Math. Ann. (2011)
DOI:10.1007/s00208-011-0693-9.

MANY PARAMETER HÖLDER PERTURBATION OF UNBOUNDED OPERATORS

ANDREAS KRIEGL, PETER W. MICHOR, AND ARMIN RAINER

ABSTRACT. If $u \mapsto A(u)$ is a $C^{0,\alpha}$-mapping, for $0 < \alpha \leq 1$, having as values unbounded self-adjoint operators with compact resolvents and common domain of definition, parametrized by u in an (even infinite dimensional) space, then any continuous (in u) arrangement of the eigenvalues of $A(u)$ is indeed $C^{0,\alpha}$ in u.

Theorem. *Let $U \subseteq E$ be a c^∞-open subset in a convenient vector space E, and $0 < \alpha \leq 1$. Let $u \mapsto A(u)$, for $u \in U$, be a $C^{0,\alpha}$-mapping with values unbounded self-adjoint operators in a Hilbert space H with common domain of definition and with compact resolvent. Then any (in u) continuous eigenvalue $\lambda(u)$ of $A(u)$ is $C^{0,\alpha}$ in u.*

Remarks and definitions. This paper is a complement to [9] and builds upon it. A function $f : \mathbb{R} \to \mathbb{R}$ is called $C^{0,\alpha}$ if $\frac{f(t)-f(s)}{|t-s|^\alpha}$ is locally bounded in $t \neq s$. For $\alpha = 1$ this is Lipschitz.

Due to [2] a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is $C^{0,\alpha}$ if and only if $f \circ c$ is $C^{0,\alpha}$ for each smooth (i.e. C^∞) curve c. [4] has shown that this holds for even more general concepts of Hölder differentiable maps.

A convenient vector space (see [8]) is a locally convex vector space E satisfying the following equivalent conditions: Mackey Cauchy sequences converge; C^∞-curves in E are locally integrable in E; a curve $c : \mathbb{R} \to E$ is C^∞ (locally Lipschitz, short Lipschitz) if and only if $\ell \circ c$ is C^∞ (Lipschitz) for all continuous linear functionals ℓ. The c^∞-topology on E is the final topology with respect to all smooth curves (Lipschitz curves). Mappings f defined on open (or even c^∞-open) subsets of convenient vector spaces E are called $C^{0,\alpha}$ (Lipschitz) if $f \circ c$ is $C^{0,\alpha}$ (Lipschitz) for every smooth curve c. A $C^{0,\alpha}$-mapping f between Banach spaces is locally Hölder-continuous of order α in the usual sense. This has been proved in [5], which is not easily accessible, thus we include a proof in the lemma below. For the Lipschitz case see [7] and [8, 12.7].

That a mapping $t \mapsto A(t)$ defined on a c^∞-open subset U of a convenient vector space E is $C^{0,\alpha}$ with values in unbounded self-adjoint operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$. And furthermore, $t \mapsto \langle A(t)u, v\rangle$ is $C^{0,\alpha}$ for each $u \in V$ and $v \in H$ in the sense of the definition given above.

This implies that $t \mapsto A(t)u$ is of the same class $U \to H$ for each $u \in V$ by [8, 2.3], [7, 2.6.2], or [5, 4.1.14]. This is true because $C^{0,\alpha}$ can be described by

Date: April 24, 2012.
2000 *Mathematics Subject Classification.* Primary 47A55, 47A56, 47B25.
Key words and phrases. many parameter perturbation theory, Hölder choice of eigenvalues.
AK was supported by FWF-Project P 23082-N13. PM was supported by FWF-Project P 21030-N13. AR was supported by FWF-Projects J 2771-N13 and P 22218-N13.

boundedness conditions only; and for these the uniform boundedness principle is valid.

