Differential Geometry, Eger (Hungary), 1989 (J. Szenthe, L. Tamássy, eds.)
Colloquia Mathematica Societatis János Bolyai, 56.
János Bolyai Math. Soc. and Elsevier, 1992, 477–489

ALL UNITARY REPRESENTATIONS ADMIT MOMENT MAPPINGS

Peter W. Michor

1. Calculus of smooth mappings

1.1. The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces a whole flock of different theories were developed, each of them rather complicated and none really convincing. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. This was the original motivation for the development of a whole new field within general topology, convergence spaces.

Then in 1982, Alfred Frölicher and Andreas Kriegl presented independently the solution to the question for the right differential calculus in infinite dimensions. They joined forces in the further development of the theory and the (up to now) final outcome is the book [F-K].

In this section I will sketch the basic definitions and the most important results of the Frölicher-Kriegl calculus.

1.2. The c^∞-topology. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(3) $\{E_B \to E : B$ bounded absolutely convex in $E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

1.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent (completeness) conditions is satisfied:

(1) Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges. This is also called c^∞-complete.
(2) If B is bounded closed absolutely convex, then E_B is a Banach space.
(3) Any Lipschitz curve in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1' = c_2$ (existence of antiderivative).

1.4. Lemma. *Let E be a locally convex space. Then the following properties are equivalent:*

(1) *E is c^∞-complete.*
(2) *If $f : \mathbb{R}^k \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.*
(3) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.*
(4) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.*

Here a mapping $f : \mathbb{R}^k \to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}^n$. f scalarwise C^∞ means that $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

This lemma says that a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

1.5. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mapping from E to F.

For E and F finite dimensional this gives the usual notion of smooth mappings: this has been first proved in [Bo]. Constant mappings are smooth. Multilinear mappings are smooth if and only if they are bounded. Therefore we denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

1.6. Structure on $C^\infty(E, F)$. We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$.

1.7. Lemma. *For locally convex spaces E and F we have:*

(1) *If F is convenient, then also $C^\infty(E, F)$ is convenient, for any E. The space $L(E, F)$ is a closed linear subspace of $C^\infty(E, F)$, so it also convenient.*
(2) *If E is convenient, then a curve $c : \mathbb{R} \to L(E, F)$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in F for all $x \in E$.*

1.8. Theorem. *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G)),$$

which is even a diffeomorphism.

Of coarse this statement is also true for c^∞-open subsets of convenient vector spaces.

1.9. Corollary. *Let all spaces be convenient vector spaces. Then the following canonical mappings are smooth.*

$$\begin{aligned}
&\mathrm{ev} : C^\infty(E, F) \times E \to F, \quad \mathrm{ev}(f, x) = f(x) \\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x, y) \\
&(\quad)^\wedge : C^\infty(E, C^\infty(F, G)) \to C^\infty(E \times F, G) \\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F, G)) \\
&\mathrm{comp} : C^\infty(F, G) \times C^\infty(E, F) \to C^\infty(E, G) \\
&C^\infty(\quad,\quad) : C^\infty(F, F') \times C^\infty(E', E) \to C^\infty(C^\infty(E, F), C^\infty(E', F')) \\
&\qquad (f, g) \mapsto (h \mapsto f \circ h \circ g) \\
&\prod : \prod C^\infty(E_i, F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

1.10. Theorem. *Let E and F be convenient vector spaces. Then the differential operator*

$$d : C^\infty(E, F) \to C^\infty(E, L(E, F)),$$
$$df(x)v := \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t},$$

exists and is linear and bounded (smooth). Also the chain rule holds:

$$d(f \circ g)(x)v = df(g(x))dg(x)v.$$

1.11. Remarks. Note that the conclusion of theorem 1.8 is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more.

If one wants theorem 1.8 to be true and assumes some other obvious properties, then the calculus of smooth functions is already uniquely determined.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

For Fréchet spaces smoothness in the sense described here coincides with the notion C^∞_c of [Ke]. This is the differential calculus used by [Mic1], [Mil], and [P-S].

