SMOOTH AND CONTINUOUS HOMOTOPIES INTO CONVENIENT MANIFOLDS AGREE

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December 6, 2002

Abstract. Continuous and smooth homotopies agree from smooth finite dimensional manifolds into infinite dimensional ones which are modeled on convenient vector spaces. Since convex charts do not exist we use radial charts.

Infinite dimensional smooth manifolds modeled on convenient vector spaces admit charts whose images are radial subsets in a convenient vector space, see [1]. 4.17. We do not know whether one can always find charts whose images are convex subsets; some radial subsets are diffeomorphic to the whole space, see [1]. 16.21.

The usual methods to approximate continuous mappings by smooth ones use integration, thus convexity. In this paper we show that one can use radial charts to obtain such approximation. We thank Anthony Blaom from Auckland for posing this problem.

1. Lemma. *Let M be a Hausdorff smooth manifold modelled on a convenient vector space E. Let $c : [0,1] \to M$ be a continuous curve.*

Then there exists a continous homotopy $h : [0,1] \times [0,1] \to M$ such that

$$h(0,s) = c(s), \quad h(t,0) = c(0), \quad h(t,1) = c(1), \quad h(1,s) \text{ smooth in } s.$$

Proof. For each $s \in [0,1]$ let $u_s : U_s \to E$ be a chart with $u_s(c(s)) = 0$ and $u_s(U_s)$ a radial open set in E; radial open sets form a neighborhood basis for 0 in E for the c^∞-topology.

There exist $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $c[s_i, s_{i+1}] \subset U_{s_i}$. Thus we may assume that $c([0,1])$ is contained in a radial open subset U in E with $c(0) = 0$. Now we choose smooth $\varphi : [0,1] \to [0,1]$ vanishing on $[0,1/2]$ with $\varphi(1) = 1$

$$h(t,s) = ((1-s)\varphi(1-t) + s)\, c((1-s)\varphi(t) + s)$$

Clearly we can modify this construction such that $h(t,s)$ is constant near $(1,0)$ and $(1,1)$. Then the pieces fit together smoothly, for $t = 1$. $\square$

PWM was supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 14195 MAT.'

2. Proposition. *Let M be a Hausdorff smooth manifold modelled on a convenient vector space E. Let $h : [0,1]\times[0,1] \to M$ be a continuous homotopy such that $h(t,s)$ is smooth in s for $t = 0,1$ and smooth in t for $s = 0,1$.*

Then there exists a continous homotopy $H : [0,1] \times [0,1] \times [0,1] \to M$ such that

$$\begin{aligned} &H(0,t,s) = h(t,s),\\ &H(r,t,s) = h(t,s) \text{ if } t = 0,1 \text{ or } s = 0,1\\ &H(1,t,s) \quad \text{is smooth in } (t,s) \end{aligned}$$

Proof. For each $(t,s) \in [0,1] \times [0,1]$ let $u_{(t,s)} : U_{(t,s)} \to E$ be a chart with $u_{(t,s)}(h(t,s)) = 0$ and $u_{(t,s)}(U_{(t,s)})$ a radial open set in E.

There exist $0 = s_0 < s_1 < \dots < s_n = 1$ and $0 = t_0 < t_1 < \dots < t_m = 1$ such that $h([t_j, t_{j+1}] \times [s_i, s_{i+1}]) \subset U_{(t_j,s_i)}$. By using a surjective mapping $\varphi : [0,1] \times [0,1] \to [0,1]\times[0,1]$ with $\varphi(t,s) = (t_i, s)$ for t near t_i and $\varphi(t,s) = (t, s_j)$ for s near s_j we can replace the homotopy by one which is constant along normal directions to the grid $\{(t,s) : t = t_i \text{ or } s = s_j\}$. By lemma 1, for each segment $h([t_i, t_{i+1}] \times \{s_j\})$ there exists a smooth curve and a connecting homotopy in $U_{(t_i,s_j)} \cap U_{(t_i,s_{j-1})}$. Similarly for $h(\{t_i\} \times [s_j, s_{j+1}])$. We use the tranversally constant period of h to insert the homotopy to the smooth curve and back and thus we may assume without loss that h is already smooth in a neighborhood of the grid and still constant normally to the grid. Along the the boundary we add a layer of cells which describe the smooth homotopy (by reparameterizations alone) between the old and the new curves on the boundary.

