also satisfies $(*)$. Then $u \in \mathcal{M}(A, J)_K$ and D_u is not surjective since the set of surjective operators is open, and hence $J \notin \mathcal{J}_{\text{reg}}(\omega)_K$.

$\mathcal{J}_{\text{reg}}(\omega)_K \subseteq \mathcal{J}(\omega)$ *is dense for all* $K \in \mathbb{N}$. Elliptic regularity 1.6 shows that $\mathcal{J}_{\text{reg}}(\omega)_K = \mathcal{J}^l_{\text{reg}}(\omega)_K \cap \mathcal{J}(\omega)$. From above follows that

$$\mathcal{J}_{\text{reg}}(\omega)_K = \mathcal{J}^l_{\text{reg}}(\omega)_K \cap \mathcal{J}(\omega) \subseteq \mathcal{J}^l(\omega) \cap \mathcal{J}(\omega) = \mathcal{J}(\omega).$$

is WO^l-dense for all $l \in \mathbb{N}$. So $\mathcal{J}(\omega)_{\text{reg}} = \bigcap_{K \in \mathbb{N}} \mathcal{J}_{\text{reg}}(\omega)_K$ is the countable intersection of WO^∞-open and -dense sets, which was to be shown. $\square$

That the space $\mathcal{M}(A,J)$ is oriented can also be shown by another argument, see McDuff [13]; she actually establishes existence of a canonical stable almost complex structure on $\mathcal{M}(A,J)$. The first to formalize this in the spirit of Donaldson's approach to the orientation problem of moduli spaces, as in [4], seems to have been Ruan in [23], and the proof above follows his line of argument.
An arc $\gamma\colon [0,1] \to \mathcal{J}^l(\omega)$ that connects two almost complex structures J_0, J_1 is called *regular* if it is transverse to the canonical projection $p_2\colon \mathcal{M}^l(A) \to \mathcal{J}^l(\omega)$, and the corresponding space of regular arcs is $\mathcal{J}^l_{\text{reg}}(J_0,J_1) := \{\gamma : [0,1] \to \mathcal{J}^l(\omega) : \gamma(0) = J_0, \gamma(1) = J_1, \gamma \text{ is regular arc}\}$. The space of regular arcs with smooth values is $\mathcal{J}_{\text{reg}}(J_0,J_1) := \bigcap_{l=1}^{\infty} \mathcal{J}^l_{\text{reg}}(J_0,J_1)$.

Theorem 4.8. *Let $g = \operatorname{genus}\Sigma$, $2n = \dim M$, and $A \in H_2(M)$. Let $J_0, J_1 \in \mathcal{J}_{\text{reg}}(\omega, A)$.*

(i) *If $\gamma\colon [0,1] \to \mathcal{J}(\omega)$ is a regular arc connecting J_0 and J_1 then $\mathcal{M}_\gamma(A) := \{(t,u) \in I \times \mathcal{N}(A) : \overline{\partial}_{\gamma(t)}u = 0\}$ is a smooth manifold with boundary, and finite dimension $n(2-2g) + 2c_1(M)(A) + 1$.*

(ii) *If $\gamma\colon [0,1] \to \mathcal{J}(\omega)$ is a regular arc connecting J_0 and J_1 then $\mathcal{M}_\gamma(A)$ carries a natural orientation, and thus $\partial\mathcal{M}_\gamma(A) = \mathcal{M}(A,J_1) - \mathcal{M}(A,J_0)$; the minus sign denotes reversed orientation.*

(iii) *$\mathcal{J}_{\text{reg}}(J_0,J_1) \subseteq \mathcal{J}(J_0,J_1)$ is WO^∞-dense in the space of all arcs connecting J_0 and J_1.*

This theorem compares moduli spaces at different regular structures, and says that they are oriented bordant. Thus the next task will be to establish compactness properties of the moduli spaces in order to make sense of that bordism.

Proof. For simplicity we shall write $\mathcal{M}_\gamma(A) = \mathcal{M}_\gamma$ throughout the proof. Let $\mathbb{R}_{\geq 0} = \{t \in \mathbb{R} : t \geq 0\}$.
(i) Let $\gamma \in \mathcal{J}_{\text{reg}}(J_0,J_1)$ then $\gamma(0,1) \subseteq \mathcal{J}(\omega) \subseteq \mathcal{J}^l(\omega)$ is a submanifold, and we denote its left inverse by $\gamma^{-1} : \gamma(0,1) \to (0,1)$. By the Sard-Smale theorem 3.18 $p_2^{-1}(\gamma(0,1)) \subseteq \mathcal{M}^l(A)$ is a smooth submanifold. Now there is a diffeomorphism $\text{flip} \circ (\text{id} \times \gamma^{-1})\colon p_2^{-1}(\gamma(0,1)) \to \{(t,u) \in (0,1) \times \mathcal{N}^{1,p}(A) : \overline{\partial}_{\gamma(t)} = 0\} =: \mathcal{M}^\circ_\gamma \subseteq [0,1] \times \mathcal{N}^{1,p}(A)$ making its image a smooth submanifold too. Consider now the topological space $\mathcal{M}_\gamma \subseteq [0,1] \times \mathcal{N}^{1,p}(A)$. A chart centered at a point $(u,0)$ in the boundary of $\mathcal{M}_\gamma$ is given by

$$[0,\varepsilon) \times U \longrightarrow \mathcal{M}_\gamma,$$

$$(t,\xi) \longmapsto (t, \exp_u t(\xi + f(t,\xi)))$$

where $U \subseteq \ker D_u$ is an open zero neighborhood, cf. 4.2, where also the smooth map $f : \mathbb{R}_{\geq 0} \times \ker D_u \to (\ker D_u)^\perp \subseteq W^{1,p}(u^*TM = T_u\mathcal{N}^{1,p}(A)$ was constructed. The same works at points $(1,u)$. By the final property $U \oplus \operatorname{im} f \subseteq W^{1,p}(u^*TM)$ is open and by elliptic regularity $u \in C^\infty(\Sigma, M)$, and hence this chart is compatible with the differentiable structure on $[0,1] \times \mathcal{N}^{1,p}(A)$ almost by definition; cf. 3.11. Thus it is also compatible with the induced structure on $\mathcal{M}_\gamma^\circ$, and M_γ becomes a smooth manifold with interior $\mathcal{M}_\gamma^\circ$ and boundary $\partial\mathcal{M}_\gamma = (\{0\} \times \mathcal{M}(A,J_0)) \cup (\{1\} \times \mathcal{M}(A,J_1))$. Its dimension is $\dim \mathcal{M}_\gamma = n(2-2g) + 2c_1(M)(A) + 1$.

A slightly different line of argument yielding the differentiable structure on $\mathcal{M}_\gamma$ is the following. Because $\mathcal{J}_{\mathrm{reg}}(\omega) \subseteq \mathcal{J}(\omega)$ is of 2^{nd} category we can assume that $I_{\mathrm{reg}} := \gamma^{-1}(\mathcal{J}_{\mathrm{reg}}(\omega)) \subseteq [0,1]$ is open dense. Then $\mathcal{M}(A,J_t)$ is a smooth manifold for all $t \in I_{\mathrm{reg}}$, where $J_t = \gamma(t)$. Furthermore, $T_{(t,u)}\mathcal{M}_\gamma = \mathbb{R}_{\geq 0} \times \ker D_u^t$, i.e. $\mathcal{M}_\gamma$ can be smoothly modelled on spaces of the type $\mathbb{R}_{\geq 0} \times \ker D_u^t$ – where $t \in I_{\mathrm{reg}}$ and D_u^t denotes the linearisation of $\overline{\partial}_{J_t}$ at u. Thus we may identify

$$T\mathcal{M}_\gamma = \coprod_{(t,u) \in I_{\mathrm{reg}} \times \mathcal{M}(A,J_t)} \mathbb{R}_{\geq 0} \times \ker D_u^t.$$

(ii.) The orientation of $\mathcal{M}_\gamma$ follows from this latter description of $T\mathcal{M}_\gamma$ together with the proof of the previous theorem.

(iii.) Note that $\mathcal{J}_{\mathrm{reg}}(J_0,J_1) = \mathcal{J}(J_0,J_1) \cap \mathcal{J}^l_{\mathrm{reg}}(J_0,J_1)$: $(\subseteq)$ is clear. For $(\supseteq)$ let $\gamma \in \mathcal{J}(J_0,J_1) \cap \mathcal{J}^l_{\mathrm{reg}}(J_0,J_1)$. Choose an almost complex structure $J = \gamma(t) \in \operatorname{im}(\gamma) \cap \operatorname{im}(p_2^k \colon \mathcal{M}^k \to \mathcal{J}^k(\omega))$ such that $J \notin \mathcal{J}_{\mathrm{reg}}(\omega)$ then

$$\begin{aligned} T_J\mathcal{J}^k(\omega) &= \operatorname{im} T_{(u,J)}p_2^k \oplus \frac{T_J\mathcal{J}^k(\omega)}{\operatorname{im} T_{(u,J)}p_2^k} \cong \operatorname{im} T_{(u,J)}p_2^k \oplus \frac{\mathcal{E}^p_{(u,J)}}{\operatorname{im} D_u} \\ &\cong \operatorname{im} T_{(u,J)}p_2^k \oplus \frac{T_J\mathcal{J}^l(\omega)}{\operatorname{im} T_{(u,J)}p_2^l} \cong \operatorname{im} T_{(u,J)}p_2^k \oplus \operatorname{im} \gamma'(t) \end{aligned}$$

for all points $(u,J) \in (p_2^k)^{-1}(J)$ and all $k \in \mathbb{N}$; indeed, because $T_{(u,J)}F : T_{(u,J)}(\mathcal{N}^{1,p}(A) \times \mathcal{J}^l(\omega)) \to \mathcal{E}^p_{(u,J)}$ is surjective for all choices of $l \in \mathbb{N}$ this follows from the discussion at the beginning of this section. At points $J \in \mathcal{J}_{\mathrm{reg}}(\omega)$ or $J \notin \operatorname{im}(\gamma) \cap \operatorname{im}(p_2^k : \mathcal{M}^k \to \mathcal{J}^k(\omega))$ transversality is fulfilled trivially. Thus the Sard-Smale theorem implies that $\mathcal{J}_{\mathrm{reg}}(J_0,J_1) = \mathcal{J}(J_0,J_1) \cap \mathcal{J}^l_{\mathrm{reg}}(J_0,J_1) \subseteq \mathcal{J}(J_0,J_1) \cap \mathcal{J}(J_0,J_1) = \mathcal{J}(J_0,J_1)$ is WO^l-dense for all $l \in \mathbb{N}$. □

As sets one has that $\mathcal{M}_\gamma(A) = \bigcup_{t\in[0,1]}\{t\} \times \mathcal{M}(A, \gamma(t))$. Unfortunately, however, this is not a very rewarding observation since $\mathcal{M}(A, \gamma(t))$ need not be a manifold for all $t \in [0, 1]$. This is due to the fact that $\mathcal{J}_{\mathrm{reg}}(\omega) \subseteq \mathcal{J}(\omega)$ is of second category but not necessarily connected. Were it connected the first part of the above proof would follow immediately from lemma 4.2.

Chapter 5

Compactness of moduli spaces

Via the non-compact bordism $V \times [0,1)$ any manifold V is bordant to the empty manifold. Thus, in order to fill theorem 4.8 with life we need to establish useful compactness properties of moduli spaces.