A prevalent opinion in contemporary mathematics is, that for infinite dimensional calculus each serious application needs its own foundation. By a serious application one obviously means some application of a hard inverse function theorem. These theorems can be proved, if by assuming enough a priori estimates one creates enough Banach space situation for some modified iteration procedure to converge. Many authors try to build their platonic idea of an a priori estimate into their differential calculus. I think that this makes the calculus inapplicable and hides the origin of the a priori estimates. I believe, that the calculus itself should be as easy to use as possible, and that all further assumptions (which most often come from ellipticity of some nonlinear partial differential equation of geometric origin) should be treated separately, in a setting depending on the specific problem. I am sure that in this sense the

Frölicher-Kriegl calculus as presented here and its holomorphic and real analytic offsprings in sections 2 and 3 below are universally usable for most applications.

Let me point out as a final remark, that also the cartesian closed calculus for holomorphic mappings along the same lines is available in [K-N], and recently the cartesian closed calculus for real analytic mapping was developed in [K-M].

2. The moment mapping for unitary representations

The following is a review of the results obtained in [Mic2]. We include only one proof, the central application of the Frölicher-Kriegl calculus.

2.1. Let G be any (finite dimensional second countable) real Lie group, and let $\rho : G \to U(\mathbf{H})$ be a unitary representation on a Hilbert space $\mathbf{H}$. Then the associated mapping $\hat{\rho} : G \times \mathbf{H} \to \mathbf{H}$ is in general *not* jointly continuous, it is only separately continuous, so that $g \mapsto \rho(g)x$, $G \to \mathbf{H}$, is continuous for any $x \in \mathbf{H}$.

Definition. A vector $x \in \mathbf{H}$ is called *smooth* (or *real analytic*) if the mapping $g \mapsto \rho(g)x$, $G \to \mathbf{H}$ is smooth (or real analytic). Let us denote by $\mathbf{H}_\infty$ the linear subspace of all smooth vectors in $\mathbf{H}$. Then we have an embedding $j : \mathbf{H}_\infty \to C^\infty(G, \mathbf{H})$, given by $x \mapsto (g \mapsto \rho(g)x)$. We equip $C^\infty(G, \mathbf{H})$ with the compact C^∞-topology (of uniform convergence on compact subsets of G, in all derivatives separately). Then it is easily seen (and proved in [Wa, p 253]) that $\mathbf{H}_\infty$ is a closed linear subspace. So with the induced topology $\mathbf{H}_\infty$ becomes a Frèchet space. Clearly $\mathbf{H}_\infty$ is also an invariant subspace, so we have a representation $\rho : G \to L(\mathbf{H}_\infty, \mathbf{H}_\infty)$. For more detailed information on $\mathbf{H}_\infty$ see [Wa, chapt. 4.4.] or [Kn, chapt. III.].

2.2. Theorem. *The mapping $\hat{\rho} : G \times \mathbf{H}_\infty \to \mathbf{H}_\infty$ is smooth.*

Proof. By cartesian closedness of the Frölicher-Kriegl calculus 1.8 it suffices to show that the canonically associated mapping

$$\hat{\rho}^\vee : G \to C^\infty(\mathbf{H}_\infty, \mathbf{H}_\infty)$$

is smooth; but it takes values in the closed subspace $L(\mathbf{H}_\infty, \mathbf{H}_\infty)$ of all bounded linear operators. So by it suffices to show that the mapping $\rho : G \to L(\mathbf{H}_\infty, \mathbf{H}_\infty)$ is smooth. But for that, since $\mathbf{H}_\infty$ is a Frèchet space, thus convenient in the sense of Frölicher-Kriegl, by 1.7(2) it suffices to show that

$$G \xrightarrow{\rho} L(\mathbf{H}_\infty, \mathbf{H}_\infty) \xrightarrow{ev_x} \mathbf{H}_\infty$$

is smooth for each $x \in \mathbf{H}_\infty$. This requirement means that $g \mapsto \rho(g)x$, $G \to \mathbf{H}_\infty$, is smooth. For this it suffices to show that

