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THE JACOBI FLOW

Peter W. Michor

Erwin Schrödinger Institut für Mathematische
Physik, Pasteurgasse 6/7, A-1090 Wien, Austria

For Wlodek Tulczyjew, on the occasion of his 65th birthday.

It is well known that the geodesic flow on the tangent bundle is the flow of a certain vector field which is called the spray $S : TM \to TTM$. It is maybe less well known that the flow lines of the vector field $\kappa_{TM} \circ TS : TTM \to TTTM$ project to Jacobi fields on TM. This could be called the 'Jacobi flow'. This result was developed for the lecture course [5], and it is the main result of this paper. I was motivated by the paper [6] of Urbanski in these proceedings to publish it, as an explanation of some of the uses of iterated tangent bundles in differential geometry.

1. The tangent bundle of a vector bundle. Let (E, p, M) be a vector bundle with fiber addition $+_E : E \times_M E \to E$ and fiber scalar multiplication $m_t^E : E \to E$. Then (TE, π_E, E), the tangent bundle of the manifold E, is itself a vector bundle, with fiber addition denoted by $+_{TE}$ and scalar multiplication denoted by m_t^{TE}.

If $(U_\alpha, \psi_\alpha : E \restriction U_\alpha \to U_\alpha \times V)_{\alpha \in A}$ is a vector bundle atlas for E, such that (U_α, u_α) is a manifold atlas for M, then $(E \restriction U_\alpha, \psi'_\alpha)_{\alpha \in A}$ is an atlas for the manifold E, where

$$\psi'_\alpha := (u_\alpha \times \mathrm{Id}_V) \circ \psi_\alpha : E \restriction U_\alpha \to U_\alpha \times V \to u_\alpha(U_\alpha) \times V \subset \mathbb{R}^m \times V.$$

Hence the family $(T(E \restriction U_\alpha), T\psi'_\alpha : T(E \restriction U_\alpha) \to T(u_\alpha(U_\alpha) \times V) = u_\alpha(U_\alpha) \times V \times \mathbb{R}^m \times V)_{\alpha \in A}$ is the atlas describing the canonical vector bundle structure of (TE, π_E, E). The transition functions are in turn:

$$\begin{aligned}
(\psi_\alpha \circ \psi_\beta^{-1})(x, v) &= (x, \psi_{\alpha\beta}(x)v) \quad \text{ for } x \in U_{\alpha\beta} \\
(u_\alpha \circ u_\beta^{-1})(y) &= u_{\alpha\beta}(y) \quad \text{ for } y \in u_\beta(U_{\alpha\beta}) \\
(\psi'_\alpha \circ (\psi'_\beta)^{-1})(y, v) &= (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v) \\
(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y, v; \xi, w) &= \big(u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v; d(u_{\alpha\beta})(y)\xi, \\
&\qquad (d(\psi_{\alpha\beta} \circ u_\beta^{-1})(y))\xi)v + \psi_{\alpha\beta}(u_\beta^{-1}(y))w\big).
\end{aligned}$$

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So we see that for fixed (y, v) the transition functions are linear in $(\xi, w) \in \mathbb{R}^m \times V$. This describes the vector bundle structure of the tangent bundle (TE, π_E, E).

For fixed (y, ξ) the transition functions of TE are also linear in $(v, w) \in V \times V$. This gives a vector bundle structure on (TE, Tp, TM). Its fiber addition will be denoted by $T(+_E) : T(E \times_M E) = TE \times_{TM} TE \to TE$, since it is the tangent mapping of $+_E$. Likewise its scalar multiplication will be denoted by $T(m_t^E)$. One may say that the second vector bundle structure on TE, that one over TM, is the derivative of the original one on E.

