Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 301 (2006) 406–424
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On polarizations in invariant theory ☆

Mark Losik [a], Peter W. Michor [b,c], Vladimir L. Popov [d,*]

[a] Saratov State University, Astrakhanskaya 83, Saratov 410026, Russia
[b] Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria
[c] Erwin Schrödinger Institut für Mathematische Physik, Boltzmanngasse 9, A-1090 Wien, Austria
[d] Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow 119991, Russia

Received 4 May 2005
Available online 12 September 2005
Communicated by Corrado de Concini

Abstract

Given a reductive algebraic group G and a finite dimensional algebraic G-module V, we study how close is the algebra of G-invariant polynomials on $V^{\oplus n}$ to the subalgebra generated by polarizations of G-invariant polynomials on V. We address this problem in a more general setting of G-actions on arbitrary affine varieties.

Keywords: Invariants; Reductive groups; Polarizations

1. Introduction

1.1. Let G be a reductive algebraic group over an algebraically closed field k of characteristic 0, and let V be a finite dimensional algebraic G-module. Given a positive integer n,

☆ Mark Losik and Peter W. Michor were supported by "Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 14195 MAT." Vladimir L. Popov was supported in part by ETH, Zürich, Switzerland, Russian grants НШ-123.2003.01 and РФФИ 05-01-00455, and Program of Mathematics Section of Russian Academy of Sciences.
* Corresponding author.
E-mail addresses: losikmv@info.sgu.ru (M. Losik), peter.michor@esi.ac.at (P.W. Michor), popovvl@orc.ru (V.L. Popov).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.07.008

consider the G-module $V^{\oplus n} := V \oplus \cdots \oplus V$ (n summands). Finding generators of the invariant algebra $k[V^{\oplus n}]^G$ of $V^{\oplus n}$ is the classical problem of invariant theory. The classical method of constructing elements of $k[V^{\oplus n}]^G$ is taking the polarizations of invariants $f \in k[V]^G$, i.e., the polynomial functions $f_{i_1,\dots,i_n}$ on $V^{\oplus n}$ given by the formal expansions

$$f(x_1 v_1 + \cdots + x_n v_n) = \sum_{i_1,\dots,i_n \in \mathbb{Z}_+} x_1^{i_1} \cdots x_n^{i_n} f_{i_1,\dots,i_n}(v_1, \dots, v_n), \tag{1}$$

where $(v_1, \dots, v_n)$ is generic element of $V^{\oplus n}$ and $x_1, \dots, x_n$ are variables. Let $\mathrm{pol}_n\, k[V]^G$ be the subalgebra of $k[V^{\oplus n}]^G$ generated by the polarizations of all the f's.

There are G-modules enjoying the property

$$\mathrm{pol}_n\, k[V]^G = k\big[V^{\oplus n}\big]^G. \tag{2}$$

For instance, (2) holds, by Study's theorem [15], for the standard action of $G = \mathbf{O}_m$ on $V = k^m$. By Weyl's theorem [17], (2) holds for $G = \mathrm{S}_m$ acting on $V = k^m$ by permuting the coordinates. In [8], (2) is established for the natural action of the Weyl group G of type B_m on $V = k^m$ and for the standard action of the dihedral group G on $V = k^2$.

However, in general, $\mathrm{pol}_n\, k[V]^G$ and $k[V^{\oplus n}]^G$ do not coincide. For instance, for the natural action of $G = \mathbf{SL}_n$ on $V = k^n$ clearly $\mathrm{pol}_n\, k[V]^G = k$ (since $k[V]^G = k$) but $k[V^{\oplus n}]^G \neq k$. It is less easy to find examples where (2) fails for finite G, but such examples exist as well: in [16] it was observed that (2) does not hold for the natural action of the Weyl group G of type D_m on $V = k^m$ ($m \geqslant 4$) for $n \geqslant 2$.

In this paper we analyze the relationship between $k[V^{\oplus n}]^G$ and $\mathrm{pol}_n\, k[V]^G$. We prove that if G is finite, then $k[V^{\oplus n}]^G$ is the integral closure of $\mathrm{pol}_n\, k[V]^G$ in its field of fractions, and the natural morphism of affine varieties determined by these algebras is bijective. Actually, instead of linear actions we consider the more general setting of actions on arbitrary affine varieties for which we define a generalization of polarizations. In this setting, we prove that if G is finite, then the invariant algebra is integral over the subalgebra generated by generalized polarizations, and the natural dominant morphism between affine varieties determined by these algebras is injective (in the graded case, bijective).

For connected G, one cannot expect such results, as the example of $\mathbf{SL}_n$ acting on k^n shows. This naturally leads to distinguishing the n's for which $k[V^{\oplus n}]^G$ is integral over $\mathrm{pol}_n\, k[V]^G$ and defining *the polarization index of* V,

$$\mathrm{pol\,ind}(V),$$

as the supremum taken over all such n's. We prove that $k[V^{\oplus m}]^G$ is integral over $\mathrm{pol}_m\, k[V]^G$ for every $m \leqslant \mathrm{pol\,ind}(V)$, and show that calculating $\mathrm{pol\,ind}(V)$ is closely related to the old problem of describing linear subspaces lying in the Hilbert nullcone of V (see [3,5,6,9,10,12], and the references therein), namely, to analyzing a certain geometric property of such subspaces.

Using this reduction, we calculate the polarization index of some G-modules V. Namely, we prove that if G is a finite group or a linear algebraic torus, then $\mathrm{pol\,ind}(V) = \infty$. For $G = \mathbf{SL}_2$, we describe all linear subspaces of V lying in the Hilbert nullcone of V and prove that $\operatorname{pol\,ind}(V) = \infty$ if V does not contain a simple 2-dimensional submodule, and $\operatorname{pol\,ind}(V) = 1$ otherwise. Finally, we calculate the polarization index of every semisimple Lie algebra $\mathfrak{g}$: we prove that $\operatorname{pol\,ind}(\mathfrak{g}) = 1$ if $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$, and $\operatorname{pol\,ind}(\mathfrak{g}) = \infty$ otherwise. As an application to the above mentioned old topic of linear subspaces lying in the Hilbert nullcone, we prove that a semisimple Lie algebra $\mathfrak{g}$ contains a 2-dimensional nilpotent nontriangularizable linear subspace if and only if $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$.

1.2. Notation

$k[X]$ is the algebra of regular functions of an algebraic variety X. If X is irreducible, $k(X)$ is the field of rational function of X.

If a group S acts on a set Z, we put $Z^S := \{z \in Z \mid s \cdot z = z \text{ for all } s \in S\}$.

Below every action of an algebraic group is algebraic (morphic).

G^0 is the identity component of an algebraic group G.

If X is an affine variety endowed with an action of a reductive algebraic group G, then $\pi_{X,G} : X \to X /\!\!/ G$ is the categorical quotient, i.e., $X /\!\!/ G$ is an affine algebraic variety and $\pi_{X,G}$ a dominant (actually, surjective) morphism such that $\pi^*_{X,G}(k[X /\!\!/ G]) = k[X]^G$.

Given a linear algebraic torus T, its character group $\operatorname{Hom}(T, \mathbf{G}_m)$ is written additively. The value of $\lambda \in \operatorname{Hom}(T, \mathbf{G}_m)$ at $t \in T$ is denoted by t^λ. For an algebraic T-module V (not necessarily finite dimensional), V_λ is the λ-isotypic component of V,

$$V_\lambda := \{ v \in V \mid t \cdot v = t^\lambda v \text{ for every } t \in T \}.$$

By $\langle v_1, \ldots, v_n \rangle$ we denote the linear span of vectors $v_1, \ldots, v_n$ of a vector space over k.

We set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$.

2. Generalized polarizations

2.1. Let a reductive algebraic group G act on the irreducible affine algebraic varieties X and Y. Let Z be an irreducible affine algebraic variety endowed with an action of a linear algebraic torus T. The set $\Lambda := \{\lambda \in \operatorname{Hom}(T, \mathbf{G}_m) \mid k[Z]_\lambda \neq 0\}$ is then a submonoid of $\operatorname{Hom}(T, \mathbf{G}_m)$ and the isotypic components yield a Λ-grading of $k[Z]$:

$$k[Z] = \bigoplus_{\lambda \in \Lambda} k[Z]_\lambda, \qquad k[Z]_\mu k[Z]_\nu \subseteq k[Z]_{\mu+\nu} \quad \text{for all } \mu, \nu \in \Lambda. \tag{3}$$

The groups G and T act on $Y \times Z$ through the first and second factors, respectively. From $k[Y \times Z] = k[Y] \otimes k[Z]$ and (3) we obtain

$$k[Y \times Z] = \bigoplus_{\lambda \in \Lambda} k[Y] \otimes k[Z]_\lambda \quad \text{and} \quad k[Y \times Z]^G = \bigoplus_{\lambda \in \Lambda} k[Y]^G \otimes k[Z]_\lambda. \tag{4}$$

We identify $k[Y]$ and $k[Z]$ respectively with the subalgebras $k[Y] \otimes 1$ and $1 \otimes k[Z]$ of $k[Y \times Z]$.