Lemma ([5])**.** *Let E and F be Banach spaces, U open in E. Then, a mapping $f : U \to F$ is $C^{0,\alpha}$ if and only if f is locally Hölder of order α, i.e., $\frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha}$ is locally bounded.*

Proof. If f is $C^{0,\alpha}$ but not locally Hölder near $z \in U$, then there are $x_n \neq y_n$ in U with $\|x_n - z\| \le 1/4^n$ and $\|y_n - z\| \le 1/4^n$, such that $\|f(y_n) - f(x_n)\| \ge n.2^n.\|y_n - x_n\|^\alpha$. Now we apply the general curve lemma [8, 12.2] with $s_n := 2^n.\|y_n - x_n\|$ and $c_n(t) := x_n - z + t\frac{y_n - x_n}{2^n\|y_n - x_n\|}$ to get a smooth curve c with $c(t + t_n) - z = c_n(t)$ for $0 \le t \le s_n$. Then $\frac{1}{s_n^\alpha}\|(f\circ c)(t_n+s_n) - (f\circ c)(t_n)\| = \frac{1}{2^{n\alpha}.\|y_n-x_n\|^\alpha}\|f(y_n) - f(x_n)\| \ge n$. The converse is obvious. □

The theorem holds for $E = \mathbb{R}$. Let $t \mapsto A(t)$ be a $C^{0,\alpha}$-curve. Going through the proof of the resolvent lemma in [9] carefully, we find that $t \mapsto A(t)$ is a $C^{0,\alpha}$-mapping $U \to L(V, H)$, and thus the resolvent $(A(t) - z)^{-1}$ is $C^{0,\alpha}$ into $L(H, H)$ in t and z jointly. There the exponential law for $\mathcal{L}\mathrm{ip}^0 = C^{0,1}$ is invoked, but one only needs that the evaluation map is bounded multilinear.

For a continuous eigenvalue $t \mapsto \lambda(t)$ as in the theorem, let the eigenvalue $\lambda(s)$ of $A(s)$ have multiplicity N for s fixed. Choose a simple closed curve γ in the resolvent set of $A(s)$ enclosing only $\lambda(s)$ among all eigenvalues of $A(s)$. Since the global resolvent set $\{(t, z) \in \mathbb{R} \times \mathbb{C} : (A(t) - z) : V \to H \text{ is invertible}\}$ is open, no eigenvalue of $A(t)$ lies on γ, for t near s. Consider

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t) - z)^{-1}dz =: P(t),$$

a $C^{0,\alpha}$-curve of projections (on the direct sum of all eigenspaces corresponding to eigenvalues in the interior of γ) with finite dimensional ranges and constant ranks. So for t near s, there are equally many eigenvalues (repeated with multiplicity) in the interior of γ. Let us order them by size, $\mu_1(t) \le \mu_2(t) \le \cdots \le \mu_N(t)$, for all t. The image of $t \mapsto P(t)$, for t near s, describes a finite dimensional $C^{0,\alpha}$ vector subbundle of $\mathbb{R} \times H \to \mathbb{R}$, since its rank is constant. The set $\{\mu_i(t) : 1 \le i \le N\}$ represents the eigenvalues of $P(t)A(t)|_{P(t)(H)}$. By the following result, it forms a $C^{0,\alpha}$-parametrization of the eigenvalues of $A(t)$ inside γ, for t near s.

The eigenvalue $\lambda(t)$ is a continuous (in t) choice among the $\mu_i(t)$, and it is $C^{0,\alpha}$ in t by the proposition below.

Result ([10], see also [1, III.2.6])**.** *Let A, B be Hermitian $N \times N$ matrices. Let $\mu_1(A) \le \mu_2(A) \le \cdots \le \mu_N(A)$ and $\mu_1(B) \le \mu_2(B) \le \cdots \le \mu_N(B)$ denote the eigenvalues of A and B, respectively. Then*

$$\max_j |\mu_j(A) - \mu_j(B)| \le \|A - B\|.$$

Here $\|.\|$ is the operator norm.

Proposition. *Let $0 < \alpha \le 1$. Let $U \ni u \mapsto A(u)$ be a $C^{0,\alpha}$-mapping of Hermitian $N \times N$ matrices. Let $u \mapsto \lambda_i(u)$, $i = 1, \dots, N$, be continuous mappings which together parametrize the eigenvalues of $A(u)$. Then each λ_i is $C^{0,\alpha}$.*

