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SMOOTH $*$-ALGEBRAS

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Abstract. Looking for the universal covering of the smooth non-commutative torus leads to a curve of associative multiplications on the space $\mathcal{O}'_M(\mathbb{R}^{2n}) \cong \mathcal{O}_C(\mathbb{R}^{2n})$ of Laurent Schwartz which is smooth in the deformation parameter $\hbar$. The Taylor expansion in $\hbar$ leads to the formal Moyal star product. The non-commutative torus and this version of the Heisenberg plane are examples of smooth *-algebras: smooth in the sense of having many derivations. A tentative definition of this concept is given.

1. Introduction

The noncommutative torus in its topological version (C^*-completion) as well as in its smooth version [6] is one of the most important examples in noncommutative geometry. Beside the fact that the classical tools of differential geometry have unambiguous generalizations to it, it provides a very nontrivial example of noncommutative geometry satisfying the axioms of [7] (see also in [8], [9]). We looked at its smooth version and asked for its universal covering. We found the Heisenberg plane as it is presented in this paper: a twisted convolution on a carefully chosen space of distributions, namely the topological dual space $\mathcal{O}'_M$ of the Schwartz space $\mathcal{O}_M$ of smooth slowly increasing functions at ∞, [29], [30]. It is large enough to contain the space of rapidly decreasing measures with support in the lattice $(2\pi\mathbb{Z})^2$ that is a space isomorphic to the space of smooth functions on the noncommutative torus (as well as on the usual commutative torus). The multiplication turns out to be a smooth curve in the deformation parameter $\hbar$. Moreover, looking at it via Fourier transform, Taylor expansion of the multiplication in the deformation parameter $\hbar$ leads to the formal Moyal star-product which is well known from deformation quantization, [24], [1].

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Then we noticed that we found examples of noncommutative $*$-algebras generalizing algebras of complex smooth functions. These $*$-algebras which can be realized as $*$-algebras of unbounded operators in Hilbert space admit "many" derivations specifying thereby the generalized smooth structure (see below). These algebras are defined in Section 2 and are tentatively called smooth $*$-algebras.

Section 3 contains our treatment of the smooth non-commutative torus, and also some related material like the smooth non-commutative circle of rational slope b/a, a quotient of the smooth non-commutative torus.

The appendix in section 5 gives an overview on convenient calculus in infinite dimensions which is necessary to obtain our results about smoothness in the deformation parameter $\hbar$, and which also gives the right setting for multilinear algebra with locally convex vector spaces.

Work on this paper started in 1996, but we were unable to prove that the Heisenberg plane is a smooth *-algebra. Finally we gave up and stated this as a conjecture. The problem is finding enough states.

2. Smooth $*$-algebras

2.1. Preliminaries. Throughout this paper by a $*$-*algebra* we always mean a complex associative algebra A with unit equipped with an antilinear involution $f \mapsto f^*$ which reverses the order of products i.e. which satisfies $(fg)^* = g^* f^*$, $\forall f, g \in A$. Given a $*$-algebra A, a *hermitian representation* [25] of A in a Hilbert space $\mathcal{H}$ is a homomorphism π of unital algebras of A into the algebra of endomorphisms of a dense subspace $D(\pi)$ of $\mathcal{H}$ satisfying $(\Psi, \pi(f)\Phi) = (\pi(f^*)\Psi, \Phi)$ for any $f \in A$ and $\Psi, \Phi \in D(\pi)$; the dense subspace $D(\pi)$ of $\mathcal{H}$ is refered to as *the domain of* π. The image of a hermitian representation in $\mathcal{H}$ is a unital subalgebra of the algebra of endomorphisms of the dense domain D of the representation which is also a $*$-algebra for an obvious involution; such a $*$-algebra will be refered to as a $*$-*algebra of (unbounded) operators in the Hilbert space* $\mathcal{H}$ *with domain* D.

A linear form φ on a $*$-algebra A is said to be *positive* if $\varphi(f^*f) \geq 0$ for all $f \in A$. Such a positive linear form satisfies $\varphi(f^*) = \overline{\varphi(f)}$ (for all $f \in A$) and $(f, g)_\omega = \varphi(f^*g)$ is a pre-Hilbert scalar product on A which induces a Hausdorff pre-Hilbert structure on the quotient $D_\varphi = A/I_\varphi$ where $I_\varphi = \{f \in A | \varphi(f^*f) = 0\}$. In view of the Schwarz inequality, I_φ is a left ideal of A so one has a homomorphism of unital algebras π_φ of A into the endomorphisms of D_φ which is in fact a hermitian representation of A in the Hilbert space $\mathcal{H}_\varphi$ obtained by completion of D_φ with domain $D(\pi_\varphi) = D_\varphi$. Let $\Omega_\varphi \in D_\varphi$ be the canonical image of the unit $1 \in A$ under the projection $A \to D_\varphi = A/I_\varphi$. Then one has $\varphi(f) = (\Omega_\varphi, \pi_\varphi(f)\Omega_\varphi)$ for any $f \in A$ and $D_\varphi = \pi_\varphi(A)\Omega_\varphi$. This construction which associates to a positive linear form φ on A the triplet $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$ of a hermitian representation π_φ of A in Hilbert space $\mathcal{H}_\varphi$ with Ω_φ in the domain of π_φ such that $\pi_\varphi(A)\Omega_\varphi$ is dense in $\mathcal{H}_\varphi$ and $\varphi = (\Omega_\varphi, \pi_\varphi(\cdot)\Omega_\varphi)$ is known as the GNS construction; given φ, the triplet $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$ is unique up to a unitary.

Given a hermitian representation π of a $*$-algebra A with domain $D(\pi)$, to each vector $\Phi \in D(\pi)$ coresponds the positive linear form φ on A defined by $\varphi(f) = (\Phi, \pi(f)\Phi)$. Conversely, the GNS construction shows that any positive linear form on A can be realized in this manner. To the action $(f, \Phi) \mapsto \pi(f)\Phi$ of A on $D(\pi)$

corresponds the action $(f,\varphi) \mapsto \varphi_f$ of A on the (strict) convex cone A^*_+ of its positive linear forms where φ_f is defined by $\varphi_f(g) = \varphi(f^*gf)$ for $f, g \in A$.

2.2. Proposition. *The following conditions* (i) *and* (ii) *are equivalent for a locally convex $*$-algebra A.*

(i) *A is a $*$-algebra of unbounded operators in Hilbert space $\mathcal{H}$ with domain D and its locally convex topology is generated by seminorms $f \mapsto \| f\Phi \|$, $\Phi \in D$.*

(ii) *There is a subset $\mathcal{S}$ of positive linear forms on A which is invariant by the action of A on A^*_+ and which is such that the locally convex topology of A is generated by the seminorms $f \mapsto (\varphi(f^*f))^{1/2}$, $\varphi \in \mathcal{S}$ and is Hausdorff.*

Proof. $(i) \Rightarrow (ii)$. This is obvious by taking $\mathcal{S} = \{f \mapsto (\Phi, f\Phi) | \Phi \in D\}$.

$(ii) \Rightarrow (i)$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$ denote the GNS triplet associated to $\varphi \in \mathcal{S}$. Take $\mathcal{H}$ to be the Hilbertian direct sum $\hat{\oplus}_{\varphi\in\mathcal{S}}\mathcal{H}_\varphi$, take $D = \oplus_{\varphi\in\mathcal{S}}\pi_\varphi(A)\Omega_\varphi$ and notice that it follows from the assumptions that $\pi = \oplus_{\varphi\in\mathcal{S}}\pi_\varphi$ is injective so A identifies canonically to the $*$-algebra $\pi(A)$ of unbounded operators in $\mathcal{H}$ with domain D. It is clear that the locally convex topology on A generated by the seminorms $f \mapsto (\varphi(f^*f))^{1/2}$, $\varphi \in \mathcal{S}$ is the same as the one generated by the seminorms $f \mapsto \| \pi(f)\Phi \|$, $\Phi \in D$. $\square$

Notice that if φ is a positive linear form on A one has

$$|\varphi(f)| \leq (\varphi(1))^{1/2}(\varphi(f^*f))^{1/2}$$

for any $f \in A$ (Schwarz inequality) so any $\varphi \in \mathcal{S}$ is automatically continuous, (notice also that the same inequality shows that $\varphi = 0$ whenever $\varphi(1) = 0$ for $\varphi \in A^*_+$).

2.3. Definition. Let A be a $*$-algebra, $\mathcal{S}$ be a subset of positive linear forms on A invariant by the action of A on A^*_+ and let $\mathcal{D}$ be a Lie subalgebra of the Lie algebra $\mathrm{Der}(A)$ of derivations of A which is also a $Z(A)$-submodule of $\mathrm{Der}(A)$ where $Z(A)$ denotes the center of A. Assume that :

(1) The locally convex topology on A generated by the semi-norms $f \mapsto \nu_\varphi(f) = (\varphi(f^*f))^{1/2}$, $\varphi \in \mathcal{S}$ is Hausdorff;

(2) $\cap\{\ker(X) | X \in \mathcal{D}\} = \mathbb{C}1$;

(3) The locally convex topology $\tau(\mathcal{S},\mathcal{D})$ on A generated by the seminorms $\nu_\varphi \circ X_1 \circ \dots X_p$, $\varphi \in \mathcal{S}$, $X_i \in \mathcal{D}$, $p \in \mathbb{N}$ is such that $(A, \tau(\mathcal{S},\mathcal{D}))$ is complete.

Then A will be said to be a *smooth $*$-algebra* relative to $\mathcal{S}$ and $\mathcal{D}$, or simply a smooth $*$-algebra when no confusion arises, the topology $\tau = \tau(\mathcal{S},\mathcal{D})$ being called *smooth topology* of A.

2.4. Commutative smooth $*$-algebras. Let M be a smooth finite dimensional manifold, let $A = C^\infty(M,\mathbb{C})$ be the $*$-algebra of all complex valued smooth functions on M. Let $\mathcal{D} = \mathrm{Der}(A) = \mathfrak{X}(M) \otimes \mathbb{C}$ be the Lie algebra of all derivations of $C^\infty(M,\mathbb{C})$, i.e. all complex valued vector fields on M. Let $\mathrm{Vol}(M) \to M$ be the real line bundle of all densities on M, and let $\Gamma^+_c(\mathrm{Vol}(M))$ be the space of all smooth non-negative densities with compact support on M. Let $\mathcal{S}$ be the space of all linear functionals of the form $f \mapsto \int_M f\mu$ for all $\mu \in \Gamma^+_c(\mathrm{Vol}(M))$.

