Proceedings of the conference
75 years of Radon transform, Vienna 1992 (S. Gindikin, P. Michor, eds.)
International Press, Cambridge, MA (1994), 249–251

RADON TRANSFORM AND CURVATURE

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Abstract. We interpret the setting for a Radon transform as a submanifold of the space of generalized functions, and compute its extrinsic curvature: it is the Hessian composed with the Radon transform.

1. The general setting. Let M and Σ be smooth finite dimensional manifolds. Let $m = \dim(M)$. A linear mapping $R : C_c^\infty(M) \to C^\infty(\Sigma)$ is called a (generalized) Radon transform if it is given in the following way: To each point $y \in \Sigma$ there corresponds a submanifold Σ_y of M and a density μ_y on Σ_y, and the operator R is given by

$$R(f)(y) := \int_{\Sigma_y} f(x)\mu_y(x).$$

We will express this situation in the following way.

Let $\mathcal{D}(M) := C_c^\infty(M)$ be the space of smooth functions with compact support on M, and let $\mathcal{D}'(M) = C_c^\infty(M)'$ be the locally convex dual space. Note that the space $C^\infty(|\Lambda^m|(M))$ of smooth densities on M is canonically contained and dense in $\mathcal{D}'(M)$.

Now suppose that we are given a smooth mapping $\sigma : \Sigma \to \mathcal{D}'(M)$. By the smooth uniform boundedness principle (see [Frölicher, Kriegl, p. 73] or [Kriegl, Michor, 4.11]) the mapping $\sigma : \Sigma \to L(\mathcal{D}(M), \mathbb{R})$ is smooth if and only if the composition with the evaluation $\mathrm{ev}_f : L(\mathcal{D}(M), \mathbb{R}) \to \mathbb{R}$ is smooth for each $f \in \mathcal{D}(M)$, i.e. $R_\sigma(f) : \Sigma \to \mathbb{R}$ is smooth for each f. Then we have an associated *Radon transform* given by

$$R_\sigma(f)(y) := \langle \sigma(y), f \rangle.$$

Clearly the Radon transform $R_\sigma : C_c^\infty(M) \to C^\infty(\Sigma)$ is injective if and only if the subset $\sigma(\Sigma) \subset \mathcal{D}'(M)$ separates points on $C_c^\infty(M)$, and the kernel of R_σ is the annihilator of $\sigma(\Sigma)$ in $C_c^\infty(M)$. We will assume in the sequel that $\sigma : \Sigma \to \mathcal{D}'(M)$ is an embedding of a smooth finite dimensional embedded submanifold of the locally convex vector space $\mathcal{D}'(M)$, but the Radon transform itself is defined also in the more general setting of a smooth mapping.

All examples of Radon transforms mentioned in these proceedings fit into the setting explained above. A trivial example is the Dirac embedding $\delta : M \to \mathcal{D}'(M)$

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associating to each point $x \in M$ the Dirac measure δ_x at that point. It's associated Radon transform is the identity for functions on M, but it's curvature (see below) is quite interesting.

2. Curvature. We now give the definition of the *second fundamental form* or the *extrinsic curvature* of a finite dimensional submanifold Σ of the locally convex space $\mathcal{D}'(M)$. Since we do not want to assume the existence of an inner product on (a certain subspace of) $\mathcal{D}'(M)$ we consider the *normal bundle* $N(\Sigma) := (T\mathcal{D}'(M)|\Sigma)/T\Sigma$ and the canonical projection $\pi : T\mathcal{D}'(M)|\Sigma \to N(\Sigma)$ of vector bundles over Σ. The linear structure of $\mathcal{D}'(M)$ gives us the obvious flat covariant derivative $\nabla_X Y$ of two vector fields X, Y on $\mathcal{D}'(M)$, which is defined by $(\nabla_X Y)(\varphi) = dY(\varphi).X(\varphi)$. For (local) vector fields $X, Y \in \mathfrak{X}(\mathcal{D}'(M))$ on $\mathcal{D}'(M)$ which along Σ are tangent to Σ we consider the section $S(X,Y)$ of $N(\Sigma)$ which is given by $S(X,Y) = \pi(\nabla_X Y)$. This section depends only on $X|\Sigma$ and $Y|\Sigma$, since we may consider the flow $\operatorname{Fl}_t^{X|\Sigma}$ of the vector field $X|\Sigma$ on the finite dimensional manifold Σ and we have $(\nabla_X Y)|\Sigma = \frac{d}{dt}|_{t=0} Y \circ \operatorname{Fl}_t^{X|\Sigma}$. Here we consider just the smooth mapping $Y : \mathcal{D}'(M) \to \mathcal{D}'(M)$. Obviously $S(X,Y)$ is $C^\infty(M)$-linear in X, and it is symmetric since $S(X,Y) - S(Y,X) = \pi(dY.X - dX.Y) = \pi([X,Y]) = 0$. So the *second fundamental form* or the *extrinsic curvature* of the submanifold Σ of $\mathcal{D}'(M)$ is given by

$$
\begin{aligned}
S : T\Sigma \times_\Sigma T\Sigma &\to N(\Sigma). \\
S(X,Y) = \pi(\nabla_X Y) &\text{ for } X, Y \in \mathfrak{X}(\Sigma).
\end{aligned}
$$