$$\begin{aligned} &G \to \mathbf{H}_\infty \xrightarrow{j} C^\infty(G, \mathbf{H}), \\ &g \mapsto \rho(g)x \mapsto (h \mapsto \rho(h)(g)x), \end{aligned}$$

is smooth. But again by cartesian closedness it suffices to show that the associated mapping

$$\begin{aligned} &G \times G \to \mathbf{H}, \\ &(g, h) \mapsto \rho(h)(g)x = \rho(hg)x, \end{aligned}$$

is smooth. And this is the case since x is a smooth vector. $\square$

2.3. we now consider $\mathbf{H}_\infty$ as a "weak" symplectic Frèchet manifold, equipped with the symplectic structure Ω, the restriction of the imaginary part of the Hermitian inner product $\langle\quad,\quad\rangle$ on $\mathbf{H}$. Then $\Omega \in \Omega^2(\mathbf{H}_\infty)$ is a closed 2-form which is non degenerate in the sense that

$$\check{\Omega} : T\mathbf{H}_\infty = \mathbf{H}_\infty \times \mathbf{H}_\infty \to T^*\mathbf{H}_\infty = \mathbf{H}_\infty \times {\mathbf{H}_\infty}'$$

is injective (but not surjective), where ${\mathbf{H}_\infty}' = L(\mathbf{H}_\infty, \mathbb{R})$ denotes the real topological dual space. This is the meaning of "weak" above.

2.4. Review. For a finite dimensional symplectic manifold (M, Ω) we have the following exact sequence of Lie algebras:

$$0 \to H^0(M) \to C^\infty(M) \xrightarrow{\operatorname{grad}^\Omega} \mathfrak{X}_\Omega(M) \xrightarrow{\gamma} H^1(M) \to 0$$

Here $H^*(M)$ is the real De Rham cohomology of M, the space $C^\infty(M)$ is equipped with the Poisson bracket $\{\quad,\quad\}$, $\mathfrak{X}_\Omega(M)$ consists of all vector fields ξ with $\mathcal{L}_\xi \Omega = 0$ (the locally Hamiltonian vector fields), which is a Lie algebra for the Lie bracket. $\operatorname{grad}^\Omega f$ is the Hamiltonian vector field for $f \in C^\infty(M)$ given by $i(\operatorname{grad}^\Omega f)\Omega = df$, and $\gamma(\xi) = [i_\xi \Omega]$. The spaces $H^0(M)$ and $H^1(M)$ are equipped with the zero bracket.

Given a symplectic left action $\ell : G \times M \to M$ of a connected Lie group G on M, the first partial derivative of ℓ gives a mapping $\ell' : \mathfrak{g} \to \mathfrak{X}_\Omega(M)$

which sends each element X of the Lie algebra $\mathfrak{g}$ of G to the fundamental vector field. This is a Lie algebra homomorphism.

$$\begin{tikzcd} H^0(M) \arrow[r, "i"] & C^\infty(M) \arrow[r, "\mathrm{grad}^\Omega"] & \mathfrak{X}_\Omega(M) \arrow[r, "\gamma"] & H^1(M) \\ & \mathfrak{g} \arrow[u, "\sigma"] \arrow[r, equal] & \mathfrak{g} \arrow[u, "\ell'"'] & \end{tikzcd}$$

A linear lift $\sigma : \mathfrak{g} \to C^\infty(M)$ of ℓ' with $\mathrm{grad}^\Omega \circ \sigma = \ell'$ exists if and only if $\gamma \circ \ell' = 0$ in $H^1(M)$. This lift σ may be changed to a Lie algebra homomorphism if and only if the 2-cocycle $\bar\sigma : \mathfrak{g} \times \mathfrak{g} \to H^0(M)$, given by $(i \circ \bar\sigma)(X, Y) = \{\sigma(X), \sigma(Y)\} - \sigma([X, Y])$, vanishes in $H^2(\mathfrak{g}, H^0(M))$, for if $\bar\sigma = \delta\alpha$ then $\sigma - i \circ \alpha$ is a Lie algebra homomorphism.