Assume now that there is an open T-orbit $\mathcal{O}$ in Z. This condition is equivalent to either of the following properties (o1), (o2), see [12, Theorem 3.3]:

$$\text{(o1)} \quad k(Z)^T = k;$$
$$\text{(o2)} \quad \dim k[Z]_\lambda = 1 \quad \text{for every } \lambda \in \Lambda. \tag{5}$$

For every $\lambda \in \Lambda$, fix a nonzero element $b_\lambda \in k[Z]_\lambda$. Multiplying every b_λ by an appropriate scalar we may assume that

$$b_\mu b_\nu = b_{\mu+\nu} \quad \text{for all } \mu, \nu \in \Lambda. \tag{6}$$

Indeed, fix a point $x_0 \in \mathcal{O}$. The definition of $k[Z]_\lambda$ implies that $b_\lambda(x_0) \neq 0$, so replacing b_λ by $b_\lambda/b_\lambda(x_0)$ we may assume that $b_\lambda(x_0) = 1$. Then (6) follows from (5), (3).

From (4) and (5) it follows that every $h \in k[Y \times Z]$ admits a unique decomposition

$$h = \sum_{\lambda \in \Lambda} p_\lambda b_\lambda, \quad p_\lambda \in k[Y] \tag{7}$$

(in (7) all but finitely many p_λ's are equal to zero), and h lies in $k[Y \times Z]^G$ if and only if $p_\lambda \in k[Y]^G$ for all λ. From (6) we obtain

$$\left(\sum_{\mu \in \Lambda} p'_\mu b_\mu\right)\left(\sum_{\nu \in \Lambda} p''_\nu b_\nu\right) = \sum_{\lambda \in \Lambda}\left(\sum_{\mu+\nu=\lambda} p'_\mu p''{}_\nu\right) b_\lambda, \quad p'_\mu, p''_\nu \in k[Y]. \tag{8}$$

Consider now a G-equivariant morphism

$$\varphi : Y \times Z \to X. \tag{9}$$

Definition 2.2. Let $f \in k[X]^G$. The invariants $p_\lambda \in k[Y]^G$ defined by (7) for $h = \varphi^*(f)$ are called *the φ-polarizations of f*. The subalgebra of $k[Y]^G$ generated by all the φ-polarizations of the elements of $k[X]^G$ is denoted by $\operatorname{pol}_\varphi k[X]^G$ and called *the φ-polarization algebra of Y*.

Remark 2.3. More generally, if Φ is a collection of G-equivariant morphisms (9) (where Z and X depend on φ), then one can define *the Φ-polarization algebra of Y* as the subalgebra of $k[Y]^G$ generated by all the φ-polarization algebras of Y for $\varphi \in \Phi$.

Since changing the b_λ's clearly replaces the φ-polarizations of $f \in k[X]^G$ by their scalar multiples, the algebra $\operatorname{pol}_\varphi k[X]^G$ does not depend on the choice of the b_λ's.

Example 2.4. If Z is a single point, (9) is a morphism $\varphi : Y \to X$, and $\operatorname{pol}_\varphi k[X]^G = \varphi^*(k[X]^G)$.

Example 2.5 *(Classical setting).* Let V be a finite dimensional algebraic G-module and let $n \in \mathbb{N}$. Take $X = V$ and $Y = V^{\oplus n}$ with the diagonal G-action. Let Z be $\mathbf{A}^n$ endowed with the natural action of the diagonal torus T of $\mathbf{GL}_n$,

$$\operatorname{diag}(t_1, \dots, t_n) \cdot (\alpha_1, \dots, \alpha_n) = (t_1\alpha_1, \dots, t_n\alpha_n).$$

Identifying $(i_1, \dots, i_n) \in \mathbb{Z}^n$ with the character $T \to \mathbf{G}_m$, $\operatorname{diag}(t_1, \dots, t_n) \mapsto t_1^{i_1} \cdots t_n^{i_n}$, we identify $\mathbb{Z}^n$ with $\operatorname{Hom}(T, \mathbf{G}_m)$. Then $\Lambda = \mathbb{Z}_+^n$. If $z_1, \dots, z_n$ are the standard coordinate functions on Z, then for every $\lambda = (i_1, \dots, i_n) \in \Lambda$, the isotypic component $k[Z]_\lambda$ is spanned by $b_\lambda := z_1^{i_1} \cdots z_n^{i_n}$. So, condition (5) holds. Clearly, (6) holds as well.

Recall that the classical n-polarizations of a polynomial $f \in k[V]$ are the polynomials $f_{i_1,\dots,i_n} \in k[V^{\oplus n}]$, where $(i_1, \dots, i_n) \in \Lambda$, such that

$$f\left(\sum_{j=1}^{n} \alpha_j v_j\right) = \sum_{i_1,\dots,i_n \in \Lambda} \alpha_1^{i_1} \cdots \alpha_n^{i_n} f_{i_1,\dots,i_n}(v_1, \dots, v_n) \quad \text{for all } v_j \in V,\ \alpha_j \in k. \tag{10}$$

Since $\alpha_1^{i_1} \cdots \alpha_n^{i_n}$ is the value of $z_1^{i_1} \cdots z_n^{i_n}$ at $(\alpha_1, \dots, \alpha_n) \in Z$, it readily follows from (10) and Definition 2.2 that the classical n-polarizations of f are the φ-polarizations of f for

$$\varphi := \tau_n : Y \times Z \to X, \quad \big((v_1, \dots, v_n), (\alpha_1, \dots, \alpha_n)\big) \mapsto \alpha_1 v_1 + \cdots + \alpha_n v_n. \tag{11}$$

In this setting, we denote the φ-polarization algebra of Y by $\operatorname{pol}_n k[V]^G$.

Example 2.6. If $G = \mathbf{O}_m$ and $V = k^m$ with the natural G-action, then $\operatorname{pol}_n k[V]^G = k[V^{\oplus n}]^G$ by Study's theorem, [15].

If $G = \mathbf{Sp}_m$, m even, and $V = k^m$ with the natural G-action, then $\operatorname{pol}_{\tau_n \times \tau_n} k[V^{\oplus 2}]^G = k[V^{\oplus n}]^G$ (see (11)) by [17].

If $G = \mathbf{SL}_m$ and $V = k^m$ with the natural G-action, then $\operatorname{pol}_{\tau_n \times \cdots \times \tau_n} k[V^{\oplus m}]^G = k[V^{\oplus n}]^G$, see [17].

From (8) we deduce that the algebra $\operatorname{pol}_\varphi k[X]^G$ is generated by all φ-polarizations of the f's for f running through the generators of $k[X]^G$. Since by Hilbert's theorem, the algebra $k[X]^G$ is finitely generated, this means that the algebra $\operatorname{pol}_\varphi k[X]^G$ is finitely generated as well. Hence there is an affine algebraic variety that we denote by $Y/\!/\varphi$, and a dominant morphism $\pi_\varphi : Y \to Y/\!/\varphi$ such that $\pi_\varphi^*(k[Y/\!/\varphi]) = \operatorname{pol}_\varphi k[X]^G$. Since $\operatorname{pol}_\varphi k[X]^G \subseteq k[Y]^G$, the definition of categorical quotient for the G-action on Y implies that there is a dominant morphism $\nu : Y/\!/G \to Y/\!/\varphi$ such that the following diagram is commutative:

$$\begin{tikzcd} Y \arrow[rr, "\pi_{Y,G}"] \arrow[dr, "\pi_\varphi"'] & & Y/\!/G \arrow[dl, "\nu"] \\ & Y/\!/\varphi. & \end{tikzcd} \tag{12}$$

The set of all morphisms from Z to X is endowed with the G-action defined by the formula $(g \cdot \psi)(z) := g \cdot (\psi(z))$ for $\psi : Z \to X$, $g \in G$, $z \in Z$. Using (9), we can consider Y as a G-stable algebraic family of such morphisms. Namely, with every $y \in Y$ we associate the morphism

$$\varphi_y : Z \to X, \quad z \mapsto \varphi(y, z). \tag{13}$$

Then for every $z \in Z$ and $g \in G$ we have $\varphi_{g \cdot y}(z) = \varphi(g \cdot y, z) = \varphi(g \cdot (y, z)) = g(\varphi(y, z)) = (g \cdot \varphi_y)(z)$, so $\varphi_{g \cdot y} = g \cdot \varphi_y$.

Lemma 2.7. *For every $y_1, y_2 \in Y$, the following properties are equivalent*:

(i) $\pi_\varphi^{-1}(\pi_\varphi(y_1)) = \pi_\varphi^{-1}(\pi_\varphi(y_2))$;
(ii) $\pi_{X,G} \circ \varphi_{y_1} = \pi_{X,G} \circ \varphi_{y_2}$.

Remark 2.8. Property (i) means that points $y_1, y_2 \in Y$ are not separated by the φ-polarization algebra $\mathrm{pol}_\varphi\, k[X]^G$.

Proof. By virtue of (13), property (ii) is equivalent to the property

$$\pi_{X,G}\bigl(\varphi(y_1, z)\bigr) = \pi_{X,G}\bigl(\varphi(y_2, z)\bigr) \quad \text{for all } z \in Z. \tag{14}$$

Since the variety X is affine, for a fixed $z \in Z$, equality in (14) holds if and only if

$$s\bigl(\pi_{X,G}\bigl(\varphi(y_1, z)\bigr)\bigr) = s\bigl(\pi_{X,G}\bigl(\varphi(y_2, z)\bigr)\bigr) \quad \text{for every } s \in k[X/\!\!/G]. \tag{15}$$

Since $\pi_{X,G}^*(k[X/\!\!/G]) = k[X]^G$, in turn, (15) is equivalent to the property

$$f\bigl(\varphi(y_1, z)\bigr) = f\bigl(\varphi(y_2, z)\bigr) \quad \text{for every } f \in k[X]^G. \tag{16}$$

Setting $h = \varphi^*(f)$ for f in (16), we thus obtain $h(y_1, z) = h(y_2, z)$ for all $z \in Z$, i.e., using the notation of (7), $\sum_{\lambda \in \Lambda} p_\lambda(y_1) b_\lambda = \sum_{\lambda \in \Lambda} p_\lambda(y_2) b_\lambda$. Since $\{b_\lambda\}$ are linearly independent, this shows that the equality in (16) is equivalent to the collection of equalities $p_\lambda(y_1) = p_\lambda(y_2)$, $\lambda \in \Lambda$. Definition 2.2 and Remark 2.8 now imply the claim. □

Lemma 2.9. *If G is a finite group, then for every two morphisms $\psi_i : Z \to X$, $i = 1, 2$, the following properties are equivalent*:

(i) $\pi_{X,G} \circ \psi_1 = \pi_{X,G} \circ \psi_2$;
(ii) *there is $g \in G$ such that $\psi_2 = g \cdot \psi_1$.*

Proof. (ii) $\Rightarrow$ (i) is clear (and holds for every reductive G, not necessarily finite). Assume now that (i) holds. Consider in $Z \times X$ the closed subset

$$\Psi := \bigl\{(z, x) \in Z \times X \bigm| \pi_{X,G}\bigl(\psi_1(z)\bigr) = \pi_{X,G}(x)\bigr\}. \tag{17}$$

Since G is finite, every fiber of $\pi_{X,G}$ is a G-orbit, see, e.g., [12, Theorem 4.10]. Hence for $(z,x) \in Z \times X$, the condition $\pi_{X,G}(\psi_1(z)) = \pi_{X,G}(x)$ in (17) is equivalent to the existence of $g \in G$ such that $x = g \cdot (\psi_1(z)) = (g \cdot \psi_1)(z)$. In turn, the last equality means that the point (z,x) lies in the graph of $g \cdot \psi_1$,

$$\Gamma_{g\cdot\psi_1} := \{(z, (g \cdot \psi_1)(z)) \in Z \times X \mid z \in Z\}. \tag{18}$$

On the other hand, (17), (18) clearly imply that $\Gamma_{g\cdot\psi_1} \subseteq \Psi$ for every g. Thus,

$$\Psi = \bigcup_{g\in G} \Gamma_{g\cdot\psi_1}. \tag{19}$$

But every $\Gamma_{g\cdot\psi_1}$ is a closed subset of $Z \times X$ isomorphic to Z. So, by (19), Ψ is a union of finitely many closed irreducible subsets of the same dimension. Hence these subsets are precisely the irreducible components of Ψ.

On the other hand, it follows from (i) that

$$\Psi := \{(z,x) \in Z \times X \mid \pi_{X,G}(\psi_2(z)) = \pi_{X,G}(x)\}. \tag{20}$$

Using the above argument, we then deduce from (20) that the graph of ψ_2,

$$\Gamma_{\psi_2} := \{(z, \psi_2(z)) \in Z \times X \mid z \in Z\},$$

is an irreducible component of Ψ as well. Therefore there is $g \in G$ such that $\Gamma_{\psi_2} = \Gamma_{g\cdot\psi_1}$. Hence $g \cdot \psi_1 = \psi_2$, i.e., (ii) holds. □

Theorem 2.10. *Maintain the notation of this section. If G is a finite group, then*

(i) *the morphism $\nu : Y/\!\!/G \to Y/\!\!/\varphi$ in* (12) *is injective*;
(ii) *$k(Y)^G$ is the field of fractions of the φ-polarization algebra* $\operatorname{pol}_\varphi k[X]^G$.

Proof. Since G is finite, fibers of $\pi_{X,G}$ are precisely G-orbits. On the other hand, by Lemmas 2.7 and 2.9, every fiber of π_φ is a G-orbit as well. This and the commutative diagram (12) yield (i). Since ν is dominant and $\operatorname{char} k = 0$, from (i) it follows that ν is a birational isomorphism. Since $Y/\!\!/\varphi$ is affine, $k(Y/\!\!/\varphi)$ is the field of fractions of $k[Y/\!\!/\varphi]$. This and (12) now imply (ii). □

2.11. Under a supplementary assumption there is a geometric criterion of finiteness of ν. It is based on a general statement essentially due to Hilbert. Namely, consider an action of a reductive algebraic group G on an irreducible affine algebraic variety M. Assume that the corresponding G-action on $k[M]$ preserves a $\mathbb{Z}_+$-grading $k[M] = \bigoplus_{n\in\mathbb{Z}_+} k[M]_n$ such that $k[M]_0 = k$ and $\dim k[M]_n < \infty$ for every n. Let A be a homogeneous subalgebra of $k[M]^G$.

Lemma 2.12. *The following properties are equivalent*:

(i) $\{x \in M \mid f(x) = 0\ \forall f \in \bigoplus_{n\in\mathbb{N}} k[M]_n^G\} = \{x \in M \mid h(x) = 0\ \forall h \in \bigoplus_{n\in\mathbb{N}} A_n\}$;
(ii) $k[M]^G$ *is integral over* A.

If these properties hold and G *is connected, then* $k[M]^G$ *is the integral closure of* A *in* $k[M]$.

Proof. For linear actions, (i) $\Rightarrow$ (ii) is proved by Hilbert in [6, §4]. In the general case the argument is the same. Implication (ii) $\Rightarrow$ (i) is clear. The last statement follows from the first since it is well known that $k[M]^G$ is integrally closed in $k[M]$ for connected G (connectedness of G implies that G acts trivially on the set of roots of the equation of integral dependence). □

Lemma 2.12 implies the following geometric criterion of finiteness of ν. Assume that the G-actions on X and Y can be extended to the $(G \times \mathbf{G}_m)$-actions such that

$$\varphi \text{ is } (G \times \mathbf{G}_m)\text{-equivariant,} \tag{21}$$

$$k[X]^{\mathbf{G}_m} = k[Y]^{\mathbf{G}_m} = k. \tag{22}$$

From (22) we then deduce that $\mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m)$ can be identified with $\mathbb{Z}$ so that the isotypic component decompositions of $k[X]$ and $k[Y]$ become the $\mathbb{Z}_+$-gradings of these algebras,

$$k[X] = \bigoplus_{n\in\mathbb{Z}_+} k[X]_n, \qquad k[Y] = \bigoplus_{n\in\mathbb{Z}_+} k[Y]_n. \tag{23}$$

Since every isotypic component is a finitely generated module over invariants, see, e.g., [12, Theorem 3.24], from (22) we deduce that these gradings enjoy the properties

$$\begin{gathered} k[X]_0 = k[Y]_0 = k \quad \text{and} \\ \dim k[X]_n < \infty, \quad \dim k[Y]_n < \infty \quad \text{for every } n \in \mathbb{Z}_+. \end{gathered} \tag{24}$$

It follows from (21) that $k[X]^G$ and $k[Y]^G$ are graded subalgebras of the graded algebras $k[X]$ and $k[Y]$ respectively, and from Definition 2.2 we deduce that $\mathrm{pol}_\varphi k[X]^G$ is a graded subalgebra of the graded algebra $k[Y]^G$.

The ideal $\bigoplus_{n\in\mathbb{N}} k[X]_n$ in $k[X]$ (respectively $\bigoplus_{n\in\mathbb{N}} k[Y]_n$ in $k[Y]$) is maximal and $\mathbf{G}_m$-stable, so the point $0_X \in X$ (respectively $0_Y \in Y$) where it vanishes, is $\mathbf{G}_m$-fixed. As invariants separate closed orbits, see, e.g., [12, Theorem 4.7], (22) implies that $X^{\mathbf{G}_m} = \{0_X\}$, $Y^{\mathbf{G}_m} = \{0_Y\}$. Hence $0_X \in X^G$, $0_Y \in Y^G$. From (21) we deduce that $\varphi(0_Y \times Z) = 0_X$. We put

$$\mathcal{N}_{Y,G} := \pi_{Y,G}^{-1}\big(\pi_{Y,G}(0_Y)\big), \quad \mathcal{P}_{Y,G} := \pi_\varphi^{-1}\big(\pi_\varphi(0_Y)\big), \quad \mathcal{N}_{X,G} := \pi_{X,G}^{-1}\big(\pi_{X,G}(0_X)\big). \tag{25}$$

By virtue of (12), the following inclusion holds:

$$\mathcal{N}_{Y,G} \subseteq \mathcal{P}_{Y,G}. \tag{26}$$

Since $\mathcal{N}_{Y,G}$ is precisely the set of points of Y whose G-orbit contains 0_Y in the closure,

$$\mathcal{N}_{S,G} = S \cap \mathcal{N}_{Y,G} \tag{27}$$

for every G-stable closed subset S of Y containing 0_Y.

Example 2.13. Maintain the notation of Example 2.5. Then the $\mathbf{G}_m$-actions on $X = V$ and $Y = V^{\oplus n}$ by scalar multiplications yield the $(G \times \mathbf{G}_m)$-extensions of G-actions such that (21), (22) hold. Thus, in the classical setting, the assumptions of Section 2.2 hold. In this case, $0_X = 0$, $0_Y = (0, \dots, 0)$. The varieties $\mathcal{N}_{V^{\oplus n},G}$ and $\mathcal{N}_{V,G}$ are respectively the Hilbert nullcones of G-modules $V^{\oplus n}$ and V, and $\mathcal{P}_{V^{\oplus n},G}$ is the locus of the maximal homogeneous ideal of $\operatorname{pol}_n k[V]^G$.

Lemma 2.14. *Maintain the assumptions of Section* 2.2*. The following properties are equivalent*:

(i) *ν is finite*;
(ii) $\mathcal{N}_{Y,G} = \mathcal{P}_{Y,G}$.

Proof. This immediately follows from Lemma 2.12. □

Theorem 2.15. *Maintain the assumptions of Section* 2.2 *and let G be finite. Then*

(i) *ν is finite and bijective*;
(ii) *if Y is normal, $\nu : Y /\!\!/ G \to Y /\!\!/ \varphi$ is the normalization of $Y /\!\!/ \varphi$, and $k[Y]^G$ is the integral closure of* $\operatorname{pol}_\varphi k[X]^G$ *in $k(Y)^G$*;
(iii) *if Y is normal,* $\operatorname{pol}_\varphi k[X]^G = k[Y]^G$ *if and only if* $\operatorname{pol}_\varphi k[X]^G$ *is integrally closed.*

Proof. Theorem 2.10(i) implies that $\mathcal{N}_{Y,G} = \mathcal{P}_{Y,G}$ $(= 0_Y)$. Hence ν is finite by Lemma 2.14. Being finite, ν is closed, and since ν is also dominant, Theorem 2.10(i) implies that ν is bijective. This proves (i). If Y is normal, then $Y /\!\!/ G$ is normal as well, see, e.g., [12, Theorem 3.16]. Since by Theorem 2.10(ii), ν is a birational isomorphism, this, (i), and the definitions of $Y /\!\!/ G$, $Y /\!\!/ \varphi$, ν prove (ii). Claim (iii) follows from (ii). □

Remark 2.16. Bijectivity of ν is equivalent to saying that $\operatorname{pol}_\varphi k[X]^G$ is the *separating set* of $k[Y]^G$ in the sense of [4, Section 2.3.2].

Corollary 2.17. *In the classical setting* (*see Example* 2.5)*, let G be finite. Then*

(i) *ν is finite and bijective*;
(ii) *$k[V^{\oplus n}]^G$ is the integral closure of* $\operatorname{pol}_n k[V]^G$ *in $k(V^{\oplus n})^G$*;
(iii) *$k[V^{\oplus n}]^G =$* $\operatorname{pol}_n k[V]^G$ *if and only if* $\operatorname{pol}_n k[V]^G$ *is integrally closed.*

Example 2.18. Maintain the notation of Example 2.5 and let $V = k^m$. If G is the symmetric group in m letters acting on V by permuting the coordinates, then $k[V^{\oplus n}]^G = \operatorname{pol}_n k[V]^G$ for every n, [17]. This equality also holds for the Weyl group of type B_m and the dihedral groups, [8]. But for the Weyl group of type D_m, $m \geqslant 4$, and $n = 2$ it does not hold, [16].

Namely, D_m acts on the standard coordinate functions $x_1, \ldots, x_m$ on V by permutations and changes of an even number of signs, and $k[V]^{\mathsf{D}_m} = k[\sigma_1, \ldots, \sigma_m]$ where

$$\sigma_s = \sum_{i=1}^{m} x_i^{2s} \quad \text{for } 1 \leqslant s \leqslant m-1, \qquad \sigma_m = x_1 \cdots x_m,$$

see, e.g., [7]. Take another copy of V with the standard coordinate functions $y_1, \ldots, y_m$, and naturally identify $x_1, \ldots, x_m, y_1, \ldots, y_m$ with the functions on $V^{\oplus 2}$. Then $k[V^{\oplus 2}]^{\mathsf{D}_m}$ is generated by $\operatorname{pol}_2 k[V]^{\mathsf{D}_m}$ and the polynomials

$$P_{r_1} \cdots P_{r_d}(\sigma_n), \quad r_1, \ldots, r_d \text{ odd}, \ \sum_{i=1}^{d} r_i \leqslant n - d, \tag{28}$$

where $P_r := \sum_{i=1}^{m} y_i^r \frac{\partial}{\partial x_i}$, see [8,16]. The group B_m is generated by D_m and the reflection w such that $w \cdot x_i = x_i$ for $i < m$ and $w \cdot x_m = -x_m$. The operators P_{r_i} from (28) commute with the diagonal action of B_m on $V^{\oplus 2}$, therefore $w(P_{r_1} \cdots P_{r_d}(\sigma_n)) = -P_{r_1} \cdots P_{r_d}(\sigma_n)$. This yields

$$\bigl(P_{r_1} \cdots P_{r_d}(\sigma_n)\bigr)^2 \in k\bigl[V^{\oplus 2}\bigr]^{\mathsf{B}_m}. \tag{29}$$

Since $k[V^{\oplus 2}]^{\mathsf{B}_m} = \operatorname{pol}_2 k[V]^{\mathsf{B}_m}$ and, clearly, $\operatorname{pol}_2 k[V]^{\mathsf{B}_m} \subseteq \operatorname{pol}_2 k[V]^{\mathsf{D}_m}$, we deduce from (29) that $k[V^{\oplus 2}]^{\mathsf{D}_m}$ is integral over $\operatorname{pol}_2 k[V]^{\mathsf{D}_m}$. This agrees with Theorem 2.15 (that gives more delicate information).

3. Polarization index

3.1. In this section we take up the classical setting and maintain the notation of Examples 2.5, 2.13, and that of (25). If $n, m \in \mathbb{N}$ and $n \leqslant m$, we naturally identify $V^{\oplus n}$ with the subspace $\{(v_1, \ldots, v_n, 0, \ldots, 0) \mid v_i \in V\}$ of $V^{\oplus m}$. It is then not difficult to see that

$$\mathcal{P}_{V^{\oplus n}, G} = V^{\oplus n} \cap \mathcal{P}_{V^{\oplus m}, G}. \tag{30}$$

Lemma 3.2. *The following properties of a point $v = (v_1, \ldots, v_n) \in V^{\oplus n}$ are equivalent*:

(i) $v \in \mathcal{P}_{V^{\oplus n}, G}$;
(ii) $\langle v_1, \ldots, v_n \rangle \subseteq \mathcal{N}_{V,G}$.

Proof. Let $f \in k[V]^G$ be a nonconstant homogeneous function. If $v \in \mathcal{P}_{V^{\oplus n},G}$, then the definition of $\mathcal{P}_{V^{\oplus n},G}$ (see (25) and Example 2.13) yields that, in the notation of (1), we have $f_{i_1,\dots,i_n}(v) = 0$ for all $i_1, \dots, i_n \in \mathbb{Z}_+$. From this and (1) we obtain

$$f(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0 \quad \text{for all } \alpha_i \in k. \tag{31}$$

So, $\langle v_1, \dots, v_n \rangle$ lies in the zero set of every f. The definition of $\mathcal{N}_{V,G}$ (see (25) and Example 2.13) now implies that $\langle v_1, \dots, v_n \rangle \subseteq \mathcal{N}_{V,G}$.

Conversely, assume that the last inclusion holds. By the definition of $\mathcal{N}_{V,G}$, this implies (31). By (1), this in turn yields that $f_{i_1,\dots,i_n}(v) = 0$ for all $i_1, \dots, i_n \in \mathbb{Z}_+$. The definition of $\mathcal{P}_{V^{\oplus n},G}$ then implies that $v \in \mathcal{P}_{V^{\oplus n},G}$. $\square$

Definition 3.3. *The polarization index of a G-module V* is

$$\operatorname{pol\,ind}(V) := \sup n$$

with the supremum taken over all n such that in (26) the equality holds, $\mathcal{N}_{V^{\oplus n},G} = \mathcal{P}_{V^{\oplus n},G}$.

From (25), Definition 3.3, and the equality $\operatorname{pol}_1 k[V]^G = k[V]^G$ we obtain

$$\operatorname{pol\,ind}(V) \geqslant 1. \tag{32}$$

It is also clear that

$$\operatorname{pol\,ind}(V \oplus U) = \operatorname{pol\,ind}(V) \quad \text{if } U \text{ is a trivial } G\text{-module}. \tag{33}$$

Lemma 3.4. *For every $n \in \mathbb{N}$,*

$$\mathcal{N}_{V^{\oplus n},G} \begin{cases} = \mathcal{P}_{V^{\oplus n},G} & \text{if } n \leqslant \operatorname{pol\,ind}(V), \\ \subsetneq \mathcal{P}_{V^{\oplus n},G} & \text{if } n > \operatorname{pol\,ind}(V). \end{cases}$$

Proof. By virtue of Definition 3.3, for $n > \operatorname{pol\,ind}(V)$, this follows from (26), and for $n \leqslant \operatorname{pol\,ind}(V)$, from (27) and (30). $\square$

Corollary 3.5. *The extension $\operatorname{pol}_n k[V]^G \subseteq k[V^{\oplus n}]^G$ is integral if and only if $n \leqslant \operatorname{pol\,ind}(V)$.*

Proof. This follows from Lemmas 3.4 and 2.14. $\square$

Call a character $\mathbf{G}_m \to \mathbf{G}_m$, $t \mapsto t^d$, *positive* if $d > 0$. Every homomorphism $\gamma : \mathbf{G}_m \to G$ endows V with the structure of $\mathbf{G}_m$-module defined by $t \cdot v := \gamma(t) \cdot v$. We denote by $V(\gamma)$ the submodule of this $\mathbf{G}_m$-module equal to the sum of all the isotypic components whose weight is positive. Clearly, if $v \in V(\gamma)$, then the closure of $\mathbf{G}_m$-orbit (and, all the more, G-orbit) of v contains 0_V. Hence $V(\gamma) \subseteq \mathcal{N}_{V,G}$. The Hilbert–Mumford theorem, [6,10] (see, e.g., [12, 5.3]), claims that

$$\mathcal{N}_{V,G} = \bigcup_{\gamma} V(\gamma). \tag{34}$$

Lemma 3.6. *The following properties of an integer $n \in \mathbb{N}$ are equivalent*:

(i) *for every linear subspace L such that $\dim L \leqslant n$ and $L \subseteq \mathcal{N}_{V,G}$, there is a homomorphism $\gamma : \mathbf{G}_m \to G$ such that $L \subseteq V(\gamma)$*;
(ii) $n \leqslant \operatorname{pol\,ind}(V)$.

Proof. Let (i) hold. Take a point $v = (v_1, \ldots, v_n) \in \mathcal{P}_{V^{\oplus n},G}$. By Lemma 3.2, $\langle v_1, \ldots, v_n \rangle$ is contained in $\mathcal{N}_{V,G}$. By (i), $\langle v_1, \ldots, v_n \rangle \subseteq V(\gamma)$ for some γ. This implies that the closure of $\mathbf{G}_m$-orbit (and, all the more, G-orbit) of v contains $0_{V^{\oplus n}}$, i.e., $v \in \mathcal{N}_{V^{\oplus n},G}$. So, by (26), we have $\mathcal{P}_{V^{\oplus n},G} = \mathcal{N}_{V^{\oplus n},G}$, whence $n \leqslant \operatorname{pol\,ind}(V)$ by Definition 3.3. This proves (i) $\Rightarrow$ (ii).

Conversely, let (ii) holds. Consider in $\mathcal{N}_{V,G}$ a linear subspace L of dimension $\leqslant n$. Then $L = \langle v_1, \ldots, v_n \rangle$ for some $v_i \in V$. By Lemma 3.2, the point $v = (v_1, \ldots, v_n)$ lies in $\mathcal{P}_{V^{\oplus n},G}$. By (ii), and Lemma 3.4, we have $v \in \mathcal{N}_{V^{\oplus n},G}$. From (34) we now deduce that $v \in V^{\oplus n}(\gamma)$ for some γ. Since $V^{\oplus n}(\gamma) = V(\gamma)^{\oplus n}$, this yields $v_i \in V(\gamma)$ for every i, or, equivalently, $L \subseteq V(\gamma)$. Thus (ii) $\Rightarrow$ (i) is proved. □

Corollary 3.7.

(i) *Let U be a submodule of V. Then*

$$\operatorname{pol\,ind}(U) \geqslant \operatorname{pol\,ind}(V).$$

(ii) *Let V_i be a finite dimensional algebraic module of a reductive algebraic group G_i, $i = 1, \ldots, m$, and let $G = G_1 \times \cdots \times G_m$, $V = V_1 \oplus \cdots \oplus V_m$. Then*

$$\operatorname{pol\,ind}(V) = \min_i \operatorname{pol\,ind}(V_i).$$

Proof. Statement (i) readily follows from Lemma 3.6. Let the assumptions of (ii) hold, and let $\pi_i : V \to V_i$, $p_i : G \to G_i$ be the natural projections. Since the Hilbert nullcone is the set of points whose orbits contain zero in the closure, we have

$$\mathcal{N}_{V,G} = \mathcal{N}_{V_1,G_1} \times \cdots \times \mathcal{N}_{V_m,G_m}.$$

A linear subspace L lying in $\mathcal{N}_{V,G}$ is contained in $V(\gamma)$ for some γ if and only if $\pi_i(L) \subseteq V_i(p_i \circ \gamma)$ for every i. Using these properties and Lemma 3.6, we deduce (ii). □

3.8. We now calculate the polarization index of some G-modules.

Theorem 3.9. *If G is a finite group, then for any G-module V,*

$$\operatorname{pol\,ind}(V)=\infty.$$

Proof. This follows from Theorem 2.15, Lemma 2.14, and Definition 3.3. □

Theorem 3.10. *If G is a linear algebraic torus, then for any G-module V,*

$$\operatorname{pol\,ind}(V)=\infty.$$

Proof. It is well known (and immediately follows from (34)) that in this case there are homomorphisms $\gamma_i:\mathbf{G}_m\to G$, $i=1,\dots,s$, such that

$$\mathcal{N}_{V,G}=V(\gamma_1)\cup\dots\cup V(\gamma_s). \tag{35}$$

Since every linear subspace L of V is an irreducible algebraic variety, (35) implies that if $L\subseteq\mathcal{N}_{V,G}$, then $L\subseteq V(\gamma_i)$ for some i, whence the claim by Lemma 3.6. □

Lemma 3.11. *Let $k[V]^G=k$.*

(i) $\operatorname{pol\,ind}(V)=\sup n$ *where the supremum is taken over all n such that $k[V^{\oplus n}]^G=k$.*
(ii) *If G^0 is semisimple, then* $\operatorname{pol\,ind}(V)$ *is equal to the generic transitivity degree of the G-action on V, see* [11]*, i.e., to the maximum n such that there is an open G-orbit in $V^{\oplus n}$. In this case,*

$$\operatorname{pol\,ind}(V)\leqslant \dim G/\dim V.$$

Proof. The condition $k[V]^G=k$ and Definition 2.2 imply that

$$\mathcal{P}_{V^{\oplus n},G}=V^{\oplus n}\quad\text{for every } n.$$

On the other hand, $k[V^{\oplus n}]^G=k$ is equivalent to

$$\mathcal{N}_{V^{\oplus n},G}=V^{\oplus n}.$$

This gives (i). Being semisimple, G^0 has no nontrivial characters, hence $k[V^{\oplus n}]^G=k$ is equivalent to the existence of an open G-orbit in $V^{\oplus n}$, see [12, Theorem 3.3 and the corollary of Theorem 2.3]. This proves (ii). □

Example 3.12. If $G=\mathbf{SL}_m$ and $V=k^m$ with the natural G-action, then Lemma 3.11 implies $\operatorname{pol\,ind}(V)=m-1$.

If $G=\mathbf{Sp}_m$ and $V=k^m$ (m even) with the natural G-action, then Lemma 3.11 implies $\operatorname{pol\,ind}(V)=1$.

If $G = \mathbf{O}_m$ and $V = k^m$ with the natural G-action, then Example 2.6 and Definition 3.3 yield $\operatorname{pol\,ind}(V) = \infty$.

If $G = \mathbf{SO}_m$ and $V = k^m$ with the natural G-action, then the classical description of $k[V^{\oplus n}]^G$, see [17], implies that $k[V^{\oplus n}]^G$ is integral over $k[V^{\oplus n}]^{\mathbf{O}_m}$. Hence in this case again $\operatorname{pol\,ind}(V) = \infty$, however, in contrast to the case of $\mathbf{O}_m$, the algebras $k[V^{\oplus n}]^G$ and $\operatorname{pol}_n k[V]^G$ do not coincide if m divides n.

We now calculate the polarization index of any $\mathbf{SL}_2$-module. Denote by R_d the $\mathbf{SL}_2$-module of binary forms in x and y of degree d, see, e.g., [12, 0.12]. Up to isomorphism, R_d is the unique simple $\mathbf{SL}_2$-module of dimension $d+1$. According to the classical Hilbert theorem, [6, §5] (see, e.g., [12, Example 1 in 5.4]),

$$\mathcal{N}_{R_d,\mathbf{SL}_2} = \bigcup_{l \in R_1} l^{[d/2]+1} R_{d-[d/2]-1}, \tag{36}$$

and for every nonzero $l \in R_1$, there is a homomorphism $\gamma : \mathbf{G}_m \to \mathbf{SL}_2$ such that

$$l^{[d/2]+1} R_{d-[d/2]-1} = R_d(\gamma) \tag{37}$$

and vice versa.

Lemma 3.13. *For $d \geqslant 2$, the following properties of a linear subspace L of R_d lying in $\mathcal{N}_{R_d,\mathbf{SL}_2}$ are equivalent*:

- (i) *L is maximal* (*with respect to inclusion*) *among the linear subspaces lying in $\mathcal{N}_{R_d,\mathbf{SL}_2}$*;
- (ii) *there is $l \in R_1$, $l \neq 0$ such that $L = l^{[d/2]+1} R_{d-[d/2]-1}$.*

Proof. Using that $k[x, y]$ is a unique factorization domain and every $l \in R_1$, $l \neq 0$ is a simple element in it, we obtain that for every nonzero $l_1, l_2 \in R_1$,

$$l_1^{[d/2]+1} R_{d-[d/2]-1} \cap l_2^{[d/2]+1} R_{d-[d/2]-1} = \{0\} \quad \text{if } l_1/l_2 \notin k. \tag{38}$$

Therefore it suffices to show that for every 2-dimensional linear subspace P lying in $\mathcal{N}_{R_d,\mathbf{SL}_2}$ there is $l \in R_1$ such that $P \subseteq l^{[d/2]+1} R_{d-[d/2]-1}$. Let f_1, f_2 be a basis of P. Then (36) implies that

$$f_i = l_i^{[d/2]+1} h_i \quad \text{for some } l_i \in R_1,\ h_i \in R_{d-[d/2]-1}. \tag{39}$$

We have to show that if $\alpha f_1 + \beta f_2 \in \mathcal{N}_{R_d,\mathbf{SL}_2}$ for every $\alpha, \beta \in k$, then $l_1/l_2 \in k$.

For contradiction, assume that l_1 and l_2 are linearly independent. Applying $\mathbf{SL}_2$, we then may assume that $l_1 = x$, $l_2 = y$. Since $P \subseteq \mathcal{N}_{R_d,\mathbf{SL}_2}$, from (36) we deduce that for every $\alpha, \beta \in k$ there are $\mu, \nu \in k$, $h \in R_{d-[d/2]-1}$ (depending on α, β) such that

$$\alpha x^{[d/2]+1} h_1 + \beta y^{[d/2]+1} h_2 = (\mu x + \nu y)^{[d/2]+1} h.$$

Using that $k[x, y]$ is unique factorization domain, we deduce from this equality and (39) that $\mu\nu \neq 0$ if $\alpha\beta \neq 0$. Hence we may assume that for every nonzero $\alpha, \beta \in k$ there are $\mu \in k$, $h \in R_{d-[d/2]-1}$ (depending on α, β) such that

$$\alpha x^{[d/2]+1} h_1 + \beta y^{[d/2]+1} h_2 = (\mu x + y)^{[d/2]+1} h. \tag{40}$$

Note that

$$\text{when } \alpha \text{ and } \beta \text{ in (40) vary, } \mu \text{ ranges over an infinite set.} \tag{41}$$

Indeed, otherwise (40) implies that there is a basis of P whose elements are divisible by some $(\mu x + y)^{[d/2]+1}$. Hence $x^{[d/2]+1} h_1$ is divisible by $(\mu x + y)^{[d/2]+1}$ as well. Since $\deg h_1 < [d/2] + 1$, this is impossible.

We now consider separately the cases of even and odd d. First, let d be even, $d = 2m$. Then

$$h_1 = \sum_{i=0}^{m-1} \eta_i x^{m-i-1} y^i, \quad \eta_i \in k. \tag{42}$$

Plugging (42) in equality (40), and then differentiating it m times with respect to x, substituting $y = -\mu x$, and dividing both sides by αx^m, we obtain the following equality:

$$\sum_{i=0}^{m-1} (-1)^i \frac{(2m-i)!}{(m-i)!} \eta_i \mu^i = 0. \tag{43}$$

Since $h_1 \neq 0$, (43) contradicts (41).

Let now d be odd, $d = 2m - 1$. Then h_1 is still given by (42) and

$$h_2 = \sum_{j=0}^{m-1} \theta_j x^j y^{m-j-1}, \quad \theta_j \in k. \tag{44}$$

Plugging (42), (44) in equality (40), and then differentiating it $m - 1$ times respectively with respect to x and y, substituting $y = -\mu x$, and dividing both sides by x^m, we obtain respectively the equalities

$$\theta_{m-1} \mu^m = \frac{\alpha}{\beta} \sum_{i=0}^{m-1} (-1)^{m+i-1} \frac{(2m-i-1)!}{(m-1)!(m-i)!} \eta_i \mu^i, \tag{45}$$

$$\alpha \eta_{m-1} + \beta \sum_{j=0}^{m-1} (-1)^{m-j} \frac{(2m-j-1)!}{(m-1)!(m-j)!} \theta_j \mu^{m-j} = 0. \tag{46}$$

Multiplying (46) by $\theta_{m-1}\mu^{m-1}$, replacing $\theta_{m-1}\mu^m$ by the right-hand side of (45), and dividing both sides by α, we obtain

$$\eta_{m-1}\theta_{m-1}\mu^{m-1}+\sum_{i,j=0}^{m-1}(-1)^{i-j-1}\frac{(2m-j-1)!(2m-i-1)!}{((m-1)!)^2(m-i)!(m-j)!}\eta_i\theta_j\mu^{m-j+i-1}=0. \quad (47)$$

