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MANIFOLDS OF SMOOTH MAPS, II: THE LIE GROUP OF DIFFEOMORPHISMS OF A NON-COMPACT SMOOTH MANIFOLD *)

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ABSTRACT.

It is shown that $Diff(X)$, the space of diffeomorphisms of a locally compact smooth manifold X, is a Lie group.

This paper is a sequel of [8], where we presented a manifold structure on the space $C^\infty(X, Y)$ of smooth mappings $X \to Y$ for (arbitrary non-compact) finite dimensional manifolds X, Y, using the notion of differentiability C^∞_π of Keller [4]. The main idea was the introduction of a new topology.

Here we show that $Diff(X)$, the space of diffeomorphisms of a locally compact manifold X, equipped with the $\mathfrak{D}^\infty$-topology of [8], is a Lie group in the same notion of differentiability C^∞_π.

In Gutknecht [3] it is shown that $Diff(X)$ for compact X admits a Lie group structure in the stronger notion C^∞_Γ. This is done by the functorial method of deriving the adjunction relation

$$C^\infty_\Gamma(X, C^\infty(Y, Z)) = C^\infty(X \times Y, Z)$$

for compact Y. An easy corollary of this is the C^∞_Γ-differentiability of the composition on $Diff(X)$. Unit and counit of this adjunction are the C^∞_Γ-differentiable mappings:

$$X \to C^\infty(Y, X \times Y) \quad \text{given by} \quad x \mapsto (y \mapsto (x, y)),$$

and evaluation

$$Ev: X \times C^\infty(X, Y) \to Y \quad \text{given by} \quad Ev(x, f) = f(x).$$

If Y is not compact, then the first mapping is not even continuous if

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$C^\infty(Y, X\times Y)$ is equipped with the Whitney-C^0-topology: a continuous curve stays constant outside some fixed compact of Y if the parameter stays within some compact set of $\mathbb{R}$. See [9] for a more detailed account of this. So the above adjunction does not exist, if Y is not compact. Therefore we are forced to prove the C^∞_π-differentiability of the composition by direct «onslaught»: the proof is complicated and heavy going and we excuse for the lack of elegance. The method of proof of [6] is of no help, since it is wrong.

The reader is assumed to be familiar with [8], especially with the sections dealing with topology (Sections 1, 2) and with the Ω-lemma (3.8). The manifold structure will be explained again (6.3) in a somewhat simpler form as presented in [9]. Sections will be numbered from 5 onwards, following those of [8] (Sections 1 to 4), citations with leading number less than 5 refer to [8] (e.g. 3.8).

5. SOME TOPOLOGY AGAIN.

Let X be a smooth finite dimensional manifold. Let $J^n(X,X)$ be the smooth fibre bundle of n-jets of smooth mappings from X to X (see 1.1). Let $J^n(X,X)_{x,y}$ be the fibre over $(x,y)\in X\times X$, i.e., the space of n-jets at x of maps $f\in C^\infty(X,X)$ with $f(x)=y$. Further, let $J^n_{inv}(X,X)_{x,y}$ be the open subset of invertible n-jets from x to y. It is clear that

$$J^n_{inv}(X,X)_{x,y} = \pi^{-1}_{1,n}(GL(X,X)_{x,y}),$$

where $\pi_{k,n}: J^n(X,X)\to J^k(X,X)$ is the canonical projection for $n\geq k$ (cf. 1.3) and $GL(X,X)_{x,y}$ denotes the open subset of invertible 1-jets from x to y in $J^1(X,X)_{x,y}$: in a canonical chart $J^1(X,X)_{x,y}$ corresponds to the space of all $\dim X\times \dim X$-matrices and $GL(X,X)_{x,y}$ corresponds to the open subset of invertible ones. By the construction of the canonical chart for $J^n(X,Y)$ it is clear that

$$J^n_{inv}(X,X) = \bigcup_{(x,y)\in X\times X} J^n_{inv}(X,X)_{x,y}$$

is a smooth subbundle of the fibre bundle $J^n(X,X)$.

5.1. LEMMA. *The mapping*

$$inv: J^n_{inv}(X, X)_{x,y} \to J^n_{inv}(X, X)_{y,x},$$

given by $inv(\sigma) = \sigma^{-1}$ *for* $\sigma \in J^n_{inv}(X, X)_{x,y}$, *is smooth and*

$$inv: J^n_{inv}(X, X) \to J^n_{inv}(X, X)$$

is a smooth fibre bundle homomorphism over

$$(x, y) \mapsto (y, x): X \times X \to X \times X.$$

PROOF. By looking at a canonical chart we see that $inv(\sigma) = \sigma^{-1}$ corresponds just to the inverse power series of the polynomial mapping $\mathbb{R}^{\dim X}, 0 \to \mathbb{R}^{\dim X}, 0$ corresponding to σ, truncated at order n. Since the coefficients of the inverse power series are rational functions of the coefficients of the polynomial, the assertion follows. QED

5.2. PROPOSITION. *The set* $Diff(X)$ *of diffeomorphisms* $X \to X$ *is an open subset of* $C^\infty(X, X)$ *in the* $\mathfrak{D}$*-topology and the* $\mathfrak{D}^\infty$*-topology.*

PROOF. These two topologies are described in Sections 1, 2 respectively. $Diff(X)$ is open in the coarser Whitney-C^∞-topology (see [7], Proposition 2.5), so the assertion follows. QED

5.3. THEOREM. *The mapping*

$$Inv: Diff(X) \to Diff(X), \text{ given by } Inv(f) = f^{-1},$$

is continuous in the $\mathfrak{D}$*-topology and the* $\mathfrak{D}^\infty$*-topology.*

PROOF. First we show that this is so for the $\mathfrak{D}$-topology. We use the base for it described in 1.5 (c): let $f \in Diff(X)$ and let $M'(L, U)$ be a basic $\mathfrak{D}$-open neighborhood of f^{-1} in $Diff(X)$, i.e., $L = (L_n)$ and $U = (U_n)$, where each L_n is compact in X with $(X \backslash L_n{}^o)$ being a locally finite family, and each U_n is open in $J^n_{inv}(X, X)$ for each $n \geq 0$. Then

$$M'(L, U) = \{ g \in C^\infty(X, X) \mid j^n(g)(X \backslash L_n{}^o) \subset U_n \text{ for all } n \}.$$

We want to construct a $\mathfrak{D}$-open neighborhood P of f such that

$$Inv(P \cap Diff(X)) \subset M'(L, U).$$

$f^{-1} \in M'(L, U)$ means

$$j^n(f^{-1})^{-1}(U_n) \supset X \backslash L_n{}^o \quad \text{for all } n.$$

Let L'_n be a sequence of compacts of X such that

$$(j^n(f^{-1}))^{-1}(U_n) \supset X \backslash L'^o_n \supset X \backslash L'_n \supset X \backslash L_n{}^o$$

and such that $X \backslash L'^o_n$ is still a locally finite family. Let

$$K'_n = f^{-1}(L'_n), \quad K_n = f^{-1}(L_n), \quad K' = (K'_n) \text{ and } K = (K_n).$$

These two are sequences of compacts and $(X \backslash K'^o_n)$, $(X \backslash K_n{}^o)$ are locally finite families. Further $K'_n \subset K_n{}^o$.