If $\sigma : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism, we may associate the *moment mapping* $\mu : M \to \mathfrak{g}' = L(\mathfrak{g}, \mathbb{R})$ to it, which is given by $\mu(x)(X) = \sigma(X)(x)$ for $x \in M$ and $X \in \mathfrak{g}$. It is G-equivariant for a suitably chosen (in general affine) action of G on $\mathfrak{g}'$. See [We] or [L-M] for all this.

2.5. We now want to carry over to the setting of 2.1 and 2.2 the procedure of 2.4. The first thing to note is that the hamiltonian mapping $\mathrm{grad}^\Omega : C^\infty(\mathbf{H}_\infty) \to \mathfrak{X}_\Omega(\mathbf{H}_\infty)$ does not make sense in general, since $\check\Omega : \mathbf{H}_\infty \to \mathbf{H}_\infty{}'$ is not invertible: $\mathrm{grad}^\Omega f = \check\Omega^{-1} df$ is defined only for those $f \in C^\infty(\mathbf{H}_\infty)$ with $df(x)$ in the image of $\check\Omega$ for all $x \in \mathbf{H}_\infty$. A similar difficulty arises for the definition of the Poisson bracket on $C^\infty(\mathbf{H}_\infty)$.

Let $\langle x, y\rangle = Re\langle x, y\rangle + \sqrt{-1}\Omega(x, y)$ be the decomposition of the hermitian inner product into real and imaginary parts. Then $Re\langle x, y\rangle = \Omega(\sqrt{-1}x, y)$, thus the real linear subspaces $\check\Omega(\mathbf{H}_\infty) = \Omega(\mathbf{H}_\infty, \quad)$ and $Re\langle \mathbf{H}_\infty, \quad \rangle$ of $\mathbf{H}_\infty{}' = L(\mathbf{H}_\infty, \mathbb{R})$ coincide.

2.6 Definition. Let $\mathbf{H}_\infty^*$ denote the real linear subspace

$$\mathbf{H}_\infty^* = \Omega(\mathbf{H}_\infty, \quad) = Re\langle \mathbf{H}_\infty, \quad \rangle$$

of $\mathbf{H}_\infty{}' = L(\mathbf{H}_\infty, \mathbb{R})$, and let us call it the *smooth dual* of $\mathbf{H}_\infty$ in view of the embedding of test functions into distributions. We have two canonical isomorphisms $\mathbf{H}_\infty^* \cong \mathbf{H}_\infty$ induced by Ω and $Re\langle \quad , \quad \rangle$, respectively. Both induce the same Fréchet topology on $\mathbf{H}_\infty^*$, which we fix from now on.

2.7 Definition. Let $C^\infty_*(\mathbf{H}_\infty,\mathbb{R})\subset C^\infty(\mathbf{H}_\infty,\mathbb{R})$ denote the linear subspace consisting of all smooth functions $f:\mathbf{H}_\infty\to\mathbb{R}$ such that each iterated derivative $d^kf(x)\in L^k_{\text{sym}}(\mathbf{H}_\infty,\mathbb{R})$ has the property that

$$d^kf(x)(\quad,y_2,\dots,y_k)\in {\mathbf{H}_\infty}^*$$

is actually in the smooth dual $\mathbf{H}^*_\infty\subset {\mathbf{H}_\infty}'$ for all $x,y_2,\dots,y_k\in\mathbf{H}_\infty$, and that the mapping

$$\begin{gathered}\prod^k\mathbf{H}_\infty\to\mathbf{H}_\infty\\ (x,y_2,\dots,y_k)\mapsto\check\Omega^{-1}(df(x)(\quad,y_2,\dots,y_k))\end{gathered}$$

is smooth. Note that we could also have used $Re\langle\quad,\quad\rangle$ instead of Ω. By the symmetry of higher derivatives this is then true for all entries of $d^kf(x)$, for all x.