From (41) we deduce that all the coefficients of the left-hand side of (47), considered as a polynomial in μ, vanish. In particular,

$$\eta_0\theta_{m-1}=\eta_{m-1}\theta_0=0. \quad (48)$$

If $\theta_{m-1}=0$, then (45), (41) imply $\eta_0=\cdots=\eta_{m-1}=0$ contrary to $h_1\neq 0$. Similarly, if $\eta_{m-1}=0$, then (46), (41) imply $\theta_0=\cdots=\theta_{m-1}=0$ contrary to $h_2\neq 0$. Thus, $\eta_{m-1}\theta_{m-1}\neq 0$, whence, by (48), $\eta_0=\theta_0=0$. From (42), (44), (40) we then deduce that for $m\geqslant 2$, the left-hand side of (40) is divisible by xy. Hence h in (40) is divisible by xy as well; in particular, $m\geqslant 3$. Thus, for $m\geqslant 3$, dividing both sides of (40) by xy, we obtain

$$\alpha x^{m-1}h_1'+\beta y^{m-1}h_2'=(\mu x+y)^{m-1}h',$$

with $h_1', h_2', h\in R_{m-2}$. This means that in considering (40) we may step down from case m to case $m-1$. Continuing this way we reduce the consideration of (40) to the case $m=2$. In this case, the above argument shows that h is a nonzero element of R_1 divisible by xy. This contradiction completes the proof. $\square$

Corollary 3.14. *The action of* $\mathbf{SL}_2$ *on the set of maximal linear subspaces of* R_d *lying in* $\mathcal{N}_{R_d,\mathbf{SL}_2}$ *is transitive. The dimension of every such subspace is equal to* $d-[d/2]$ *and* 2 *respectively for* $d\neq 1$ *and* $d=1$.

Theorem 3.15. *For* $G=\mathbf{SL}_2$ *and* $V=R_{d_1}\oplus\cdots\oplus R_{d_m}$,

$$\operatorname{pol\,ind}(V)=\begin{cases}1, & \text{if } d_i=1 \text{ for some } i,\\ \infty, & \text{otherwise.}\end{cases}$$

Proof. Since the G-module R_0 is trivial, by (33) we may assume that $d_i\geqslant 1$ for every i. Since $k[R_1]^{\mathbf{SL}_2}=k$ and $k[R_1^{\oplus 2}]^{\mathbf{SL}_2}\neq k$, Lemma 3.11 implies that $\operatorname{pol\,ind}(R_1)=1$. From this, Corollary 3.7, and (32) we deduce the claim for the cases where $d_i=1$ for some i.

Assume now that $d_i\geqslant 2$ for every i, and let L be a linear subspace of V lying in $\mathcal{N}_{V,G}$. Let $\pi_i: V=R_{d_1}\oplus\cdots\oplus R_{d_m}\to R_{d_i}$ be the natural projection to the ith summand, and let $L_i:=\pi_i(L)$. Since $\pi_i(\mathcal{N}_{V,G})=\mathcal{N}_{R_i,G}$, we have $L_i\subset\mathcal{N}_{R_i,G}$ for every i. Hence by Lemma 3.13 and (37), for every i, there is homomorphism $\gamma_i:\mathbf{G}_m\to G$ such that $L_i\subseteq R_{d_i}(\gamma_i)$. Take now a point $v\in L$ such that $\pi_i(v)\neq 0$ for every i. Since L lies in $\mathcal{N}_{V,G}$, it follows from (34) that $v\in V(\gamma)$ for some γ. Hence $\pi_i(v)\in R_{d_i}(\gamma)$. But $\pi_i(v)\in R_{d_i}(\gamma_i)$ as well. By (37), (38), this yields $R_{d_i}(\gamma_i)=R_{d_i}(\gamma)$. Hence $L\subseteq V(\gamma)$. From Lemma 3.6 we now deduce that $\operatorname{pol\,ind}(V)=\infty$ completing the proof. $\square$

Theorem 3.16. *Let G be a connected semisimple algebraic group and let $\mathfrak{g}$ be its Lie algebra endowed with the adjoint G-action. Then*

$$\operatorname{pol\,ind}(\mathfrak{g}) = \begin{cases} 1, & \text{if } \mathfrak{g} \text{ is not isomorphic to } \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. In this case, $\mathcal{N}_{\mathfrak{g},G}$ is the cone of all nilpotent elements in $\mathfrak{g}$, see, e.g., [12, 5.1]. Every subspace $\mathfrak{g}(\gamma)$ is the unipotent radical of a parabolic subalgebra of $\mathfrak{g}$, see [2, VIII, 4.4], [14, 8.4.5], and hence lies in a maximal (with respect to inclusion) unipotent subalgebra of $\mathfrak{g}$. Maximal unipotent subalgebras of $\mathfrak{g}$ are precisely the unipotent radicals of Borel subalgebras of $\mathfrak{g}$, and G acts transitively on the set of such subalgebras, see, e.g., [14, Chapter 6]. This implies that for a linear subspace L of $\mathfrak{g}$ lying in $\mathcal{N}_{\mathfrak{g},G}$ the following properties are equivalent:

(i) the subalgebra of $\mathfrak{g}$ generated by L is unipotent (i.e., lies in $\mathcal{N}_{\mathfrak{g},G}$);
(ii) there is a homomorphism $\gamma : \mathbf{G}_m \to G$ such that $L \subseteq \mathfrak{g}(\gamma)$.

From this, (32), and Lemma 3.6 we deduce that equality $\operatorname{pol\,ind}\mathfrak{g} = 1$ is equivalent to the following property: there is a 2-dimensional linear subspace L of $\mathfrak{g}$ such that L lies in $\mathcal{N}_{\mathfrak{g},G}$ but the subalgebra of $\mathfrak{g}$ generated by L does not lie in $\mathcal{N}_{\mathfrak{g},G}$. If this property holds, we say, for brevity, that $\mathfrak{g}$ is a 2-algebra.

We shall show now that if $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$, then $\mathfrak{g}$ is a 2-algebra. To this end we remark that if a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is a 2-algebra, then $\mathfrak{g}$ is a 2-algebra as well: since the cone of nilpotent elements of $\mathfrak{s}$ lies in $\mathcal{N}_{\mathfrak{g},G}$, this readily follows from the definition of a 2-algebra. Given this remark, we see that the following two statements immediately imply our claim:

(a) if $\mathfrak{g} \ncong \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$, then $\mathfrak{g}$ contains a subalgebra isomorphic to either $\mathfrak{sl}_3$ or $\mathfrak{so}_5$;
(b) $\mathfrak{sl}_3$ and $\mathfrak{so}_5$ are 2-algebras

(note that in $\mathfrak{so}_5$ there are no subalgebras isomorphic to $\mathfrak{sl}_3$, and vice versa).

To prove (a), denote by Φ the root system of $\mathfrak{g}$ with respect to a fixed maximal torus. Let $\alpha_1, \ldots, \alpha_l$ be a system of simple roots in Φ (enumerated as in [1]). Fix a Chevalley basis $\{X_\alpha, X_{-\alpha}, H_\alpha\}_{\alpha \in \Phi}$ of $\mathfrak{g}$, [2]. We may assume that $\mathfrak{g}$ is simple, $\mathfrak{g} \ncong \mathfrak{sl}_2, \mathfrak{so}_5$. For such $\mathfrak{g}$, it is easily seen that there are two roots $\lambda, \mu \in \Phi$ such that the subalgebra of $\mathfrak{g}$ generated by X_λ and X_μ is isomorphic to $\mathfrak{sl}_3$: for $\mathfrak{g}$ of types A_l $(l \geqslant 2)$, B_l $(l \geqslant 3)$, C_l $(l \geqslant 3)$, D_l $(l \geqslant 4)$, F_4, one can take $\lambda = \alpha_1$, $\mu = \alpha_2$; for types E_6, E_7, E_8, take $\lambda = \alpha_1$, $\mu = \alpha_3$; for type G_2, take $\lambda = \alpha_2$, $\mu = 3\alpha_1 + \alpha_2$. This proves (a).

We turn now to the proof of (b). In $\mathfrak{sl}_3$ we explicitly present a subspace L enjoying the desired properties (we are grateful to H. Radjavi for this example, [13]). Namely, take

$$L := \langle X_{\alpha_1} + X_{\alpha_2}, X_{-\alpha_1} - X_{-\alpha_2} \rangle = \left\{ \begin{bmatrix} 0 & a & 0 \\ b & 0 & a \\ 0 & -b & 0 \end{bmatrix} \,\middle|\, a, b \in k \right\}. \tag{49}$$

Then (49) implies that the subalgebra generated by L contains the element $H_{\alpha_1} - H_{\alpha_2}$. Since it is semisimple, this subalgebra does not lie in $\mathcal{N}_{\mathfrak{g},G}$. On the other hand, the matrix in the right-hand side of (49) is nilpotent (this is equivalent to the property that the sums of all its principal minors of orders 2 and 3 are equal to 0, and this is immediately verified). So, $L \subseteq \mathcal{N}_{\mathfrak{g},G}$. This proves (b) for $\mathfrak{sl}_3$.