Throughout this chapter (M, ω) will be a compact symplectic manifold with fixed dimension $\dim M = 2n$, and $\Sigma = \mathbb{C}P^1 = S^2 = \mathbb{C}_\infty$ will be the source of the pseudoholomorphic curves. As in section 1.B., $G = \mathrm{PSL}(2, \mathbb{C})$ will act on $\mathcal{M}(A, J)$ by reparametrization. It will be crucial to assume certain properties of the homology class A that the curves are required to represent, and these properties will be realized by requiring (M, ω) to be weakly monotone – as defined in 5.B. In Ruan [24] a useful compactification is constructed for general symplectic manifolds.

5.A. Framing

This section somewhat continues the spirit of the previous chapter.

The reduction of a cusp curve was established in 2.B.(2) and its intersection pattern in 2.B.(3).

Definition (Framed class). Let $A \in H_2(M)$, $p \in \mathbb{N}$, and $J \in \mathcal{J}(\omega)$. A set of data $D_p = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ is called a *framing* of A if the following are fulfilled.

(i) $A^1, \ldots, A^a$ are classes in $H_2(M)$ and are *effective*, that is they can be realized by a reduced tuple $(u^1, \ldots, u^a)$ of J-curves in the sense that $u^i_*[\mathbb{C}P^1] = A^i$ for all $i \in \{1, \ldots, a\}$, and $(u^1, \ldots, u^a)$ can be realized as the reduction of a cusp map representing A.

(ii) $j_2, \ldots, j_a$ describe the intersection pattern of the necessarily connected cusp map $(u^1, \ldots, u^a)$. This means that $1 \leq j_i < i$ and there exists

$(\alpha_2, \dots, \alpha_a, \beta_2, \dots, \beta_a) \in (\mathbb{C}P^1)^{2a-2}$ such that $u^{j_i}(\alpha_i) = u^i(\beta_i)$ for all $i \in \{2, \dots, a\}$.

(iii) $\sigma \colon \{1, \dots, p\} \to \{1, \dots, a\}$ is a map.

(iv) If $a = 1$ then we require that $A = \lambda A^1$ with $\lambda > 1$.

The set $\mathcal{D} = \mathcal{D}(p, A)$ is the set of all possible frames of A. If $p = 1$ we also write $D_1 = D$.

The label $\sigma : \{1, \dots, p\} \to \{1, \dots, a\}$ will be needed to define meaningful evaluation maps in section 5.C. Until then, however, the development will not depend on the index p and shall consequently be omitted for now.

1. Since reduction may change the homology class, $A \neq A^1 + \ldots + A^a$, in general. However, there still will be numbers $\lambda_i \in \mathbb{N}$ such that $A = \sum_{i=1}^{a} \lambda_i A^i$. □

2. *The set of all framings* $\mathcal{D} = \{D_p$ frames $A\}$ *is finite.* Without loss we can assume $p = 1$. Necessarily one has the inequality $\omega(A^i) \leq \omega(A)$. But now 2.6 says that $\{A^i \in H_2(M)$ has a J-holomorphic representative with $\omega(A^i) \leq \omega(A)\}$ is finite. □

$\boldsymbol{\mathcal{M}^l(D)}$**.** Let $A \in H_2(M)$ and $D = \{A^1, \dots, A^a, j_2, \dots, j_a, \sigma\}$ a framing of A. Then the *universal moduli space* of simple cusp curves of *type* D is

$$\begin{aligned} \mathcal{M}^l(D) := \{(u, J, \alpha, \beta) \in \mathcal{M}^l(A^1, \dots, A^a) \times (\mathbb{C}P^1)^{2a-2} :& \\ u^{j_k}(\alpha_k) = u^k(\beta_k) \text{ for all } k \in \{2, \dots, a\}\}.& \end{aligned}$$

The space $\mathcal{M}^l(A^1, \dots, A^a)$ was introduced in section 4.C. □

Proposition 5.1. *For all framings D of $A \in H_2(M)$ the evaluation map*

$$\begin{aligned} \mathrm{ev} \colon \mathcal{M}^l(A^1, \dots, A^a) \times (\mathbb{C}P^1)^{2a-2} &\longrightarrow M^{2a-2}, \\ (u^1, \dots, u^a, J, \alpha_2, \dots, \alpha_a, \beta_2, \dots, \beta_a) &\longmapsto (u^{j_k}(\alpha_k), u^k(\beta_k))_{k=2}^a \end{aligned}$$

is transversal to the multi diagonal $\Delta = \{(x_k, x_k) : x_k \in M, k \in \{2, \dots, a\}\} \subseteq M^{2a-2}$*. Hence* $\mathrm{ev}^{-1}(\Delta) = \mathcal{M}^l(D)$ *is a smooth Banach manifold.*

Proof. Fix a tuple $(\alpha_k, \beta_k)_{k=2}^a \in (\mathbb{C}P^1)^{2a-2}$. We claim even more, namely that

$$\mathrm{ev} \colon \mathcal{M}^l(A^1, \dots, A^a) \longmapsto M^{2a-2}, (u_k, J)_{k=1}^a \longmapsto (u^{j_k}(\alpha_k), u^k(\beta_k))_{k=2}^a$$

is transverse to Δ. Take an element $(u_k, J)_{k=1}^a$ in the inverse image with $(u^{j_k}(\alpha_k), u^k(\beta_k))_{k=2}^a = (x_k, x_k)_{k=2}^a \in (M \times M)^{a-1}$. The general case being analogous, we will restrict attention to the case $a = 3$ and $j_3 = 1$. The tangent mapping to ev then appears as

$$
\begin{aligned}
T_{(u^1,u^2,u^3,J)}\mathcal{M}^l(A^1, A^2, A^3) &\longrightarrow M^4, \\
(\xi^1, \xi^2, \xi^3, \eta) &\longmapsto (\xi^1(\alpha_2), \xi^2(\beta_2), \xi^1(\alpha_3), \xi^3(\beta_3)).
\end{aligned}
$$

Now let $(X_2, Y_2, X_3, Y_3) \in (T_{x_2}M)^2 \times (T_{x_3}M)^2$ arbitrary, and suppose that $\alpha_2 = \alpha_3$. Proposition 4.5 implies that there are $(\xi^i, \eta^i) \in T_{(u^i,J)}\mathcal{M}^l(A^i)$ for $i \in \{1, 2, 3\}$ with

(i) $\xi^1(\alpha_2) = X_2, \xi^2(\beta_2) = Y_2, \xi^3(\beta_3) = Y_3 - X_3 + X_2$. Thus $(X_2, Y_2, X_3, Y_3) = (0, 0, X_3 - X_2, X_3 - X_2) + \mathrm{ev}(\xi^1, \xi^2, \xi^3, \eta^1 + \eta^2 + \eta^3)$.

(ii) $\mathrm{supp}\eta^i$ is contained in a small annulus centered at $u^i(\beta_i)$, and does not intersect any $\mathrm{im}\, u^j$ for $i \neq j$.

(iii) Hence $T_{(u^i,J)}F \cdot (\xi^i, \eta^1 + \eta^2 + \eta^3) = T_{(u^i,J)}F \cdot (\xi^i, \eta^i)$, that is $(\xi^1, \xi^2, \xi^3, \eta^1 + \eta^2 + \eta^3) \in T_{(u^1,u^2,u^3,J)}\mathcal{M}^l(A^1, A^2, A^3)$.

Point (ii) follows by combining 4.5 and 1.5. If $\alpha_2 \neq \alpha_3$ the result is even easier for then we may find ζ^1 with $\zeta^1(\alpha_3) = X_3$, and $\mathrm{supp}\xi^1 \cap \mathrm{supp}\zeta^1 = \emptyset$, and work with the sum $\xi^1 + \zeta^1$. □

$\mathcal{M}(D, J)$. For $J \in \mathcal{J}(\omega)$ the *moduli space* of reduced cusp curves that are of *type* $D = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ is the space

$$
\begin{aligned}
\mathcal{M}(D, J) := \{&(u, \alpha, \beta) \in (C^\infty(\Sigma, M))^a \times (\mathbb{C}P^1)^{2a-2} : \\
&u^k \in \mathcal{M}(A^k, J), u^{j_k}(\alpha_k) = u^k(\beta_k), (u^1, \ldots, u^a) \text{ is reduced}\}
\end{aligned}
$$

The last condition makes sure that there is no $\phi \in G$ with $u^i \circ \phi = u^j$ for $i \neq j$. Since we do not know whether $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A^k)$ the space $\mathcal{M}(A^k, J)$ is merely considered as a set. The following proposition asserts that this space is a manifold for generic J. It may be empty, which, however, could be considered good news, for then $\mathcal{M}(A, J)/G$ will enjoy better compactness properties. □

An ω-tame structure $J \in \mathcal{J}^l(\omega)$ is called *(D-)regular* if $\mathrm{pr}\colon T_{(u,J)}\mathcal{M}^l(D) \to T_J\mathcal{J}^l(\omega)$, $(\xi, \eta) \mapsto \eta$ is surjective for all $(u, J) \in (\mathrm{pr}\colon \mathcal{M}^l(D) \to \mathcal{J}^l(\omega))^{-1}(J)$. The set of all regular structures is denoted by $\mathcal{J}^l_{\mathrm{reg}}(\omega, D)$, and the set of smooth regular structures is $\mathcal{J}_{\mathrm{reg}}(\omega, D) = \bigcap_{l \in \mathbb{N}} \mathcal{J}^l_{\mathrm{reg}}(\omega, D)$. A $J \in \mathcal{J}(\omega)$ is said to be *generic* if $\mathcal{M}(D, J)$ is a smooth manifold. The following proposition asserts that the words generic and regular can be used synonymously.

Theorem 5.2. *Let $2n = \dim M$, and $D = (A^1, \dots, A^a, j_2, \dots, j_a, \sigma)$ be the framing data of a class $A \in H_2(M)$. With the notation from above the following hold.*

(i) *Let $J \in \mathcal{J}_{\text{reg}}(\omega, D)$ then $\mathcal{M}(D, J)$ is a smooth manifold of finite dimension*

$$\dim \mathcal{M}(D, J) = 2n + 2\sum_{i=1}^{a} c_1(M)(A^i) + 4(a-1).$$

Moreover, it carries an almost complex structure.