2.8 Lemma. *For $f\in C^\infty(\mathbf{H}_\infty,\mathbb{R})$ the following assertions are equivalent:*

(1) *$df:\mathbf{H}_\infty\to{\mathbf{H}_\infty}'$ factors to a smooth mapping $\mathbf{H}_\infty\to\mathbf{H}^*_\infty$.*
(2) *f has a smooth Ω-gradient $\operatorname{grad}^\Omega f\in\mathfrak{X}(\mathbf{H}_\infty)=C^\infty(\mathbf{H}_\infty,\mathbf{H}_\infty)$ such that $df(x)y=\Omega(\operatorname{grad}^\Omega f(x),y)$.*
(3) *$f\in C^\infty_*(\mathbf{H}_\infty,\mathbb{R})$.*

2.9. Theorem. *The mapping*

$$\operatorname{grad}^\Omega:C^\infty_*(\mathbf{H}_\infty,\mathbb{R})\to\mathfrak{X}_\Omega(\mathbf{H}_\infty),\qquad \operatorname{grad}^\Omega f:=\check\Omega^{-1}\circ df,$$

is well defined; also the Poisson bracket

$$\begin{aligned}\{\quad,\quad\}&:C^\infty_*(\mathbf{H}_\infty,\mathbb{R})\times C^\infty_*(\mathbf{H}_\infty,\mathbb{R})\to C^\infty_*(\mathbf{H}_\infty,\mathbb{R}),\\ \{f,g\}&:=i(\operatorname{grad}^\Omega f)i(\operatorname{grad}^\Omega g)\Omega=\Omega(\operatorname{grad}^\Omega g,\operatorname{grad}^\Omega f)=\\ &=(\operatorname{grad}^\Omega f)(g)=dg(\operatorname{grad}^\Omega f)\end{aligned}$$

is well defined and gives a Lie algebra structure to the space $C^\infty_*(\mathbf{H}_\infty,\mathbb{R})$.

We also have the following long exact sequence of Lie algebras and Lie algebra homomorphisms:

$$0\to H^0(\mathbf{H}_\infty)\to C^\infty_*(\mathbf{H}_\infty,\mathbb{R})\xrightarrow{\operatorname{grad}^\Omega}\mathfrak{X}_\Omega(\mathbf{H}_\infty)\xrightarrow{\gamma}H^1(\mathbf{H}_\infty)=0$$

2.10. We consider now again as in 2.1 a unitary representation $\rho : G \to U(\mathbf{H})$. By theorem 2.2 the associated mapping $\hat{\rho} : G \times \mathbf{H}_\infty \to \mathbf{H}_\infty$ is smooth, so we have the infinitesimal mapping $\rho' : \mathfrak{g} \to \mathfrak{X}(\mathbf{H}_\infty)$, given by $\rho'(X)(x) = T_e(\hat{\rho}(\quad, x))$ for $X \in \mathfrak{g}$ and $x \in \mathbf{H}_\infty$. Since ρ is a unitary representation, the mapping ρ' has values in the Lie subalgebra of all linear hamiltonian vector fields $\xi \in \mathfrak{X}(\mathbf{H}_\infty)$ which respect the symplectic form Ω, i.e. $\xi : \mathbf{H}_\infty \to \mathbf{H}_\infty$ is linear and $\mathcal{L}_\xi \Omega = 0$.

Now let us consider the mapping $\check{\Omega} \circ \rho'(X) : \mathbf{H}_\infty \to T(\mathbf{H}_\infty) \to T^*(\mathbf{H}_\infty)$. We have $d(\check{\Omega} \circ \rho'(X)) = d(i_{\rho'(X)}\Omega) = \mathcal{L}_{\rho'(X)}\Omega = 0$, so the linear 1-form $\check{\Omega} \circ \rho'(X)$ is closed, and since $H^1(\mathbf{H}_\infty) = 0$, it is exact. So there is a function $\sigma(X) \in C^\infty(\mathbf{H}_\infty, \mathbb{R})$ with $d\sigma(X) = \check{\Omega} \circ \rho'(X)$, and $\sigma(X)$ is uniquely determined up to addition of a constant. If we require $\sigma(X)(0) = 0$, then $\sigma(X)$ is uniquely determined and is a quadratic function. In fact we have $\sigma(X)(x) = \int_{c_x} \check{\Omega} \circ \rho'(X)$, where $c_x(t) = tx$. Thus