For $y \in \Sigma$ the convenient vector space $N_y(\Sigma) = \mathcal{D}'(M)/T_y\Sigma$ is the dual space of the closed linear subspace $\{f \in \mathcal{D}(M) : \langle T_y\sigma.X, f\rangle = 0 \text{ for all } X \in T_y\Sigma\}$.

3. Theorem. *Let $\sigma : \Sigma \to \mathcal{D}'(M)$ be a smooth embedding of a finite dimensional smooth manifold Σ into the space of distributions on a manifold M, and let $R_\sigma : C_c^\infty(M) \to C^\infty(\Sigma)$ be the associated Radon transform. Then the extrinsic curvature of $\sigma(\Sigma)$ in $\mathcal{D}'(M)$ is the Hessian composed with the Radon transform in the sense explained in the proof.*

Proof. Since $\sigma(\Sigma)$ is an embedded submanifold of finite dimension in $\mathcal{D}'(M)$, it is also splitting, and thus for each vector field $X \in \mathfrak{X}(\Sigma)$ there exists a (local) smooth extension $\tilde{X} \in \mathfrak{X}(\mathcal{D}'(M))$. It is not known whether $\mathcal{D}'(M)$ admits smooth partitions of unity. The space $C_c^\infty(M)$ of test functions admits smooth partitions of unity, see [Kriegl, Michor]. So we have $T\sigma \circ X = \tilde{X} \circ \sigma$.

For $y \in \Sigma$ the normal space $N_y(\Sigma) = \mathcal{D}'(M)/T_y\sigma(T_y\Sigma)$ is the dual space of the annihilator of $T_y\sigma(T_y\Sigma)$ in $C_c^\infty(M)$. A test function $f \in C_c^\infty(M)$ is in this annihilator if and only if $\langle T_y\sigma.X, f\rangle = 0$ for all $X \in T_y\Sigma$. Let us choose a smooth curve $c : \mathbb{R} \to \Sigma$ with $c(0) = y$ and $c'(0) = X$. Then we have

$$
\begin{aligned}
\langle T_y\sigma.X, f\rangle &= \langle \tfrac{d}{dt}|_0 \sigma(c(t)), f\rangle = \tfrac{d}{dt}|_0 \langle \sigma(c(t)), f\rangle \\
&= \tfrac{d}{dt}|_0 R_\sigma f(c(t)) = d(R_\sigma f)_y(X).
\end{aligned}
$$

So we have $N_y(\Sigma) = \{f \in C_c^\infty(M) : d(R_\sigma f)_y = 0\}'$.

Now we will compute the extrinsic curvature. Let $X, Y \in \mathfrak{X}(\Sigma)$ be vector fields, let $\tilde{X}, \tilde{Y}$ be smooth extensions to $\mathcal{D}'(M)$, let $y \in \Sigma$, and choose $f \in C_c^\infty(M)$ with $d(R_\sigma f)_y = 0$. Then we have

$$\begin{aligned}
\langle S(X,Y)(y), f\rangle &= \langle (\nabla_{\tilde{X}}\tilde{Y})(\sigma(y)), f\rangle \\
&= \langle d\tilde{Y}(\sigma(y)).\tilde{X}(\sigma(y)), f\rangle \\
&= \langle d\tilde{Y}(\sigma(y)).d\sigma(y).X(y), f\rangle \\
&= \langle d(\tilde{Y}\circ\sigma)(y).X(y), f\rangle \\
&= \langle d(d\sigma.Y)(y).X(y), f\rangle, \\
Y(R_\sigma f) &= d(R_\sigma f).Y = \tfrac{d}{dt}|_0 R_\sigma f \circ \mathrm{Fl}^Y_t \\
&= \tfrac{d}{dt}|_0 \langle \sigma\circ \mathrm{Fl}^Y_t, f\rangle = \langle d\sigma.Y, f\rangle, \\
XY(R_\sigma f)(y) &= \tfrac{d}{dt}|_0 (Y(R_\sigma f))(\mathrm{Fl}^X_t(y)) = \tfrac{d}{dt}|_0 \langle (d\sigma.Y)(\mathrm{Fl}^X_t(y)), f\rangle \\
&= \langle d(d\sigma.Y).X(y), f\rangle = \langle S(X,Y)(y), f\rangle.
\end{aligned}$$

So $\langle S(X,Y)(y), f\rangle$ is the Hessian of $R_\sigma f$ at y applied to $(X(y), Y(y))$. □

References

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