Then the locally convex topology on $C^\infty(M,\mathbb{C})$ described in 2.3.1 is the compact open topology which is Hausdorff. Condition 2.3.2 is obviously satisfied. The topology $\tau(\mathcal{S},\mathcal{D})$ from 2.3.3 is equivalent to the compact C^∞-topology, i.e. the topology of uniform convergence on compact subsets in all derivatives separately. Thus $C^\infty(M,\mathbb{C})$ is a smooth $*$-algebra relative to $\mathcal{S}$ and $\mathcal{D}$.

3. The non-commutative torus

3.1. The non-commutative torus. By Fourier expansion the algebra $C^\infty(S^1\times S^1,\mathbb{C})$ of all smooth functions on the torus consists of all

$$f=\sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} f_{k,l}u^kv^l, \tag{1}$$

where $(f_{k,l})$ is any rapidly decreasing sequence of complex numbers, i.e. for each $m\in\mathbb{N}$ the seminorm

$$\|f\|_m := \sup_{k,l\in\mathbb{Z}} |f_{k,l}|\,(1+|k|+|l|)^m<\infty, \tag{2}$$

and where $u=\exp(2\pi it)$ and $v=\exp(2\pi is)$ are the coordinates on the torus.

Let us fix a complex number q with $|q|=1$. Then the smooth q-torus $C^\infty(T_q^2)$ is the convenient associative algebra (in fact a Fréchet algebra) which is given by all elements of the form (1), but where we assume now that U, V are two indeterminates which satisfy

$$UV=qVU \tag{3.}$$

Defining

$$U^*:=U^{-1},\quad V^*:=V^{-1} \tag{4}$$

makes $C^\infty(T_q^2)$ into a $*$-algebra. Note that $U^kV^l=q^{kl}V^lU^k$ and hence

$$fg=\Big(\sum_{k,l}f_{k,l}U^kV^l\Big)\Big(\sum_{m,n}g_{m,n}U^mV^n\Big)=\sum_{k,l}\Big(\sum_{m,n}f_{m,n}g_{k-m,l-n}q^{-n(k-m)}\Big)U^kV^l$$

$$f^*=\Big(\sum_{k,l}f_{k,l}U^kV^l\Big)^*=\sum_{k,l}\bar f_{k,l}V^{-l}U^{-k}=\sum_{k,l}\bar f_{-k,-l}q^{-kl}U^kV^l.$$

Using the convention

$$f=\sum_{k,l}f'_{k,l}q^{-\frac{kl}{2}}U^kV^l,\qquad \text{so } f'_{k,l}=f_{k,l}q^{\frac{kl}{2}}$$

we get nicer descriptions for the product and the adjoint f^*:

$$\begin{aligned}fg&=\Big(\sum_{k,l}f'_{k,l}q^{-\frac{kl}{2}}U^kV^l\Big)\Big(\sum_{k,l}g'_{m,n}q^{-\frac{mn}{2}}U^mV^n\Big)\\&=\sum_{k,l}\Big(\sum_{m,n}f'_{m,n}g'_{k-m,l-n}q^{-\frac12(kn-ml)}\Big)q^{-\frac{kl}{2}}U^kV^l\\f^*&=\sum_{k,l}\bar f'_{-k,-l}q^{-kl/2}U^kV^l.\end{aligned}$$

If (the argument of) q is rational (mod 2π), let $N\in\mathbb{N}$ be the smallest positive natural number such that $q^N=1$. If q is irrational, we put $N=0$.

3.2. Proposition. *If q is rational, then there exits a smooth vector bundle $A_q \to S^1 \times S^1$ with standard fiber the algebra $\mathrm{Mat}_N(\mathbb{C})$ of all complex $(N \times N)$-matrices and with transition functions in $GL(n, \mathbb{C})$ acting on Mat_N by conjugation, such that the non-commutative torus $C^\infty(T_q^2)$ is isomorphic to the algebra $\Gamma(A_q)$ of all smooth sections of the algebra bundle $A_q \to S^1 \times S^1$. The center of $C^\infty(T_q^2)$ is isomorphic to $C^\infty(S^1 \times S^1, \mathbb{C})$. The first Chern class of the complex vector bundle A_q vanishes.*

Moreover, there is a smooth vector bundle $E_q \to S^1 \times S^1$ with standard fiber $\mathbb{C}^N$ such that A_q is the full endomorphism bundle $\mathrm{End}(E_q)$. The first Chern class of E_q also vanishes.

Proof. We first claim that the algebra Mat_N is the unique algebra generated by two unitary elements U_0 and V_0 which are subject to the relations

$$U_0.V_0 = qV_0.U_0, \quad U_0^N = V_0^N = \mathbb{I}. \tag{1}$$

To see this note that each element in the algebra generated by U_0 and V_0 may be written in the form $\sum_{0 \le k,l \le N-1} a_{k,l} U_0^k V_0^l$, so this algebra is of dimension $\le N^2$. On the other hand we consider the matrices in Mat_N,

$$U_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & & 0 & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1 & 0 & 0 & \dots & & \\ 0 & q & 0 & \dots & & \\ 0 & 0 & q^2 & 0 & \dots & \\ \vdots & & & \ddots & & \\ 0 & \dots & & 0 & q^{N-1} & \end{pmatrix},$$

which satisfy relations (1) and thus generate a C^*-subalgebra which clearly commutes only with the multiples of the identity, so it has to be the full matrix algebra.

Now we consider the trivial bundle $S^1 \times S^1 \times \mathrm{Mat}_N \xrightarrow{pr_{1,2}} S^1 \times S^1$. The space of smooth section is then $C^\infty(S^1 \times S^1, \mathrm{Mat}_N) = C^\infty(S^1 \times S^1, \mathbb{C}) \otimes \mathrm{Mat}_N$, which is generated by the unitary central elements u, v, and unitary U_0, V_0 with the relations (1), where the coefficients are again rapidly decreasing with respect to the powers of u and v.

Consider now the cyclic group $\mathbb{Z}_N = \mathbb{Z}/N.\mathbb{Z}$, the q-action of $(m, n) \in \mathbb{Z}_N \times \mathbb{Z}_N = \mathbb{Z}_N^2$ on $S^1 \times S^1$ given by $(u, v) \mapsto (q^m.u, q^n.v)$, and the q-action on Mat_N given by $A \mapsto U_0^n.V_0^{-m}.A.V_0^m.U_0^{-n}$. Note that inside the adjoint action of $GL(n, \mathbb{C})$ the matrices U_0 and V_0 commute, since they do so in $PGL(n, \mathbb{C})$, and that (m, n) maps U_0 to $q^m.U_0$, and maps V_0 to $q^n.V_0$. We may consider the following diagram, where the horizontal arrows are covering mappings since all involved actions are strictly discontinuous, and where the left vertical arrow is $\mathbb{Z}_N^2$-equivariant.

$$\begin{array}{ccc} S^1 \times S^1 \times \mathrm{Mat}_N & \xrightarrow{\mathbb{Z}_N^2} & A_q \\ {\scriptstyle pr_{1,2}} \downarrow & & \downarrow {\scriptstyle p_q} \\ S^1 \times S^1 & \xrightarrow[\mathbb{Z}_N^2]{\pi} & S^1 \times S^1 \end{array}$$

Since the action of $\mathbb{Z}_N^2$ on Mat_N is by algebra automorphisms, the resulting smooth mapping $A_q \to S^1 \times S^1$ is a smooth algebra bundle. The sections of A_q correspond exactly to the $\mathbb{Z}_N^2$-equivariant sections of the left hand side. A section $f : S^1 \times S^1 \to \mathrm{Mat}_N$,

$$f = \sum_{\substack{k,l\in\mathbb{Z}\\ 0\le s,t\le N-1}} c_{k,l,s,t}\, u^k v^l U_0^s V_0^t$$

is $\mathbb{Z}_N^2$-equivariant if and only if the following condition is satisfied:

$$c_{k,l,s,t} \neq 0 \text{ only if } k \equiv s \mod N \text{ and } l \equiv t \mod N.$$

But then we may put $c_{k,l} = c_{k,l,s,t}$, where $s \equiv k \mod N$ and $t \equiv l \mod N$, and the section f can be written as

$$f = \sum_{k,l\in\mathbb{Z}} c_{k,l}(uU_0)^k(vV_0)^l.$$

We just have to note that $U = uU_0$ and $V = vV_0$ satisfy only the relations 3.1.3 of the noncommutative torus.

The first Chern class $c_1(A_q)$ of the complex vector bundle A_q vanishes, by the following argument: The mapping $\pi : S^1 \times S^1 \to S^1 \times S^1$ in the diagram above is an N^2-sheeted covering, has mapping degree N^2. Thus the mapping in cohomology is $H^2(\pi) = N^2 : H^2(S^1\times S^1,\mathbb{Z}) = \mathbb{Z} \to H^2(S^1\times S^1,\mathbb{Z}) = \mathbb{Z}$. We have $H^2(\pi)c_1(A_q) = c_1(\pi^* A_q) = 0$ since $\pi^* A_q$ is a trivial bundle. Thus also $c_1(A_q) = 0$.

Now we will construct the bundle $E_q \to S^1 \times S^1$. We cannot push it down from a trivial bundle via the group action by $\mathbb{Z}_N^2$ since $w \mapsto U_0^n.V_0^{-m}.w$ is not a representation of $\mathbb{Z}_N^2$ on $\mathbb{C}^N$. We have to absorb the non-commutativity into a larger group acion. Thus we consider the following semidirect product group, its action on $S^1 \times S^1 \times S^1$, and its unitary representation on $\mathbb{C}$:

$$\begin{gathered}
S^1 \to (\mathbb{Z}_N \times \mathbb{Z}_N) \ltimes S^1 \to \mathbb{Z}_N \times \mathbb{Z}_N\\
(m,n,\theta).(m',n',\theta') = (m+m', n+n', \theta\theta' q^{mn'})\\
((\mathbb{Z}_N \times \mathbb{Z}_N) \ltimes S^1) \times (S^1\times S^1\times S^1) \to S^1\times S^1\times S^1\\
(m,n,\theta).(\varphi,\psi,\nu) = (q^m\varphi, q^n\psi, \theta\nu\psi^m)\\
(m,n,\theta).w = \theta U_0^n V_0^{-m}.w, \quad w \in \mathbb{C}^N.
\end{gathered}$$

Using the actions we can define the bundle $E_q \to S^1 \times S^1$ as follows:

$$\begin{array}{ccc}
S^1\times S^1\times S^1\times\mathbb{C}^N & \xrightarrow{\mathbb{Z}_N\times\mathbb{Z}_N\times S^1} & E_q\\
{\scriptstyle pr_{1,2}}\downarrow & & \downarrow{\scriptstyle p_q}\\
S^1\times S^1\times S^1 & \xrightarrow{\mathbb{Z}_N\times\mathbb{Z}_N\times S^1} & S^1\times S^1
\end{array}$$

It is easy to check that all these actions are compatible with each other in such a way that we get a free fiberwise action of the algebra bundle A_q on the vector bundle E_q. By counting dimensions we see that $A_q = \mathrm{End}(E_q)$. For the first Chern class we can repeat the argument from above. □

3.3. Corollary. *Let q be a primitive N-th root of unity. Then the noncommutative torus algebra $C^\infty(T_q^2)$ is Morita equivalent to the commutative torus algebra $C^\infty(T^2)$.*

Proof. By theorem 3.2 we have the algebra isomorphism $C^\infty(T_q^2) \cong \Gamma(\mathrm{End}(E_q))$. But for any vector bundle the full automorphism algebra, which acts from the left on the space of smooth sections of the vector bundle, is Morita equivalent to the algebra of smooth functions on the base, which we may view as acting from the right. □

3.4. Derivations of the non-commutative torus. Let $D \in \mathrm{Der}(C^\infty(T_q^2))$, let us assume that D is bounded.