$$\begin{aligned}\sigma(X)(x) &= \textstyle\int_0^1 \Omega(\rho'(X)(tx), \frac{d}{dt}tx)dt = \\ &= \Omega(\rho'(X)(x), x)\textstyle\int_0^1 dt \\ &= \tfrac{1}{2}\Omega(\rho'(X)(x), x).\end{aligned}$$

2.11. Lemma. *The mapping*

$$\sigma : \mathfrak{g} \to C^\infty_*(\mathbf{H}_\infty, \mathbb{R}), \qquad \sigma(X)(x) = \tfrac{1}{2}\Omega(\rho'(X)(x), x)$$

for $X \in \mathfrak{g}$ and $x \in \mathbf{H}_\infty$, is a Lie algebra homomorphism and $\operatorname{grad}^\Omega \circ \sigma = \rho'$.

For $g \in G$ we have $\rho(g)^*\sigma(X) = \sigma(X) \circ \rho(g) = \sigma(Ad(g^{-1})X)$, so σ is G-equivariant.

2.12. The moment mapping. For a unitary representation $\rho : G \to U(\mathbf{H})$ we can now define the *moment mapping*

$$\begin{aligned}\mu : \mathbf{H}_\infty &\to \mathfrak{g}' = L(\mathfrak{g}, \mathbb{R}), \\ \mu(x)(X) &:= \sigma(X)(x) = \tfrac{1}{2}\Omega(\rho'(X)x, x),\end{aligned}$$

for $x \in \mathbf{H}_\infty$ and $X \in \mathfrak{g}$.

2.13 Theorem. *The moment mapping $\mu : \mathbf{H}_\infty \to \mathfrak{g}'$ has the following properties:*

(1) $(d\mu(x)y)(X) = \Omega(\rho'(X)x, y)$ *for* $x, y \in \mathbf{H}_\infty$ *and* $X \in \mathfrak{g}$, *so* $\mu \in C^\infty_*(\mathbf{H}_\infty, \mathfrak{g}')$.

(2) *For* $x \in \mathbf{H}_\infty$ *the image of* $d\mu(x) : \mathbf{H}_\infty \to \mathfrak{g}'$ *is the annihilator* $\mathfrak{g}_x^\Omega$ *of the Lie algebra* $\mathfrak{g}_x = \{X \in \mathfrak{g} : \rho'(X)(x) = 0\}$ *of the isotropy group* $G_x = \{g \in G : \rho(g)x = x\}$ *in* $\mathfrak{g}'$.

(3) *For* $x \in \mathbf{H}_\infty$ *the kernel of* $d\mu(x)$ *is*

$$(T_x(\rho(G)x))^\Omega = \{y \in \mathbf{H}_\infty : \Omega(y, T_x(\rho(G)x)) = 0\},$$

the Ω-annihilator of the tangent space at x of the G-orbit through x.

(4) *The moment mapping is equivariant:* $Ad'(g) \circ \mu = \mu \circ \rho(g)$ *for all* $g \in G$, *where* $Ad'(g) = Ad(g^{-1})' : \mathfrak{g}' \to \mathfrak{g}'$ *is the coadjoint action.*

(5) *The pullback operator* $\mu^* : C^\infty(\mathfrak{g}, \mathbb{R}) \to C^\infty(\mathbf{H}_\infty, \mathbb{R})$ *actually has values in the subspace* $C^\infty_*(\mathbf{H}_\infty, \mathbb{R})$. *It also is a Lie algebra homomorphism for the Poisson brackets involved.*

2.14. Let again $\rho : G \to U(\mathbf{H})$ be a unitary representation of a Lie group G on a Hilbert space $\mathbf{H}$.