The space $\{\Xi \in TE : Tp.\Xi = 0 \text{ in } TM\} = (Tp)^{-1}(0)$ is denoted by VE and is called the *vertical bundle* over E. The local form of a vertical vector Ξ is $T\psi'_\alpha.\Xi = (y, v; 0, w)$, so the transition functions are $(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y, v; 0, w) = (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v; 0, \psi_{\alpha\beta}(u_\beta^{-1}(y))w)$. They are linear in $(v, w) \in V \times V$ for fixed y, so VE is a vector bundle over M. It coincides with $0_M^*(TE, Tp, TM)$, the pullback of the bundle $TE \to TM$ over the zero section. We have a canonical isomorphism $\mathrm{Vl}_E : E \times_M E \to VE$, called the *big vertical lift*, given by $\mathrm{Vl}_E(u_x, v_x) := \partial_t|_0(u_x + tv_x)$, which is fiber linear over M. We will mainly use the *small vertical lift* $\mathrm{vl}_E : E \to TE$, given by $\mathrm{vl}_E(v_x) = \partial_t|_0 t.v_x = \mathrm{Vl}_E(0_x, v_x)$. The local representation of the vertical lift is $(T\psi'_\alpha \circ \mathrm{vl}_E \circ (\psi'_\alpha)^{-1})(y, v) = (y, 0; 0, v)$.

If $\varphi : (E, p, M) \to (F, q, N)$ is a vector bundle homomorphism, then we have $\mathrm{vl}_F \circ \varphi = T\varphi \circ \mathrm{vl}_E : E \to VF \subset TF$. So vl is a natural transformation between certain functors on the category of vector bundles and their homomorphisms. The mapping $\mathrm{vrp}_E := pr_2 \circ \mathrm{Vl}_E^{-1} : VE \to E$ is called the *vertical projection*.

2. The second tangent bundle of a manifold. All of 1 is valid for the second tangent bundle TTM of a manifold, but here we have one more natural structure at our disposal. The *canonical flip* or *involution* $\kappa_M : TTM \to TTM$ is defined locally by

$$(TTu \circ \kappa_M \circ TTu^{-1})(x, \xi; \eta, \zeta) = (x, \eta; \xi, \zeta),$$

where (U, u) is a chart on M. Clearly this definition is invariant under changes of charts (Tu_α equals ψ'_α from 1).

The flip κ_M has the following properties:

(1) $\kappa_N \circ TTf = TTf \circ \kappa_M$ for each $f \in C^\infty(M, N)$.
(2) $T(\pi_M) \circ \kappa_M = \pi_{TM}$ and $\pi_{TM} \circ \kappa_M = T(\pi_M)$.
(3) $\kappa_M^{-1} = \kappa_M$.
(4) κ_M is a linear isomorphism from the vector bundle $(TTM, T(\pi_M), TM)$ to the bundle (TTM, π_{TM}, TM), so it interchanges the two vector bundle structures on TTM.
(5) It is the unique smooth mapping $TTM \to TTM$ which satisfies

$$\partial_t \partial_s c(t, s) = \kappa_M \partial_s \partial_t c(t, s)$$

for each $c : \mathbb{R}^2 \to M$.

All this follows from the local formula given above. A quite early use of κ_M is in [4].

3. Lemma. *For vector fields X, $Y \in \mathfrak{X}(M)$ we have*

$$\begin{aligned}
[X,Y] &= \mathrm{vrp}_{TM} \circ (TY \circ X - \kappa_M \circ TX \circ Y),\\
TY \circ X -_{TM} \kappa_T \circ TX \circ Y &= \mathrm{Vl}_{TM}(Y, [X,Y])\\
&= (\mathrm{vl}_{TM} \circ [X,Y])\; T(+_{TM})\; (0_{TM} \circ Y).
\end{aligned}$$

See [3] 6.13, 6.19, or 37.13 for different proofs of this well known result.

4. Linear connections and their curvatures. Let (E, p, M) be a vector bundle. Recall that a linear connection on the vector bundle E can be described by specifying its *connector* $K : TE \to E$. This notions seems to be due to [2]. Any smooth mapping $K : TE \to E$ which is a (fiber linear) homomorphism for both vector bundle structures on TE,

$$\begin{tikzcd}
TE \arrow[r, "K"] \arrow[d, "\pi_E"'] & E \arrow[d, "p"] \\
E \arrow[r, "p"] & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TE \arrow[r, "K"] \arrow[d, "Tp"'] & E \arrow[d, "p"] \\
TM \arrow[r, "\pi_M"] & M
\end{tikzcd}$$