We treat each cell of the grid separately. So we may assume without loss that the whole $h([0,1] \times [0,1])$ is contained in a radial open set U in E, with $h(0,0) = 0$, is smooth near the boundary, and constant normally to the boundary.

Now we use lemma 1 along each ray from $(0,0)$ exactly as described in the proof of 1 to get the required homotopy. □

3. Proposition. *Let M be a finite dimensional smooth manifold, even with boundary or corners. Let N be a smooth manifold which is modelled on a covenient vector space F. If smooth mappings $f, g : M \to N$ are continuously homotopic then they are also smoothly homotopic.*

Proof. Let $\dim M = m$. Let $h : [0,1] \times M \to N$ be the continuous homotopy. We may assume (by a homotopy using only reparameterization) that $h(t,x)$ is constant in t near $t = 0$ and $t = 1$.

We cover N by smooth charts which are diffeomorphic to radial open sets in F. There exists a smooth triangulation of $[0,1]\times M$ respecting all boundary faces such that each simplex and fixed vertex is mapped by h into one chart such that the vertex goes to the center of the chart.

There exists a smooth homotopy k on $[0,1] \times M$ which for each vertex in the 0-skeleton contracts a small open ball around this vertex into the vertex. and remaint the identity outside of larger balls. Composing h with k we can replace h by homotopic h_1 which is constant near the 0-skeleton, and is called again h.

Next we use a smooth homotopy k on $[0,1] \times M$ which contracts a tubular neighbourhood of each 1-simplex into the 1-simplex and remains the identity outside of larger tubular neighbourhoods. Composing h (which is already constant near the 0-skeleton) with k we replace h by a homotopy h_1 which is locally constant in normal directions to the 1-skeleton (in some Riemannian metric, e.g.). Call this again h.

By induction we can replace h by a homotopic one which is locally constant near the m-skeleton in normal directions to it. Note that h then still maps each simplex into some chart of N.

By lemma 1, for each 1-simplex σ^1 we have a homotopy relative ends between $h|\sigma^1$ and a smooth mapping $\sigma^1 \to N$ which stays completely in all radial charts corresponding to each m-simplex containing σ^1, for both vertices. We use the normally constant period of h near σ^1 to insert the homotopy to the smooth curve, rest there, and to go back, and thus we may assume without loss that h is already smooth in a neighborhood of the m-skeleton, and still locally constant near the 1-skeleton in normal directions.

Next, for each 2-simplex σ^2 choose a vertex p, connect it with the rays to the points of the opposite face, and apply the construction in the proof of lemma 1 to replace h with a homotopic one which is smooth near the 2-skeleton, and still constant in normal directions near the m-skeleton.

By iteration we can replace h by a homotopic mapping which is smooth. Note that each application of lemma 1 furnishes a smooth homotopy where h was already smooth (near $\{0,1\} \times M$, since $h(0,x) = f(x)$ and $h(1,x) = g(x)$, for example). Thus we may adjust the ends of the resulting smooth homotopy so that they are between f and g. $\square$

The proof of proposition 3 can easily be adapted to show the following:

4. Corollary. *Let $f : M \to N$ be a continuous mapping from a finite dimensional smooth manifold which may have corners, into a smooth manifold modelled on convenient vector spaces.*

Then f is continuously homotopic to a smooth mapping. This homotopy can be chosen constant on a submanifold where f is already smooth.

References

[1] Kriegl, A.; Michor, Peter W., *The Convenient Setting of Global Analysis*, Surveys and Monographs, Vol. 53, AMS, Providence, 1997, http://www.ams.org/online_bks/surv53/.

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