Definition. A vector $x \in \mathbf{H}$ is called it real analytic if the mapping $g \mapsto \rho(g)x$, $G \to \mathbf{H}$ is a real analytic mapping, in the real analytic structure of the Lie group G.

We will use from now on the theory of real analytic mappings in infinite dimensions as developed in [K-M]. So the following conditions on $x \in \mathbf{H}$ are equivalent:

(1) x is a real analytic vector.

(2) $\mathfrak{g} \ni X \mapsto \rho(\exp X)x$ is locally near 0 given by a converging power series.

(3) For each $y \in \mathbf{H}$ the mapping $\mathfrak{g} \ni X \mapsto \langle \rho(\exp X)x, y\rangle \in \mathbb{C}$ is smooth and real analytic along affine lines in $\mathfrak{g}$, locally near 0.

The only nontrivial part is (3) $\Rightarrow$ (1), and this follows from [K-M, 1.6 and 2.7] and the fact, that ρ is a representation.

Let $\mathbf{H}_\omega$ denote the vector space of all real analytic vectors in $\mathbf{H}$. Then we have a linear embedding $j : \mathbf{H}_\omega \to C^\omega(G, \mathbf{H})$ into the space of real analytic mappings, given by $x \mapsto (g \mapsto \rho(g)x)$. We equip $C^\omega(G, \mathbf{H})$ with the convenient vector space structure described in [K-M, 5.4, see also 3.13]. Then $\mathbf{H}_\omega$ consists of all equivariant functions in $C^\omega(G, \mathbf{H})$ and is therefore a closed subspace. So it is a convenient vector space with the induced structure.

The space $\mathbf{H}_\omega$ is dense in the Hilbert space $\mathbf{H}$ by [Wa, 4.4.5.7] and an invariant subspace, so we have a representation $\rho : G \to L(\mathbf{H}_\omega, \mathbf{H}_\omega)$.

2.15. Theorem. *The mapping $\hat{\rho} : G \times \mathbf{H}_\omega \to \mathbf{H}_\omega$ is real analytic in the sense of* [K-M].

Proof. Similar to the proof of theorem 2.2. □

2.16. Again we consider now $\mathbf{H}_\omega$ as a "weak" symplectic real analytic Fréchet manifold, equipped with the symplectic structure Ω, the restriction of the imaginary part of the hermitian inner product $\langle\quad,\quad\rangle$ on $\mathbf{H}$. Then again $\Omega \in \Omega^2(\mathbf{H}_\omega)$ is a closed 2-form which is non degenerate in the sense that $\check{\Omega} : \mathbf{H}_\omega \to \mathbf{H}'_\omega = L(\mathbf{H}_\omega, \mathbb{R})$ is injective. Let

$$\mathbf{H}^*_\omega := \check{\Omega}(\mathbf{H}_\omega) = \Omega(\mathbf{H}_\omega, \quad) = Re\langle \mathbf{H}_\omega, \quad \rangle \subset \mathbf{H}'_\omega = L(\mathbf{H}_\omega, \mathbb{R})$$

again denote the *analytic dual* of $\mathbf{H}_\omega$, equipped with the topology induced by the isomorphism with $\mathbf{H}_\omega$.

2.17 Remark. All the results leading to the smooth moment mapping can now be carried over to the real analytic setting with *no* changes in the proofs. So all statements from 2.9 to 2.13 are valid in the real analytic situation. We summarize this in one more result:

2.18 Theorem. *Consider the injective linear continuous G-equivariant mapping $i : \mathbf{H}_\omega \to \mathbf{H}_\infty$. Then for the smooth moment mapping $\mu : \mathbf{H}_\infty \to \mathfrak{g}'$ from 2.13 the composition $\mu \circ i : \mathbf{H}_\omega \to \mathbf{H}_\infty \to \mathfrak{g}'$ is real analytic. It is called the real analytic moment mapping.*

2.19. Remarks. It is my conjecture that for an irreducible representation which is constructed by geometric quantization of an coadjoint orbit (the Kirillov method), the restriction of the moment mapping to the intersection of the unit sphere with the space of smooth vectors takes values exactly in the orbit one started with, if the construction is suitably normalized.