and which is a left inverse to the vertical lift, $K \circ \mathrm{vl}_E = \mathrm{Id}_E : E \to TE \to E$, specifies a linear connection. Namely: The inverse image $H := K^{-1}(0_E)$ of the zero section $0_E \subset E$, it is a subvector bundle for both vector bundle structures, and for the vector bundle stucture $\pi_E : TE \to E$ the subbundle H turns out to be a complementary bundle for the vertical bundle $VE \to E$. We get then the associated *horizontal lift mapping*

$$C : TM \times_M E \to TE, \quad C(\quad, u) = \Big(Tp|\ker(K : T_uE \to E_{p(u)})\Big)^{-1}$$

which has the following properties

$$\begin{aligned}
&(Tp, \pi_E) \circ C = \mathrm{Id}_{TM \times_M E},\\
&C(\quad, u) : T_{p(u)}M \to T_uE \text{ is linear for each } u \in E,\\
&C(X_x, \quad) : E_x \to (Tp)^{-1}(X_x) \text{ is linear for each } X_x \in T_xM.
\end{aligned}$$

Conversely given a smooth horizontal lift mapping C with these properties one can reconstruct a connector K.

For any manifold N, smooth mapping $s : N \to E$ along $f = p \circ s : N \to M$, and vector field $X \in \mathfrak{X}(N)$ a connector $K : TE \to E$ defines the *covariant derivative* of s along X by

(1) $$\nabla_X s := K \circ Ts \circ X : N \to TN \to TE \to E.$$

See the following diagram for all the mappings.

(2) $$\begin{tikzcd}
 & TE \arrow[d, "\pi_E"] \arrow[dr, "K"] & \\
TN \arrow[ur, "Ts"] & E & E \\
N \arrow[u, "X"] \arrow[ur, "s"] \arrow[urr, "\nabla_X s"'] \arrow[r, "f"'] & M &
\end{tikzcd}$$

In canonical coordinates as in 1 we have then

$$\begin{aligned}
&C((y,\xi),(y,v)) = (y,v;\xi,\Gamma_y(v,\xi)),\\
&K(y,v;\xi,w) = (y,w-\Gamma_y(v,\xi)),\\
&\nabla_{(y,\xi)}(\mathrm{Id},s) = (\mathrm{Id},ds(y)\xi-\Gamma_y(s(y),\xi)),
\end{aligned}$$

where the *Christoffel symbol* $\Gamma_y(v,\xi)$ is smooth in y and bilinear in (v,ξ). Here the sign is the negative of the one in many more traditional approaches, since Γ parametrizes the horizontal bundle.

Let $C^\infty_f(N,E)$ denote the space of all sections along f of E, isomorphic to the space $C^\infty(f^*E)$ of sections of the pullback bundle. The covariant derivative may then be viewed as a bilinear mapping $\nabla : \mathfrak{X}(N)\times C^\infty_f(N,E)\to C^\infty_f(N,E)$. It has the following properties which follow directly from the definitions:

(3) $\nabla_X s$ is $C^\infty(N,\mathbb{R})$-linear in $X\in\mathfrak{X}(N)$. For $x\in N$ also we have $\nabla_{X(x)}s = K.Ts.X(x) = (\nabla_X s)(x)\in E$.
(4) $\nabla_X(h.s) = dh(X).s + h.\nabla_X s$ for $h\in C^\infty(N,\mathbb{R})$.
(5) For any manifold Q, smooth mapping $g:Q\to N$, and $Y_y\in T_yQ$ we have $\nabla_{Tg.Y_y}s = \nabla_{Y_y}(s\circ g)$. If $Y\in\mathfrak{X}(Q)$ and $X\in\mathfrak{X}(N)$ are g-related, then we have $\nabla_Y(s\circ g) = (\nabla_X s)\circ g$.

For vector fields X, $Y\in\mathfrak{X}(M)$ and a section $s\in C^\infty(E)$ the curvature $R\in\Omega^2(M,L(E,E))$ of the connection is given by

(6)
$$R(X,Y)s = ([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})s$$

Theorem. *Let $K:TE\to E$ be the connector of a linear connection on a vector bundle (E,p,M). If $s:N\to E$ is a section along $f:=p\circ s:N\to M$ then we have for vector fields X, $Y\in\mathfrak{X}(N)$*

(7)
$$\begin{aligned}
\nabla_X\nabla_Y s-\nabla_Y\nabla_X s-\nabla_{[X,Y]}s &=\\
&=(K\circ TK\circ\kappa_E - K\circ TK)\circ TTs\circ TX\circ Y =\\
&= R\circ(Tf\circ X, Tf\circ Y)s : N\to E,
\end{aligned}$$

where $R\in\Omega^2(M;L(E,E))$ is the curvature.

Proof. Let first $m^E_t : E\to E$ denote the scalar multiplication. Then we have $\partial_t|_0 m^E_t = \mathrm{vl}_E$ where $\mathrm{vl}_E : E\to TE$ is the vertical lift. We use then lemma 3 and some obvious commutation relations to get in turn:

$$\begin{aligned}
&\mathrm{vl}_E\circ K = \partial_t|_0 m^E_t\circ K = \partial_t|_0 K\circ m^{TE}_t = TK\circ\partial_t|_0 m^{TE}_t = TK\circ\mathrm{vl}_{(TE,\pi_E,E)}\,.\\
&\nabla_X\nabla_Y s-\nabla_Y\nabla_X s-\nabla_{[X,Y]}s\\
&\qquad = K\circ T(K\circ Ts\circ Y)\circ X - K\circ T(K\circ Ts\circ X)\circ Y - K\circ Ts\circ[X,Y]\\
&K\circ Ts\circ[X,Y] = K\circ\mathrm{vl}_E\circ K\circ Ts\circ[X,Y]\\
&\qquad = K\circ TK\circ\mathrm{vl}_{TE}\circ Ts\circ[X,Y] = K\circ TK\circ TTs\circ\mathrm{vl}_{TN}\circ[X,Y]\\
&\qquad = K\circ TK\circ TTs\circ((TY\circ X-\kappa_N\circ TX\circ Y)\ (T-)\ 0_{TN}\circ Y)\\
&\qquad = K\circ TK\circ TTs\circ TY\circ X - K\circ TK\circ TTs\circ\kappa_N\circ TX\circ Y - 0.
\end{aligned}$$

Now we sum up and use $TTs \circ \kappa_N = \kappa_E \circ TTs$ to get the first result. If in particular we choose $f = \mathrm{Id}_M$ so that s is a section of $E \to M$ and X, Y are vector fields on M, then we get the curvature R.

To see that in the general case $(K \circ TK \circ \kappa_E - K \circ TK) \circ TTs \circ TX \circ Y$ coincides with $R(Tf \circ X, Tf \circ Y)s$ one has to write out (1) and $(TTs \circ TX \circ Y)(x) \in TTE$ in canonical charts induced from vector bundle charts of E. $\square$

5. Torsion. *Let $K : TTM \to M$ be a linear connector on the tangent bundle, let $X, Y \in \mathfrak{X}(M)$. Then the torsion is given by*

$$\mathrm{Tor}(X, Y) = (K \circ \kappa_M - K) \circ TX \circ Y.$$

If moreover $f : N \to M$ is smooth and $U, V \in \mathfrak{X}(N)$ then we get also

$$\begin{aligned}\mathrm{Tor}(Tf.U, Tf.V) &= \nabla_U(Tf \circ V) - \nabla_V(Tf \circ U) - Tf \circ [U, V] \\ &= (K \circ \kappa_M - K) \circ TTf \circ TU \circ V.\end{aligned}$$

Proof. (9) We have in turn

$$\begin{aligned}\mathrm{Tor}(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= K \circ TY \circ X - K \circ TX \circ Y - K \circ \mathrm{vl}_{TM} \circ [X, Y] \\ K \circ \mathrm{vl}_{TM} \circ [X, Y] &= K \circ ((TY \circ X - \kappa_M \circ TX \circ Y)\ (T-)\ 0_{TM} \circ Y) \\ &= K \circ TY \circ X - K \circ \kappa_M \circ TX \circ Y - 0.\end{aligned}$$

An analogous computation works in the second case, and that $(K \circ \kappa_M - K) \circ TTf \circ TU \circ V = \mathrm{Tor}(Tf.U, Tf.V)$ can again be checked in local coordinates. $\square$