Let now $\mathfrak{g} = \mathfrak{so}_5$. In this case, an explicit construction of the desired subspace L is unknown to us, so we shall use an indirect argument. The underlying space of $\mathfrak{g}$ is the space of all skew-symmetric (5×5)-matrices. Let $x_{ij} \in k[\mathfrak{g}]$, $1 \leqslant i, j \leqslant 5$, be the standard coordinate functions on $\mathfrak{g}$ given by $x_{ij}((a_{pq})) = a_{ij}$. Then $x_{ij} = -x_{ji}$. Consider the matrix $A := (x_{ij})$. Then $k[\mathfrak{g}]^G = k[f_2, f_4]$ where f_2, f_4 are the coefficients of the characteristic polynomial of A, i.e., $\det(tI_5 - A) = t^5 + f_2 t^3 + f_4 t$, see, e.g., [12]. The Newton formulas expressing the sums of squares of eigenvalues of A via the elementary symmetric functions of them imply that $\operatorname{tr}(A^2) = -2f_2$, $\operatorname{tr}(A^4) = 2f_2^2 - 4f_4$. Hence

$$k[\mathfrak{g}]^G = k\bigl[\operatorname{tr}\bigl(A^2\bigr), \operatorname{tr}\bigl(A^4\bigr)\bigr]. \tag{50}$$

Let now $y_{ij}, z_{ij} \in k[\mathfrak{g}^{\oplus 2}]$, $1 \leqslant i, j \leqslant 5$, be the standard coordinate functions on $\mathfrak{g}^{\oplus 2}$ given by $y_{ij}((a_{pq}), (b_{rs})) = a_{ij}$, $z_{ij}((a_{pq}), (b_{rs})) = b_{ij}$. Then $y_{ij} = -y_{ji}$ and $z_{ij} = -z_{ji}$. Consider the matrices $B := (y_{ij})$, $C := (z_{ij})$. Taking into account that $\operatorname{tr}(PQ) = \operatorname{tr}(QP)$ for any square matrices P, Q, it is not difficult to deduce that for every $\alpha_1, \alpha_2 \in k$, the following equalities hold:

$$\operatorname{tr}\bigl((\alpha_1 B + \alpha_2 C)^2\bigr) = \alpha_1^2 \operatorname{tr}\bigl(B^2\bigr) + 2\alpha_1\alpha_2 \operatorname{tr}(BC) + \alpha_2 \operatorname{tr}\bigl(C^2\bigr), \tag{51}$$

$$\begin{aligned} \operatorname{tr}\bigl((\alpha_1 B + \alpha_2 C)^4\bigr) = {} & \alpha_1^4 \operatorname{tr}\bigl(B^4\bigr) + 4\alpha_1^3\alpha_2 \operatorname{tr}\bigl(B^3 C\bigr) + 2\alpha_1^2\alpha_2^2\bigl(2\operatorname{tr}\bigl(B^2C^2\bigr) + \operatorname{tr}\bigl((BC)^2\bigr)\bigr) \\ & + 4\alpha_1\alpha_2^3 \operatorname{tr}\bigl(BC^3\bigr) + \alpha_2^4 \operatorname{tr}\bigl(C^4\bigr). \end{aligned} \tag{52}$$

From (50), the definition of $\operatorname{pol}_2 k[\mathfrak{g}]^G$ (see Example 2.5 and the first paragraph right after it), and (51), (52) we deduce that $\operatorname{pol}_2 k[\mathfrak{g}]^G$ is the algebra

$$k\bigl[\operatorname{tr}\bigl(B^2\bigr), \operatorname{tr}(BC), \operatorname{tr}\bigl(C^2\bigr), \operatorname{tr}\bigl(B^4\bigr), \operatorname{tr}\bigl(B^3C\bigr), 2\operatorname{tr}\bigl(B^2C^2\bigr) + \operatorname{tr}\bigl((BC)^2\bigr), \operatorname{tr}\bigl(BC^3\bigr), \operatorname{tr}\bigl(C^4\bigr)\bigr].$$

This shows that the transcendence degree of $\operatorname{pol}_2 k[\mathfrak{g}]^G$ over k is not bigger than 8. On the other hand, since $\dim \mathfrak{g}^{\oplus 2} = \dim G = 20$, the transcendence degree of $k[\mathfrak{g}^{\oplus 2}]^G$ over k is not smaller than $\dim \mathfrak{g}^{\oplus 2} - \dim G = 10$, see, e.g., [12, Theorem 3.3 and the corollary of Lemma 2.4] (actually it is equal to 10 since, as one easily proves, the generic G-stabilizer of the G-module $\mathfrak{g}^{\oplus 2}$ is finite). Therefore $k[\mathfrak{g}^{\oplus 2}]^G$ is not integral over $\operatorname{pol}_2 k[\mathfrak{g}]^G$. By (32) and Corollary 3.5 we now deduce that $\operatorname{pol\,ind}(\mathfrak{g}) = 1$, i.e., $\mathfrak{g}$ is a 2-algebra.

To complete the proof we have to calculate $\operatorname{pol\,ind}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ where $\mathfrak{g}_i = \mathfrak{sl}_2$ for every i. We may assume that $G = G_1 \times \cdots \times G_m$ where $G_i = \mathbf{SL}_2$ for every i. Corollary 3.7 then reduces the proof to the case $m = 1$. Since the $\mathbf{SL}_2$-modules $\mathfrak{sl}_2$ and R_2 are isomorphic, the claim now follows from Theorem 3.15. $\square$

Call a linear subspace L of a reductive Lie algebra $\mathfrak{g}$ *triangularizable* if there is a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that L lies in the unipotent radical of $\mathfrak{b}$ (for $\mathfrak{g} = \mathrm{Mat}_{n\times n}$, this means that L is conjugate to a subspace of the space of upper triangular matrices, see [3,5,9]). Call L *nilpotent* if every element of L is nilpotent.

Corollary 3.17. *A semisimple Lie algebra* $\mathfrak{g}$ *contains a* 2*-dimensional nilpotent nontriangularizable linear subspace if and only if* $\mathfrak{g}$ *is not isomorphic to* $\mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$.

References

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres IV–VI, Hermann, Paris, 1968.

[2] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres VII, VIII, Hermann, Paris, 1975.

[3] A. Causa, R. Re, T. Teodorescu, Some remarks on linear spaces of nilpotent matrices, Matematiche (Suppl.) 53 (1998) 23–32.

[4] H. Derksen, G. Kemper, Computational Invariant Theory, Invariant Theory and Algebraic Transformation Groups, vol. 1, Encyclopaedia Math. Sci., vol. 130, Springer, Berlin, 2002.

[5] M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices I, Amer. J. Math. 80 (1958) 614–622;
M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices II, Duke Math. J. 27 (1960) 21–31;
M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices III, Ann. of Math. (2) 70 (1959) 167–205;
M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices IV, Ann. of Math. (2) 75 (1962) 382–418.

[6] D. Hilbert, Über die vollen Invariantensysteme, Math. Ann. 42 (1893) 313–373.

[7] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge Univ. Press, Cambridge, 1990.

[8] M. Hunziker, Classical invariant theory for finite reflection groups, Transform. Groups 2 (2) (1997) 147–163.

[9] B. Mathes, M. Omladič, H. Radjavi, Linear spaces of nilpotent matrices, Linear Algebra Appl. 149 (1991) 215–225.

[10] D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory, third enlarged ed., Ergeb. Math. Grenzgeb. (2), vol. 34, Springer, 1993.

[11] V. Popov, Generically multiple transitive algebraic group actions, in: Proc. Internat. Colloquium "Algebraic Groups and Homogeneous Spaces", 6–14 January, 2004, Tata Inst. of Fund. Research, Mumbai, in press; math.AG/0409024.

[12] V.L. Popov, E.B. Vinberg, Invariant Theory, Encyclopaedia Math. Sci., vol. 55, Springer, Heidelberg, 1994, pp. 123–284.

[13] H. Radjavi, Letter to V.L. Popov, 24 March 2005.

[14] T.A. Springer, Linear Algebraic Groups, second ed., Progress in Math., vol. 9, Birkhäuser, Boston, 1998.

[15] E. Study, Ber. Sächs. Akad. Wissensch. (1897) 443.

[16] N.R. Wallach, Invariant differential operators on a reductive Lie algebra and Weyl group representations, J. Amer. Math. Soc. 6 (4) (1993) 779–816.

[17] H. Weyl, The Classical Groups. Their Invariants and Representations, Princeton Univ. Press, 1946.