Then D is uniquely determined by the values

$$D(U) = \sum_{k,l} u_{k,l} U^k V^l, \quad D(V) = \sum_{k,l} v_{k,l} U^k V^l. \tag{1}$$

The relation $D(U).V + U.D(V) = qD(V).U + qV.D(U)$, by comparison of coefficients, leads quickly to

$$u_{k,l-1}(1 - q^{1-k}) + v_{k-1,l}(1 - q^{1-l}) = 0. \tag{2}$$

Now let N be the smallest integer with $q^N = 1$ for rational q, let $N = 0$ for irrational q.

Then for $k \equiv 1(\mod N)$ equation (2) implies that we have $v_{k-1,l} = 0$ for $l \not\equiv 1 \mod N$, and that $v_{k-1,l}$ can be prescribed arbitrarily (but rapidly decreasing) for $l \equiv 1 \mod N$. This means that we may prescribe $D(V) = g(U^N, V^N)V$ for arbitrary $g \in C^\infty(S^1 \times S^1, \mathbb{R})$.

Similarly for $l \equiv 1 \mod N$ equation (2) implies that we have $u_{k,l-1} = 0$ for $k \not\equiv 1 \mod N$, and that $u_{k,l-1}$ can be prescribed arbitrarily (but rapidly decreasing) for $k \equiv 1(\mod N)$. This means that we may prescribe $D(U) = f(U^N, V^N)U$ for arbitrary $f \in C^\infty(S^1 \times S^1, \mathbb{R})$.

Let us write D_U for the derivation given by $D_U(U) = U$ and $D_U(V) = 0$, similarly $D_V \in \mathrm{Der}(C^\infty(T_q^2))$ is given by $D_V(U) = 0$ and $D_V(V) = V$. Thus for any $f, g \in C^\infty(S^1 \times S^1, \mathbb{C})$, in the center of $C^\infty(T_q^2)$, the expression

$$f(U^N, V^N)D_U + g(U^N, V^N)D_V \tag{3}$$

describes a derivation which is not inner, since it acts on the center (if $N > 0$).

On the other hand for any $a = \sum a_{k,l} U^k V^l$ the inner derivation $\mathrm{ad}(a)b = a.b - b.a$ satisfies

$$\begin{aligned}
\mathrm{ad}(a)U &= \sum_{k,l} a_{k-1,l}(q^{-l} - 1)U^k V^l, \\
\mathrm{ad}(a)V &= \sum_{k,l} a_{k,l-1}(1 - q^{-k})U^k V^l,
\end{aligned}$$

so that all other derivations specified by (2) are inner derivations.

So we see that $\operatorname{Der}(C^\infty(T_q^2)) = \operatorname{Inn}(C^\infty(T_q^2)) \rtimes \operatorname{Out}(C^\infty(T_q^2))$, a semidirect product with $\operatorname{Inn}(C^\infty(T_q^2))$ an ideal, where the action of $\operatorname{Out}(C^\infty(T_q^2))$ on $\operatorname{Inn}(C^\infty(T_q^2))$ (the same as on $C^\infty(T_q^2)$) is given by the expression (2).

For q rational, the description (3) corresponds to the covariant derivative ∇_X along the vectorfield $X = f(u,v)\frac{\partial}{\partial t} + g(u,v)\frac{\partial}{\partial s}$ on $S^1 \times S^1$, where $u = e^{2\pi i t}$ and $v = e^{2\pi i s}$, with respect to the unique flat connection on the algebra bundle $A_q \to S^1 \times S^1$, which is induced by the description in 3.2, and which respects the fiberwise 'matrix'-multiplication. In this case the outer derivations correspond exactly to the derivations of the center.

For q irrational this is not the case. Here $\operatorname{Out}(C^\infty(T_q^2))$ is linearly generated by the two derivations D_U and D_V.

3.5. Conjecture. It might be the case that every (algebraic) derivation of the non-commutative torus is automatically bounded. This would follow from an automatic continuity result for algebra homomorphisms. One can find such results in the literature but they have too strong assumptions to be immediately applicable.

The following argument shows how to carry over continuity from algebra homomorphisms to derivations: A linear mapping $D : C^\infty(T_q^2) \to C^\infty(T_q^2)$ is a derivation if and only if the mapping $(\mathrm{Id}, D\varepsilon) : C^\infty(T_q^2) \to C^\infty(T_q^2) \times C^\infty(T_q^2)\varepsilon$ is an algebra homomorphism, where ε is in the center and $\varepsilon^2 = 0$ so that the multiplication in $C^\infty(T_q^2) \times C^\infty(T_q^2)\varepsilon$ is given by $(f + g\varepsilon)(f' + g'\varepsilon) = ff' + (fg' + gf')\varepsilon$.

3.6. The non-commutative torus is a smooth $*$-algebra. In fact we will show that the topology described in 2.3.3 is the one we started with in 3.1.

What are the states on $C^\infty(T_q^2)$? We consider first the trace $\operatorname{tr}(\sum_{k,l} c_{k,l}U^kV^l) = c_{0,0}$. We will use only states of the form

$$f \mapsto \omega_g(f) = \operatorname{tr}(g^* f g)$$

for some $g \in C^\infty(T_q^2)$, and indeed $g = 1$ will suffice. We start to check that we can reproduce a generating system of seminorms. For that it suffices to consider

$$f = \sum_{k,l} c_{k,l}U^kV^l \mapsto \omega_1(f) = \operatorname{tr}(f^*f)^{1/2} =$$

$$= \operatorname{tr}\Big(\sum_{k,l,m,n} \overline{c_{m,n}} V^{-n}U^{-m} c_{k,l}U^kV^l\Big)^{1/2} = \Big(\sum_{k,l} \overline{c_{k,l}} c_{k,l}\Big)^{1/2} = \|f\|_{\ell^2}$$

and to compose it with an appropriate composition of the two basic derivations D_U and D_V from 3.4 which give us:

$$D_U^m D_V^n \Big(\sum_{k,l} c_{k,l}U^kV^l\Big) = \sum_{k,l} c_{k,l}k^m l^n U^kV^l.$$

It remains to show that an arbitrary state ω on $C^\infty(T_q^2)$ is bounded: We use the Gelfand-Naimark-Segal construction. The subspace $I_\omega := \{f \in C^\infty(T_q^2) : \omega(f^*f) = 0\}$ is a left ideal, since by the Cauchy-Schwarz inequality we have $\omega((gf)^*gf) = \omega((f^*g^*g)f) \le \omega(g^*gff^*g^*g)\omega(f^*f) = 0$. Then $D_\omega := C^\infty(T_q^2)/I_\omega$ is a pre-Hilbert space with the inner product $\omega(f^*g)$ which is positively defined by the definition of I_ω. We get a *-representation $\pi_\omega : C^\infty(T^2_q) \to L(D_\omega, D_\omega)$. Since $\mathrm{Id}_{D_\omega} = \pi_\omega(U^*U) = \pi_\omega(U)^*\pi_\omega(U) = \pi_\omega(U)\pi_\omega(U)^*$, the operators $\pi_\omega(U)$ and $\pi_\omega(V)$ are unitary. Since the coefficients in $C^\infty(T^2_q)$ are rapidly decreasing,

$$\pi_\omega(f) = \pi_\omega(\sum_{k,l} c_{k,l}U^kV^l) = \sum_{k,l} c_{k,l}\pi_\omega(U)^k\pi_\omega(V)^l$$

is a bounded operator for each $f \in C^\infty(T^2_q)$, and π_ω is bounded. Thus the representation π_ω and the state ω can be extended to the 'C^*-algebra completion' $C(T^2_q)$ of $C^\infty(T^2_q)$ and ω has norm 1 on $C(T^2_q)$. Since $C^\infty(T^2_q) \to C(T^2_q)$ is continuous, ω is bounded on $C^\infty(T^2_q)$.

3.7. Higher dimensional non-commutative tori. Let us fix a complex number q with $|q| = 1$, and let us consider the algebra $C^\infty(T^n_q)$ consisting of all

$$f = \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n} f_k S_1^{k_1} S_2^{k_2} \dots S_n^{k_n} \tag{1}$$

where (f_k) is any rapidly decreasing sequence of complex numbers so that for each $m \in \mathbb{N}$ the seminorm

$$\|f\|_m := \sup_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n} |f_k|(1 + |k_1| + \dots + |k_n|)^m < \infty,$$

and where the generators $S_1, \dots, S_n$ satisfy the commutation rules

$$\begin{cases} S_iS_{i+1} = qS_{i+1}S_i & \text{for } i = 1, \dots, n-1 \\ S_iS_j = S_jS_i & \text{for } |i-j| \geq 2 \end{cases}$$

This looks like an interesting generalization of the non-commutatve torus $C^\infty(T^2_q)$. But it is not so interesting as the following result shows:

Lemma. *For $n = 2p$ we have*

$$C^\infty(T^{2p}_q) = C^\infty(T^2_q)\hat{\otimes}\dots\hat{\otimes}C^\infty(T^2_q) \quad p \text{ times},$$

where we may use the projective tensor product.

For $n = 2p+1$ we have

$$C^\infty(T^{2p+1}_q) = C^\infty(T^2_q)\hat{\otimes}\dots\hat{\otimes}C^\infty(T^2_q)\hat{\otimes}C^\infty(S^1),$$

the projective tensor product of 2p copies of the non-commutative 2-torus with one 1-torus.

Proof. Let first $n = 2p$. Consider the new set of generators of the algebra T^{2p}_q

$$\begin{cases} U_j := S_1S_3\dots S_{2j-1} & \text{for } j = 1, \dots, p \\ V_j := S_{2j} & \text{for } j = 1, \dots, p \end{cases}$$

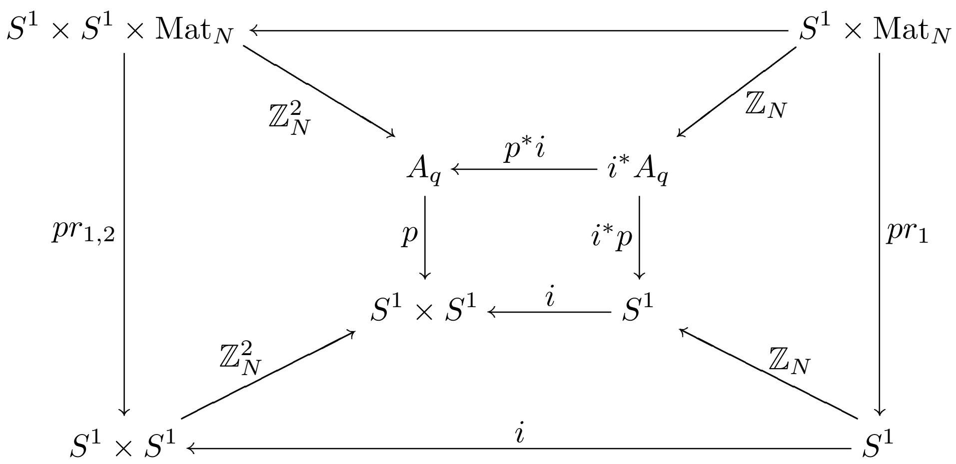
Then obviously $U_jV_j = qV_jU_j$ and all other pairs commute so that the first result follows.

If we have moreover an element S_{2p+1} then we also consider the last generator $Z = S_1S_3\dots S_{2p+1}$ which lies in the center of T^{2p+1}_q (it even generates the center if q is irrational) and thus splits off a central subalgebra isomorphic to $C^\infty(S^1)$. □

3.8. The non-commutative circle. We look for the non-commutative circle as a smooth algebra which is a quotient of the non-commutative torus. Since $C^\infty(T^2_q)$ is a simple algebra for irrational q we will succeed only for rational q, thus let us take $q \in S^1 \subset \mathbb{C}$ with $q^N = 1$ for minimal N. As in 3.1 let $u = \exp(2\pi i t)$ and $v = \exp(2\pi i s)$ be the coordinates on the torus $S^1 \times S^1$, and let $z = \exp(2\pi i x)$ be the coordinate on S^1. Let us consider the embedding

$$i : S^1 \to S^1 \times S^1, \quad i(z) = (z^a, z^b),$$

where $a, b \in \mathbb{Z}$ are relatively prime. Then we consider the algebra bundle $A_q \to S^1 \times S^1$ with typical fiber Mat_N constructed in the proof of proposition 3.2, and take the pullback bundle $i^* A_q \to S^1$ and the space of smooth sections is then viewed as the *non-commutative q-circle*. We want to describe it by generators and relations. For that consider the following diagram



where all diagonal mappings are covering maps with the groups of covering transformations indicated: $(m, n) \in \mathbb{Z}_N^2$ acts on $S^1 \times S^1$ by $(u, v) \mapsto (q^m u, q^n v)$ and on Mat_N by $A \mapsto U_0^n.V_0^{-m}.A.V_0^m.U_0^{-n}$; $p \in \mathbb{Z}_N$ acts on S^1 by $z \mapsto q^p z$ and on Mat_N by $A \mapsto U_0^{bp}.V_0^{-ap}.A.V_0^{ap}.U_0^{-bp}$. The outer horizontal mappings are equivariant with respect to the homomorphism $\mathbb{Z}_N \to \mathbb{Z}_N^2$ which is given by $p \mapsto (ap, bp)$. So the smooth sections of the algebra bundle $i^* A_q \to S^1$ correspond to the $\mathbb{Z}_N$-equivariant smooth functions $S^1 \to \mathrm{Mat}_N$. A smooth function

$$f = \sum_{\substack{k \in \mathbb{Z} \\ 0 \le s,t \le N-1}} c_{k,s,t}\, z^k U_0^s V_0^t$$

is $\mathbb{Z}_N$-equivariant if and only if the following condition is satisfied:

$$c_{k,s,t} \neq 0 \text{ only if } k \equiv as + bt \mod N$$

But then the function f can be written as

$$\begin{aligned} f &= \sum_{\substack{j \in \mathbb{Z} \\ 0 \le s,t \le N-1}} c_{as+bt+jN,s,t} z^{jN} (z^a U_0)^s (z^b V_0)^t \\ &= \sum_{\substack{j \in \mathbb{Z} \\ 0 \le s,t \le N-1}} c_{as+bt+jN,s,t} Z^j U^s V^t, \end{aligned}$$

where $Z := z^N$, $U := z^a U_0$ and $V := z^b V_0$ satisfy the relations

$$UV = qVU, \quad Z \text{ is central}, \quad U^N = Z^a, \quad V^N = Z^b. \tag{1}$$

We also have $Z = U^{Na'} V^{Nb'}$ where $a', b' \in \mathbb{Z}$ satisfy $aa' + bb' = 1$. So the *non-commutative q-circle of slope b/a* in the non-commutative q-torus is the associative algebra generated by two elemets U, V with the relations (1), and with rapidly decreasing coefficients.

If $q = 1$ we have $N = 1$, thus $U = Z^a$, $V = Z^b$, and clearly we just have the algebra of smooth functions on S^1.

4. The smooth Heisenberg algebra

4.1. We recall here (see [22], [29], or [30]) some wellknown results from the theory of distributions which we shall need in the following. We consider the following spaces of smooth functions on $\mathbb{R}^n$:

The space $\mathcal{S}(\mathbb{R}^n)$ of all rapidly decreasing smooth functions f for which $x \mapsto (1 + |x|^2)^k \partial^\alpha f(x)$ is bounded for all $k \in \mathbb{N}$ and all multiindices $\alpha \in \mathbb{N}_0^n$, with the locally convex topology described by these conditions, a nuclear Fréchet space. Its dual space $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

The space $\mathcal{O}_C(\mathbb{R}^n)$ of all smooth functions f on $\mathbb{R}^n$ for which there exists $k \in \mathbb{Z}$ such that $x \mapsto (1 + |x|^2)^k \partial^\alpha f(x)$ is bounded for each multiindex $\alpha \in \mathbb{N}_0^n$, with the locally convex topology described by this condition (a nuclear LF space). Its dual space $\mathcal{O}_C'(\mathbb{R}^n)$ is usually called the space of rapidly decreasing distributions (see [29]).

The space $\mathcal{O}_M(\mathbb{R}^n)$ of all smooth functions f on $\mathbb{R}^n$ such that for each multiindex $\alpha \in \mathbb{N}_0^n$ there exists $k \in \mathbb{Z}$ such that $x \mapsto (1 + |x|^2)^k \partial^\alpha f(x)$ is bounded, with the locally convex topology described by this condition (a nuclear space). This is the space of tempered smooth functions. Its dual space $\mathcal{O}_M'(\mathbb{R}^n)$ will be called the space of *speedily decreasing distributions.*

There are the following inclusions between these spaces:

$$\mathcal{S} \subset \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{S}', \qquad \mathcal{S} \subset \mathcal{O}_M' \subset \mathcal{O}_C' \subset \mathcal{S}'.$$

The Fourier transform of functions $f \in \mathcal{S}$ and its inverse,

$$\mathcal{F}f(y) := \int_{\mathbb{R}^n} e^{-i\langle x,y\rangle} f(x)\,dx, \quad \mathcal{F}^{-1}f(x) := \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} f(y)\,dy$$

extend to isomorphisms of $\mathcal{S}'$, which induce isomorphisms $\mathcal{O}_M \to \mathcal{O}_C'$ and $\mathcal{O}_C \to \mathcal{O}_M'$. Under the convolution product $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$ the space $\mathcal{S}$ is a commutative algebra and the Fourier transform is an isomorphism between this and the pointwise multiplication. The convolution carries over to distributions as follows: It induces an associative commutative product on $\mathcal{O}_C'$ and makes $\mathcal{S}'$ into an $\mathcal{O}_C'$-module. The Fourier transform is an algebra isomorphism $\mathcal{F} : (\mathcal{O}_C', *) \to (\mathcal{O}_M, \cdot)$. The convolution $*$ is jointly continuous on $\mathcal{O}_C'$. Moreover $\mathcal{S} * \mathcal{S}' \subset \mathcal{O}_M$ and $\mathcal{S} * \mathcal{O}_C' \subset \mathcal{S}$. See [29], pp. 246ff and 268. The space $\mathcal{O}_M \cong \mathcal{O}_C'$ is a complete bornological nuclear locally convex vector space, and the dual $\mathcal{O}_M' \cong \mathcal{O}_C$ is a complete nuclear (LF)-space, thus also bornological, see [15], II, §4,4, théorème 16 (page 131). Let us summarize the embeddings and isomorphisms in the following diagram:

$$\begin{array}{ccccc} & \mathcal{O}'_M & \longrightarrow & \mathcal{O}'_C & \\ \nearrow & \mathcal{F}\Big\uparrow\cong & & \mathcal{F}\Big\uparrow\cong & \searrow \\ \mathcal{S} & & & & \mathcal{S}' \\ \searrow & & & & \nearrow \\ & \mathcal{O}_C & \longrightarrow & \mathcal{O}_M & \end{array}$$

Moreover we have $\mathcal{O}_C(\mathbb{R}^n)\hat{\otimes}\mathcal{O}_C(\mathbb{R}^m) \cong \mathcal{O}_C(\mathbb{R}^{n+m})$ *and* $\mathcal{O}_M(\mathbb{R}^n)\hat{\otimes}\mathcal{O}_M(\mathbb{R}^m) \cong \mathcal{O}_M(\mathbb{R}^{n+m})$, *for the completed projective tensor product which agrees with the injective one.*

Since we have been unable to locate this result in the literature we sketch a proof: We start with $\mathcal{O}_M$. By ([29], p. 246) the space $\mathcal{O}_M(\mathbb{R}^n)$ is the space of the multipliers in $L_b(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, with the induced topology, where L_b denotes the space of continuous linear mappings with the topology of uniform convergence on bounded sets (i.e. on compact sets, since $\mathcal{S}$ is Montel), whose bornology is the same as that from 5.5.1. It is well known that $L_b(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)) \cong \mathcal{S}(\mathbb{R}^n)'\hat{\otimes}\mathcal{S}(\mathbb{R}^n)$. Thus we have the following diagrams of embeddings:

$$\begin{array}{ccc} \mathcal{O}_M(\mathbb{R}^n)\hat{\otimes}\mathcal{O}_M(\mathbb{R}^m) & \dashrightarrow & \mathcal{O}_M(\mathbb{R}^{n+m}) \\ \Big\downarrow & & \Big\downarrow \\ L_b(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))\hat{\otimes}L_b(\mathcal{S}(\mathbb{R}^m), \mathcal{S}(\mathbb{R}^m)) & & L_b(\mathcal{S}(\mathbb{R}^{n+m}), \mathcal{S}(\mathbb{R}^{n+m})) \\ \Big\| & & \Big\| \\ \mathcal{S}(\mathbb{R}^n)'\hat{\otimes}\mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^m)'\hat{\otimes}\mathcal{S}(\mathbb{R}^m) & = & \mathcal{S}(\mathbb{R}^{n+m})'\hat{\otimes}\mathcal{S}(\mathbb{R}^{n+m}) \end{array}$$

It remains to check that the spaces of smooth functions with compact support are dense in $\mathcal{O}_M$, which is easy, and that the trace topology on subspaces of functions with fixed compact support is the usual Fréchet topology, so that $C_c^\infty(\mathbb{R}^n) \otimes C_c^\infty(\mathbb{R}^m)$ is dense in $\mathcal{O}_M(\mathbb{R}^n)\hat{\otimes}\mathcal{O}_M(\mathbb{R}^m)$. Thus the result for $\mathcal{O}_M$ follows. For $\mathcal{O}_C$ we get then the result by $\mathcal{O}_C(\mathbb{R}^{n+m}) \cong \mathcal{O}_M(\mathbb{R}^{n+m})' \cong (\mathcal{O}_M(\mathbb{R}^n)\hat{\otimes}\mathcal{O}_M(\mathbb{R}^m))' \cong \mathrm{Bilin}_{\mathrm{cont}}(\mathcal{O}_M(\mathbb{R}^n), \mathcal{O}_M(\mathbb{R}^m); \mathbb{R}) \cong \mathcal{O}_M(\mathbb{R}^n)'\hat{\otimes}\mathcal{O}_M(\mathbb{R}^m)' \cong \mathcal{O}_C(\mathbb{R}^n)\hat{\otimes}\mathcal{O}_C(\mathbb{R}^m)$.

4.2. The Heisenberg relation. Let Q, P be two generators which satisfy the Heisenberg relation

$$[Q, P] = QP - PQ = i\hbar. \tag{1}$$

We suppose that they are hermitian: $Q^* = Q$ and $P^* = P$, which implies that $\hbar$ should be real.

Lemma. *Then the unitary generators* e^{iQ} *and* e^{iP} *satisfy the Weyl relation*

$$e^{itQ}.e^{isP} = e^{-its\hbar}.e^{isP}.e^{itQ} \text{ for } (t,s) \in \mathbb{R}^2 \tag{2}$$

Algebraic proof. We claim that the Heisenberg relations imply that for all $m, n \in \mathbb{N}_0$ we have

$$Q^n P^m = \sum_{k=0}^{\infty} \binom{n}{k}\binom{m}{k} k!(i\hbar)^k P^{m-k} Q^{n-k}, \tag{3}$$

which is in fact a finite sum. In the simplest cases (3) boils down to $QP^m = P^m Q + mi\hbar P^{m-1}$ and $Q^n P = PQ^n + ni\hbar Q^{n-1}$ which follow easily from (1). From these simple cases one may then prove (3) by induction. Finally (2) follows from (3) by a simple power series calculation. $\square$

Analytic proof. Another proof of (2) goes as follows. Let Q and P act on the space $\mathcal{S}(\mathbb{R})$ of all rapidly decreasing functions, by $(Qf)(u) = uf(u)$ and $(Pf)(u) = \frac{\hbar}{i}\partial_u f(u)$. Then the operators Q and P satisfy the Heisenberg relation (1), and they are selfadjoint with respect to the inner product $\int_{\mathbb{R}} \overline{f(u)} g(u) du$. It is more difficult to see that there are no other relations between these operators. Let us consider the smooth 1-parameter subgroups of isomorphisms e^{isP} and e^{itQ} with infinitesimal generators iP and iQ:

$$\begin{aligned} \left(e^{isP} f\right)(u) &:= f(u + s\hbar), \\ \left(e^{itQ} f\right)(u) &:= e^{itu} f(u), \\ \left(e^{isP} e^{itQ} f\right)(u) &= \left(e^{itz} f(z)\right)_{z=u+s\hbar} = e^{ist\hbar} e^{itu} f(u + s\hbar) \\ &= (e^{ist\hbar}.e^{itQ}.e^{isP} f)(u). \quad \square \end{aligned} \tag{4}$$

Using the Baker-Campbell-Hausdorff formula. Recall that (for finite dimensional matrices) we have $e^Q e^P = e^{C(Q,P)}$ where

$$\begin{aligned} C(Q, P) &= P + \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (e^{t.\operatorname{ad} Q}.e^{\operatorname{ad} P})^n.Q\, dt \\ &= Q + P + \tfrac{1}{2}[Q, P] + \tfrac{1}{12}\Big([Q, [Q, P]] - [P, [P, Q]]\Big) + \cdots \end{aligned}$$

Since we have $[Q, P] = i\hbar$, we see that $C(Q, P) = Q + P + \frac{i}{2}\hbar$. Thus we may use formally new generating elements

$$e^{itQ} e^{isP} = e^{itQ + isP - \frac{i}{2}\hbar ts} = e^{-\frac{i}{2}\hbar ts} e^{i(tQ+sP)} \tag{5}$$

and we see that the multiplication then will be

$$\begin{aligned} e^{i(x_1 Q + y_1 P)} e^{i(x_2 Q + y_2 P)} &= e^{-\frac{i}{2}\hbar(x_1 y_2 - x_2 y_1)} e^{i((x_1+x_2)Q + (y_1+y_2)P)} \\ &= e^{-\frac{i}{2}\hbar\omega(x,y)} e^{i((x_1+x_2)Q + (y_1+y_2)P)}, \end{aligned} \tag{6}$$

where $\omega(x, y) = x_1 y_2 - x_2 y_1$ is the symplectic form on $\mathbb{R}^2$.

4.3. The twisted convolution in two versions. Let Q, P be hermitian generators with $[Q, P] = i\hbar$ as in 4.2. For a rapidly decreasing distribution $a(t,s) \in \mathcal{O}'_C(\mathbb{R}^2)$ we consider the formal expression

$$\int_{\mathbb{R}^2} a(t,s) e^{itQ} e^{isP}\, dt\, ds. \tag{1}$$

If we multiply two such expressions and compute (formally, but see below) in the space of endomorphisms of $\mathcal{S}(\mathbb{R})$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} a(t,s) e^{itQ} e^{isP}\, dt\, ds. \int_{\mathbb{R}^2} b(u,v) e^{iuQ} e^{ivP}\, du\, dv = \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} a(t,s) b(u,v) e^{itQ} e^{isP} e^{iuQ} e^{ivP}\, dt\, ds\, du\, dv = \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} a(t,s) b(u,v) e^{isu\hbar} e^{i(t+u)Q} e^{i(s+v)P}\, dt\, ds\, du\, dv = \\
&\quad = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} a(t'-u, s'-v) b(u,v) e^{i(s'u - vu)\hbar}\, du\, dv \right) e^{it'Q} e^{is'P}\, dt'\, ds'
\end{aligned}$$

so that we may consider the 'twisted convolution' (formally, but see below)

$$(a *_\hbar b)(t,s) = \int_{\mathbb{R}^2} a(t-u, s-v) b(u,v) e^{i(su - vu)\hbar}\, du\, dv. \tag{1}$$

For a speedily decreasing distribution $a(t,s) \in \mathcal{O}'_M(\mathbb{R}^2)$ we consider the formal expression

$$\int_{\mathbb{R}^2} a(t,s) e^{i(tQ+sP)}\, dt\, ds = \int_{\mathbb{R}^2} a(t,s) e^{\frac{i\hbar}{2} ts} e^{itQ} e^{isP}\, dt\, ds. \tag{2}$$

If we multiply two such expressions and compute as above we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} a(t,s) e^{i(tQ+sP)}\, dt\, ds. \int_{\mathbb{R}^2} b(u,v) e^{i(uQ+vP)}\, du\, dv = \\
&\quad = \int_{\mathbb{R}^4} a(t,s) b(u,v) e^{i(tQ+sP)} e^{i(uQ+vP)}\, dt\, ds\, du\, dv = \\
&\quad = \int_{\mathbb{R}^4} a(t,s) b(u,v) e^{-\frac{i\hbar}{2}(tv - su)} e^{i((t+u)Q + (s+v)P)}\, dt\, ds\, du\, dv = \\
&\quad = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} a(t'-u, s'-v) b(u,v) e^{-\frac{i\hbar}{2}(t'v - s'u)}\, du\, dv \right) e^{i(t'Q + s'P)}\, dt'\, ds',
\end{aligned}$$

which motivates the 'other twisted convolution' for speedily decreasing distributions $a, b \in \mathcal{O}'_M(\mathbb{R}^{2n})$

$$(a \hat{*}_\hbar b)(x) = \int_{\mathbb{R}^{2n}} a(x-y) b(y) e^{-\frac{i\hbar}{2}\omega(x,y)}\, dy \tag{3}$$

where $\omega(x,y) = \sum_{i=1}^n (x_{2i-1} y_{2i} - y_{2i-1} x_{2i})$ is the symplectic form on $\mathbb{R}^{2n}$.

Theorem. *The 'twisted convolution'*

$$(a *_\hbar b)(t,s) = \int_{\mathbb{R}^2} a(t-u, s-v)b(u,v)e^{i(su-vu)\hbar}\, du\, dv \tag{4}$$

is a well defined, jointly continuous, and associative product on the space $\mathcal{O}'_M(\mathbb{R}^2)$ of speedily decreasing distributions. It is smooth in the variable $\hbar \in \mathbb{R}$. The convenient algebra $(\mathcal{O}'_M)_\hbar := (\mathcal{O}'_M, *_\hbar)$ is called the smooth Heisenberg plane with parameter $\hbar \in \mathbb{R}$. The noncommutative torus $T^2_{e^{i\hbar}}$ with rotation parameter $q = e^{i\hbar}$ is a closed subalgebra with unit of $(\mathcal{O}'_M)_\hbar$, it corresponds to the subspace of all rapidly decreasing measures on $\mathbb{R}^2$ with support in the lattice $(2\pi\mathbb{Z})^2$. The generalization of this to $\mathbb{R}^{2n}$ also holds.

The 'other twisted convolution'

$$(a \hat{*}_\hbar b)(x) = \int_{\mathbb{R}^{2n}} a(x-y)b(y)e^{-\frac{i\hbar}{2}\omega(x,y)}\, dy \tag{5}$$

is an associative bounded multiplication on the space $\mathcal{O}'_M(\mathbb{R}^{2n})$ of speedily decreasing distributions, and the algebras $(\mathcal{O}'_M(\mathbb{R}^{2n}), *_\hbar)$ and $(\mathcal{O}'_M(\mathbb{R}^{2n}), \hat{*}_\hbar)$ are isomorphic under the mapping

$$a(x) \mapsto e^{-\frac{i\hbar}{2}\sum_{i=1}^n x_{2i-1}x_{2i}}a(x).$$

Moreover, for both multiplications the algebras $\mathcal{O}'_M(\mathbb{R}^{2n})$ decompose as (bornological or projective or injective) tensorproduct of n commuting factors

$$\mathcal{O}'_M(\mathbb{R}^{2n}) = \mathcal{O}'_M(\mathbb{R}^2)\tilde{\otimes}\dots\tilde{\otimes}\mathcal{O}'_M(\mathbb{R}^2).$$

Formula (1) *defines a bounded linear mapping $\mathcal{O}'_M(\mathbb{R}^2) \to L(\mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$ which is injective if $\hbar \neq 0$, and is an algebra homomorphism from the twisted convolution* (4) *to the composition. Likewise formula* (2) *defines a bounded linear mapping $\mathcal{O}'_M(\mathbb{R}^2) \to L(\mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$ which is injective if $\hbar \neq 0$, and is an algebra homomorphism from the other twisted convolution* (5) *to the composition. The analoga on $\mathbb{R}^{2n}$ also hold.*

Proof. We have to check that $a *_\hbar b$, given by (4), defines a distribution in $\mathcal{O}'_M(\mathbb{R}^2)$. So let $g \in \mathcal{O}_M(\mathbb{R}^2)$, then

$$\begin{aligned}\langle a *_\hbar b, g\rangle &= \int\int a(t-u, s-v)b(u,v)e^{i(su-vu)\hbar}g(t,s)\; du\, dv\, dt\, ds\\ :&= \int\int a(t,s)b(u,v)e^{isu\hbar}g(t+u, s+v)\; du\, dv\, dt\, ds,\end{aligned}$$

which makes sense since we shall see that $(t,s,u,v) \mapsto e^{isu\hbar}g(t+u,s+v)$ is an element in $\mathcal{O}_M(\mathbb{R}^4) = \mathcal{O}_M(\mathbb{R}^2)\hat{\otimes}\mathcal{O}_M(\mathbb{R}^2)$, and moreover that $\hbar \mapsto ((t,s,u,v) \mapsto e^{isu\hbar}g(t+u,s+v))$ is a smooth curve $\mathbb{R} \to \mathcal{O}_M(\mathbb{R}^4)$. All this is a consequence of the following facts:

(6) $\mathcal{O}_M(\mathbb{R}^4)$ is a bounded algebra for the pointwise multiplication.
(7) For a polynomial $p : \mathbb{R}^n \to \mathbb{R}^m$ the mapping $p^* : \mathcal{O}_M(\mathbb{R}^m) \to \mathcal{O}_M(\mathbb{R}^n)$ is bounded linear.
(8) $x \mapsto e^{ix}$ belongs to $\mathcal{O}_M(\mathbb{R})$.
(9) $\hbar \mapsto ((s,u) \mapsto e^{isu\hbar})$ is a smooth curve in $\mathcal{O}_M(\mathbb{R}^2)$ since obviously the mapping $\mathcal{O}_M(\mathbb{R}^3) \to C^\infty(\mathbb{R}, \mathcal{O}_M(\mathbb{R}^2))$ is bounded linear.

This shows that $a *_\hbar b$ is a bounded (thus continuous, since $\mathcal{O}_M$ is bornological by [15], II, §4,4, théorème 16 (page 131)) linear functional on $\mathcal{O}_M(\mathbb{R}^4)$, and that $(a,b) \mapsto a *_\hbar b$ is bounded.

It is easy to see that $*_\hbar$ is an associative product, since this is clear for $\hbar = 0$ and for $\hbar \neq 0$ we have an injective algebra homomorphism $(\mathcal{O}'_M(\mathbb{R}^2), *_\hbar) \to L(\mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$, see below.

The statement about the noncommutative torus is clear.

The statement about the other twisted convolution follows via the isomorphism. The extension to $\mathbb{R}^{2n}$ is obvious and the decomposition into the tensorproduct follows from the considerations in 4.1.

Finally, on $\mathbb{R}^2$, the statement about the representation on $\mathcal{S}(\mathbb{R})$ can be proved as follows. Using 4.2.4 for $f \in \mathcal{S}(\mathbb{R})$ we have

$$\left(\left(\int_{\mathbb{R}^2} a(t,s)e^{itQ}e^{isP}\,dt\,ds\right) f\right)(u) := \int_{\mathbb{R}^2} a(t,s)(e^{itQ}e^{isP}f)(u)\,dt\,ds = $$
$$= \int_{\mathbb{R}^2} a(t,s)e^{itu}f(u+s\hbar)\,dt\,ds.$$

We observe that for $u \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$ the mapping $(t,s,u) \mapsto e^{itu}f(u+s\hbar)$ belongs to $\mathcal{O}_M(\mathbb{R})\hat{\otimes}\mathcal{S}(\mathbb{R}^2)$, but not to $\mathcal{O}_C(\mathbb{R}^2)\hat{\otimes}\mathcal{S}(\mathbb{R})$. This follows from (6)-(9) and from the fact that for a polynomial $p : \mathbb{R}^n \to \mathbb{R}^m$ the mapping $p^* : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^n)$ is bounded linear. This implies the result, since the extension to $\mathbb{R}^{2n}$ is again obvious. □

4.4. Remarks. The twisted convolution $*_\hbar$ is not well defined on the classical space $\mathcal{O}'_C \supset \mathcal{O}'_M$ of rapidly decreasing distributions, since $e^{isu\hbar}g(t+u, s+v)$ is not in $\mathcal{O}_C$, even if g is in $\mathcal{O}_C$, because $(s,u) \mapsto e^{isu\hbar}$ is not in $\mathcal{O}_C$, see [29], p. 245. Property 4.3.7 is wrong for $\mathcal{O}_C$, but it holds for linear mappings. Is it true that $\mathcal{O}'_M$ is the optimal space of distributions on which the twisted convolution defines an algebra structure?

The statement that $a *_\hbar b$ is smooth in $\hbar$ cannot be improved to real analytic $\mathbb{R} \to \mathcal{O}'_M(\mathbb{R}^2)$ in the weak sense of [20]. The source of this is the fact that $\hbar \mapsto (x \mapsto e^{ix\hbar})$ is not real analytic $\mathbb{R} \to \mathcal{O}_M(\mathbb{R})$, even after composing with a linear functional: Let $f \in \mathcal{S}(\mathbb{R}) \subset \mathcal{O}'_M(\mathbb{R})$ be such that the Fourier transform $\mathcal{F}f \in \mathcal{S}(\mathbb{R})$ is not real analytic. Then

$$\hbar \mapsto \langle f, e^{i(\quad)\hbar}\rangle = \int f(x)e^{ix\hbar}dx = (\mathcal{F}f)(-\hbar)$$

is not real analytic. This is related to the fact that the Moyal $*$-product is only formal in $\hbar$, although there exist integral expressions in the sense of distributions which are smooth in $\hbar$, see 4.5 and 4.6 below.

In [23] J. Maillard defined spaces of distributions $\mathcal{O}'_\hbar(\mathbb{R}^2)$ as follows, depending on $\hbar$: $\mathcal{O}'_\hbar(\mathbb{R}^2)$ consists of all distributions $a \in \mathcal{S}'(\mathbb{R}^2)$ such that the formal expression from above

$$\int_{\mathbb{R}^2} a(t,s)e^{i(tQ+sP)}\,dt\,ds$$

defines a linear mapping $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, which then turns out to be bounded. From 4.3 it follows that $\mathcal{O}'_M(\mathbb{R}^2) \subseteq \mathcal{O}'_\hbar(\mathbb{R}^2)$.

So for the twisted convolution as in 4.3 the (possibly) different spaces $\mathcal{O}'_\hbar(\mathbb{R}^2)$ stabilize to (or at least contain) a fixed space $\mathcal{O}'_M(\mathbb{R}^2)$. Also Kammerer in [18] gives many results on the space $\mathcal{O}'_\hbar(\mathbb{R}^2)$.

4.5. The Fourier transform of the twisted convolution. Suppose that $a = \mathcal{F}f$ and $b = \mathcal{F}g$ for $f, g \in \mathcal{O}_C(\mathbb{R}^2)$. Then we have in the weak sense (as distributions)

$$
\begin{aligned}
&\mathcal{F}^{-1}((\mathcal{F}f) *_\hbar (\mathcal{F}g))(x) = \\
&= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (\mathcal{F}f)(y-z)(\mathcal{F}g)(z) e^{i(\langle x,y\rangle + (z_1 y_2 - z_1 z_2)\hbar)} dy\, dz \\
&= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (\mathcal{F}f)(y)(\mathcal{F}g)(z) e^{i(\langle x,y+z\rangle + z_1 y_2 \hbar)} dy\, dz \\
&= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^4} f(u)g(v) \left(\int_{\mathbb{R}^4} e^{i(\langle y,x-u\rangle + \langle z,u-v\rangle + (z_1 y_2 - z_1 z_2)\hbar)} dy\, dz \right) du\, dv
\end{aligned}
$$

Let us now use $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$, the composition of the two one dimensional Fourier transforms in both variables separately, and recall that the integals above are weak, are in $\mathcal{O}_M(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$, so they make sense only when applied to test functions in $\mathcal{S}$. Then the last but one expression becomes

$$
\begin{aligned}
&= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (\mathcal{F}_1\mathcal{F}_2 f)(y_1,y_2)(\mathcal{F}_1\mathcal{F}_2 g)(z_1,z_2) \\
&\qquad\qquad e^{i(x_1y_1 + x_2y_2 + x_1z_1 + x_2z_2 + z_1y_2\hbar)} dy_1\, dy_2\, dz_1\, dz_2 \\
&= \int_{\mathbb{R}^2} \tfrac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{F}_1\mathcal{F}_2 f)(y_1,y_2) e^{ix_1y_1} dy_1 \\
&\qquad\qquad \tfrac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{F}_2\mathcal{F}_1 g)(z_1,z_2) e^{ix_2z_2} dz_2 e^{i(x_2y_2 + x_1z_1 + z_1y_2\hbar)} dy_2\, dz_1 \\
&= \int_{\mathbb{R}^2} (\mathcal{F}_2 f)(x_1,y_2)(\mathcal{F}_1 g)(z_1,x_2) e^{i(x_2y_2 + x_1z_1 + z_1y_2\hbar)} dy_2\, dz_1 \\
&= \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \int_{\mathbb{R}} y_2^k (\mathcal{F}_2 f)(x_1,y_2) e^{ix_2y_2} dy_2 \int_{\mathbb{R}} z_1^k (\mathcal{F}_1 g)(z_1,x_2) e^{ix_1z_2} dz_2 \\
&= 4\pi^2 \sum_{k=0}^{\infty} \frac{(-i\hbar)^k}{k!} \partial_2^k f(x_1,x_2) \partial_1^k g(x_1,x_2),
\end{aligned}
$$

where we used $i\partial_x f(x) = \mathcal{F}_y^{-1}(y(\mathcal{F}f)(y))(x)$. The last expression is half of the Moyal star product, represented by a convergent integral. Obviously the series can only be interpreted as a formal power series in $\hbar$. But note that the divergence appears only after the interchange of the sum with the integral; before the expressions are bounded bilinear in f and g, and even smooth in $\hbar$.

Also one should compare this result with the treatment of the Weyl calculus in [21], III, 18.5.

4.6. The Fourier transform of the other twisted convolution. Let us apply the other twisted convolution to $a = \mathcal{F}f, b = \mathcal{F}g \in \mathcal{O}'_M(\mathbb{R}^2)$ for $f, g \in \mathcal{O}_C(\mathbb{R}^2)$:

$$\begin{aligned}
&\mathcal{F}^{-1}((\mathcal{F}f)\hat{*}_\hbar(\mathcal{F}g))(x) = \\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^2}((\mathcal{F}f)\hat{*}_\hbar(\mathcal{F}g))(y)e^{i\langle x,y\rangle}dy \\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^4}(\mathcal{F}f)(y-z)(\mathcal{F}g)(z)e^{i(\langle x,y\rangle - \frac{\hbar}{2}\omega(y,z))}dy\,dz \\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^4}(\mathcal{F}f)(y)(\mathcal{F}g)(z)e^{i(\langle x,y+z\rangle - \frac{\hbar}{2}\omega(y,z))}dy\,dz \\
&= \tfrac{1}{(2\pi)^2}\int_{\mathbb{R}^4}(\mathcal{F}f)(y)e^{i\langle x,y\rangle}(\mathcal{F}g)(z)e^{i\langle x,z\rangle}\left(\sum_{k=0}^{\infty}\frac{(-i\hbar)^k}{2^k\,k!}(y_1x_2 - y_2z_1)^k\right)dy\,dz
\end{aligned}$$

Let us now use $(i\partial_1)^m(i\partial_2)^n f(x) = \mathcal{F}_y^{-1}(y_1^m y_2^n(\mathcal{F}f)(y))(x)$, which also holds in the weak sense for tempered distributions. Then we may continue to compute in the weak sense of distributions:

$$\begin{aligned}
&= \tfrac{1}{(2\pi)^2}\sum_{k=0}^{\infty}\frac{(-i\hbar)^k}{2^k\,k!}\int_{\mathbb{R}^4}(\mathcal{F}f)(y)e^{i\langle x,y\rangle}(\mathcal{F}g)(z)e^{i\langle x,z\rangle}(y_1x_2 - y_2z_1)^k dy_1\,dy_2\,dz_1\,dz_2 \\
&= (2\pi)^2\sum_{k=0}^{\infty}\frac{(-i\hbar)^k}{2^k\,k!}(\partial_{y_2}\partial_{z_1} - \partial_{y_1}\partial_{z_2})^k(f(y_1,y_2)g(z_2,z_2))\Big|_{\substack{y_1=z_1=x_1\\ y_2=z_2=x_2}}.
\end{aligned}$$

This is now really the Moyal star product, expressed as a sum of bidifferential operators.

4.7. Convolution algebras on the Heisenberg group. Let us consider the Heisenberg group in the following form: $\mathrm{He}_2^\hbar = \mathbb{R}^2 \times S^1$ with multiplication

$$(x,\alpha).(y,\beta) = (x+y, \alpha\beta e^{\frac{i\hbar}{2}\omega(x,y)}) = (x+y, \alpha\beta e^{\frac{i\hbar}{2}(x_1y_2 - x_2y_1)}).$$

Let us consider the bounded linear mapping between the spaces of speedily decreasing distributions

$$\tilde{}\ : \mathcal{O}'_M(\mathbb{R}^2) \to \mathcal{O}'_M(\mathrm{He}_2^\hbar), \quad \tilde{a}(x_1,x_2,\alpha) = a(x_1,x_2)\alpha$$

Since the Haar measure on $\mathrm{He}_2^\hbar$ is just the usual measure $dx_1 \wedge dx_2 \wedge d\alpha$, where we choose $\int_{S_1} d\alpha = 1$, we can then compute the convolution as a weak integral (in the sense of tempered distributions):

$$\begin{aligned}
(\tilde{a} * \tilde{b})(x,\alpha) &= \int_{\mathrm{He}_2^\hbar}\tilde{a}(u,\beta)\tilde{b}((x,\alpha)(u,\beta)^{-1})du d\beta \\
&= \int_{\mathrm{He}_2^\hbar}\tilde{a}(u,\beta)\tilde{b}(x-u, \alpha\beta^{-1}e^{\frac{i\hbar}{2}(u_1x_2 - u_2x_1)})du_1\,du_2\,d\beta \\
&= \int_{\mathrm{He}_2^\hbar}a(u_1,u_2)\beta b(x_1-u_1, x_2-u_2)\alpha\beta^{-1}e^{\frac{i\hbar}{2}(u_1x_2 - u_2x_1)}du_1\,du_2\,d\beta \\
&= \int_{\mathbb{R}^2}a(u_1,u_2)b(x_1-u_1,x_2-u_2)e^{\frac{i\hbar}{2}(u_1x_2-u_2x_1)}du_1\,du_2\,\alpha \\
&= \widetilde{(a\hat{*}_\hbar b)}(x,\alpha)
\end{aligned}$$

The groups $\mathrm{He}_2^\hbar$ are all isomorphic for $\hbar \neq 0$, an isomorphism $\mathrm{He}_2^\hbar \to \mathrm{He}_2^1$ is given by $(x_1, x_2, \alpha) \mapsto (\hbar x_1, \hbar x_2, \alpha)$. Thus all the algebras $(\mathcal{O}'_M, \hat{*}_\hbar)$ are isomorphic for $\hbar \neq 0$, in strong contrast to the behaviour of the subalgebras $T^2_{e^{i\hbar}}$, the noncommutative tori.

4.8. Derivations. Let us determine all derivations of the smooth Heisenberg plane. We use the form $(\mathcal{O}'_M, \hat{*}_\hbar)$ from 4.3.5, and we start with the inner derivations. We have for $a, b \in \mathcal{O}'_M(\mathbb{R}^2)$ in the weak sense

(1)
$$\begin{aligned}\mathrm{ad}(a)b &= \int_{\mathbb{R}^2} a(x-y)b(y)e^{-\frac{i\hbar}{2}\omega(x,y)}\,dy - \int_{\mathbb{R}^2} b(x-y)a(y)e^{-\frac{i\hbar}{2}\omega(x,y)}\,dy \\ &= \int_{\mathbb{R}^2} a(x-y)b(y)\left(e^{-\frac{i\hbar}{2}\omega(x,y)} + e^{\frac{i\hbar}{2}\omega(x,y)}\right)\,dy \\ &= \int_{\mathbb{R}^2} a(x-y)b(y)2\cos\left(\tfrac{\hbar}{2}\omega(x,y)\right)\,dy\end{aligned}$$

Proposition. *Every bounded derivation of $(\mathcal{O}'_M(\mathbb{R}^2), *_\hbar)$ is inner, if $\hbar \neq 0$.*

Proof. Let us note first that Q and P are elements of $\mathcal{O}'_M$, namely we have

$$Q = \int \delta'_t(0)\delta_s(0)e^{itQ}e^{isP}\,dt\,ds, \quad \text{etc.}$$

Let $D : (\mathcal{O}'_M(\mathbb{R}^2), *_\hbar) \to (\mathcal{O}'_M(\mathbb{R}^2), *_\hbar)$ be a bounded derivation. Then we let

$$D(Q) = \int a_Q(t,s)e^{itQ}e^{isP}dt\,ds, \quad D(P) = \int a_P(t,s)e^{itQ}e^{isP}dt\,ds$$

We want to find a distribution $b \in \mathcal{O}'_M$ such that $B = \int b(t,s)e^{itQ}e^{isP}dt\,ds$ satisfies $D(Q) = [B,Q]$ and $D(P) = [B,P]$. We have (using formulas from the analytic proof of lemma 4.2) for $f \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}([e^{isP},Q]f)(u) &= (zf(z))|_{z=u+s\hbar} - uf(u+s\hbar) = (s\hbar e^{isP}f)(u) \\ [B,Q] &= \int b(t,s)e^{itQ}[e^{isP},Q]\,dt\,ds = \int b(t,s)s\hbar e^{itQ}e^{isP}\,dt\,ds,\end{aligned}$$

and similarly

$$\begin{aligned}([e^{itQ},P]f)(u) &= e^{itu}\tfrac{\hbar}{i}\partial_u f(u) - \tfrac{\hbar}{i}\partial_u(e^{itu}f(u)) = -(t\hbar e^{itQ}f)(u) \\ [B,P] &= \int b(t,s)[e^{itQ},P]e^{isP}\,dt\,ds = -\int b(t,s)t\hbar e^{itQ}e^{isP}\,dt\,ds,\end{aligned}$$

so that we have to solve

$$b(t,s)s\hbar = a_Q(t,s), \quad -b(t,s)t\hbar = a_P(t,s).$$

Applying the Fourier transform we have to find $\hat{b} \in \mathcal{O}_C$ which satisfies

$$\begin{aligned}i\hbar\partial_s\hat{b}(t,s) &= \hat{a}_Q(s,t), \quad i\hbar\partial_t\hat{b}(t,s) = -\hat{a}_P(s,t), \\ d\hat{b} &= \tfrac{1}{i\hbar}(\hat{a}_Q ds - \hat{a}_P dt).\end{aligned}$$

This can be solved by the Lemma of Poincaré in $\mathcal{O}_C(\mathbb{R}^2)$ if and only if

$$d(\hat{a}_Q ds - \hat{a}_P dt) = (\partial_t \hat{a}_Q + \partial_s \hat{a}_P) dt \wedge ds = 0.$$

But this is the case since we have in turn, using the results from above,

$$\begin{aligned} D(Q)P + QD(P) - D(P)Q - PD(Q) = D([Q,P]) = D(i\hbar.1) = 0 \\ [D(Q), P] = [D(P), Q] \\ -a_Q(t,s)t\hbar = a_P(t,s)s\hbar \\ i\hbar\partial_t \hat{a}_Q + i\hbar \partial_s \hat{a}_P = 0, \end{aligned}$$

as required. The lemma of Poincaré has the form: $d\varphi = 0$ implies $\varphi = d\psi$ where $\psi(x) = \int_0^1 \sum_i \varphi_i(tx) x_i dt$. Thus $\varphi_i \in \mathcal{O}_C(\mathbb{R}^2)$ implies $\psi \in \mathcal{O}_C(\mathbb{R}^2)$ by a simple estimation. Thus on Q and P the bounded derivation D agrees with an inner derivation.

It remains to show that a bounded derivation D which vanishes on Q and on P must vanish on $\mathcal{O}'_M$. For that we note the following facts:

The curve $t \mapsto e^{itQ}$ is a smooth 1-parameter group of isomorphisms of $\mathcal{S}(\mathbb{R})$ with infinitesimal generator iQ, and it is the unique 1-parameter group with this generator, since for any other $C(t)$ we have $\partial_t(e^{itQ})C(-t) = e^{itQ} iQC(t) - e^{itQ} iQC(t) = 0$, so that $e^{itQ}C(-t)$ is the constant Id.

Consider the semidirect product $(\mathcal{O}'_M(\mathbb{R}^2), *_\hbar) \ltimes \mathcal{O}'_M(\mathbb{R}^2)\varepsilon$ where ε is in the center and $\varepsilon^2 = 0$, with the multiplication $(a + b\varepsilon).(a' + b'\varepsilon) = aa' + (ab' + ba')\varepsilon$. Obviously D is a derivation if and only if $a \mapsto a + D(a)\varepsilon$ is a homomorphism of algebras.

Thus $t \mapsto e^{itQ} + D(e^{itQ})\varepsilon$ is a smooth 1-parameter group in the semidirect product with infinitesimal generator $iQ + D(iQ)\varepsilon = iQ + 0$ and with second 1-parameter group $e^{itQ} + 0$, thus $D(e^{itQ}) = 0$ for all t.

Similarly $D(e^{isP}) = 0$ for all s. Thus D vanishes on $e^{itQ}e^{isP}$ for each t and s. And if $a \in C_c^\infty(\mathbb{R}^2) \subset \mathcal{O}'_M(\mathbb{R}^2)$, then

$$D\left(\int a(t,s) e^{itQ} e^{isP} dt\, ds\right) = \int a(t,s) D\left(e^{itQ}e^{isP}\right) dt\, ds = 0,$$

since Riemann sums converge Mackey to the integral. Finally one should note that C_c^∞ is dense in $\mathcal{O}'_M$, so the result follows. □

4.9. Conjecture. *The smooth Heisenberg plane $\mathcal{O}'_M(\mathbb{R}^{2n})$ is a smooth $*$-algebra with derivation space the space of all bounded derivations in the given topology, and a suitable state space.*

In fact we think that the topology described in 2.3.3 is the one of $\mathcal{O}'_M(\mathbb{R}^{2n}) \cong \mathcal{O}_C(\mathbb{R}^{2n})$. One has to show that each state is a bounded linear functional, and that we are able to find enough states and derivations in order to generate the topology described in 4.1.

5. APPENDIX: Calculus in infinite dimensions and convenient vector spaces

5.1. The notion of convenient vector spaces arose in the quest for the right setting for differential calculus in infinite dimensions: The traditional approach to differential calculus works well for Banach spaces, but for more general locally convex spaces there are difficulties. The main one is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology, so that even the chain rule is not valid without further assumptions. In addition to their importance for differential calculus convenient vector spaces together with bounded linear mappings and the appropriate tensor product form a monoidally closed category, the only useful one which functional analysis offers beyond Banach spaces.

In this section we sketch the basic definitions and the most important results concerning calculus for convenient vector spaces. All locally convex spaces will be assumed to be Hausdorff. Proofs for the results sketched here can be found in [12] (sauf for 5.8 which was proved in [5]). A complete coverage is in the book [20]; [5] contains an overview and a presentation of non-commutative geometry based on convenient vector spaces.

5.2. Smooth curves. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist (and are continuous) - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth curves. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does depend on the locally convex topology of E only through its underlying bornology (system of bounded sets).

5.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(1) Any Mackey-Cauchy-sequence (so that there are scalars $\lambda_{n,m} \to \infty$ such that $\{\lambda_{n,m}(x_n - x_m) : n, m \in \mathbb{N}\}$ is bounded) converges.
(2) If B is bounded closed and absolutely convex, then the linear span E_B of B is a Banach space with respect to the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(3) Any Lipschitz curve (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded) in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1 = c_2'$ (existence of antiderivative).
(5) If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
(6) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.
(7) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. To be scalarwise C^∞ means for a curve f that $\lambda \circ f$ is C^∞ for all continuous (equivalently: all bounded) linear functionals λ on E. Obviously c^∞-completeness is weaker than sequential completeness, so any sequentially complete locally convex vector space is convenient. From 5.2.4 one easily sees that (sequentially) closed linear subspaces of convenient vector spaces are again convenient. We always assume that a convenient vector space is equipped with its bornological topology. All spaces which a working mathematician needs in daily life are convenient. For any locally convex space E there is a convenient vector space $\tilde{E}$ called the completion of E, and a bornological embedding $i : E \to \tilde{E}$, which is characterized by the property that any bounded linear map from E into an arbitrary convenient vector space extends to $\tilde{E}$.

5.4. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mappings from E to F. For E and F finite dimensional (or even Fréchet spaces) this gives the usual notion of smooth mappings (Already for $E = \mathbb{R}^2$ this is a non-trivial statement). Multilinear mappings are smooth if and only if they are bounded. We denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

5.5. Differential calculus. We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$. We have the following results:

(1) *If F is convenient, then also $C^\infty(E, F)$ is convenient, for any E. The space $L(E, F)$ is a closed linear subspace of $C^\infty(E, F)$, so it is convenient also.*

(2) *The smooth uniform boundedness principle: If E is convenient, then a curve $c : \mathbb{R} \to L(E, F)$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in F for all $x \in E$.*

(3) *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G)),$$

which is even a homeomorphism. Note that this result, for $E = \mathbb{R}$, is the prime assumption of variational calculus. As a consequence evaluation mappings, insertion mappings, and composition are smooth.

(4) *The differential $d : C^\infty(E, F) \to C^\infty(E, L(E, F))$, given by $df(x)v := \lim_{t \to 0} \frac{1}{t}(f(x + tv) - f(x))$, exists and is linear and bounded (smooth). Also the chain rule holds: $d(f \circ g)(x)v = df(g(x))dg(x)v$.*

5.6. The category of convenient vector spaces and bounded linear maps is complete and cocomplete, so all categorical limits and colimits can be formed. In particular we can form products and direct sums of convenient vector spaces.

For convenient vector spaces E_1, ... ,E_n, and F we can now consider the space of all bounded n-linear maps, $L(E_1, \dots, E_n; F)$, which is a closed linear subspace of $C^\infty(\prod_{i=1}^n E_i, F)$ and thus again convenient. It can be shown that multilinear maps are bounded if and only if they are partially bounded, i.e. bounded in each coordinate and that there is a natural isomorphism (of convenient vector spaces) $L(E_1, \dots, E_n; F) \cong L(E_1, \dots, E_k; L(E_{k+1}, \dots, E_n; F))$

5.7. Result. *On the category of convenient vector spaces there is a unique tensor product* $\tilde{\otimes}$ *which makes the category symmetric monoidally closed, i.e. there are natural isomorphisms of convenient vector spaces* $L(E_1; L(E_2, E_3)) \cong L(E_1 \tilde{\otimes} E_2, E_3)$, $E_1 \tilde{\otimes} E_2 \cong E_2 \tilde{\otimes} E_1$, $E_1 \tilde{\otimes} (E_2 \tilde{\otimes} E_3) \cong (E_1 \tilde{\otimes} E_2) \tilde{\otimes} E_3$ *and* $E \tilde{\otimes} \mathbb{R} \cong E$.

5.8. Result. [5], 2.7. *Let* A *be a convenient algebra,* M *a convenient right* A*-module and* N *a convenient left* A*-module. This means that all structure mappings are bounded bilinear.*

(1) *There is a convenient vector space* $M \tilde{\otimes}_A N$ *and a bounded bilinear map* $b : M \times N \to M \tilde{\otimes}_A N$, $(m, n) \mapsto m \otimes_A n$ *such that* $b(ma, n) = b(m, an)$ *for all* $a \in A$, $m \in M$ *and* $n \in N$ *which has the following universal property: If* E *is a convenient vector space and* $f : M \times N \to E$ *is a bounded bilinear map such that* $f(ma, n) = f(m, an)$ *then there is a unique bounded linear map* $\tilde{f} : M \tilde{\otimes}_A N \to E$ *with* $\tilde{f} \circ b = f$.

(2) *Let* $L^A(M, N; E)$ *denote the space of all bilinear bounded maps* $f : M \times N \to E$ *having the above property, which is a closed linear subspace of* $L(M, N; E)$. *Then we have an isomorphism of convenient vector spaces* $L^A(M, N; E) \cong L(M \tilde{\otimes}_A N, E)$.

(3) *If* B *is another convenient algebra such that* N *is a convenient right* B*-module and such that the actions of* A *and* B *on* N *commute, then* $M \tilde{\otimes}_A N$ *is in a canonical way a convenient right* B*-module.*

(4) *If in addition* P *is a convenient left* B*-module then there is a natural isomorphism of convenient vector spaces*

$$M \tilde{\otimes}_A (N \tilde{\otimes}_B P) \cong (M \tilde{\otimes}_A N) \tilde{\otimes}_B P$$

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