6. Sprays. Given a linear connector $K : TTM \to M$ on the tangent bundle with its horizontal lift mapping $C : TM \times_M TM \to TTM$, then $S := C \circ \mathrm{diag} : TM \to TM \times_M TM \to TTM$ is called the *spray*. This notion is due to [1]. The spray has the following properties:

$\pi_{TM} \circ S = \mathrm{Id}_{TM}$	a vector field on TM,
$T(\pi_M) \circ S = \mathrm{Id}_{TM}$	a second order differential equation,
$S \circ m_t^{TM} = T(m_t^{TM}) \circ m_t^{TTM} \circ S$	'quadratic',

where m_t^E is the scalar multiplication by t on a vector bundle E. From S one can reconstruct the torsion free part of C. The following result is well known:

Lemma. *For a spray $S : TM \to TTM$ on M, for $X \in TM$*

$$\mathrm{geo}^S(X)(t) := \pi_M(\mathrm{Fl}_t^S(X))$$

defines a geodesic structure on M, where Fl^S is the flow of the vector field S.

The abstract properties of a geodesic structure are obvious:

$$\begin{aligned}&\mathrm{geo} : TM \times \mathbb{R} \supset U \to M \\ &\mathrm{geo}(X)(0) = \pi_M(X), \quad \partial_t|_0\, \mathrm{geo}(X)(t) = X \\ &\mathrm{geo}(tX)(s) = \mathrm{geo}(X)(ts) \\ &\mathrm{geo}(\mathrm{geo}(X)'(t))(s) = \mathrm{geo}(X)(t+s)\end{aligned}$$

From a geodesic structure one can reconstruct the spray by differentiation.

7. Theorem. *Let $S : TM \to TTM$ be a spray on a manifold M. Then $\kappa_{TM} \circ TS : TTM \to TTTM$ is a vector field. Consider a flow line*

$$Y(t) = \mathrm{Fl}_t^{\kappa_{TM} \circ TS}(Y(0))$$

of this field. Then we have:

$c := \pi_M \circ \pi_{TM} \circ Y$ is a geodesic on M.
$\dot{c} = \pi_{TM} \circ Y$ is the velocity field of c.
$J := T(\pi_M) \circ Y$ is a Jacobi field along c.
$\dot{J} = \kappa_M \circ Y$ is the velocity field of J.
$\nabla_{\partial_t} J = K \circ \kappa_M \circ Y$ is the covariant derivative of J.
The Jacobi equation is given by:

$$\begin{aligned} 0 &= \nabla_{\partial_t} \nabla_{\partial_t} J + R(J, \dot{c})\dot{c} + \nabla_{\partial_t} \operatorname{Tor}(J, \dot{c}) \\ &= K \circ TK \circ TS \circ Y. \end{aligned}$$

This implies that in a canonical chart induced from a chart on M the curve $Y(t)$ is given by

$$(c(t), c'(t); J(t), J'(t)).$$

Proof. Consider a curve $s \mapsto X(s)$ in TM. Then each $t \mapsto \pi_M(\mathrm{Fl}_t^S(X(s)))$ is a geodesic in M, and in the variable s it is a variation through geodesics. Thus $J(t) := \partial_s|_0 \pi_M(\mathrm{Fl}_t^S(X(s)))$ is a Jacobi field along the geodesic $c(t) := \pi_M(\mathrm{Fl}_t^S(X(0)))$, and each Jacobi field is of this form, for a suitable curve $X(s)$. We consider now the curve $Y(t) := \partial_s|_0 \mathrm{Fl}_t^S(X(s))$ in TTM. Then by 2.(6) we have

$$\begin{aligned} \partial_t Y(t) &= \partial_t \partial_s|_0 \mathrm{Fl}_t^S(X(s)) = \kappa_{TM} \partial_s|_0 \partial_t \mathrm{Fl}_t^S(X(s)) = \kappa_{TM} \partial_s|_0 S(\mathrm{Fl}_t^S(X(s))) \\ &= (\kappa_{TM} \circ TS)(\partial_s|_0 \mathrm{Fl}_t^S(X(s))) = (\kappa_{TM} \circ TS)(Y(t)), \end{aligned}$$

so that $Y(t)$ is a flow line of the vector field $\kappa_{TM} \circ TS : TTM \to TTTM$. Moreover using the properties of κ from section 2 and of S from section 6 we get

$$\begin{aligned} T\pi_M.Y(t) &= T\pi_M.\partial_s|_0 \mathrm{Fl}_t^S(X(s)) = \partial_s|_0 \pi_M(\mathrm{Fl}_t^S(X(s))) = J(t), \\ \pi_M T\pi_M Y(t) &= c(t), \text{ the geodesic,} \\ \partial_t J(t) &= \partial_t T\pi_M.\partial_s|_0 \mathrm{Fl}_t^S(X(s)) = \partial_t \partial_s|_0 \pi_M(\mathrm{Fl}_t^S(X(s))), \\ &= \kappa_M \partial_s|_0 \partial_t \pi_M(\mathrm{Fl}_t^S(X(s))) = \kappa_M \partial_s|_0 \partial_t \pi_M(\mathrm{Fl}_t^S(X(s))) \\ &= \kappa_M \partial_s|_0 T\pi_M.\partial_t \mathrm{Fl}_t^S(X(s)) = \kappa_M \partial_s|_0 (T\pi_M \circ S) \mathrm{Fl}_t^S(X(s)) \\ &= \kappa_M \partial_s|_0 \mathrm{Fl}_t^S(X(s)) = \kappa_M Y(t), \\ \nabla_{\partial_t} J &= K \circ \partial_t J = K \circ \kappa_M \circ Y. \end{aligned}$$

Finally let us express the well known Jacobi expression, where we put $\gamma(t, s) := \pi_M(\mathrm{Fl}_t^S(X(s)))$ for short and use most of the expressions from above:

$$\begin{aligned} &\nabla_{\partial_t} \nabla_{\partial_t} J + R(J, \dot{c})\dot{c} + \nabla_{\partial_t} \operatorname{Tor}(J, \dot{c}) = \\ &\quad = \nabla_{\partial_t} \nabla_{\partial_t}.T\gamma.\partial_s + R(T\gamma.\partial_s, T\gamma.\partial_t)T\gamma.\partial_t + \nabla_{\partial_t} \operatorname{Tor}(T\gamma.\partial_s, T\gamma.\partial_t) \\ &\quad = K.T(K.T(T\gamma.\partial_s).\partial_t).\partial_t \\ &\qquad + (K.TK.\kappa_{TM} - K.TK).TT(T\gamma.\partial_t).T\partial_s.\partial_t \\ &\qquad + K.T((K.\kappa_M - K).TT\gamma.T\partial_s.\partial_t).\partial_t \end{aligned}$$

Note that for example for the term in the second summand we have

$$TTT\gamma.TT\partial_t.T\partial_s.\partial_t = T(T(\partial_t\gamma).\partial_s).\partial_t = \partial_t\partial_s\partial_t\gamma = \partial_t.\kappa_M.\partial_t.\partial_s\gamma = T\kappa_M.\partial_t.\partial_t.\partial_s\gamma$$

which at $s=0$ equals $T\kappa_M\ddot{J}$. Using this we get for the Jacobi expression at $s=0$:

$$\begin{aligned}
&\nabla_{\partial_t}\nabla_{\partial_t}J + R(J,\dot c)\dot c + \nabla_{\partial_t}\operatorname{Tor}(J,\dot c) = \\
&\quad = (K.TK + K.TK.\kappa_{TM}.T\kappa_M - K.TK.T\kappa_M + K.TK.T\kappa_M - K.TK).\partial_t\partial_t J = \\
&\quad = K.TK.\kappa_{TM}.T\kappa_M.\partial_t\partial_t J = K.TK.\kappa_{TM}.\partial_t Y = K.TK.TS.Y,
\end{aligned}$$

where we used $\partial_t\partial_t J = \partial_t(\kappa_M.Y) = T\kappa_M\partial_t Y = T\kappa_M.\kappa_{TM}.TS.Y$. Finally the validity of the Jacobi equation $0 = K.TK.TS.Y$ follows trivially from $K\circ S = 0_{TM}$. $\square$

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Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: `Peter.Michor@esi.ac.at`