(ii) *The set $\mathcal{J}_{\text{reg}}(\omega, D) \subseteq \mathcal{J}(\omega)$ is of second category. Furthermore, it is true that $\mathcal{J}_{\text{reg}}(\omega, D) \subseteq \mathcal{J}_{\text{reg}}(\omega, A^i)$.*

Proof. For shortness sake the notation $u = (u^1, \dots, u^a)$, $(\alpha, \beta) = (\alpha_i, \beta_i)_{i=2}^a$, and the like shall be used throughout this proof.
(i.) Assume $J \in \mathcal{J}_{\text{reg}}(\omega, D)$, and consider the projection onto the i-th factor $\text{pr}_i : T_{(u,J,\alpha,\beta)}(\mathcal{M}^l(A^1, \dots, A^a) \times (\mathbb{C}P^1)^{2a-2}) \to T_{(u^i,J)}\mathcal{M}^l(A^i)$, $(\xi, \eta, v, w) \mapsto (\xi^i, \eta)$ then the right hand side vertical line in

$$\begin{array}{ccc} T_{(u,J,\alpha,\beta)}(\mathcal{M}^l(A^1, \dots, A^a) \times (\mathbb{C}P^1)^{2a-2}) & \xleftarrow{\supseteq} & T_{(u,J)}\mathcal{M}^l(D) \\ \text{pr}_i \big\downarrow & & \big\downarrow \text{pr} \\ T_{(u^i,J)}\mathcal{M}^l(A^i) & \xrightarrow{\text{p}_2} & T_J\mathcal{J}^l(\omega) \end{array}$$

is surjective by assumptiuon. Hence also the bottom line must be surjective. By the beginning of section 4.D. this map is Fredholm and its being surjective is equivalent to $J \in \mathcal{J}_{\text{reg}}(\omega, A^i)$, thus proving the second part of (ii). In particular, $\mathcal{M}(A^i, J)$ is a finite dimensional manifold, implying

$$\dim \ker \text{pr}_i = \dim \{(\xi, 0, v, w) : \xi^i = 0\} \le 2na + 2\sum_{i=1}^{a} c_1(M)(A^i) + 4(a-1),$$

$$\dim \operatorname{coker} \text{pr}_i \le 2n(a-1);$$

Indeed, $J \in \mathcal{J}_{\text{reg}}(\omega, A^i)$ and hence $D_{u^i} : T_{u^i}\mathcal{N}^{1,p}(A^i) \to \mathcal{E}^p_{(u^i,J)}$ must be surjective. As in the proof of lemma 4.2 we can decompose $\xi^i = \xi^i_1 \oplus \frac{1}{2}(D_{u^i}|_{V_i})^{-1}(-\eta(u^i) \cdot Tu^i \cdot j)$ and the dimensions are governed only by the equations stemming from the evaluation map. Thus $\text{p}_2 \circ \text{pr}_1$ is Fredholm, and the fiber $(\text{p}_2 \circ \text{pr}_1)^{-1}(J) =: \mathcal{M}(A^1, \dots, A^a, J)$ will be a smooth manifold with $T_{(u,\alpha,\beta)}\mathcal{M}(A^1, \dots, A^a, J) = T_{u^1}\mathcal{M}(A^1, J) \times \dots \times T_{u^a}\mathcal{M}(A^a, J) \times \mathbb{R}^{4a-4}$. This space has finite dimension $\dim \mathcal{M}(A^1, \ldots, A^a, J) = 2na + 2\sum_{i=1}^a c_1(M)(A^i) + 4(a-1)$. There is an evaluation map

$$\mathrm{ev}\colon \mathcal{M}(A^1, \ldots, A^a, J) \longrightarrow M^{2a-2}, \qquad (u, \alpha, \beta) \longmapsto (u^{j_i}(\alpha_i), u^i(\beta_i))_{i=2}^a.$$

The goal is to show that $\mathcal{M}(D, J) = \mathrm{ev}^{-1}(\Delta)$ is a manifold; Δ again is the multi diagonal. Let $x_i := u^i(\beta_i)$ and $x = (x_i)_{i=2}^a$. By proposition 5.1 above the modified evaluation map

$$\begin{aligned} \overline{\mathrm{ev}}\colon T_{(u,J,\alpha,\beta)}(\mathcal{M}^l(A^1, \ldots, A^a) \times (\mathbb{C}P^1)^{2a-2}) &\longrightarrow T_x M^{a-1}, \\ (\xi, \eta, v, w) &\longmapsto (\xi^{j_i}(\alpha_i) - \xi^i(\beta_i))_{i=2}^a \end{aligned}$$

is surjective whenever $(u, \alpha, \beta) \in \mathcal{M}(D, J)$. Let $X = \overline{\mathrm{ev}}(\xi, \eta, v, w) \in T_x M^{a-1}$ arbitrary. Since J was assumed a regular structure there is (ξ', v', w') such that $(\xi', \eta, v', w') \in T_{(u,J,\alpha,\beta)}\mathcal{M}^l(D)$, that is $\overline{\mathrm{ev}}(\xi', \eta, v', w') = 0$. Then

$$X = \overline{\mathrm{ev}}(\xi, \eta, v, w) = \overline{\mathrm{ev}}(\xi, \eta, v, w) - \overline{\mathrm{ev}}(\xi', \eta, v', w') = \overline{\mathrm{ev}}(\xi - \xi', 0, v - v', w - w'),$$

and hence the restricted evaluation map $\overline{\mathrm{ev}}_{(u,\alpha,\beta)}\colon T_{(u,\alpha,\beta)}\mathcal{M}(A^1, \ldots, A^a, J) \to T_x M^{a-1}$ is surjective too. This implies that $\mathrm{ev}^{-1}(\Delta) = \mathcal{M}(D, J)$ is a smooth manifold with $T_{(u,\alpha,\beta)}\mathcal{M}(D, J) = \ker \overline{\mathrm{ev}}_{(u,\alpha,\beta)}$. The dimension computes as follows.

$$\begin{aligned} \dim T_{(u,\alpha,\beta)}\mathcal{M}(D, J) &= \dim \{(\xi, v, w) : \xi^{j_i}(\alpha_i) = \xi^i(\beta_i)\} \\ &= 2na + 2\sum_{i=1}^a c_1(M)(A^i) + 4(a-1) - 2n(a-1) \\ &= 2n + 2\sum_{i=1}^a c_1(M)(A^i) + 4(a-1). \end{aligned}$$

$\mathcal{M}(D, J)$ carries an almost complex structure. Let i denote the product complex structure on $(\mathbb{C})^{2a-2}$. Each tangent space $T_{(u,\alpha,\beta)}\mathcal{M}(D, J) = \ker \overline{\mathrm{ev}}_{(u,\alpha,\beta)}$ is a complex vector space with respect to $J(u^1) \times \ldots \times J(u^a) \times i$, and varying this argument smoothly yields the claim.

(ii.) □

$\mathcal{M}_\gamma(D)$. Let $J_0, J_1 \in \mathcal{J}_{\mathrm{reg}}(\omega, D)$ and $D = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ the framing of a class $A \in H_2(M)$. A smooth arc $\gamma\colon [0,1] \to \mathcal{J}^l(\omega)$ with $\gamma(0) = J_0$ and $\gamma(1) = J_1$ is called *regular* or *generic* if $\operatorname{im}\gamma \subseteq \mathcal{J}(\omega)$ and γ is transversal to $\mathrm{pr}_2\colon \mathcal{M}^l(D) \to \mathcal{J}^l(\omega)$, $(u, J, \alpha, \beta) \mapsto J$ for some $l \in \mathbb{N}$. As in the proof of 4.8 it then follows that γ is transversal to $\mathrm{pr}_2\colon \mathcal{M}^l(D) \to \mathcal{J}^l(\omega)$ for all $l \in \mathbb{N}$. The space

$$\mathcal{M}_\gamma(D) := \{(t,u,\alpha,\beta) \in [0,1] \times C^\infty(\Sigma, M) \times (\mathbb{C}P^1)^{2a-2} : \\ (u,\alpha,\beta) \in \mathcal{M}(\gamma(t), D)\}$$

will serve to compare moduli spaces of framed classes at different generic structures. $\mathcal{M}(\gamma(t), D)$ is regarded as a set only. □

Theorem 5.3. *Let $J_0, J_1 \in \mathcal{J}_{\mathrm{reg}}(\omega, D)$, and $D = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ the framing of a class $A \in H_2(M)$, and* $\dim M = 2n$.

(i) *If $\gamma\colon [0,1] \to \mathcal{J}^l(\omega)$ is generic such that $\gamma(0) = J_0$ and $\gamma(1) = J_1$ then $\mathcal{M}_\gamma(D)$ is a smooth manifold with boundary of dimension*

$$\dim \mathcal{M}_\gamma(D) = 2n + \sum_{i=1}^{a} c_1(M)(A^i) + 4a - 3.$$

Moreover, $\mathcal{M}_\gamma(D)$ carries a natural orientation, and thus $\partial\mathcal{M}_\gamma(D) = \mathcal{M}(D, J_1) - \mathcal{M}(D, J_0)$.

(ii) *The set of regular arcs is WO^∞-dense in the set of all arcs connecting J_0 and J_1.*

Proof. *(i.)* Let J_0, J_1 regular, and choose a regular arc $\gamma\colon [0,1] \to \mathcal{J}^l(\omega)$ connecting these. Since J_0 is assumed regular $\mathrm{pr}_2 : \mathcal{M}^l(D) \to \mathcal{J}^l(\omega)$ is a submersion at J_0, and hence in a whole neighborhood of J_0. By the implicit function theorem there is $\varepsilon > 0$ such that γ can be lifted to a curve $\tilde{\gamma}\colon [0,\varepsilon) \to \mathcal{M}^l(D)$ starting at some chosen point $(u, J_0, \alpha, \beta) \in \mathrm{pr}_2^{-1}(J_0)$. The same can be done at J_1 or at any other regular structure $J_t = \gamma(t) \in \mathcal{J}_{\mathrm{reg}}(\omega, D)$. As in the proof of theorem 4.8 the space $\mathcal{M}_\gamma(D)$ can therefore be modelled on

$$\mathbb{R}_{\geq 0} \times T_{(u,\alpha,\beta)}\mathcal{M}(D, J_t)$$

where $J_t = \gamma(t) \in \mathcal{J}_{\mathrm{reg}}(\omega, D)$. The dimension computes to $\dim \mathcal{M}_\gamma(D) = \dim \mathcal{M}(D, J_0) + 1 = 2n + \sum_{i=1}^{a} c_1(M)(A^i) + 4a - 3$, as claimed. This also proves the assertion on the orientation as established in the previous theorem. Statement *(ii)* again follows from the Sard-Smale theorem. □

5.B. Weak monotonicity

A homology class $A \in H_2(M)$ is said to be *spherical* if it lies in the image of the Hurewicz homomorphism $h_2\colon \pi_2(M) \to H_2(M)$, $[u] \mapsto u_*[S^2]$.

Definition. (M, ω) is called *weakly monotone* if the inequality $c_1(M)(A) \geq 0$ holds for all spherical homology classes $A \in H_2(M)$ with $\omega(A) > 0$ and $c_1(M)(A) \geq 3 - n$. It is called *monotone* if there is a number $\lambda > 0$ such that $\omega(A) = \lambda c_1(M)(A)$.

Definition. For $K > 0$ an ω-tame almost complex structure J on (M, ω) is called K-*semi positive* if $c_1(u_*[\mathbb{C}P^1]) \geq 0$ holds for all simple J-curves $u \colon \mathbb{C}P^1 \to M$ with $E(u) = \omega(u_*[\mathbb{C}P^1]) \leq K$. The set of all K-semi positive structures on (M, ω) is denoted by $\mathcal{J}_K(\omega)$. A structure J is *semi positive* if it is so for all $K \in \mathbb{N}$, and the corresponding set is $\mathcal{J}_+(\omega) = \bigcap_{K \in \mathbb{N}} \mathcal{J}_K(\omega)$. $J \in \mathcal{J}(\omega)$ is called A-semi positive if $c_1(M)(A) \geq 0$ holds for all J-holomorphic $u \in \mathcal{N}(A)$ such that $E(u) \leq K$ for all $K \in \mathbb{N}$. Accordingly there is a set $\mathcal{J}_+(\omega, A)$, and also $\mathcal{J}_+(\omega) = \bigcap_{A \in H_2(M)} \mathcal{J}_+(\omega, A)$.

The compactification of the moduli space $\mathcal{M}(A, J)/G$ of unparametrized curves strongly depends upon the chosen class A. For example, theorem 2.4 shows that this space already is compact if A is indecomposable. In general, by Gromov's theorem 2.5 one expects that a compactification should be obtained by adding sufficiently many cusp curves. However, in order to ensure not having to add 'too much' one has to make certain positivity assumptions on A.
The notion of a weakly monotone manifold was first introduced by McDuff in [15], wherein these were called semi positive manifolds. The permutation of names came about in [16], and now it is the structures that are said to be semi positive, and the condition of weak monotonicity makes sure that enough of these exist, see below.