Proof. It suffices to check that λ_i is $C^{0,\alpha}$ along each smooth curve in U, so we may assume without loss that $U = \mathbb{R}$. We have to show that each continuous eigenvalue $t \mapsto \lambda(t)$ is a $C^{0,\alpha}$-function on each compact interval I in U. Let $\mu_1(t) \le \cdots \le \mu_N(t)$ be the increasingly ordered arrangement of eigenvalues. Then each μ_i is a $C^{0,\alpha}$-function on I with a common Hölder constant C by the result above. Let $t < s$ be in I. Then there is an i_0 such that $\lambda(t) = \mu_{i_0}(t)$. Now let t_1 be the maximum of all $r \in [t, s]$ such that $\lambda(r) = \mu_{i_0}(r)$. If $t_1 < s$ then $\mu_{i_0}(t_1) = \mu_{i_1}(t_1)$ for some $i_1 \neq i_0$. Let t_2 be the maximum of all $r \in [t_1, s]$ such that $\lambda(r) = \mu_{i_1}(r)$. If $t_2 < s$ then $\mu_{i_1}(t_2) = \mu_{i_2}(t_2)$ for some $i_2 \notin \{i_0, i_1\}$. And so on until $s = t_k$ for some $k \leq N$. Then we have (where $t_0 = t$)

$$\frac{|\lambda(s) - \lambda(t)|}{(s-t)^\alpha} \leq \sum_{j=0}^{k-1} \frac{|\mu_{i_j}(t_{j+1}) - \mu_{i_j}(t_j)|}{(t_{j+1} - t_j)^\alpha} \cdot \left(\frac{t_{j+1} - t_j}{s - t}\right)^\alpha \leq Ck \leq CN. \quad \square$$

Proof of the theorem. For each smooth curve $c : \mathbb{R} \to U$ the curve $\mathbb{R} \ni t \mapsto A(c(t))$ is $C^{0,\alpha}$, and by the 1-parameter case the eigenvalue $\lambda(c(t))$ is $C^{0,\alpha}$. But then $u \mapsto \lambda(u)$ is $C^{0,\alpha}$. $\square$

Remark. Let $u \mapsto A(u)$ be $C^{0,1}$. Choose a fixed continuous ordering of the eigenvalues, e.g., by size. We claim that along a smooth or Lipschitz curve $c(t)$ in U, none of these can accelerate to ∞ or $-\infty$ in finite time. Thus we may denote them as $\dots \lambda_i(u) \leq \lambda_{i+1}(u) \leq \dots$, for all $u \in U$. Then each λ_i is $C^{0,1}$.

The claim can be proved as follows: Let $t \mapsto A(t)$ be a Lipschitz curve. By reducing to the projection $P(t)A(t)|_{P(t)(H)}$, we may assume that $t \mapsto A(t)$ is a Lipschitz curve of Hermitian $N \times N$ matrices. So $A'(t)$ exists a.e. and is locally bounded. Let $t \mapsto \lambda(t)$ be a continuous eigenvalue. It follows that λ satisfies [9, (6)] a.e. and, as in the proof of [9, (7)], one shows that for each compact interval I there is a constant C such that $|\lambda'(t)| \leq C + C|\lambda(t)|$ a.e. in I. Since $t \mapsto \lambda(t)$ is Lipschitz, in particular, absolutely continuous, Gronwall's lemma (e.g. [3, (10.5.1.3)]) implies that $|\lambda(s) - \lambda(t)| \leq (1 + |\lambda(t)|)(e^{a|s-t|} - 1)$ for a constant a depending only on I.

References

[1] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997.

[2] J. Boman, *Differentiability of a function and of its compositions with functions of one variable*, Math. Scand. **20** (1967), 249–268.

[3] J. Dieudonné, *Foundations of modern analysis*, Pure and Applied Mathematics, Vol. X, Academic Press, New York, 1960.

[4] C.-A. Faure, *Sur un théorème de Boman*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 20, 1003–1006.

[5] ———, *Théorie de la différentiation dans les espaces convenables*, Ph.D. thesis, Université de Genéve, 1991.

[6] C.-A. Faure and A. Frölicher, *Hölder differentiable maps and their function spaces*, Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988), World Sci. Publ., Teaneck, NJ, 1989, pp. 135–142.

[7] A. Frölicher and A. Kriegl, *Linear spaces and differentiation theory*, Pure and Applied Mathematics (New York), John Wiley & Sons Ltd., Chichester, 1988, A Wiley-Interscience Publication.

[8] A. Kriegl and P. W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997, `http://www.ams.org/online_bks/surv53/`.

[9] ———, *Differentiable perturbation of unbounded operators*, Math. Ann. **327** (2003), no. 1, 191–201.

[10] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung).*, Math. Ann. **71** (1912), 441–479 (German).

Andreas Kriegl: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: `Andreas.Kriegl@univie.ac.at`

Peter W. Michor: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: Peter.Michor@univie.ac.at

Armin Rainer: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: armin.rainer@univie.ac.at