Let d be a metric on X and let $\mathfrak{p}$ be a strictly positive continuous function on X such that

$$0 < max\{\mathfrak{p}(x) \mid x \in K_n\} < \text{distance between the compact } L'_n \text{ and the disjoint closed set } X \backslash L_n{}^o,$$

for each $n \in \mathbb{N}$. Such a function may be found since $(X \backslash L'^o_n)$ is locally finite (cf. Proof of 1.4). Let

$$V_n = inv(U_n) \subset J^n_{inv}(X, X),$$

then V_n is open by 5.1, and let $V = (V_n)$.

Consider the basic $\mathfrak{D}$-open set $M'(K', V)$. We claim that it contains f. For let $n \in \mathbb{N}$ and $x \in X \backslash K'^o_n$, then

$$j^n(f)(x) = inv(j^n(f^{-1})(f(x)));$$

but

$$f(x) \in f(X \backslash K'^o_n) = X \backslash L'^o_n, \quad \text{so} \quad j^n(f^{-1})(f(x)) \in U_n$$

by the choice of L'_n. This implies

$$j^n(f)(x) \in inv(U_n) = V_n \quad \text{and so} \quad f \in M'(K', V).$$

Now let $V_{\mathfrak{p} \circ f}(f)$ be the $\mathfrak{D}$-open neighborhood

$$\{g \in C^\infty(X, X) \mid d(f(x), g(x)) < \mathfrak{p}(f(x)) \text{ for all } x \in X\}$$

(cf. 1.5: $(supp\ 1/\mathfrak{p} \circ f)$ is locally finite), and let

$$P = M'(K', V) \cap V_{\mathfrak{p} \circ f}(f).$$

We claim that $Inv(P \cap Diff(X)) \subset M'(L, U)$.

Let $g \in P \cap Diff(X)$. Then $g(K'_n) \subset L_n$ for all n, since for $x \in K'_n \subset K_n$

we have

$$d(f(x), g(x)) < \mathfrak{p}(f(x)), \quad f(x) \in L'_n$$

and $\mathfrak{p}(f(x))$ is less than the distance between L'_n and $X \backslash L_n{}^o$. So

$$g^{-1}(X \backslash L_n{}^o) = X \backslash g^{-1}(L_n{}^o) \subset X \backslash g^{-1} g(K'^o_n) = X \backslash K'^o_n .$$

Take $n \in \mathrm{N}$ and $x \in X \backslash L_n{}^o$, then

$$j^n(g^{-1})(x) = inv(j^n(g)(g^{-1}(x))).$$

$x \in X \backslash L_n{}^o$ implies $g^{-1}(x) \in X \backslash K'^o_n$ as we saw above, so

$$j^n(g)(g^{-1}(x)) \in V_n = inv(U_n)$$

since $g \in M'(K', V)$, so $j^n(g^{-1})(x) \in U_n$. This shows $g^{-1} \in M'(L, U)$. To see that Inv is continuous for the $\mathfrak{D}^\infty$-topology too it suffices to note that Inv is compatible with the equivalence relation from 2.1. QED

5.4. PROPOSITION. *Let X, Y, Z be smooth locally compact manifolds. Then the canonical identification*

$$C^\infty(X, Y \times Z) = C^\infty(X, Y) \times C^\infty(X, Z)$$

is a homeomorphism for the $\mathfrak{D}$- and the $\mathfrak{D}^\infty$-topologies.

REMARK. A direct proof of this fact can be given along the lines of ([2], Chapter II, Proposition 3.6). Lemma 1.9 plays a vital role in it. The assertion for the $\mathfrak{D}^\infty$-topology will be a consequence of 6.4 below; we will not need more than that later on.

6. DIFFERENTIATION.

6.1. *The notion of differentiation:* We use the notion of differentiability C^∞_π of Keller [4], but in the formally weaker form of C^∞_c. In [4] it is shown that $C^\infty_\pi = C^\infty_c$ holds in general.

Let E, F be locally convex linear spaces, let $f: E \to F$ be a mapping. f is said to be of class C^1_c if, for all $x, y \in E$ and $\lambda \in \mathrm{R}$, we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(f(x + \lambda y) - f(x)) = Df(x)y$$

in F, where $Df(x)$ is a linear mapping $E \to F$ for each $x \in E$, and the mapping

$$(x,y) \mapsto Df(x)y: E \times E \to F$$

is jointly continuous.

This concept is applicable with the obvious changes if f is only defined on some open subset of E.

f is said to be of class C_c^2 if Df is of class C_c^1 as a mapping $E \times E \to F$, and so on for the higher derivatives.

We refer to [4] for more information.

The notion C_c^p was introduced by Ehresmann (Bastiani [1]), who also showed the following - we include a proof for completeness sake.

6.2. PROPOSITION *(partial derivatives). Let E_1, E_2, F be locally convex linear spaces. Let $f: E_1 \times E_2 \to F$ be a mapping. f is of class C_c^1 iff the mappings $x_1 \mapsto f(x_1, x_2)$, $x_2 \mapsto f(x_1, x_2)$ are of class C_c^1 for each fixed x_2, x_1 respectively, with derivatives $D_1 f(x_1, x_2)y_1$ and $D_2 f(x_1, x_2)y_2$ respectively, which are jointly continuous in all appearing variables. The derivative of f is then given by*

$$Df(x_1, x_2)(y_1, y_2) = D_1 f(x_1, x_2)y_1 + D_2 f(x_1, x_2)y_2.$$

The same is true if f is defined on an open subset of $E_1 \times E_2$ only.

PROOF. Necessity:

$$D_1 f(x_1, x_2)y_1 = \lim_{\lambda \to 0} \frac{1}{\lambda}(f(x_1 + \lambda y_1, x_2) - f(x_1, x_2)) =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda}(f((x_1, x_2) + \lambda(y_1, 0)) - f(x_1, x_2)) = Df(x_1, x_2)(y_1, 0).$$

So $D_1 f$ is jointly continuous in all appearing variables. Analogously for $D_2 f$.

Sufficiency:

$$\begin{aligned} & D_1 f(x_1, x_2)y_1 + D_2 f(x_1, x_2)y_2 = \\ & = \lim_{\lambda \to 0} D_1 f(x_1, x_2 + \lambda y_2)y_1 + D_2 f(x_1, x_2)y_2 = \\ & = \lim_{\lambda \to 0} \lim_{\mu \to 0} \frac{1}{\mu}(f(x_1 + \mu y_1, x_2 + \lambda y_2) - f(x_1, x_2 + \lambda y_2)) + \\ & \qquad + \lim_{\mu \to 0} \frac{1}{\mu}(f(x_1, x_2 + \mu y_2) - f(x_1, x_2)) = \end{aligned}$$

$$= \lim_{\mu \to 0} \frac{1}{\mu}(f(x_1 + \mu y_1, x_2 + \mu y_2) - f(x_1, x_2 + \mu y_2)) +$$
$$+ \frac{1}{\mu}(f(x_1, x_2 + \mu y_2) - f(x_1, x_2)) =$$
$$= \lim_{\mu \to 0} \frac{1}{\mu}(f((x_1, x_2) + \mu(y_1, y_2)) - f(x_1, x_2)) = Df(x_1, x_2)(y_1, y_2).$$

The joint continuity of Df in all appearing variables is clear from the same property of $D_1 f$ and $D_2 f$. QED

6.3. *The manifold structure.* We give a short review of the manifold structure on $C^\infty(X, Y)$, X, Y being locally compact smooth manifolds. We use a simplified form of the manifold structure set up in 3.3, 3.4 and 3.6; this version is described in ([9], 8).

Let $\tau: TY \to Y$ be a smooth mapping such that for each y in Y the mapping $\tau_y: TY \to Y$ is a diffeomorphism onto an open neighborhood of y in Y, and $\tau_y(0_y) = y$. Such a map may be constructed by an exponential map following an appropriate fibre respecting diffeomorphism from TY onto the open neighborhood of the zero section, on which the exponential map is diffeomorphic. If $\pi_Y: TY \to Y$ denotes the canonical projection, then the mapping $(\tau, \pi_Y): TY \to Y \times Y$ is a diffeomorphism onto an open neighborhood of the diagonal in $Y \times Y$. In Seip [11] these maps (which need not be defined globally there) are called «*local additions*». We will adopt the same name for convenience sake.

If $f \in C^\infty(X, Y)$, consider the pullback f^*TY which is a vector bundle over X, and the space $\mathfrak{D}(f^*TY)$ of all smooth sections with compact support of this bundle, equipped with the $\mathfrak{D}^\infty$-topology (which coincides with the $\mathfrak{D}$-topology here). This is a locally convex dually nuclear (LF)-space, being the straightforward generalization of the space $\mathfrak{D}$ of test functions with compact support in distribution theory. See 2.7 for further information.

Let $\psi_f: \mathfrak{D}(f^*TY) \to C^\infty(X, Y)$ be the mapping

$$\psi_f(s)(x) = \tau_{f(x)} s(x) \in Y.$$

Denote by U_f the image of ψ_f, which is an open subset of $C^\infty(X, Y)$; for let $Z_f = \bigcup_{x \in X} (\{x\} \times \tau_{f(x)}(T_{f(x)} Y))$ be the open neighborhood of the graph $\{(x, f(x)) \mid x \in X\}$ of f in $X \times Y = J^0(X, Y)$ (in fact a tubular neighborhood with vertical projection, cf. 3.3-3.5). Then U_f consists of all $g \in C^\infty(X, Y)$ such that the graph of g is contained in Z_f and $g \sim f$ (i.e., g and f differ only on a relatively compact subset of X), so U_f is $\mathfrak{D}^\infty$-open.

ψ_f is continuous by 2.5 and has a continuous inverse

$$\phi_f : U_f \to \mathfrak{D}(f^* TY) \text{ given by } \phi_f(g)(x) = \tau_{f(x)}^{-1}(g(x)) = \\ = (\tau, \pi_Y)^{-1}(g, f)(x),$$

as is easily checked up.

$(U_f, \phi_f, \mathfrak{D}(f^* TY))$ will serve as canonical chart centered at f. Now let us check the coordinate change. Let $f, g \in C^\infty(X, Y)$ with: $U_f \cap U_g \neq \emptyset$. For $s \in \phi_f(U_f \cap U_g)$ we have

$$\phi_g \psi_f(s)(x) = \tau_{g(x)}^{-1}(\psi_f(s)(x)) = \tau_{g(x)}^{-1} \tau_{f(x)}(s(x)),$$

so the map

$$\phi_g \psi_f : \phi_f(U_f \cap U_g) \subset \mathfrak{D}(f^* TY) \to \mathfrak{D}(g^* TY)$$

is given by

$$(\tau_g^{-1} \circ \tau_f)_* = ((\tau, \pi_Y)^{-1} \circ (Id_Y, g) \circ (\tau, \pi_{f^* TY}))_*,$$

by pushing forward sections by a fibre preserving (locally defined) diffeomorphism. So the coordinate change is continuous and of differentiability class C_π^∞ by the Ω-Lemma 3.8.

6.4 PROPOSITION. *Let X, Y, Z be smooth locally compact manifolds. Then the canonical identification*

$$C^\infty(X, Y \times Z) = C^\infty(X, Y) \times C^\infty(X, Z)$$

is of class C_π^∞. The identification is compatible with the choice of canonical charts.

PROOF. Let $(f, g) \in C^\infty(X, Y) \times C^\infty(X, Z)$. We write again (f, g) for the corresponding element of $C^\infty(X, Y \times Z)$ given by

$$(f, g)(x) = (f(x), g(x)).$$

Let $\tau: TY \to Y$ be a local addition on Y, $\rho: TZ \to Z$ be one on Z, then $\tau \times \rho$ is a local addition on $Y \times Z$. We have

$$\mathfrak{D}((f,g)^*T(Y\times Z)) = \mathfrak{D}(f^*TY \oplus g^*TZ) = \mathfrak{D}(f^*TY)\times\mathfrak{D}(g^*TZ)$$

and $U_{(f,g)} = U_f \times U_g$ for the canonical charts, and the following diagram commutes:

$$\begin{tikzcd}
U_{(f,g)} \arrow[r, "\phi_{(f,g)}"] \arrow[d, equal] & \mathfrak{D}((f,g)^*T(Y\times Z)) \arrow[d, equal] \\
U_f \times U_g \arrow[r, "\phi_f \times \phi_g"'] & \mathfrak{D}(f^*TY)\times\mathfrak{D}(g^*TZ).
\end{tikzcd}$$

QED

6.5. PROPOSITION. *For each $n \geq 0$, the mapping*

$$j^k: C^\infty(X,Y) \to C^\infty(X, J^k(X,Y))$$

is of class C^∞_π.

PROOF. Let

$$\tau: TY \to Y \quad \text{and} \quad \rho: TJ^k(X,Y) \to J^k(X,Y)$$

be local additions for Y and $J^k(X,Y)$ respectively. Let $f \in C^\infty(X,Y)$, then $j^k f \in C^\infty(X, J^k(X,Y))$; consider the canonical chart (U_f, ϕ_f) centered at f of $C^\infty(X,Y)$ and the canonical chart $(U_{j^k f}, \phi_{j^k f})$ centered at $j^k f$ of $C^\infty(X, J^k(X,Y))$. We have to check wether the mapping

$$\phi_{j^k f} \circ j^k \circ \psi_f : \mathfrak{D}(f^*TY) \to \mathfrak{D}((j^k f)^* TJ^k(X,Y))$$

is of class C^∞_π. For $s \in \mathfrak{D}(f^*TY)$ we have

$$((\phi_{j^k f} \circ j^k \circ \psi_f)(s))(x) = (\rho_{j^k f(x)})^{-1}(j^k(\psi_f(s))(x)) =$$
$$= (\rho_{j^k f(x)})^{-1}(j^k(\tau \circ s)(x)) = (\rho_{j^k f(x)})^{-1} J^k(X,\tau)(j^k s(x)),$$

where

$$J^k(X,\tau): J^k(X,TY) \to J^k(X,Y)$$

is the (functorially) induced mapping. Now $j^k s \in C^\infty(X, J^k(X,Y))$ by definition but in fact it is an element of the closed subspace $\mathfrak{D}J^k(f^*TY)$ of smooth sections with compact support of the vector bundle $J^k(f^*TY)$

over X, which consists of k-jets of sections of the bundle f^*TY (cf. [10]). We will write $\bar{j}^k s$ if we consider it to be a section of $J^k(f^*TY)$. The mapping

$$\bar{j}^k : \mathfrak{D}(f^*TY) \to \mathfrak{D}(J^k(f^*TY))$$

is linear and continuous (being a complex of partial differential operators on a space of test functions - in a canonical chart) so it is trivially of class C^∞_π. Therefore we have

$$(\phi_{j^k f} \circ j^k \circ \psi_f)(s)(x) = (\rho_{j^k f})^{-1}(J^k(X,\tau)|J^k(f^*TY))_x \bar{j}^k(s)(x),$$

and it is easily seen that

$$(\rho_{j^k f})^{-1}(J^k(X,\tau)|J^k(f^*TY)) : J^k(f^*TY) \to (j^k f)^* T J^k(X,Y)$$

is a fibre preserving smooth mapping, so pushing forward sections by it is of class C^∞_π by the Ω-Lemma 3.8. The assertion follows by the chain rule. QED

6.6. REMARK. The mapping $T: C^\infty(X,Y) \to C^\infty(TX,TY)$ is not even continuous in the $\mathfrak{D}$-topology, neither in the $\mathfrak{D}^\infty$-topology: let f_n be a sequence converging to f in $C^\infty(X,Y)$. Then f_n equals f off some fixed compact set in X for all but finitely many n (2.3 or [9]). But if f_n differs from f for infinitely many n at some $x \in X$, then Tf_n differs from Tf on the whole fiber $T_x X$, so Tf_n cannot converge to Tf in general. Thus there is no chance for T to be differentiable. But it can be shown that the mapping

$$(s,f) \mapsto Tf.s, \quad \mathfrak{D}(TX) \times C^\infty(X,Y) \to C^\infty(X,TY)$$

is continuous and even differentiable (compare [7], 2, Proposition 6), since we may write

$$(Tf.s)(x) = (T_x f)(s(x)) = (j^1 f(x)).s(x) = comp(j^1 f, s)(x),$$

where $comp: TX \times_X J^1(X,Y) \to TY$ is just composition of matrices and vectors locally, which is smooth. We will use this technique in a much more complicated situation later on.

6.7. THEOREM. *Let X be a locally compact smooth manifold. Then the* *space $Diff(X)$ of all diffeomorphisms of X is a Lie group in the $\mathfrak{D}^\infty$-topology.*

$Diff(X)$ is open in $C^\infty(X, Y)$ in the $\mathfrak{D}$- and the $\mathfrak{D}^\infty$-topology, composition is continuous by 2.5 and inversion is continuous by 5.3, so $Diff(X)$ is a topological group in the $\mathfrak{D}$-topology and in the $\mathfrak{D}^\infty$-topology. In the latter it is an open submanifold of $C^\infty(X, Y)$, and we will show that it is a Lie group in this induced manifold structure. The proof of this will occupy Sections 7 and 8.

7. THE COMPOSITION IS DIFFERENTIABLE.

7.1. Before we can begin with the proof, we need some preparation. If $p: E \to X$ is a vector bundle then let us denote by $V(E) = \ker T(p)$ the vertical subbundle of the bundle $TE \to E$. If $E_x = p^{-1}(x)$ is the fibre over $x \in X$ and $i_x: E_x \hookrightarrow E$ is the embedding, then we may identify $T_v(E_x)$ with E_x itself for $v \in E_x$ via the affine structure of E_x and define

$$V(v, w) \in V(E) \quad \text{for } v, w \in E_x \quad \text{by} \quad V(v, w) = T_v(i_x)w.$$

It is clear that $V = V_E: E \oplus E \to V(E)$ is an isomorphism of vector bundles over X. $V(v, w)$ will be called the *vertical lift of w over v*. The mapping:

$$\xi_E = pr_2 \circ V_E^{-1}: V(E) \to E \oplus E \to E$$

is called the *vertical projection.*

LEMMA. *1° $V(E)$, V_E, ξ_E commute with pullbacks of vector bundles.*

2° Let $a: E \to E'$ be a smooth fibre mapping between vector bundles as given by the diagram

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "\beta"] & Y \end{tikzcd}$$

then the fibre derivative of a, the mapping $d_f a: E \oplus E \to E'$ over β is *given by* $d_f a = \xi_{E'} \circ (Ta \mid V(E)) \circ V_E$.

PROOF. If $f: X' \to X$ is a mapping, then the pullback bundle f^*E is given as the following categorical pullback:

$$\begin{array}{ccc} f^*E = E \underset{(p,f,X)}{\times} X' & \xrightarrow{pr_2} & X' \\ \Big\downarrow{\scriptstyle pr_1} & & \Big\downarrow{\scriptstyle f} \\ E & \xrightarrow{p} & X \end{array}$$

In view of this, we have: $f^*(E \oplus E) = f^*E \oplus f^*E$,

$$V(f^*E) = V(E \underset{(p,f,X)}{\times} X') = V(E) \underset{(Tp, Tf \mid 0_{X'}, 0_X)}{\times} 0_{X'},$$

where 0_X is the zero section of X, both as a manifold and a mapping, and $V_{f^*E} = V_E \times 0_{X'}$. Assertion 2 of the lemma is clear. QED

REMARK. If $f: X \to Y$ is a smooth mapping and $\tau: TY \to Y$ is a local addition, then we will denote by $\tau_f: f^*TY \to Y \times X$ the diffeomorphism into given by

$$f^*TY = TY \underset{(\pi_Y, f, Y)}{\times} X \xrightarrow{\tau \times Id_X} Y \times X.$$

Clearly we have $(\tau_f)^{-1}(y,x) = \tau_{f(x)}{}^{-1}(y)$, if we identify $(f^*TY)_x$ with $T_{f(x)}Y$.

7.2. THEOREM. *Let* X, Y, Z *be locally compact smooth manifolds. Then the composition mapping*

$$Comp: C^\infty(Y,Z) \times C^\infty_{prop}(X,Y) \to C^\infty(X,Z),$$

given by $Comp(g,f) = g \circ f$, *is of class* C^∞_π .

Here $C^\infty_{prop}(X,Y)$ denotes the open subset of proper mappings of $C^\infty(X,Y)$ (cf. 1.9, 2.5).

PROOF. Let $g \in C^\infty(Y,Z)$, $f \in C^\infty_{prop}(X,Y)$. We will show that *Comp* is differentiable in the canonical charts centered at g, f and $g \circ f$ respectively, as described in 6.3. Let

$$\tau: TY \to Y \quad \text{and} \quad \rho: TZ \to Z$$

be local additions, inducing the canonical charts.
We will suppose that U_f and U_g are small enough such that

$$Comp\,(U_g \times U_f) \subset U_{g \circ f}$$

- to be precise we should restrict to open subsets of U_f and U_g respectively, using continuity of the composition (2.5), but we will not specify this to save notation. So by some abuse of language we consider the mapping (1):

$$c = \phi_{g \circ f} \circ Comp \circ (\psi_g \times \psi_f) : \mathfrak{D}(g^*TZ) \times \mathfrak{D}(f^*TY) \to \mathfrak{D}((g \circ f)^*TZ).$$

We have to show that c is of class C^∞_π and we will do this by showing that it is of class C^∞_c, using 6.2.
The mapping c is given for $t \in \mathfrak{D}(g^*TZ)$, $s \in \mathfrak{D}(f^*TY)$, by

$$\begin{aligned}(2) \qquad c(t,s)(x) &= \phi_{gf} \circ Comp \circ (\psi_g \times \psi_f)(t,s)(x) = \\ &= \phi_{gf}(\psi_g(t) \circ \psi_f(s))(x) = \rho^{-1}_{gf(x)}(\rho\, t\, \tau s(x)) = \\ &= (\rho, \pi_Z)^{-1}(\rho\, t\, \tau s(x), gf(x)) = \rho^{-1}_{gf}(\rho\, t\, \tau s(x), x).\end{aligned}$$

There is some abuse of notation in this formula too: we did not distinguish vector fields along f with compact support from sections of the vector bundle f^*TY, i.e., we have identified $\mathfrak{D}_f(X, TY)$ with the isomorphic space $\mathfrak{D}(f^*TY)$, to save notation. Let us first look at the differentiability of the mapping

$$(3) \quad t \mapsto c(t,s), \ t \in \mathfrak{D}(g^*TZ), \ \text{ for fixed } s \in \mathfrak{D}(f^*TY).$$

Since $\tau s \sim f$ and f is proper, τs is proper too, so the mapping

$$(4) \quad (\tau s)^* : \mathfrak{D}(g^*TZ) \to \mathfrak{D}((g\tau s)^*TZ), \quad (\tau s)^* t = t\,\tau s,$$

is continuous and linear.
Then we consider the fibre respecting (but not everywhere defined) diffeomorphism over Id_X:

$$\begin{array}{llll}(5) & (g\tau s)^*TZ = TZ \underset{(\pi_Z, g\tau s, Z)}{\times} X & \xrightarrow{\ \rho_{g\tau s} = \rho \times Id_X\ } & Z \times X \\ & & & \big\downarrow \rho^{-1}_{gf} \\ & & & TZ \underset{(\pi_Z, gf, Z)}{\times} X = (gf)^*TZ.\end{array}$$

It is clear that

$$(6)\quad [(\rho_{gf}^{-1}\circ\rho_{g\tau s})_*\circ(\tau s)^*t](x) = \rho_{gf}^{-1}(\rho t\tau s(x), x) = \\ = \rho_{gf(x)}^{-1}\rho t\tau s(x) = c(t,s)(x).$$

By the chain rule and the Ω-lemma the mapping (3) is therefore of class C_π^∞ and its derivative is given by

$$\begin{aligned}(7)\quad D_1 c(t,s)(t') &= D((\rho_{gf}^{-1}\circ\rho_{g\tau s})_*(\tau s)^*)(t)(t') = \\ &= D((\rho_{gf}^{-1}\circ(\rho\times Id_X))_*)((\tau s)^*t)(D(\tau s)^*(t)(t')) = \\ &= d_f(\rho_{gf}^{-1}\circ(\rho\times Id_X))(t\tau s, t'\tau s) = \\ &= \xi_{(gf)^*TZ}\circ T(\rho_{gf}^{-1}\circ(\rho\times Id_X))\circ V_{(g\tau s)^*TZ}(t\tau s, t'\tau s) = \\ &= \xi_{(gf)^*TZ}\circ T(\rho_{gf}^{-1})\circ T(\rho\times Id_X)\circ(V_{TZ}(t\tau s, t'\tau s)\times 0_X) \\ &= \xi_{(gf)^*TZ}\circ T(\rho_{gf}^{-1})\circ(T(\rho)\circ V_{TZ}(t\tau s, t'\tau s), 0_X).\end{aligned}$$

Here we again considered $t\tau s$, $t'\tau s$ both as sections of the bundle $(g\tau s)^*TZ$ and as mappings $X\to TZ$. We used heavily Lemma 7.1. The last line of formula (7) shows that $D_1 c(t,s)t'$ is jointly continuous in s, t, t' (use 6.4, the continuity of the composition and the fact that the mapping

$$s\mapsto\tau s,\quad \mathfrak{D}(f^*TY)\to C^\infty_{prop}(X,Y)$$

is continuous).

Now we investigate the differentiability of the mapping

$$(8)\quad s\mapsto c(t,s),\quad s\in\mathfrak{D}(f^*TY),\quad \text{for fixed } t\in\mathfrak{D}(g^*TZ).$$

For fixed t we define the mapping

(9) $a(t): f^*TY\to(gf)^*TZ$ by:

$$\begin{array}{ccccc} f^*TY = TY \underset{(\pi_Y,f,Y)}{\times} X & \xrightarrow{\tau_f=\tau\times Id_X} & Y\times X & \xrightarrow{(\rho t)\times Id_X} & Z\times X \\ & & & & \Big\downarrow \rho_{gf}^{-1} \\ & & & & TZ \underset{(\pi_Z,gf,Z)}{\times} X = (gf)^*TZ. \end{array}$$

Then we have

$$(10)\quad a(t)_*s = \rho_{gf}^{-1}\circ(\rho t\times Id_X)\circ(\tau\times Id_X)\circ s =$$

$$= \rho_{gf}^{-1}(\rho\, t\, \tau\, s,\, Id_X) = c(t, s),$$

where $a(t)$ is a fibre mapping over Id_X. So by the Ω-Lemma and by the chain rule the mapping (8) is of class C^∞_π and the derivative is given by:

$$(11)\quad D_2 c(t,s)s' = D(a(t)_*)(s)(s') = d_f a(t) \circ (s, s').$$

But the mapping $t \mapsto a(t)$ is *not* continuous (there is a non proper open embedding on the right of t) neither is $t \mapsto d_f a(t)$. So we have to rearrange the expression (11) in such a way as to see the joint continuity in t, s, s'. We compute as follows, again using Lemma 7.1 :

$$\begin{aligned}
(12)\quad D_2 c(t,s)s' &= d_f a(t) \circ (s, s') = \\
&= \xi_{(gf)^* TZ} \circ T(a(t)) \circ V_{f^* TY}(s, s') = \\
&= \xi_{(gf)^* TZ} \circ T(\rho_{gf}^{-1}) \circ T(\rho\, t \times Id_X) \circ T(\tau \times Id_X) \circ V_{f^* TY}(s, s') = \\
&= \xi_{(gf)^* TZ} \circ T(\rho_{gf}^{-1}) \circ (T(\rho\, t\, \tau) \times T Id_X) \circ (V_{TY} \times 0_X)(s, s') = \\
&= \xi_{(gf)^* TZ} \circ T(\rho_{gf}^{-1}) \circ (T(\rho\, t\, \tau) \circ V_{TY}(s, s'),\ 0_X).
\end{aligned}$$

So it remains to show that the mapping

$$(13)\qquad (t, s, s') \mapsto T(\rho\, t\, \tau) V_{TY}(s, s')$$

is continuous. For that we use the manifold

$$M := TTY \underset{TY}{\times} J^1(TY, Y) \underset{Y}{\times} J^1(Y, TZ) \underset{TZ}{\times} J^1(TZ, Z),$$

$$\begin{tikzcd}
TTY \arrow[dr, "\pi_{TY}"'] & & J^1(TY,Y) \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & & J^1(Y,TZ) \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & & J^1(TZ,Z) \arrow[dl, "\alpha"] \\
& TY & & Y & & TZ &
\end{tikzcd}$$

which is a submanifold of the product, and the following composition evaluation mapping $\mu : M \to TZ$ given by:

$$(14)\qquad \mu(v, \sigma_1, \sigma_2, \sigma_3) = \sigma_3 \circ \sigma_2 \circ \sigma_1(v).$$

Since μ is locally just multiplication of matrices, it is C^∞. Then we have:

$$\begin{aligned}
(15)\quad & T(\rho\, t\, \tau) V_{TY}(s, s') = \\
&\qquad = \mu \circ (V_{TY}(s, s'),\ j^1(\tau) \circ s,\ j^1(t) \circ \tau \circ s,\ j^1(\rho) \circ t \circ \tau \circ s).
\end{aligned}$$

This expression is jointly continuous in t, s, s' by 5.4, 6.4, 6.5 and by

the fact that $s \mapsto \tau s$ is continuous $\mathfrak{D}(f^*TY) \to C^\infty_{prop}(X, Y)$.
In view of 6.2 we have shown that c is of class C^1_c and that

(16) $Dc(t,s)(t',s') = D_1c(t,s)t' + D_2c(t,s)s' = (7)+(12)$.

(17) *The higher derivatives:* If we want to show that c is of class C^2_c we have to check that Dc is of class C^1_c. In order to apply 6.2 again we have to compute all partial derivatives of Dc and have to show that they are jointly continuous in all appearing variables. Now (7) and (15) are composita of expressions that look like (2) again and by 6.5 $j^1(t)$ is of class C^∞_π in t. So we may apply what we have already proved and get C^2_c. By induction we get $C^\infty_c = C^\infty_\pi$. QED

7.3. COROLLARY. *Let X, Y be locally compact smooth manifolds. Then the evaluation mapping $Ev: X \times C^\infty(X,Y) \to Y$ is of class C^∞_π (and consequently $\mathfrak{D}^\infty$-continuous).*

PROOF. First we show that

$$X = C^\infty(*, X) = C_{prop}(*, X)$$

diffeomorphically, where $*$ denotes the one-point manifold.
Let $\tau: TX \to X$ be some local addition. Then the canonical chart (U_f, ϕ_f) centered at $f: * \to X$ corresponds to the chart $(Im\,\tau_{f(*)}, \tau_{f(*)}^{-1})$ centered at $f(*)$ of X.
Now the following diagram commutes and so the assertion follows from the differentiability of the composition:

$$\begin{array}{ccc} X \times C^\infty(X,Y) & \xrightarrow{\quad Ev \quad} & Y \\ \| & & \| \\ C^\infty_{prop}(*, X) \times C^\infty(X,Y) & \xrightarrow{\quad Comp \quad} & C^\infty(*, Y). \end{array}$$

QED

7.4. COROLLARY. *Let X, Y, Z be locally compact smooth manifolds. Then the canonical mapping*

$$C^\infty(X, C^\infty(Y,Z)) \subset (Z^Y)^X = Z^{Y\times X}$$

takes its values in $C^\infty(Y\times X, Z)$.

REMARKS. 1° Since X is finite dimensional we need not specify the notion of differentiability in $C^\infty(X, C^\infty(Y,Z))$, since all reasonable notions coincide (see [4]); $C^\infty(Y,Z)$ is equipped with its canonical C^∞_π-manifold structure.

2° This canonical mapping is not surjective on $C^\infty(Y\times X, Z)$ by topological reasons, as we already mentioned in the introduction. It is surjective, however, if Y is compact. That has been shown by Gutknecht [3] for the stronger notion of differentiability C^∞_Γ ; in order to be complete we will prove this fact in our setting too (7.5).

PROOF OF 7.4. Let $f\in C^\infty(X, C^\infty(Y,Z))$. Then the canonical mapping associates to f the mapping $\hat{f}: Y\times X \to Z$ given by

$$\hat{f}(y,x) = f(x)(y) = Ev(y, f(x)) = Ev\circ(Id_Y\times f)(y,x).$$

So $\hat{f} = Ev\circ(Id_Y\times f)$ is of class C^∞. QED

7.5. THEOREM. *Let X, Y, Z be smooth manifolds, X, Z locally compact and Y compact. Then*

$$C^\infty(X, C^\infty(Y,Z)) = C^\infty(Y\times X, Z)$$

via the canonical identification.

PROOF. In view of 7.4 it remains to show that the canonical identification mapping is onto $C^\infty(Y\times X, Z)$, and by abstract non-sense it suffices to show that the mapping

$$\eta: X \to C^\infty(Y, Y\times X), \quad \eta(x)(y) = (y,x),$$

is of class C^∞_π. For if $f\in C^\infty(Y\times X, Z)$, then

$$f_*: C^\infty(Y, Y\times X) \to C^\infty(Y,Z)$$

is of class C^∞_π by the Ω-Lemma or by 7.2, so $f_*\circ\eta: X\to C^\infty(Y,Z)$ is of class C^∞_π too; this latter mapping is easily seen to be the canonical associate to f (η is the so-called unit of the adjunction, in categorical terms).

Now fix $x_0\in X$ and let $\tau: TX\to X$ and $\rho: TY\to Y$ be local additions. Then

$$\rho\times\tau: TY\times TX = T(Y\times X) \to Y\times X$$

is a local addition; let

$$\phi_{\eta(x_0)}: U_{\eta(x_0)} \to \mathfrak{D}(\eta(x_0)^* T(Y \times X))$$

be the canonical chart of $C^\infty(Y, Y \times X)$, centered at $\eta(x_0)$, which comes from $\rho \times \tau$. Since $\eta(x_0)$ is given by $y \mapsto (y, x_0)$ we see that

$$\eta(x_0)^* T(Y \times X) = TY \times T_{x_0} X = TY \oplus (Y \times T_{x_0} X)$$

as bundle over Y, so

$$\mathfrak{D}(\eta(x_0)^* T(Y \times X)) = \mathfrak{D}(TY) \times \mathfrak{D}(Y \times T_{x_0} X).$$

In view of this identification we have for $x \in X$ near x_0 and $y \in Y$:

$$((\phi_{\eta(x_0)} \circ \eta)(x))(y) = (\rho \times \tau)_{\eta(x_0)(y)}^{-1}(x)(y) =$$
$$= (\rho \times \tau)_{(y, x_0)}^{-1}(y, x) = (\rho_y^{-1}(y), \tau_{x_0}^{-1}(x)) = (0_y, \tau_{x_0}^{-1}(x)).$$

So $\phi_{\eta(x_0)} \circ \eta$ is given by the sequence

$$X \supset V \xrightarrow{\tau_{x_0}^{-1}} T_{x_0} X \xrightarrow{(0, B)} \mathfrak{D}(TY) \times \mathfrak{D}(Y \times T_{x_0} X)$$

which is clearly differentiable in any sense, where V is a suitable neighborhood of x_0 in X and where B is the continuous linear mapping which maps each point of $T_{x_0} X$ into the constant function $Y \to T_{x_0} X$. B is well defined and continuous iff Y is compact. QED

7.6. PROPOSITION. *The tangent mapping of the composition*

$$T\,Comp: \mathfrak{D}(Y, TZ) \times \mathfrak{D}_{prop}(X, TY) \to \mathfrak{D}(X, TZ)$$

is given by

$$T_{(g,f)} Comp(t, s) = (j^1 g) \circ s + t \circ f,$$

$t \in \mathfrak{D}_g(Y, TZ)$, $s \in \mathfrak{D}_f(X, TY)$.

PROOF. Since we know already that $Comp$ is differentiable we may compute the tangent mapping by considering one parameter variations through g and f with «tangent vectors» t and s and differentiating their composition pointwise, i.e., for fixed $x \in X$, using Lemma 4.4.

8. THE INVERSION IS DIFFERENTIABLE.

8.1. THEOREM. *Let X be a locally compact smooth manifold and let $Diff(X)$ be the open subset of diffeomorphisms of $C^\infty(X, X)$. Then in-*

version $Inv: Diff(X) \to Diff(X)$ *is of class* C_π^∞.

PROOF. (1) It suffices to show that $Inv: Diff(X) \to Diff(X)$ is of class C_π^∞ in a neighborhood U of the identity Id_X of X. For let $f \in Diff(X)$, then

$$V_f = ((f^{-1})_*)^{-1}(U) = \{ f^{-1} \circ g \mid g \in U \}$$

is a neighborhood of f by 2.5. For any $g \in V_f$ we have $f^{-1} \circ g \in U$ and

$$g^{-1} = (f^{-1} \circ g)^{-1} \circ f^{-1} = (f^{-1})^* \circ Inv \mid U \circ (f^{-1})_*(g),$$

thus

$$Inv \mid V_f = (f^{-1})^* \circ Inv \mid U \circ (f^{-1})_* .$$

Since $(f^{-1})^*$ and $(f^{-1})_*$ are of class C_π^∞ (by 7.1 or by the A- and Ω-Lemma respectively) the chain rule implies that $Inv \mid V_f$ is of class C_π^∞ too.

(2) Now let $\tau: TX \to X$ be a local addition and let $U = U_{Id}$ be a canonical chart centered at $Id \in Diff(X)$, and let

$$\phi = \phi_{Id}: U \to \mathfrak{D}(TX), \quad \phi(f)(x) = \tau_x^{-1}(f(x))$$

and

$$\psi = \psi_{Id}: \mathfrak{D}(TX) \to U, \quad \psi(s)(x) = \tau_x s(x),$$

be the chart maps. We have to show that the mapping

$$(3) \qquad i = \phi \circ Inv \circ \psi : \mathfrak{D}(TX) \to \mathfrak{D}(TX)$$

is of class C_π^∞. There is again some abuse of notation involved: to be precise we have to restrict the domain of the mapping $Inv \circ \psi$ to an open subset of $\mathfrak{D}(TX)$ so that its image is contained in the domain U of ϕ. This is possible by 5.3, and we will silently assume this in the following to save notation. Consider the mapping

$$c = \phi \circ Comp \circ (\psi \times \psi) : \mathfrak{D}(TX) \times \mathfrak{D}(TX) \to \mathfrak{D}(TX)$$

of 7.2 (1) (here too we have some silent restrictions involved). Then for any $s \in \mathfrak{D}(TX)$ we have:

$$(4) \quad c(s, i(s)) = \phi \circ Comp(\psi(s), \psi(\phi \circ Inv \circ \psi(s))) =$$
$$= \phi(\psi(s) \circ \psi(s)^{-1}) = \phi(Id) = 0 .$$

Experience with finite dimensional Lie groups or the formula of Ver

Eecke about the derivative of implicitly given functions suggests to try the following ansatz:

$$(5) \qquad Di(s) = -D_2 c(s, i(s))^{-1} \circ D_1 c(s, i(s)).$$

That this is indeed the derivative of i will be shown in Lemma 8.2 below. For the moment we will take it for granted and we will investigate this formula. Recall from 7.2 (11) that

$$(6) \quad D_2 c(s, i(s)) = D(a(i(s))_*)(s) = (d_f a(i(s)))_*(s),$$

where d_f denotes the fibre derivative and where

$$a(i(s)) = (\tau, \pi_X)^{-1} \circ ((\tau \circ i(s)) \times Id_X) \circ (\tau, \pi_X): TX \to TX$$

is a fibre preserving smooth diffeomorphism, given by 7.2 (9). Now

$$\tau \circ i(s) = \psi(i(s)) = \psi \circ \phi \circ Inv \circ \psi(s) = \psi(s)^{-1},$$

so $a(i(s))$ is invertible and (6) is invertible too, and we have

$$a(i(s))^{-1} = (\tau, \pi_X)^{-1} \circ (\psi(s)^{-1} \times Id_X)^{-1} \circ (\tau, \pi_X) =$$
$$= (\tau, \pi_X)^{-1} \circ (\psi(s) \times Id_X) \circ (\tau, \pi_X) = a(s): TX \to TX.$$

So by the implicit function theorem in finite dimensions we have

$$\begin{aligned} (d_f a(i(s)))_x (s(x))^{-1} &= d_f(a(i(s))^{-1})_x (a(i(s))_x s(x)) = \\ &= d_f a(s)_x (c(i(s), s)(x)) && \text{by 7.2 (10)} \\ &= d_f a(s)_x (0_x) && \text{by (4)} \\ &= D_2 c(s, 0_X)_x && \text{by 7.2 (11).} \end{aligned}$$

Putting this back into (5) we get

$$(7) \qquad Di(s)s' = -D_2 c(s, 0_X) \circ D_1 c(s, i(s))s'.$$

From this formula and from 7.2 (7) and (15) it is clear that $Di(s)s'$ is jointly continuous in s and s', so i is of class C^1_c. This implies in turn that Di is of class C^1_c again, applying the chain rule and 7.2 (17) to the right-hand side of (7), so i is of class C^2_c. Now a straightforward induction shows that i is of class $C^\infty_c = C^\infty_\pi$. QED

8.2. LEMMA. *With the notation of 8.1 we have*

$$(4) \qquad supp\left(\frac{i(s+\lambda s')-i(s)}{\lambda}\right) \subset K \quad \text{for all } 0<|\lambda|\leq 1$$

and that

$$(5)\ j^k\left(\frac{i(s+\lambda s')-i(s)}{\lambda}\right)(x) \text{ is «uniformly bounded» for } (x,\lambda) \text{ in } K\times[-1,1]\setminus\{0\}) \text{ for any } k.$$

(4) is easily checked:

$$\{\, i(s+\lambda s')-i(s) \mid -1\leq\lambda\leq 1 \,\}$$

is a continuous compact path in $\mathfrak{D}(TX)$, so it can move only inside some compact $K\subset X$ (compare the fact mentioned in the introduction); this K satisfies (4).

To prove (5) we note that it suffices to show that for each k and x_0 in K there is a neighborhood U_{x_0} of x_0 in X such that the expression (5) stays «uniformly bounded» for $(x,\lambda)\in U_{x_0}\times([-1,1]\setminus\{0\})$.

We choose $U=U_{x_0}$ to be so small that $TX|U \approx U\times\mathrm{R}^n$. For $x\in U$ we may write

$$s(x)=(x,\bar{s}(x)),\quad a(i(s))(x,y)=(x,\bar{a}(x,y)),\quad etc...$$

Then (3) looks like

$$(6)\ \lim_{\lambda\to 0}\int_0^1 d_2\bar{a}(x,\bar{s}(x)+\mu\lambda\bar{s}'(x)).$$
$$.(\overline{i(s)}(x)+\mu(\overline{i(s+\lambda s')}(x)-\overline{i(s)}(x))).\frac{\overline{i(s+\lambda s')}(x)-\overline{i(s)}(x)}{\lambda}d\mu$$
$$=-(\overline{D_1 c(s,i(s)).s'})(x),$$

uniformly for $x\in U$ and for each derivative with respect to x. Let us write (6) for short in the form

$$(7) \qquad \lim_{\lambda\to 0}\int_0^1 A(x,\lambda,\mu).B(x,\lambda)\,d\mu = C(x),$$

so that $B(x,\lambda)$ is the local representative over U of (1). Then let $G:L(\mathrm{R}^n)\to\mathrm{R}^+$ be the continuous mapping

$$G(l)=inf\{\,|l(v)| \mid |v|=1\,\},\quad l\in L(\mathrm{R}^n).$$

Since $A(x,\lambda,\mu):\mathrm{R}^n\to\mathrm{R}^n$ is invertible (cf. 8.1) and continuous in x, λ,μ, we conclude that $G(A(x,\lambda,\mu))\geq\epsilon$ for all relevant x,λ,μ. Therefore

$$| A(x,\lambda,\mu).B(x,\lambda)| \geq \epsilon . | B(x,\lambda) | ,$$

and by (7) $| B(x,\lambda) |$ has to be bounded for $\lambda \to 0$.

Let d_x denote the derivative with respect to x. Then we have

$$d_x(A(x,\lambda,\mu).B(x,\lambda)) =$$
$$d_x A(x,\lambda,\mu).B(x,\lambda) + A(x,\lambda,\mu).d_x B(x,\lambda).$$

The first summand is already bounded and for the second we may repeat the above argument. A simple induction then shows that $d_x^k B(x,\lambda)$ is bounded for $\lambda \to 0$ for any k, so (5) is proved.

Now we proceed to prove the Lemma. Since

$$M = \{ \frac{i(s+\lambda s') - i(s)}{\lambda} \mid 0 < |\lambda| \leq 1 \}$$

is bounded and $\mathfrak{D}(TX)$ is a Montel space, M is precompact, so there are cluster points of M for $\lambda \to 0$. Let t be such a cluster point, then there is a net

$$(t_\alpha) = (\frac{i(s+\lambda_\alpha s') - i(s)}{\lambda_\alpha}) \text{ with } \lambda_\alpha \to 0$$

such that $t_\alpha \to t$ in $\mathfrak{D}(TX)$. By the joint continuity of $D_2 c$ in all variables we conclude that

$$\lim_\alpha \int_0^1 D_2 c(s+\mu\lambda_\alpha s', i(s)+\mu(i(s+\lambda_\alpha s') - i(s))).t_\alpha \, d\mu =$$
$$= D_2 c(s,i(s)).t,$$

since $t_\alpha \to t$ and $\lambda_\alpha \to 0$. By (3) again we conclude that

$$D_2 c(s,i(s)).t = -D_1 c(s,i(s)).s',$$

Since $D_2 c(s,i(s))$ is invertible we get

$$t = -D_2 c(s,i(s))^{-1} D_1 c(s,i(s)).s'.$$

This holds for any cluster point of M for $\lambda \to 0$, so the lemma is proved.

QED

8.3. PROPOSITION. *The tangent mapping of the inversion*

$$TInv: \mathfrak{D}_{Diff(X)}(X, TX) \to \mathfrak{D}_{Diff(X)}(X, TX)$$

is given by

$$T_f Inv(s) = -(Tf^{-1}.s)\circ f^{-1} = -j^1(f)^{-1}.(s\circ f^{-1}) = -f_* s .$$

PROOF. As in 7.6 one may compute the tangent mapping by differentiating the inverse of a one parameter variation through f with «tangent vector» s. The computation is a little more difficult than in 7.6.

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