1. All symplectic manifolds of dimension $\dim M = 2n \leq 6$ or verifying $\pi_2(M) = \{0\}$ are weakly monotone. □

2. *If* (M, ω) *is weakly monotone then* $\mathcal{J}_{\mathrm{reg}}(\omega) := \bigcap_{A \in H_2(M)} \mathcal{J}_{\mathrm{reg}}(\omega, A) \subseteq \mathcal{J}_+(\omega)$. Let $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A)$ with $c_1(M)(A) < 0$. Since $\omega(A) = E(u) > 0$ for all $u \in \mathcal{M}(A, J)$ weak monotonicity implies that $c_1(M)(A) < 3 - n$. By 4.7 it follows that

$$\dim \frac{\mathcal{M}(A, J)}{G} = 2n + 2c_1(M)(A) - 6 < 2n + 6 - 2n + 6 = 0.$$

Thus $\mathcal{M}(A, J) = \emptyset$ and every A-regular structure $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A)$ has to be A-semi positive. Also note that $H_2(M)$ is a discrete lattice, thus $\mathcal{J}_{\mathrm{reg}}(\omega) \subseteq \mathcal{J}(\omega)$ is again a set of second category being the countable intersection of sets of second category, cf. theorem 4.7. □

3. *Let* (M,ω) *be weakly monotone,* $A \in H_2(M)$ *and suppose that* $D = (A^1,\dots,A^a,j_2,\dots,j_a,\sigma)$ *is a framing of* A *which is the type of a reduced* J*-cusp curve such that* $J \in \mathcal{J}_{\mathrm{reg}}(\omega, D)$*. Then*

$$\sum_{i=1}^{a} c_1(M)(A^i) \le c_1(M)(A).$$

Indeed, by proposition 5.2 it is true that $\mathcal{J}_{\mathrm{reg}}(\omega, D) \subseteq \mathcal{J}_{\mathrm{reg}}(\omega, A^i) \subseteq \mathcal{J}_+(\omega, A^i)$. Hence $c_1(M)(A^i) \ge 0$ by the above. $\square$

In general, $\mathcal{J}_+(\omega) = \bigcap_{K\in\mathbb{N}} \mathcal{J}_K(\omega)$ need neither be nonempty nor open.

Lemma 5.4. *Let* M *be a compact (even dimensional) manifold, and denote by* $\Omega^2_{\mathrm{sp}}(M)$ *the set of all symplectic forms on* M*. Let* $K \in \mathbb{N}$ *be arbitrarily fixed then the subset*

$$\{(\omega, J) \in \Omega^2_{\mathrm{sp}}(M) \times \mathcal{J} : J \in \mathcal{J}_K(\omega)\} \subseteq \{(\omega, J) \in \Omega^2_{\mathrm{sp}}(M) \times \mathcal{J} : J \in \mathcal{J}(\omega)\}$$

is open with respect to the WO^1*-topology. In particular,* $\mathcal{J}_K(\omega) \subseteq \mathcal{J}(\omega)$ *is open for the* WO^1*-topology for all* $\omega \in \Omega^2_{\mathrm{sp}}(M)$*.*

Proof. We show that the complement is closed. Let $(\omega_n)_n$ be a sequence of symplectic structures converging to $\omega \in \Omega^2_{\mathrm{sp}}(M)$ in the WO^1-topology. Let $(J_n)_n$ be a sequence of ω_n-tame but not (ω_n, K)-semi positive almost complex structures converging to $J \in \mathcal{J}(\omega)$ in WO^1. Then there is a sequence of simple curves $(u_n)_n$ with $\overline{\partial}_{J_n} u_n = 0$, $E(u_n) \le K$, and $c_1((u_n)_*[\mathbb{C}P^1]) < 0$. By theorem 2.5 there is a subsequence $(u_{n_k})_k$ converging in the weak C^1-topology to a cusp curve with parametrizing components $(u^1,\dots,u^a)$. These components have the property that

$$\overline{\partial}_J u^i = 0, \qquad E(u^i) \le K, \qquad \sum_{i=1}^{a} c_1(u^i_*[\mathbb{C}P^1]) = c_1((u_n)_*[\mathbb{C}P^1]) < 0$$

for large $n \in \mathbb{N}$. Thus at least one u^i must have $c_1(u^i_*[\mathbb{C}P^1]) < 0$ and J cannot be (ω, K)-semi positive. The second statement is true since projections are open mappings. $\square$

Lemma 5.5. *Let* (M,ω) *be a compact, weakly monotone symplectic manifold, and* $A \in H_2(M)$*. Then* $\mathcal{J}_+(\omega, A)$ *contains a path-connected subset, which is of second category in* $\mathcal{J}(\omega)$*.*

Proof. By theorem 4.7 the second statement is clear since $\mathcal{J}_{\mathrm{reg}}(\omega, A) \subseteq \mathcal{J}_+(\omega, A)$. For the first part choose $J_i \in \mathcal{J}_{\mathrm{reg}}(\omega, A) \subseteq \mathcal{J}_+(\omega, A)$ for $i \in \{0, 1\}$. Then theorem 4.8 implies existence of a path $\gamma\colon [0,1] \to \mathcal{J}(\omega)$ connecting J_0 and J_1 such that $\mathcal{M}_\gamma(A) = \{(t,u) : u \in \mathcal{M}(A, \gamma(t))\}$ is a manifold of dimension $\dim \mathcal{M}_\gamma(A) = 2n + 2c_1(M)(A) + 1$. Assume that $c_1(M)(A) < 0$. Then weak monotonicity implies $c_1(M)(A) \leq 2 - n$, since $\omega(A) = E(u) > 0$. Now

$$\dim \mathcal{M}_\gamma(A)/G = 2n + 2c_1(M)(A) + 1 - 6 \leq 2n + 4 - 2n - 5 = -1,$$

and $\mathcal{M}(A, \gamma(t))$ has to be empty. Thus $\gamma(t) \in \mathcal{J}_+(\omega, A)$ for all $t \in [0,1]$. $\square$

5.C. Structure theorem

Subsequently various actions by the 6-dimensional reparametrization group G will be introduced. These will all be free, and in each case this will follow from section 1.B. Furthermore, $A \in H_2(M)$ shall denote a fixed class, $D_p = (A^1, \dots, A^a, j_2, \dots, j_a, \sigma)$ its framing data, and J a generic almost complex structure, i.e. $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A) \cap \mathcal{J}_{\mathrm{reg}}(\omega, D_p)$.

$\boldsymbol{\mathcal{W}(p, A, J)}$**.** As stated the action $G \times \mathcal{M}(A, J) \times (\mathbb{C}P^1)^p \to \mathcal{M}(A, J) \times (\mathbb{C}P^1)^p$, $(\phi, u, z^1, \dots, z^p) \mapsto (u \circ \phi^{-1}, \phi(z^1), \dots, \phi(z^p))$ is free, and thus the following are well defined.

$$\begin{aligned} \mathcal{C}(A, J) &:= \mathcal{M}(A, J)/G, \\ \mathrm{ev}\colon \mathcal{W}(p, A, J) &:= \mathcal{M}(A, J) \times_G (\mathbb{C}P^1)^p \longrightarrow M^p, \\ &\qquad\qquad [u, z^1, \dots, z^p] \longmapsto (u(z^1), \dots, u(z^p)). \end{aligned}$$

The dimensions of the orbit spaces are $\dim \mathcal{C}(A, J) = 2n + 2c_1(M)(A) - 6$ and $\dim \mathcal{W}((, J)p, A) = 2n + 2c_1(M)(A) + 2p - 6$. $\square$

$\boldsymbol{\mathcal{C}(D_p, J)}$**.** Analogously, there is a reparametrization action

$$\begin{aligned} G^a \times \mathcal{M}(D_p, J) &\longrightarrow \mathcal{M}(D_p, J), \\ \big((\phi_i, u^i)_{i=1}^a, (\alpha_i, \beta_i)_{i=2}^a\big) &\longmapsto \big((u^i \circ \phi_i^{-1})_{i=1}^a, (\phi_{j_i}(\alpha_i), \phi_i(\beta_i))_{i=2}^a\big). \end{aligned}$$

$\mathcal{M}(D_p, J)$ was just defined so as to ensure that this is a free action. The orbit space is

$$\mathcal{C}(D_p, J) := \mathcal{M}(D_p, J)/G^a, \qquad \dim \mathcal{C}(D_p, J) = 2n + \sum_{i=1}^a c_1(M)(A^i) - 2a - 4.$$

If we assume (M, ω) to be weakly monotone then $a > 1$ implies the inequality $\dim \mathcal{C}(D_p, J) = 2n + \sum_{i=1}^a c_1(M)(A^i) - 2a - 4 \leq \dim \mathcal{C}(A, J) - 2$. This follows from 5.A.(3). $\square$

***$\mathcal{W}(D_p, J)$*.** Let $\sigma\colon \{1.\dots,p\} \to \{1,\dots,a\}$ be the label specifying on which of the components of the cusp curve the evaluation map should act. Consider the free action

$$\begin{aligned}G^a \times \mathcal{M}(D_p,J) \times (\mathbb{C}P^1)^p &\longrightarrow \mathcal{M}(D_p,J) \times (\mathbb{C}P^1)^p,\\ \big((\phi_i,u^i)_{i=1}^a,(\alpha_i,\beta_i)_{i=2}^a,(z^i)_{i=1}^p\big) &\longmapsto \big((u^i\circ\phi_i^{-1})_{i=1}^a,(\phi_{j_i}(\alpha_i),\phi_i(\beta_i))_{i=2}^a,\\ &\qquad\qquad (\phi_{\sigma(i)}(z^i))_{i=1}^p\big).\end{aligned}$$

The evaluation map is

$$\begin{aligned}\mathrm{ev}_\sigma\colon \mathcal{W}(D_p,J) := \mathcal{M}(D_p,J)\times_{G^a}(\mathbb{C}P^1)^p &\longrightarrow M^p,\\ [u,\alpha,\beta,z] &\longmapsto (u^{\sigma(i)}(z^i))_{i=1}^p\end{aligned}$$

where $[u,\alpha,\beta,z] = [(u^i)_{i=1}^a,(\alpha_i,\beta_i)_{i=2}^a,(z^i)_{i=1}^p]$. The thus defined space is a smooth manifold. If (M,ω) is weakly monotone and $a>1$ its dimension is

$$\begin{aligned}\dim\mathcal{W}(D_p,J) &= \dim\mathcal{M}(D_p,J) + 2p - 6a\\ &= 2n + 2\sum_{i=1}^a c_1(M)(A^i) + 4(a-1) + 2p - 6a\\ &= 2n + 2\sum_{i=1}^a c_1(M)(A^i) + 2p - 2a - 4\\ &\le 2n + 2c_1(M)(A) + 2p - 2a - 4,\end{aligned}$$

by 5.A.(3). (It is only the inequality that needs the above assumptions.) Thus $\dim\mathcal{W}(D_p,J)\le\dim\mathcal{W}(p,A,J)-2$, and the following theorem shows, in particular, that this continues to hold under more general circumstances. □

Theorem 5.6 (Structure theorem). *Let (M,ω) be a compact symplectic manifold, $A\in H_2(M)$, and $p\in\mathbb{N}$. Then the following hold true.*

(i) *If $J\in\mathcal{J}_{\mathrm{reg}}(\omega,D_p)$ and $D_p=(A^1,\dots,A^a,j_2,\dots,j_a,\sigma)$ is a frame of A then $\mathcal{W}(D_p,J)$ is a smooth manifold of dimension*

$$\dim\mathcal{W}(D_p,J) = 2n + 2\sum_{i=1}^a c_1(M)(A^i) + 2p - 2a - 4.$$

Moreover, $\mathcal{W}(D_p,J)$ carries a natural orientation.

(ii) *Assume that* $A \in H_2(M)$ *is not a multiple class* $A = \lambda B$ *with* $\lambda > 1$ *and* $c_1(M)(B) = 0$, *and let* $D_p = (A^1, \dots, A^a, j_2, \dots, j_a, \sigma)$ *be a framing of A. Then*

$$\dim \mathcal{W}(D_p, J) \leq \dim \mathcal{W}(p, A, J) - 2\max\{1, a-1\}$$

whenever $J \in \mathcal{J}_{\text{reg}}(\omega, A) \cap \mathcal{J}_{\text{reg}}(\omega, D_p) \cap \mathcal{J}_K(\omega)$ *with* $K \geq \omega(A)$.

(iii) *If* $J \in \mathcal{J}_{\text{reg}}(\omega, A) \cap \mathcal{J}_{\text{reg}}(\omega, D_p)$ *then the set* $\mathcal{D}$ *of all frames* D_p *of* A *is finite, and*

$$O_{\text{ev}} := \bigcap_{\substack{K \subseteq \mathcal{W}(p,A,J) \\ \text{cp}}} \overline{\text{ev}(\mathcal{W}(p, D, J) \setminus K)} \subseteq \bigcup_{D_p \in \mathcal{D}} \text{ev}_\sigma(\mathcal{W}(D_p, J)).$$

(iv) $\mathcal{J}_{\text{reg}}(\omega, A) \cap \mathcal{J}_{\text{reg}}(\omega, D_p) \subseteq \mathcal{J}(\omega)$ *is of second category, and* $\mathcal{J}_K(\omega) \subseteq \mathcal{J}(\omega)$ *is open. If* (M, ω) *is furthermore assumed weakly monotone then* $\mathcal{J}_{\text{reg}}(\omega, A) \cap \mathcal{J}_{\text{reg}}(\omega, D_p) \cap \mathcal{J}_K(\omega, A)$ *is of second category in* $\mathcal{J}(\omega)$, *and any two elements in this set can be connected by a generic path (i.e. regular in the sense of 4.8 and 5.3); moreover, every generic path is completely contained in* $\mathcal{J}_K(\omega, A)$.

This theorem says that cusp curves are rather singular objects, like the cusp of a cone.

Proof. *(i.)* Since $J \in \mathcal{J}_{\text{reg}}(\omega, D_p)$ the space $\mathcal{W}(D_p, J)$ is a smooth manifold as the quotient under a free action, and the dimension formula follows from the discussion above. Since the action $G^a \times \mathcal{M}(D_p, J) \times (\mathbb{C}P^1)^p \to \mathcal{M}(D_p, J) \times (\mathbb{C}P^1)^p$ is free we may view

$$\mathcal{M}(D_p, J) \times (\mathbb{C}P^1)^p \longrightarrow \mathcal{W}(D_p, J) = \mathcal{M}(D_p, J) \times_{G^a} (\mathbb{C}P^1)^p$$

as a principal bundle with structure group G^a. Thus there is a principal bundle atlas such that the transition functions take values in $G^a \subseteq \{\phi \in \text{GL}(4, \mathbb{R}) : \det \phi > 0\}^a$, and G^a is connected containing the orientation preserving identity. Hence $\mathcal{M}(D_p, J) \times (\mathbb{C}P^1)^p \to \mathcal{W}(D_p, J) = \mathcal{W}(D_p, J)$ is orientable as a bundle, and a natural orientation is obtained by requiring the fibers to carry the induced orientation from $\mathcal{M}(D_p, J) \times (\mathbb{C}P^1)^p$, cf. theorem 5.2. Now bundle and total space are oriented, thus inducing an orientation on the base $\mathcal{W}(D_p, J)$.
(ii) $a = 1$. By definition it follows that $A = \lambda A^1$ with $\lambda > 1$. Then $c_1(A^1) > 0$ by assumption, and because Chern classes evaluated on homology classes take integral values it follows that $c_1(M)(A) = c_1(\lambda A^1) = \lambda c_1(A^1) \geq 2c_1(A^1) \geq c_1(A^1) + 1$.

$a > 1$. In this case the statement follows as in the discussion above. It is only to remark that $J \in \mathcal{J}_K(\omega)$ implies $c_1(M)(A^i) \geq 0$. Suppose $c_1(M)(A^i) < 0$ then $\omega(A) \geq \omega(A^i) > K \geq \omega(A)$ which is absurd.

(iii.) This is a consequence of Gromov's theorem. If $([u_n, z_n^1, \dots, z_n^p])_n$ is a sequence lifted to $\mathcal{W}(p, A, J)$, corresponding to a sequence in $\operatorname{im} \operatorname{ev} \subseteq M^p$ that converges to a point in O_{ev}, then we may rid $([u_n, z_n^1, \dots, z_n^p])_n$ of all convergent subsequences without changing its limit behavior in M^p.

Let $([u_n, z_n^1, \dots, z_n^p])_n$ be a sequence in $\mathcal{W}(p, A, J)$ that does not have a convergent subsequence in $\mathcal{W}(p, A, J)$. By theorem 2.5 any $([u_n])_n$ representing sequence $(u_n)_n$ has a subsequence weakly converging to a cusp curve $\overline{c} = (\overline{c}^1, \dots, \overline{c}^{\overline{a}})$. By 2.B.(2) $\overline{c}$ may be replaced by a reduced cusp curve $c = (c^1, \dots, c^a)$ such that $\operatorname{im} c = \operatorname{im} \overline{c}$. Since weak convergence is stronger than pointwise convergence there is a diagonal subsequence $([u_{n_k}, z_{n_k}^1, \dots, z_{n_k}^p])_k$ such that $\operatorname{ev}[u_{n_k}, z_{n_k}^1, \dots, z_{n_k}^p] \longrightarrow (x^1, \dots, x^p) \in M^p$ where each x^i lies in a component of the reduced cusp curve c. This means that there exists a point $s = (s^1, \dots, s^p) \in (\mathbb{C}P^1)^p$ and a label $\sigma \colon \{1, \dots, p\} \to \{1, \dots, a\}$ such that $(x^i = u^{\sigma(i)}(s^i))_{i=1}^p$ or $(x^1, \dots, x^p) = \operatorname{ev}_\sigma[u, \alpha, \beta, s]$; here $u = (u^1, \dots, u^a)$ is a parametrization of $c = (c^1, \dots, c^a)$ and $(\alpha, \beta) = (\alpha_i, \beta_i)_{i=2}^{2a-2}$ are auxiliary points used in the construction of $\mathcal{W}(D_p, J)$. Thus $(x^1, \dots, x^p) = \operatorname{ev}_\sigma[u, \alpha, \beta, s] \in \operatorname{ev}_\sigma(\mathcal{W}(D_p, J))$ for some, not necessarily unique, framing data $D_p = (A^1, \dots, A^a, j_2, \dots, j_a, \sigma)$.

(iv.) The first part follows from theorem 4.7 and proposition 5.2 since countable intersections of second category sets are again of second category. Part two is lemma 5.5. □

$\boldsymbol{\mathcal{W}_\gamma(p, A)}$**.** Let $A \in H_2(M)$, $J_i \in \mathcal{J}_{\mathrm{reg}}(\omega, A)$ for $i \in \{0, 1\}$, and $\gamma \colon [0,1] \to \mathcal{J}(\omega)$ a generic arc connecting these. From theorem 4.8 recall the space $\mathcal{M}_\gamma(A)$. Consider the following.

$$
\begin{aligned}
G \times \mathcal{M}_\gamma(A) \times (\mathbb{C}P^1)^p &\longrightarrow \mathcal{M}_\gamma(A) \times (\mathbb{C}P^1)^p, \\
(\phi, t, u, z^1, \dots, z^p) &\longmapsto (t, u \circ \phi^{-1}, \phi(z^1), \dots, \phi(z^p)), \\
\operatorname{ev}^\gamma \colon \mathcal{W}_\gamma(p, A) := \mathcal{M}_\gamma(A) \times_G (\mathbb{C}P^1)^p &\longrightarrow M^p, \\
[t, u, z^1, \dots, z^p] &\longmapsto (u(z^1), \dots, u(z^p)).
\end{aligned}
$$

The action is free, thus $\mathcal{W}_\gamma(p, A)$ is a manifold, and by the same reasoning as above it is naturally oriented. Its dimension is $\dim \mathcal{W}_\gamma(p, A) = 2n + 2c_1(M)(A) + 2p - 5$. Moreover, this space is a manifold with boundary $\partial \mathcal{W}_\gamma(p, A) = \mathcal{W}(p, A, J_1) - \mathcal{W}(p, A, J_0)$. □

$\boldsymbol{\mathcal{W}_\gamma(D_p)}$**.** Let $A \in H_2(M)$ be framed by $D_p = (A^1, \dots, A^a, j_2, \dots, j_a, \sigma)$, and J_0, J_1 generic structures connected by a generic path $\gamma \colon [0,1] \to \mathcal{J}(\omega)$. Then there is a free action

$$\begin{aligned} G^a \times \mathcal{M}_\gamma(D_p) \times (\mathbb{C}P^1)^p &\longrightarrow \mathcal{M}_\gamma(D_p) \times (\mathbb{C}P^1)^p, \\ \big((\phi_i)_{i=1}^a, t, (u^i)_{i=1}^a, (\alpha_i, \beta_i)_{i=2}^a, (z^i)_{i=1}^p\big) &\longmapsto \big(t, (u^i \circ \phi_i^{-1})_{i=1}^a, \\ &\qquad (\phi_{j_i}(\alpha_i), \phi_i(\beta_i))_{i=2}^a, (\phi_{\sigma(i)}(z^i))_{i=1}^p\big) \end{aligned}$$

with an evaluation map which is defined on the appropriate orbit space

$$\begin{aligned} \mathrm{ev}_\sigma^\gamma \colon \mathcal{W}_\gamma(D_p) := \mathcal{M}_\gamma(D_p) \times_{G^a} (\mathbb{C}P^1)^p &\longrightarrow M^p, \\ [t, u, \alpha, \beta, z] &\longmapsto (u^{\sigma(i)}(z^i))_{i=1}^p = \mathrm{ev}_\sigma[u, \alpha, \beta, z]. \end{aligned}$$

□

Lemma 5.7. *Let (M, ω) be a compact, weakly monotone, symplectic manifold with $2n = \dim M$, $p \in \mathbb{N}$, $A \in H_2(M)$, and $D_p = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ a framing of A. Assume that A is not a multiple class $A = \lambda B$ for $\lambda > 1$ and $B \in H_2(M)$ with $c_1(M)(B) = 0$. If $J_i \in \mathcal{J}_{\mathrm{reg}}(\omega, A) \cap \mathcal{J}_{\mathrm{reg}}(\omega, D_p) \cap \mathcal{J}_K(\omega)$ for $i \in \{0, 1\}$ and some $K \geq \omega(A)$ then $\mathcal{W}_\gamma(D_p)$ is a smooth, oriented manifold of dimension*

$$\dim \mathcal{W}_\gamma(D_p) = 2n + 2\sum_{i=1}^a c_1(M)(A^i) + 2p - 2a - 3 \leq \dim \mathcal{W}_\gamma(p, A) - 2.$$

Moreover, the set $\mathcal{D}$ of all framings of A is finite and

$$\bigcap_{\substack{K \subseteq \mathcal{W}_\gamma(p,A) \\ \mathrm{cp}}} \overline{\mathrm{ev}^\gamma(\mathcal{W}_\gamma(p, A) \setminus K)} \subseteq \bigcup_{D_p \in \mathcal{D}} \mathrm{ev}_\sigma^\gamma(\mathcal{W}_\gamma(D_p)).$$

Proof. By citing theorems 4.8 and 5.3 this is proved just like the theorem above. This is the point where weak monotonicity really is essential. In order to obtain the statement on the dimension we should know that any two points in the set $\mathcal{J}_{\mathrm{reg}}(\omega, A) \cap \mathcal{J}_{\mathrm{reg}}(\omega, D_p) \cap \mathcal{J}_K(\omega)$ can be connected by a generic path completely contained in $\mathcal{J}_K(\omega)$, and this indeed is true for weakly monotone (M, ω), by the last theorem. □

5.D. Gromov invariant Φ

The first task of this section is to reformulate the properties of moduli spaces as established in theorem 5.6. (M, ω) still denotes a compact symplectic manifold of dimension $\dim M = 2n$.

Definition (Pseudo cycles). A *d-dimensional pseudo cycle* in M is a pair (F, f) such that F is an oriented smooth manifold without boundary, $f \in C^\infty(F, M)$, $\dim F = d$, and

$$O_f := \bigcap_{\substack{K \subseteq F \\ \mathrm{cp}}} \overline{f(F \setminus K)} \qquad \text{has dimension at most } d-2.$$

A subset $X \subseteq M$ is said to be of *dimension at most k* if there is a k-dimensional manifold L and a smooth mapping $l\colon L \to M$ such that $X \subseteq \operatorname{im} l$.

A *d-dimensional pseudo chain* in M is a pair (F, f) such that F is a smooth, oriented d-dimensional manifold with boundary, $f\colon F \to M$ a smooth mapping, and O_f is of dimension at most $d-2$, and $O_{f|_{\partial F}}$ of dimension at most $d-3$.

Definition (Bordant pseudo cycles). A d-pseudo cycle (F, f) *bords* if there is a $d+1$-pseudo chain (K, k) such that $\partial K = F$ and $g|_F = f$. If F is an oriented manifold then $-F$ is F with reversed orientation. Two d-pseudo cycles (F_1, f_1) and (F_2, f_2) are said to be *bordant* or *bordant equivalent* if $(F_2 - F_1, f_2 \sqcup f_1) = (F_2 \sqcup (-F_1), f_2 \sqcup f_1)$ bords.

1. Since M is compact O_f is compact too. If (F, f) is the interior of a manifold with boundary $(\overline{F}, \overline{f})$ in the sense that $F \subseteq \overline{F}$ and $\overline{f}|_F = f$ then $O_f = \overline{f}(\partial\overline{F})$. □

2. Being bordant is an equivalence relation on the set of all d-pseudo cycles. Reflexivity: take the product with $[0, 1]$. Symmetry: by reversing the orientation of the $d+1$-pseudo cycle that is being borded. Transitivity: by reversing the orientation of the second pseudo cycle bordism, and gluing along the common border. Let $\Omega_d(M)$ denote the set of all equivalence classes of d-pseudo cycles in M. □

3. Let (F, f) be a d-dimensional pseudo cycle in $\mathbb{R}^{2n}$, and (L, l) a pair such that $\dim L = d-2$ and $O_f \subseteq \operatorname{im} l$. Consider the smooth homotopy h : $[0, 1] \times (F \sqcup L) \to \mathbb{R}^{2n}$, $(t, x) \mapsto ((f \sqcup l)(x) + t)$. Then (F, f^h) is a d-pseudo cycle, where $f^h(x) := f(x) + 1$, and $O_{f^h} \subseteq \operatorname{im} l^h$, where $l^h(x) := l(x) + 1$. Moreover (F, f) and (F, f^h) are bordant via $([0, 1] \times F, h)$. □

4. The operation $[F, f] + [G, g] := [F \sqcup G, f \sqcup g]$ is well defined, and $(\Omega_d(M), +)$ is an Abelian group. The neutral element is the empty manifold (which has every dimension) together with the empty map; the inverse of $[F, f]$ is given by reversing the orientation of F, i.e. by $[-F, f]$. □

5. Let V, W be compact, oriented, smooth k-dimensional manifolds without boundary. V, W are called *compactly bordant* if there is a $k+1$-dimensional compact, oriented manifold with boundary $W - V$. The set under this equivalence relation shall be Ω_k. The weak direct sum $\Omega_* := \oplus_{k\in\mathbb{Z}}\Omega_k$ is a $\mathbb{Z}$-graded ring defined by the following operations. $[V]+[W] := [V \sqcup W]$ with the empty manifold as neutral element; if V, W are nonempty then $[V]\cdot[W] := [V \times W]$, and these operations are well defined by computations like $\partial(B\times W) = (V' - V)\times W = (V'\times W) - (V\times W)$ if $\partial B = V' - V$. □

6. $\Omega_*(M) := \oplus_{d\in\mathbb{Z}}\Omega_d(M)$ is a $\mathbb{Z}$-graded module over Ω_*. Let $[F,f]\in\Omega_d(M)$, $[V]\in\Omega_k$, and define $f\times 1 : F\times V \to M$ by $(f\times 1)(x,y) = f(x)$ then $[F,f][V] := [F\times V, f\times 1]\in\Omega_{d+k}(M)$ is well defined. □

Pseudo cycles (F_i, f_i) of dimension d_i for $i\in\{1,\dots,m\}$ are said to be *in general position* if

(i) $f_1\times\dots\times f_m$ is transverse to the diagonal $\Delta\subseteq M^m$, and

(ii) $\overline{\operatorname{im} f_1}\cap\dots\cap\overline{\operatorname{im} f_{k-1}}\cap O_{f_k}\cap\overline{\operatorname{im} f_{k+1}}\cap\dots\cap\overline{\operatorname{im} f_m} = \emptyset$ for all $k\in\{1,\dots,m\}$.

Under these circumstances $(f_1\times\dots\times f_m)^{-1}(\Delta)$ will be a compact, smooth submanifold of dimension $d_1+\dots+d_m-2n$.

Lemma 5.8. *Let $[F_i,f_i]\in\Omega_{d_i}(M)$ for $i\in\{1,\dots,m\}$ such that $d_1+\dots+d_m = 2n$. Then there exist representatives (F_i,f_i) of $[F_i,f_i]$ which are in general position. Hence $(f_1\times\dots\times f_m)^{-1}(\Delta)$ is finite.*

Proof. Choose arbitrary representatives $(F_i,f_i)_{i=1}^m$. By definition there are pairs $(L_i,l_i)_{i=1}^m$ consisting of d_i-2-dimensional manifolds and smooth mappings $l_i\colon L_i\to M$ such that $O_{f_i}\subseteq\operatorname{im} l_i$. Let $H := (F_1\sqcup L_1)\times\dots\times(F_m\sqcup L_m)$ and $h := (f_1\sqcup l_1)\times\dots\times(f_m\sqcup l_m)\colon H\to M^m$.
With high enough $k\in\mathbb{N}$ embed $M^m\hookrightarrow\mathbb{R}^k$, and choose a tubular neighborhood $\pi\colon T\to M^m$ of M^m in $\mathbb{R}^k$. Let $B\subseteq\mathbb{R}^k$ denote the open unit ball, and consider the mapping

$$\alpha\colon H\times B\longrightarrow M^m,\qquad (x,t)\longmapsto \pi(h(x)+\varepsilon t)$$

with small enough $\varepsilon>0$ such that $\varepsilon B+\operatorname{im} h\subseteq T$. Fix arbitrary $x\in H$, and note that $\alpha_x = \alpha(x,\_) : B\to M^m$, $t\mapsto h(x)+\varepsilon t\mapsto\pi(h(x)+\varepsilon t)$ is the composite of two submersions. Thus also $\alpha : H\times B\to M^m$ is a submersion, and in particular α is transverse to the diagonal $\Delta\subseteq M^m$. Therefore its inverse image under α is a smooth submanifold, and we define $g = h|_G\colon\alpha^{-1}(\Delta) = G\to M^m$.

If $t \in B$ is a regular value of the projection $\mathrm{pr}_2 : G \to B$, $(x,t) \mapsto t$ *then the map* $\alpha_t = \alpha(\_\,,t)\colon H \to M^m$ *is transverse to the diagonal* Δ. There are two possibilities, namely $(x,t) \in G$ and $(x,t) \notin G$; if $(x,t) \notin G$ then α_t is transverse to Δ at x trivially. Let $(x,t) \in G$, $p = \alpha(x,t)$, and assume $t \in B$ is a regular value of $\mathrm{pr}_2\colon G \to B$; choose $X \in T_pM^m$ arbitrarily fixed. Since $\alpha : H \times B \to M^m$ is a submersion there is a vector $(\xi, v) \in T_{(x,t)}(H \times B)$ with $T_{(x,t)}\alpha \cdot (\xi, v) - X \in T_p\Delta$. By regularity of t we can find ξ' such that $(\xi', v) \in T_{(x,t)}G$, or equivalently $T_{(x,t)}\alpha \cdot (\xi', v) \in T_p\Delta$. Thus

$$T_{(x,t)}\alpha \cdot (\xi, v) - X \in T_p\Delta \iff T_{(x,t)}\alpha \cdot (\xi - \xi', 0) - X \in T_p\Delta.$$

Hence $\alpha_t\colon H \to M^m$ intersects the diagonal transversally for all elements t in the second category set of regular values of the projection $\mathrm{pr}_2\colon H \times B \to B$. Suppose t_0 is such a regular value. Then the map $\eta : [0,1] \times H \to M^m$, $(\lambda, x) \mapsto \pi(h(x) + \varepsilon\lambda t_0)$ is a smooth homotopy from h to $\eta(1,\_) =: h_1$; indeed, $\pi|_{M^m} = \mathrm{id}$. Now $h_1 =: \prod_{i=1}^m (f_i^h \sqcup l_i^h)$ is transverse to the diagonal and each f_i, l_i is homotopic to f_i^h, l_i^h respectively. Moreover, transversality of $\prod_{i=1}^m (f_i^h \sqcup l_i^h)$ with Δ implies that also

$$\prod_{i=1}^m f_i^h \qquad \text{and} \qquad f_1^h \times \ldots \times l_k^h \times \ldots \times f_m^h \qquad \text{and} \qquad \prod_{i=1}^m l_i^h$$

all are tranverse to the diagonal for $k \in \{1, \ldots, m\}$. By reason of dimension it follows that

$$\begin{aligned}\operatorname{im} f_1^h \cap \ldots \cap \operatorname{im} f_{k-1}^h \cap \operatorname{im} l_k^h \cap \operatorname{im} f_{k+1}^h \cap \ldots \cap \operatorname{im} f_m^h& \\ = \operatorname{im} l_1^h \cap \ldots \cap \operatorname{im} l_m^h = \emptyset.&\end{aligned}$$

The thus obtained (F_i, f_i^h) *are* d_i*-dimensional pseudo cycles that are bordant equivalent to* (F_i, f_i)*, and fulfill* $O_{f_i^h} \subseteq \operatorname{im} l_i^h$. We need to check that $O_{f_i^h} = \bigcap_{K \underset{\mathrm{cp}}{\subseteq} F_i} \overline{f_i^h(F_i \setminus K)} \subseteq \operatorname{im} l_i^h$, and it is easier to show this for all $i \in \{1, \ldots, m\}$ simultaneously. Let $(x_n)_n$ be a sequence in $\prod_{i=1}^m F_i$ such that all subsequences are divergent in all of the m coordinates. Then there is a diagonal subsequence $(x_{n_k})_k$ with the property that

$$\begin{aligned}&f(x_{n_k}) = (f_i(x_{n_k}^i))_{i=1}^m \to (l_i(x^i))_{i=1}^m = l(x) = \pi(l(x) + 0), \quad \text{and hence}\\ &f^h(x_{n_k}) = \pi(f(x_{n_k}) + \varepsilon t_0) \to \pi(l(x) + \varepsilon t_0) = l^h(x)\end{aligned}$$

by continuity. That is $\prod_{i=1}^m O_{f_i^h} \subseteq \prod_{i=1}^m \operatorname{im} l_i^h$. By transversality the space $D := (f_1^h \times \ldots \times f_m^h)^{-1}(\Delta)$ is a zero dimensional manifold, i.e. discrete. By point (3) above (F_i, f_i) are bordant to (F_i, f_i^h), and thus really are representatives of $[F_i, f_i]$.

D is compact. Choose an arbitrary sequence $(x_n)_n$ in D. If $(x_n)_n$ had a subsequence such that all its m components were convergent when projected to F_i then this subsequence would also converge in $\prod_{i=1}^m F_i$, and hence also in D since this is a closed subset. Suppose now that the i-th component $(x_n^i)_n = (\mathrm{pr}_i(x_n))_n$ does not posses a subsequence convergent in F_i. By the pseudo cycle property there is a subsequence $(x_{n_k}^i)_k$ such that $f_i^h(x_{n_k}^i) \to p^i \in O_{f_i^h}$. Now, $(x_{n_k}^i)_k$ is the i-th component of a sequence $(x_{n_k}^1, \ldots, x_{n_k}^m)_k$ in D. Since this is the inverse image of Δ we have $f_1^h(x_{n_k}^1) = \ldots = f_m^h(x_{n_k}^m)$ for all $k \in \mathbb{N}$. This, however, implies

$$\begin{aligned} f_l^h(x_{n_k}^l) = f_i^h(x_{n_k}^i) \to p^i &\in \overline{\operatorname{im} f_1^h} \cap \ldots \cap O_{f_i^h} \cap \ldots \cap \overline{\operatorname{im} f_m^h} \\ &\subseteq (\operatorname{im} f_1^h \cup \operatorname{im} l_1^h) \cap \ldots \cap \operatorname{im} l_i^h \cap \ldots \cap (\operatorname{im} f_m^h \cup \operatorname{im} l_m^h) = \emptyset \end{aligned}$$

which is absurd. □

Φ. Let $d_i \in \mathbb{Z}$ and $i \in \{1, \ldots, m\}$. The invariant Φ we want to establish can be introduced as an intersection form

$$\begin{aligned} \Phi \colon \Omega_{d_1}(M) \times \ldots \times \Omega_{d_m}(M) &\longrightarrow \mathbb{Z}, \\ ([F_1, f_1], \ldots, [F_m, f_m]) &\longmapsto \sum_{x \in F} \varepsilon(x) \end{aligned}$$

which is defined as follows. If $d_1 + \ldots + d_m \neq 2n$ put $\Phi([F_1, f_1], \ldots, [F_m, f_m]) = 0$. If $d_1 + \ldots + d_m = 2n$ we apply the above lemma to find representatives (F_i, f_i) in general position, and define $F := (f_1 \times \ldots \times f_m)^{-1}(\Delta)$; $\Delta \subseteq M^m$ still denotes the diagonal. This is a compact, zero dimensional manifold, thus finite. If now $x = (x_1, \ldots, x_m) \in F$ and $p = f_i(x_i) \in M$ then transversality and equality of dimensions imply that $T_pM = \bigoplus_{i=1}^m \operatorname{im} T_{x_i} f_i$, and each $T_{x_i} f_i : T_{x_i} F_i \to \operatorname{im} T_{x_i} f_i$ is an isomorphism; hence also $T_x f := \bigoplus_{i=1}^m T_{x_i} f_i : \bigoplus_{i=1}^m T_{x_i} F_i \to \bigoplus_{i=1}^m \operatorname{im} T_{x_i} f_i = T_pM$ is an isomorphism. The mapping $\varepsilon \colon F_1 \times \ldots \times F_m \to \{-1, 0, 1\}$ is defined by

$$\varepsilon(x) := \begin{cases} \operatorname{sign} \det T_x f, & \text{if } x \in F \\ 0, & \text{else} \end{cases}$$

where the determinant is computed with respect to a choice of positively oriented bases on the $T_{x_i} F_i$ and T_pM, and $x = (x_1, \ldots, x_m)$. □

Lemma 5.9. *The map $\Phi : \Omega_{d_1}(M) \times \ldots \times \Omega_{d_m}(M) \to \mathbb{Z}$ is well defined. Moreover, it is multi linear with respect to the group structures on $\Omega_{d_i}(M)$ and $\mathbb{Z}$.*

Proof. Let $[F_i, f_i]$, $[G_i, g_i]$ be bordism classes of d_i-dimensional pseudo cycles. With a suitable choice of representatives in general position it follows that $\Phi(\dots, [F_i, f_i] + [G_i, g_i], \dots) = \Phi(\dots, [F_i, f_i], \dots) + \Phi(\dots, [G_i, g_i], \dots)$; to save on typing assume that $i = 1$ and $m = 2$, then

$$\begin{aligned}((f_1 \sqcup g_1) \times f_2)^{-1}(\Delta) &= ((f_1 \times f_2) \sqcup (g_1 \times f_2))^{-1}(\Delta) \\ &= (f_1 \times f_2)^{-1}(\Delta) \sqcup (g_1 \times f_2)^{-1}(\Delta) =: F \sqcup G.\end{aligned}$$

indeed implies that

$$\begin{aligned}\Phi([F_1, f_1] + [G_1, g_1], [F_2, f_2]) &= \sum_{x \in F \sqcup G} \varepsilon(x) = \sum_{x \in F} \varepsilon(x) + \sum_{x \in G} \varepsilon(x) \\ &= \Phi([F_1, f_1], [F_2, f_2]) + \Phi([G_1, g_1], [F_2, f_2]).\end{aligned}$$

To see independence of representatives proceed as follows. Without loss of generality we continue to assume $m = 2$. Choose representatives (F_1, f_1), (G_1, g_1) of $[F_1, f_1]$, and (F_2, f_2) of $[F_2, f_2]$ such that (F_1, f_1), (F_2, f_2) and (G_1, g_1), (F_2, f_2) both are in general position. Let $F := (f_1 \times f_2)^{-1}(\Delta)$, $G := (g_1 \times f_2)^{-1}(\Delta)$, and $\Phi_F := \sum_{x \in F} \varepsilon(x)$, $\Phi_G := \sum_{x \in G} \varepsilon(x)$. We have to show that $\Phi_F = \Phi_G$. By the same arguments as in the lemma above there is a bordism (B, b) from (F_1, f_1) to (G_1, g_1) such that (B, b), (F_2, f_2) are in general position. (The modifications to obtain a homotopy with fixed end points are the same as in the proof of the usual transversality theorem, and the result still applies to our case since the arguments go through with $d_1 + \ldots + d_2 = 2n + 1$ as well.) Thus $A := (b \times f_2)^{-1}(\Delta)$ is a compact, one-dimensional manifold with boundary $\partial A = G - F$; i.e. A is a disjoint union of finitely many compact intervals.

It suffices to consider those intervals that have one endpoint in F and the other in G; it could also happen that two points in, say, F are connected by an interval but these two points then cancel out when counted with sign, i.e. do not contribute to the number Φ_F. By a homotopy argument we can assume that we have an embedding $a\colon A \hookrightarrow M$. Now $\varepsilon(x) = 1$ for a point $x \in G$ implies $\varepsilon(-y) = -1$ for the corresponding point $-y \in -F$. Thus $\Phi_G = -\Phi_{-F} = \Phi_F$. □

Assumption. As we aim to count the number of pseudo-holomorphic curves intersecting certain singular homology classes we make the following assumption, which might be false in general. *For all $d \in \mathbb{Z}$ there exists a well-defined mapping $H_d(M) \to \Omega_d(M)$.*

In order to prove this one could try to proceed as follows. Let $A \in H_d(M)$. Then there is a finite, oriented simplicial complex S such that the formal sum over all $d-1$-dimensional simplices counted with signs gives 0. This means that S is a fundamental cycle itself determining a fundamental class $[S]$; and we let $s\colon S \to M$ be a map such that $s_*[S] = A$. Now one needs to show that S can be endowed with a topological structure so that it becomes a closed topological manifold, and $s\colon S \to M$ a (deformation of a) continuous mapping. The next step would then be to smooth out the edges, thereby making S into a closed smooth manifold, and $s\colon S \to M$ a (deformation of a) smooth mapping. This should yield $[S, s] \in \Omega_d(M)$ and $s_*(S) = A$. □

Let $A \in H_2(M)$ and consider a framing $D_p = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ of A. An almost complex, ω-tame structure $J \in \mathcal{J}(\omega)$ is called *generic* if $J \in \mathcal{J}_{\mathrm{reg}}(\omega, A) \cap \mathcal{J}_{\mathrm{reg}}(\omega, D_p) \cap \mathcal{J}_K(\omega)$ where $K \geq \omega(A)$; if (M, ω) is assumed weakly monotone then this set is of second category in $\mathcal{J}(\omega)$, and any two generic structures can be connected by a generic path completely contained in $\mathcal{J}_K(\omega)$.
The relevant definitions come from the previous section.

Lemma 5.10. *Assume (M, ω) is weakly monotone. Suppose $A \in H_2(M)$ is not a multiple class $A = \lambda B$ for $B \in H_2(M)$ with $c_1(M)(B) = 0$ and $\lambda > 1$, and let $D_p = (A^1, \ldots, A^a, j_2, \ldots, j_a, \sigma)$ be a framing of A.*

(i) *If J is generic then $(\mathcal{W}(p, A, J), \mathrm{ev})$ is a pseudo cycle in M^p of dimension $2n + 2c_1(M)(A) + 2p - 6$.*

(ii) *If J_0, J_1 are generic then $\mathcal{W}(p, A, J_0)$ and $\mathcal{W}(p, A, J_1)$ are bordant as pseudo cycles.*

Proof. *(i.)* This is a reformulation of theorem 5.6. Let $\mathcal{D}$ denote the finite set of all framings of A. Then the manifold $\mathcal{W} := \coprod_{D_p \in \mathcal{D}} \mathcal{W}(D_p, J)$ is of dimension $\dim \mathcal{W} \leq \dim \mathcal{W}(p, A, J) - 2$, and hence

$$O_{\mathrm{ev}} \subseteq \mathrm{im}\,\Big(\coprod_{D_p \in \mathcal{D}} \mathrm{ev}_{D_p} \colon \mathcal{W} \to M^p \Big)$$

is of dimension at most $\dim \mathcal{W}(p, A, J) - 2$. *(ii.)* Let $\gamma : [0, 1] \to \mathcal{J}(\omega)$ be a generic path with $J_0 = \gamma(1)$, $J_1 = \gamma(1)$ such that $\mathcal{W}_\gamma(p, A)$ is $2n + 2c_1(M)(A) + 2p - 5 = \dim \mathcal{W}(p, A, J) + 1$-dimensional manifold. By lemma 5.7 and the discussion preceding it $(\mathcal{W}_\gamma(p, A), \mathrm{ev}^\gamma)$ then is a $\dim \mathcal{W}(p, A, J) + 1$-dimensional pseudo chain. It determines the asserted bordism since it is true that $\partial \mathcal{W}_\gamma(p, A) = \mathcal{W}(p, A, J_1) - \mathcal{W}(p, A, J_0)$. □

Definition (Gromov invariant Φ). Under the assumptions of the above lemma, given $p \in \mathbb{N}$, the triple (ω, A, J) determines a unique bordism class $[\mathcal{W}(p, A, J), \mathrm{ev}]$ which is independent of the particular generic structure J

used to define it. It is here that we make use of the above assumption, and suppose that every homology class can be represented by a unique bordism class. The *Gromov invariant* Φ is the well-defined homomorphism

$$\Phi^p_{(\omega,A)} := \Phi([\mathcal{W}(p,A,J),\mathrm{ev}],\_,\ldots,\_)\colon H_{d_1}(M)\times\ldots\times H_{d_p}(M)\longrightarrow\mathbb{Z}.$$

By definition of the intersection form Φ this can be nonzero if and only if

$$\begin{aligned} 2np &= \dim\mathcal{W}(p,A,J)+d_1+\ldots+d_p \\ \Longleftrightarrow d_1+\ldots+d_p &= 2(n-1)(p-1)-2c_1(M)(A)+4. \end{aligned}$$

Let $\Omega^2_{\mathrm{sp}}(M)$ denote the space of all symplectic forms on M. A *deformation* of a symplectic structure $\omega_0\in\Omega^2_{\mathrm{sp}}(M)$ is a smooth path $[0,1]\to\Omega^2_{\mathrm{sp}}(M)$, $t\mapsto\omega_t$. An *isotopic deformation* or *isotopy* of ω_0 is a deformation that does not change the cohomology class, i.e. $[\omega_0]=[\omega_t]\in H^2(M)$ for all $t\in[0,1]$; by definition, ω is closed.

Theorem 5.11. *Assume (M,ω) is weakly monotone. Suppose $A\in H_2(M)$ is not a multiple class $A=\lambda B$ for $B\in H_2(M)$ with $c_1(M)(B)=0$ and $\lambda>1$, and let $D_p=(A^1,\ldots,A^a,j_2,\ldots,j_a,\sigma)$ be a framing of A.*

(i) *$\Phi^p_{(\omega,A)}$ is an invariant of (M,ω,A) that depends only on the weakly monotone deformation class of ω. That is, if $[0,1]\to\Omega^2_{\mathrm{sp}}(M)$, $t\mapsto\omega_t$ is a deformation of $\omega=\omega_0$ such that (M,ω_t) is weakly monotone for all $t\in[0,1]$ then $\Phi^p_{(\omega_0,A)}=\Phi^p_{(\omega_1,A)}$.*

(ii) *If $\dim M\leq 6$ or $\pi_2(M)=\{0\}$ then $\Phi^p_{(\omega,A)}$ is an invariant of (M,ω,A) that depends only on the deformation class of ω.*

Proof. *(i.)* Let $[0,1]\to\Omega^2_{\mathrm{sp}}(M)$, $t\mapsto\omega_t$ be a deformation of $\omega=\omega_0$ such that (M,ω_t) is weakly monotone for all $t\in[0,1]$. Since the taming condition is open there is an open neighborhood of t_0 in $[0,1]$ such that $\mathcal{J}(\omega_{t_0})=\mathcal{J}(\omega_t)$ for all t in this neighborhood. By connectedness of $[0,1]$ it therefore follows that $\mathcal{J}(\omega_0)=\mathcal{J}(\omega_1)$. The same argument repeated together with lemma 5.4 thus implies that ω_0 and ω_1 also have the same generic structures. This proves $\Phi^p_{(\omega_0,A)}=\Phi^p_{(\omega_1,A)}$.
(ii.) Under these assumptions every symplectic structure on M is weakly monotone. □

5.E. An application (Squeezing)

Let V be a compact connected symplectic manifold such that $\pi_2(V)=\{0\}$, and equip $M:=\mathbb{C}P^1\times V$ with a product symplectic structure ω. For a point $v_0\in V$ consider the class $A:=[\mathbb{C}P^1\times\{v_0\}]\in H_2(M)$.

1. The thus chosen class A has the property that $c_1(M)(A) = 2$. For the dimension of $\mathcal{W}(1, A, J) = \mathcal{W}(A, J)$ this means that

$$\dim \mathcal{W}(A, J) = \dim M + 4 + 2 - 6,$$

where we have chosen to consider the generic structure $J = i \times J'$ with some generic J' on V. (McDuff shows in [14, Lemma 2.3.4] that the standard complex structure i on $\mathbb{C}P^1$ indeed is sufficiently regular, i.e. generic.) By the dimension condition in the definition of the Gromov invariant Φ it makes sense to consider the (not a priori trivial) homomorphism

$$\Phi_{(\omega,A)} = \Phi^1_{(\omega,A)} \colon H_0(M) \longrightarrow \mathbb{Z}$$

which we can evaluate at a point $p \in M$. Note that the J-holomorphic maps $u \in \mathcal{M}(A, J)$ all are of the form $u = (u^1, u^2) = (\phi, v)$ where $\phi \in G = \mathrm{PSL}(2, \mathbb{C})$ and v denotes the constant map $v : \mathbb{C}P^1 \ni z \mapsto v \in V$; indeed, $E(u_2) = (\mathrm{pr}_2 \circ u)_*[\mathbb{C}P^1] = 0$, and hence $u_1 : \mathbb{C}P^1 \to \mathbb{C}P^1$ must by biholomorphic since the set of injective points is dense. For this choice of J the inverse image

$$\mathrm{ev}^{-1}(p) = \{[u, z] \in \mathcal{W}(A, J) : u(z) = (\phi(z), v) = p\}$$

consists of one element only; if $\phi_1(z_1) = \phi_2(z_2)$ then $(\phi_2, z_2) = (\phi_1 \circ (\phi_2^{-1} \circ \phi_1)^{-1}, (\phi_2^{-1} \circ \phi_1)(z_2))$ which means that $[(\phi_2, v), z_2] = [(\phi_1, v), z_1] \in \mathcal{W}(A, J)$. Since $\mathrm{ev} \colon \mathcal{W}(A, J) \to M$ is orientation preserving this shows that

$$\Phi_{(\omega,A)}(p) = 1,$$

and by the results from the last section this computation is independent of the choices. □

2. The result of the above calculation may be rephrased in saying that the mapping degree of $\mathrm{ev} : \mathcal{W}(A, J) \to M$ equals 1. However, this only makes sense if ev is a proper mapping. Therefore, we shall assume that $\pi_2(V) = \{0\}$. Then A necessarily is indecomposable, and hence theorem 2.4 implies that $\mathcal{W}(A, J)$ is compact. It is now a general fact that mappings with nonzero degree are surjective which means that there is a J-holomorphic curve through every point in M – and this is true for arbitrary generic J. □

3. Assume now $J \in \mathcal{J}(\omega)$ is not generic. Because $\mathcal{J}_{\mathrm{reg}}(\omega) \subseteq \mathcal{J}(\omega)$ is WO^∞-dense there is a sequence $(J_n)_n$ of generic structures WO^∞-converging to J. Let $p \in M$ arbitrary. Because $\mathrm{ev}_n : \mathcal{W}(A, J_n) \to M$ is surjective there is a sequence $([u_n, z_n])_n$ with $[u_n, z_n] \in \mathcal{W}(A, J_n)$ and $u_n(z_n) = p$ for all $n \in \mathbb{N}$. By theorem 2.4 there is a subsequence $([u_{n_k}, z_{n_k}])_k$ converging to an element $[u, z]$ in the set $\mathcal{W}(A, J)$; i.e. there exist $(\phi_{n_k})_k$ in G such that $u_{n_k} \circ \phi_{n_k}^{-1} \to u$ in the WO^∞-topology and $\phi_{n_k}(z_{n_k}) \to z$. In particular $p = u_{n_k}(z_{n_k}) \to u(z) = p$. Thus we have shown that the set theoretical mapping

$$\mathrm{ev}\colon \mathcal{W}(A, J) \longrightarrow M = \mathbb{C}P^1 \times V$$

is surjective for all choices of $J \in \mathcal{J}(\omega)$. $\square$

Let $B^{2k}(r) \subseteq \mathbb{R}^{2k}$ denote the open ball in $\mathbb{R}^{2k}$ of radius r.

Theorem 5.12 (Gromov's non-squeezing theorem). *Let (V, ω_V) be a compact connected symplectic manifold of dimension $\dim V = 2k-2$ such that $\pi_2(V) = \{0\}$. If $\psi\colon B^{2k}(r) \hookrightarrow B^2(s) \times V$ is a symplectic embedding then $r \leq s$.*

Proof. Let $\varepsilon > 0$ and $A = [\mathbb{C}P^1 \times \{v_0\}] \in H_2(\mathbb{C}P^1 \times V)$ for $v_0 \in V$. Denote the standard symplectic structure on $B^2(s) \subseteq \mathbb{R}^2$, $B^{2k}(r) \subseteq \mathbb{R}^{2k}$ by ω_2, ω_{2k} respectively, and the standard complex structure on $B^{2k}(r) \subseteq \mathbb{R}^{2k}$ by J_{2k}. On $\mathbb{C}P^1$ we pick a symplectic structure (volume form) $\omega_{\mathbb{C}P^1}$ such that $\pi s^2 + \varepsilon = \mathrm{vol}(\mathbb{C}P^1) = \int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^1} = \omega(A)$, where $\omega = \omega_{\mathbb{C}P^1} \times \omega_V$ is the product symplectic structure on $M := \mathbb{C}P^1 \times V$. Moreover, we choose a symplectic (volume preserving) embedding $B^2(s) \hookrightarrow \mathbb{C}P^1$.
By assumption, there exists a symplectic embedding $\psi : (B^{2k}(r), \omega_{2k}) \hookrightarrow (M, \omega)$, and we call its restriction to a closed ball $\psi^\varepsilon : \overline{B^{2k}(r-\varepsilon)} \hookrightarrow M$. Since embeddings are immersions, on the compact subset $\mathrm{im}\,\psi^\varepsilon$ the push forward $\psi^\varepsilon_* J_{2k}$ is well defined. By lemma 1.4 there exists $J \in \mathcal{J}(\omega)$ which equals $\psi^\varepsilon_* J_{2k}$ when restricted to $\mathrm{im}\,\psi^\varepsilon$.
Since the above computation of the Gromov invariant Φ implies that $\mathrm{ev} : \mathcal{W}(A, J) \to M$ is surjective there exists a J-holomorphic map $u \in \mathcal{M}(A, J)$ which meets $\psi^\varepsilon(0)$. Let $S := (\psi^\varepsilon)^{-1}(\mathrm{im}\, u)$. Because J_{2k} was chosen standard the surface $S \subseteq \overline{B^{2k}(r-\varepsilon)}$ is a minimal surface through the origin with respect to the Euclidean metric. Therefore, lemma 3.15 in Lawson [12] implies the first inequality in

$$\pi(r-\varepsilon)^2 \leq \mathrm{vol} S = \int_S \omega_{2k} = \int_S (\psi^\varepsilon)^* \omega = \int_{\psi^\varepsilon(S)} \omega < \int_{\mathrm{im}\, u} \omega = \omega(A) = \pi s^2 + \varepsilon$$

which holds for all $\varepsilon > 0$. The middle inequality is truly strict since the image of a map $u \in \mathcal{M}(A, J)$ can never be fully contained in $\mathrm{im}\,\psi^\varepsilon$; otherwise $A = (\psi^\varepsilon \circ (\psi^\varepsilon)^{-1} \circ u)_*[\mathbb{C}P^1] = (\psi^\varepsilon)_*(0) = 0$, since $H_2(\mathbb{R}^{2k}) = 0$. $\square$

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