I have been unable to prove this conjecture in general, but Herbert Wiklicky [Wi] has checked that this is true for the Heisenberg group. He also checked that this moment mapping produces the expectation value for the (angular) momentum in physically relevant situations and he claims that this moment mapping describes a sort of classical limit for the quantum theory described by the unitary representation in question.

Let me add some thoughts on the rôle of the moment mapping in the study of unitary representations. I think that its restriction to the intersection of the unit sphere with the space of smooth vectors maps to one coadjoint orbit, if the representation is irreducible (I was unable to prove this). It is known that not all irreducible representations come from line bundles over coadjoint orbits (alias geometric quantization), but there might be a higher dimensional vector bundle over this coadjoint orbit, whose space of sections contains the space of smooth vectors as subspace of sections which are covariantly constant along some complex polarization.

References

[A-K] Auslander, Louis; Kostant, Bertram, *Polarization and unitary representations of solvable Lie groups*, Inventiones Math. **14** (1971), 255–354.

[Bo] Boman, Jan, *Differentiability of a function and of its compositions with functions of one variable*, Math. Scand. **20** (1967), 249–268.

[F-K] Frölicher, Alfred; Kriegl, Andreas, *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.

[L-M] Libermann, Paulette; Marle, C. M., *Symplectic geometry and analytical mechanics*, Mathematics and its applications, D. Reidel, Dordrecht, 1987.

[Ke] Keller, Hans H., *Differential calculus in locally convex spaces*, Springer Lecture Notes 417, 1974.

[Ki1] Kirillov, A. A., *Elements of the theory of representations*, Springer-Verlag, Berlin, 1976.

[[**Ki2**] Kirillov, A. A., *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17** (1962), 53–104.

[Kn] Knapp, Anthony W., *Representation theory of semisimple Lie groups*, Princeton University Press, Princeton, 1986.

[Ko] Kostant, Bertram, *Quantization and unitary representations*, Lecture Notes in Mathematics, Vol. 170,, Springer-Verlag, 1970, pp. 87–208.

[Kr1] Kriegl, Andreas, *Die richtigen Räume für Analysis im Unendlich - Dimensionalen*, Monatshefte Math. **94** (1982), 109–124.

[Kr2] Kriegl, Andreas, *Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokalkonvexen Vektorräumen*, Monatshefte für Math. **95** (1983), 287–309.

[K-M] Kriegl, Andreas; Michor, Peter W., *The convenient setting for real analytic mappings*, Acta Mathematica (1990).

[K-N] Kriegl, Andreas; Nel, Louis D., *A convenient setting for holomorphy*, Cahiers Top. Géo. Diff. **26** (1985), 273–309.

[Mic1] Michor, Peter W., *Manifolds of differentiable mappings*, Shiva Mathematics Series 3, Orpington, 1980.

[Mic2] Michor, Peter W., *The moment mapping for unitary representations*, J. Global Anal. Geo (1990).

[Mil] Milnor, John, *Remarks on infinite dimensional Lie groups*, Relativity, Groups, and Topology II, Les Houches, 1983, B.S. DeWitt, R. Stora, Eds., Elsevier, Amsterdam, 1984.

[P-S] Pressley, Andrew; Segal, Graeme, *Loop groups*, Oxford Mathematical Monographs, Oxford University Press, 1986.

[Wa] Warner, Garth, *Harmonic analysis on semisimple Lie groups, Volume I*, Springer-Verlag, New York, 1972.

[We] Weinstein, Alan, *Lectures on symplectic manifolds*, Regional conference series in mathematics **29** (1977), Amer. Math. Soc..

[Wi] Wiklicky, Herbert, *Physical interpretations of the moment mapping for unitary representations*, Diplomarbeit, Universität Wien, 1989.

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria.