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LIFTING SMOOTH CURVES OVER INVARIANTS FOR REPRESENTATIONS OF COMPACT LIE GROUPS

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Abstract. We show that one can lift locally real analytic curves from the orbit space of a compact Lie group representation, and that one can lift smooth curves even globally, but under an assumption.

1. Introduction

In [1] we investigated the following problem. Let $P(t) = x^n - \sigma_1(t)x^{n-1} + \dots + (-1)^n\sigma_n(t)$ be a polynomial with all roots real, smoothly parameterized by t near 0 in $\mathbb{R}$. Can we find n smooth functions $x_1(t), \dots, x_n(t)$ of the parameter t defined near 0, which are the roots of $P(t)$ for each t? We showed that this is possible under quite general conditions: real analyticity or no two roots should meet of infinite order. Some applications to perturbations of unbounded operators in Hilbert space are also given in [1].

This problem can be reformulated in the following way. Let the symmetric group S_n act in $\mathbb{R}^n$ by permuting the coordinates (the roots), and consider the polynomial mapping $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{R}^n \to \mathbb{R}^n$ whose components are the elementary symmetric polynomials (the coefficients). Given a smooth curve $c : \mathbb{R} \to \sigma(\mathbb{R}^n) \subset \mathbb{R}^n$, is it possible to find a smooth lift $\bar{c} : \mathbb{R} \to \mathbb{R}^n$ with $\sigma \circ \bar{c} = c$?

In this paper we tackle the following generalization of this problem. Consider an orthogonal representation of a compact Lie group G on a real vector space V. Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. Then the mapping $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$ defines a bijection of the orbit space V/G to the semialgebraic set $\sigma(V) \subseteq \mathbb{R}^n$. A curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space V/G is called smooth if it is smooth as a curve in $\mathbb{R}^n$. This is well defined, i.e., does not depend on the choice of generators.

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Problem. Given a smooth curve $c : \mathbb{R} \to V/G$ in the orbit space, does there exist a smooth lift to V, i.e., a smooth curve $\bar{c} : \mathbb{R} \to V$ with $c = \sigma \circ \bar{c}$?

Our main results are the following. We show that

(1) A real analytic curve $c : \mathbb{R} \to V/G$ admits a real analytic lift $\bar{c} : \mathbb{R} \to V$, at least locally.
(2) A smooth curve c admits a global smooth lift $\bar{c}$ if it satisfies the genericity conditions 3.5.
(3) If the representation of G on V is polar then a real analytic curve or a smooth one with the genericity conditions admits an orthogonal lift $\bar{c}$, i.e., a lift meeting orbits orthogonally, which is unique up to a transformation from G.

For a general representation we did not succeed to prove the existence of an orthogonal lift and we suspect that it is not true in general.

Note that in [1], 7.4 we showed the existence of a smooth curve in the orbit space of a (polar) representation which admits a smooth lift but no orthogonal lift. Similar lifting problems have been treated for smooth homotopies in [12].

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2. Preliminaries

2.1. The setting. Let G be a compact Lie group and $\rho : G \to O(V)$ an orthogonal representation in a real finite dimensional Euclidean vector space V with an inner product $\langle \; | \; \rangle$. By a classical theorem of Hilbert and Nagata the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \ldots, d_n$. We may assume that $\sigma_1 = \langle v|v \rangle$ is the Euclidean metric. Consider the *orbit map* $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. Note that if $(y_1, \ldots, y_n) = \sigma(v)$ for $v \in V$, then $(t^{d_1} y_1, \ldots, t^{d_n} y_n) = \sigma(tv)$ for $t \in \mathbb{R}$. The image $\sigma(V)$ is a semialgebraic set in the categorical quotient $V/\!\!/G := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$ where I is the ideal of relations between $\sigma_1, \ldots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, thus it induces a homeomorphism between V/G and $\sigma(V)$.

2.2. Description of $\sigma(V)$.

Let $\langle \; | \; \rangle$ denote also the G-invariant dual inner product on V^*. The differentials $d\sigma_i : V \to V^*$ are G-equivariant, and the polynomials $v \mapsto \langle d\sigma_i(v) | d\sigma_j(v) \rangle$ are in $\mathbb{R}[V]^G$ and are entries of an $n \times n$ symmetric matrix valued polynomial

$$B(v) := \begin{pmatrix} \langle d\sigma_1(v)|d\sigma_1(v)\rangle & \ldots & \langle d\sigma_1(v)|d\sigma_n(v)\rangle \\ \vdots & \ddots & \vdots \\ \langle d\sigma_n(v)|d\sigma_1(v)\rangle & \ldots & \langle d\sigma_n(v)|d\sigma_n(v)\rangle \end{pmatrix}.$$

There is a unique matrix valued polynomial $\tilde{B}$ on $V/\!\!/G$ such that $B = \tilde{B} \circ \sigma$. Denote by $\tilde{b}_{ij}$ the entries of the matrix $\tilde{B}$.

For a real symmetric matrix A let $A \geq 0$ indicate that A is positive semidefinite.

Theorem. (Procesi–Schwarz [8]) $\sigma(V) = \{z \in V/\!\!/G : \tilde{B}(z) \geq 0\}$.

For each $1 \leq i_1 < \cdots < i_s \leq n, 1 \leq j_1 < \cdots < j_s \leq n, (s \leq n)$ consider the matrix with entries $\langle d\sigma_{i_p} | d\sigma_{j_q} \rangle$ for $1 \leq p, q \leq s$, a minor of B. Denote its determinant by $\Delta^{j_1,\dots,j_s}_{i_1,\dots,i_s}$. Since $\Delta^{j_1,\dots,j_s}_{i_1,\dots,i_s}$ is a G-invariant polynomial on V there is a unique polynomial $\tilde{\Delta}^{j_1,\dots,j_s}_{i_1,\dots,i_s}$ on $V/\!\!/G$ such that $\Delta^{j_1,\dots,j_s}_{i_1,\dots,i_s} = \tilde{\Delta}^{j_1,\dots,j_s}_{i_1,\dots,i_s} \circ \sigma$.

2.3. The slice theorem. For a point $v \in V$ we denote by G_v its isotropy group and by $N_v = T_v(G.v)^\perp$ the normal subspace of the orbit $G.v$ at v. It is well known that there exists a G-invariant neighborhood U of v which is real analytically G-isomorphic to the crossed product (or associated bundle) $G \times_{G_v} S_v = (G \times S_v)/G_v$, where S_v is a ball in N_v with center at the origin. The quotient U/G is homeomorphic to S_v/G_v.

More precisely, $G \times_{G_v} N_v$ carries the structure of an affine real algebraic variety as the categorical (and geometrical) quotient $(G \times N_v)/\!\!/G_v$ with respect to the action $G_v : G \times N_v$ defined by $h(g, x) = (gh^{-1}, hx)$. Denote by $[g, x]$ the point of $G \times_{G_v} N_v$ represented by the pair $(g, x) \in G \times N_v$. The group G acts on $G \times_{G_v} N_v$ via left multiplication of the first component. There is a G-equivariant polynomial map $\phi : G \times_{G_v} N_v \longrightarrow V$, $[g, x] \mapsto g(v + x)$. It induces a polynomial map $\psi : (G \times_{G_v} N_v)/\!\!/G \longrightarrow V/\!\!/G$ mapping $(G \times_{G_v} N_v)/G$ into V/G.

The G-equivariant embedding $\alpha : N_v \hookrightarrow G \times_{G_v} N_v$, $x \mapsto [e, x]$, induces an isomorphism $\beta : N_v/\!\!/G_v \xrightarrow{\sim} (G \times_{G_v} N_v)/\!\!/G$ mapping N_v/G_v onto $(G \times_{G_v} N_v)/G$.

Set $\eta = \phi \circ \alpha$ (so $\eta(x) = v + x$) and $\theta = \psi \circ \beta$. We have the following commutative diagram

$$\begin{array}{ccccc} N_v & \xrightarrow{\tau} & N_v/G_v & \subset & N_v/\!\!/G_v \\ \eta\downarrow & & \downarrow & & \downarrow\theta \\ V & \xrightarrow[\sigma]{} & V/G & \subset & V/\!\!/G \end{array}$$

where τ is the orbit map for the action $G_v : N_v$.

Theorem. (Cf. [7], [11]) 1) *There is a ball $S_v \subset N_v$ centered at 0 such that the restriction of ϕ to $G \times_{G_v} S_v$ is an analytic G-isomorphism onto a (G-invariant) neighbourhood of v in V.*

2) *The map θ is a local analytic isomorphism at 0.*

Obviously θ induces a local homeomorphism of N_v/G_v and V/G.

It follows that the problem of local lifting curves in V/G passing through $\sigma(v)$ reduces to the same problem for curves in N_v/G_v passing through 0.

A point $v \in V$ (and its orbit $G.v \in V/G$) is called regular if the representation of G_v in the normal space N_v is trivial. Hence a neighborhood of this point is analytically G-isomorphic to $G/G_v \times S_v$. The set V_{reg} of regular points is open and dense in V, and the projection $V_{\text{reg}} \to V_{\text{reg}}/G$ is a locally trivial fiber bundle. A nonregular orbit or point is called singular.

2.4. Theorem. [11] *For $v \in V$, let $N_v^{G_v}$ be the subspace of G_v-invariant vectors of N_v. Then $\operatorname{grad}\sigma_1(v), \dots, \operatorname{grad}\sigma_n(v)$ span $N_v^{G_v}$ as a real vector space.*

2.5. Stratification of the orbit space.

Let G_v be the isotropy group of $v \in V$ and (G_v) the conjugacy class of G_v which is called the type of the orbit $G.v$. The union V_H of orbits of type (H), where H is a subgroup of G, is called an isotropy stratum of the representation ρ and the image $\sigma(V_H)$ is called an isotropy stratum of $V/G = \sigma(V)$. It is known (see, for example, [2] or [12]), that the isotropy strata of $\sigma(V)$ are real analytic manifolds and their collection gives a stratification of $\sigma(V)$. All the regular points of $\sigma(V)$ constitute a single stratum, called the principal one.

It follows from 2.4 that the dimension of the stratum of V/G of type (G_v) equals $\dim N_v^{G_v} = \operatorname{rk} d\sigma(v) = \operatorname{rk} B(v) = \operatorname{rk}\ \tilde{B}(\sigma(v))$.

Note that by 2.3 the stratification of V/G in a neighborhood of each $\sigma(v) \in V/G$ is naturally isomorphic to the stratification of N_v in a neighborhood of 0.

3. Local lifting curves over invariants

3.1. Lemma. Lifting at regular orbits. *A smooth (real analytic) curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ admits a smooth (real analytic) orthogonal lift $\bar{c}$ in a neigbourhood of a regular point $c(t_0) \in V_{\text{reg}}/G$. It is unique up to a transformation from G.*

Proof. The orthogonal distribution $V_{\text{reg}} \ni v \mapsto N_v$ of the fiber bundle $\pi : V_{\text{reg}} \to V_{\text{reg}}/G$ defines a real analytic Ehresmann connection in π. A local orthogonal lift of the curve c is the same as a horizontal lift with respect to this connection, near t_0. See [5], section 9. □

To lift a curve in the neighborhood of a singular point we shall need two lemmas.

3.2. Removing fixed points. Let V^G be the space of G-invariant vectors, and let V' be its orthogonal subspace in V. Then $V = V^G \oplus V'$, $\mathbb{R}[V]^G = \mathbb{R}[V^G] \otimes \mathbb{R}[V']^G$, and $V/G = V^G \times V'/G$. We need the following obvious lemma.

Lemma. *Any lift of a smooth (real analytic) curve $c = (c_0, c_1)$ in $V^G \times V'/G$ has the form $\bar{c} = (c_0, \bar{c}_1)$, where $\bar{c}_1$ is a smooth (real analytic) lift of c_1 to V'. The lift $\bar{c}$ is orthogonal if and only if the lift $\bar{c}_1$ is orthogonal.*

3.3. For a smooth function f defined near 0 in $\mathbb{R}$ let the *multiplicity* or *order of flatness* $m(f)$ at 0 be the supremum of all integer p such that $f(t) = t^p g(t)$ near 0 for a smooth function g. If $m(f) < \infty$ then $f(t) = t^{m(f)} g(t)$ where now $g(0) \neq 0$. A nonzero function f is called *flat* if $m(f) = \infty$. Similarly one can define a function flat at $t \in \mathbb{R}$.

Let $c = (c_1, \ldots, c_n)$ be a smooth curve in $\sigma(V) \subseteq \mathbb{R}^n$ with $c(0) = 0$. Since $c(t) \geqslant 0$ for all t, we have $m(c_1) = 2r > 0$, where $r \in \mathbb{N}$ or $r = \infty$.

Multiplicity Lemma. *We have $m(c_i) \geq r d_i$ $(1 \leq i \leq n)$.*

Proof. Suppose that for some $k > 1$ we have $m(c_k) < r d_k$. Then $m := \min\{m(c_1)/d_1, \ldots, m(c_n)/d_n\} < r$. We consider the following continuous curve in $\mathbb{R}^n$ for $t \geq 0$: $c_{(m)}(t) := (t^{-2m} c_1(t), t^{-d_2 m} c_2(t), \ldots, t^{-d_n m} c_n(t))$. By 2.1, $c_{(m)}(t) \in \sigma(V)$ for $t > 0$, and since $\sigma(V)$ is closed in $\mathbb{R}^n$ by 2.2, also $c_{(m)}(0) \in \sigma(V)$. Since $m < r$ the first coordinate of $c_{(m)}(t)$ vanishes at $t = 0$. Then $\sigma^{-1}(c_{(m)}(0)) = \{0\}$ and therefore $c_{(m)}(0) = 0$. In particular, for those k with $c(m_k) = m d_k$ we get a contradiction. □

If $r<\infty$, one can consider the curve

$$c_{(r)}(t)=(t^{-2r}c_1(t),t^{-d_2 r}c_2(t),\dots,t^{-d_n r}c_n(t))\in\sigma(V).$$

We have $c_{(r)}(0)\neq 0$. If $c_{(r)}$ is liftable at 0 and $\overline{c_{(r)}}$ is its smooth (real analytic) lift, then $\bar c(t):=t^r\overline{c_{(r)}}(t)$ is a smooth (real analytic) lift of c. If $\overline{c_{(r)}}$ is an orthogonal lift, then also $\bar c$, and conversely, since the action of G commutes with homotheties of V. Moreover the orthogonal lift of c is uniquely determined up to the action of a constant element in G if and only if the orthogonal lift of $c_{(r)}$ has this property.

3.4. Theorem. Local real analytic lifts. *Let $c:\mathbb{R}\to\sigma(V)\subset\mathbb{R}^n$ be a real analytic curve. Then there exists a real analytic lift $\bar c$ in V of c, locally near each $t\in\mathbb{R}$.*

Proof. We show that there exist lifts of c locally near each point $t_0\in\mathbb{R}$, without loss $t_0=0$, through any $v\in\sigma^{-1}(c(0))$. We do this by the following algorithm in 4 steps, which always stops, since each step either gives a local lift, or reduces the lifting problem to a smaller group.

Step 1. If $c(0)\neq 0$ corresponds to a regular orbit, unique orthogonal real analytic lifts exist through all $v\in\sigma^{-1}(c(0))$, by 3.1.

Step 2. If $V^G\neq 0$ we remove fixed points by 3.2.

Step 3. If $V^G=0$, $c(0)\neq 0$ corresponds to a singular orbit, we consider the isotropy representation $G_v\to O(N_v)$ with the orbit map $\tau:N_v\to\mathbb{R}^m$ for $v\in V$ such that $\sigma(v)=c(0)$. By Theorem 2.3 the lifting problem reduces to the smaller (since $V^G=0$) group G_v, acting on N_v.

If $N_v^{G_v}\neq 0$ we can continue in step 2. If $N_v^{G_v}=0$ we can continue in step 4.

Step 4. If $V^G=0$ and $c(0)=0$ then $m(c_1)=2r$ for some integer $r\geq 1$ or $r=\infty$. In the latter case $c_1=0$. This implies that $c=0$ is constant, which clearly can be lifted. In the former case by the multiplicity lemma 3.3 we have $m(c_i)\geq rd_i$ and the lifting problem reduces to the curve $c_{(r)}$ (see 3.3), for which $c_{(r)}(0)\neq 0$. Now we can continue in step 1, step 2, or step 3. □

3.5. Genericity conditions.

Let s be a nonnegative integer. Denote by A_s the union of all the strata S of V/G with $\dim S\leqslant s$, and by I_s the ideal of $\mathbb{R}[V/\!\!/G]=\mathbb{R}[V]^G$ consisting of polynomials vanishing on A_{s-1}.

Let $c:\mathbb{R}\to\sigma(V)\subset\mathbb{R}^n$ be a smooth curve, $t\in\mathbb{R}$, and $s=s(c,t)$ a minimal integer such that for a neighborhood J of t we have $c(J)\subset A_s$. The curve c is *normally nonflat at t* if there is $f\in I_s$ such that $f\circ c$ is nonflat at t, i.e., the Taylor series of $f\circ c$ at t is not identically zero. This automatically holds if $c(t)\notin A_{s-1}$.

A smooth curve $c:\mathbb{R}\to\sigma(V)\subset\mathbb{R}^n$ is called *generic* if c is normally nonflat at t for each $t\in\mathbb{R}$. A real analytic curve is automatically generic.

Proposition. *If a smooth curve $c:\mathbb{R}\to\sigma(V)$ is normally nonflat at $t\in\mathbb{R}$, curves which are obtained from it in the above reduction process, i.e., removing of fixed points, passing to the slice representation or replacing c by the curve $c_{(r)}$ (see 3.3), are normally nonflat at t as well.*

Proof. Let $V^G\neq 0$. In the notation of 3.2, each stratum S of V/G has the form $V^G\times S_1$, where S_1 is a stratum of V'/G.

Let $c = (c_0, c_1)$ be a smooth curve in $\sigma(V)$. If $f \in I_s$ is a function such that $f \circ c$ is nonflat at t, then $f = \sum_i \phi_i \otimes f_i$, where $\phi_i \in \mathbb{R}[V^G]$, $f_i \in I'_{s-k}$ (the ideal of $\mathbb{R}[V']^G$ consisting of polynomials vanishing on all strata of V'/G of dimension $< s - k$), and $f_i \circ c_1$ is nonflat at t for some i.

If $V^G = 0$ and $c(t) \neq 0$ the statement of the proposition follows from 2.4 and 2.5 since the notion of normal nonflatness is local.

Let $V^G = 0$, $c(t) = 0$, $s = s(c,t)$, and $f \in I_s$ be such that $f \circ c$ is nonflat at t. We may suppose that $t = 0$ and f is homogeneous. Then the function $f \circ c_{(r)}$ is nonflat at 0. $\square$

3.6. Theorem. *Let $c : \mathbb{R} \to \sigma(V) \subset \mathbb{R}^n$ be a smooth curve. Then c is normally nonflat at $t \in \mathbb{R}$ if*

(1) *The functions $\tilde{\Delta}^{j_1,\dots,j_s}_{i_1,\dots,i_s} \circ c$ vanish in a neighborhood of t whenever $s > r$.*
(2) *There exists a minor $\tilde{\Delta}^{j_1,\dots,j_r}_{i_1,\dots,i_r}$ such that $\tilde{\Delta}^{j_1,\dots,j_r}_{i_1,\dots,i_r} \circ c$ is nonflat at t.*

Proof. By 2.5, $r = s(c,t)$ and $\tilde{\Delta}^{j_1,\dots,j_r}_{i_1,\dots,i_r} \in I_r$. $\square$

This theorem gives the best practical way to check the normal nonflatness of a curve c.

3.7. Theorem. Local smooth lifts. *Let $c : \mathbb{R} \to \sigma(V) \subset \mathbb{R}^n$ be a smooth curve which is normally nonflat at $t_0 \in \mathbb{R}$. Then there exists a smooth lift $\bar{c}$ in V of c, locally near t_0.*

Proof. The proof is the same as one of Theorem 3.4 since by Proposition 3.5 one can use the normal nonflatness of c at t_0 instead of the analyticity of c. $\square$

3.8. Lemma. *Let $c : \mathbb{R} \to \sigma(V) \subset \mathbb{R}^n$ be a smooth curve which is normally nonflat at t_0. Suppose that $\bar{c}_1, \bar{c}_2 : I \to V$ are smooth lifts of c on an open interval I containing t_0. Then there exists a smooth curve g in G defined near t_0 such that $\bar{c}_1(t) = g(t).\bar{c}_2(t)$ for all t near t_0. The real analytic version of this result is also true.*

Proof. We prove this by induction on the size (dimension, and number of connected components in the case of the same dimension) of G and use Proposition 3.5 in each step of the next induction process.

Without loss let $t_0 = 0$ and $\bar{c}_1(0) = \bar{c}_2(0)$.

Step 1. If $V^G \neq 0$ we remove the fixed points by 3.2.

Step 2. Let $V^G = 0$ and $c(0) = 0$. If $c(t) \equiv 0$, the statement is trivial. If $c(t) \not\equiv 0$, then $r = m(c) < \infty$ since c is normally nonflat at 0, and $t^{-r}\bar{c}_1(t)$, $t^{-r}\bar{c}_2(t)$ are smooth lifts of $c_{(r)}$. If we can find $g(t) \in G$ taking $t^{-r}\bar{c}_2(t)$ to $t^{-r}\bar{c}_1(t)$, then we also have $g(t).\bar{c}_2(t) = \bar{c}_1(t)$. Thus we may assume that $c(0) \neq 0$.

Step 3. If $V^G = 0$ and $c(0) \neq 0$, then for a normal slice S_v at $v = \bar{c}_1(0)$ we know that $p : G.S_v \cong G \times_{G_v} S_v \to G/G_v \cong G.v$ is the projection of a fiber bundle associated to the principal bundle $G \to G/G_v$. Then $p \circ \bar{c}_1$ and $p \circ \bar{c}_2$ are two smooth curves in G/G_v defined near $t = 0$, which admit smooth lifts g_1 and g_2 into G (via the horizontal lift of a principal connection, say), and $t \mapsto g_j(t)^{-1}.\bar{c}_j(t)$ are two smooth curves in S_v, lifts of c. Thus we reduced our problem to the smaller group G_v. If v is a regular point then G_v acts trivially on N_v and these two lifts are automatically the same. If v is a singular point and $N_v^{G_v} \neq 0$ we apply step 1. If v is a singular point and $N_v^{G_v} = 0$ we apply step 2.

In the real analytic situation the proof is the same: one has to use a real analytic principal connection in step 3. □

4. Global lifting curves over invariants

4.1. Theorem. Global smooth lifts. *Let $c : \mathbb{R} \to \sigma(V) \subset \mathbb{R}^n$ be a generic smooth curve. Then there exists a global smooth lift $\bar{c} : \mathbb{R} \to V$ with $\sigma \circ \bar{c} = c$.*

Proof. By 3.7 there exist local smooth lifts near any $t \in \mathbb{R}$. It is sufficient to prove that each local smooth lift of c defined on an open interval I can be extended to a larger interval whenever $I \neq \mathbb{R}$.

Suppose $\bar{c}_1 : I \to V$ is a local smooth lift of c, the open interval I is bounded from above (say), and t_0 is its upper boundary point. By 3.7 there exists a local smooth lift $\bar{c}_2$ of c near t_0, and a $t_1 < t_0$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_1. By Lemma 3.8 there exists a smooth curve g in G, locally defined near t_1, such that $\bar{c}_1(t) = g(t).\bar{c}_2(t)$. We consider the right logarithmic derivative $X(t) = g'(t).g(t)^{-1} \in \mathfrak{g}$ and choose a smooth function $\chi(t)$ which is 1 for $t \leq t_1$ and becomes 0 before g ceases to exist. Then $Y(t) = \chi(t)X(t)$ is smooth and defined near $[t_1, \infty)$. The differential equation $h'(t) = Y(t).h(t)$ with initial condition $h(t_1) = g(t_1)$ then has a solution h in G defined near $[t_1, \infty)$ which coincides with g below t_1. Then $\bar{c}(t) := \bar{c}_1(t)$ for $t \leq t_1$ and $\bar{c}(t) := h(t).\bar{c}_2(t)$ for $t \geq t_1$ is a smooth lift of c on a larger interval. □

4.2. Theorem. Polar representations. *Let $\rho : G \to O(V)$ be a polar orthogonal representation of a compact Lie group G (see* [3], [4]*) and $\sigma : V \to \mathbb{R}^n$ the corresponding orbit map. Let $c : \mathbb{R} \to \sigma(V) \subset \mathbb{R}^n$ be a curve which is either real analytic, or smooth but generic. Then there exists a global orthogonal real analytic or smooth lift $\bar{c} : \mathbb{R} \to V$ with $\sigma \circ \bar{c} = c$ which is unique up to the action of a constant in G.*

A representation is *polar* if there exists a linear subspace $\Sigma \subset V$, called a *section* or a *Cartan subspace*, which meets each orbit orthogonally. See [3], [4], and [9]. The trace of the G-action is the action of the *generalized Weyl group* $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ on Σ, which is a finite group, and is a reflection group for connected G. We shall also need the following generalization of the Chevalley restriction theorem, which is due to Dadok and Kac [4] and independently to C. L. Terng, see [8], 4.12, or [13], theorem D: *For a polar representation the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^{W(\Sigma)}$ of Weyl group-invarint polynomials on Σ, via restriction.*

Proof. Let Σ be a section. By the above theorem $\sigma|\Sigma : \Sigma \to \mathbb{R}^n$ is the orbit map for the representation $W = W(\Sigma) \to O(\Sigma)$. If c is a smooth curve satisfying the assumption of the theorem, by Theorem 4.1 there exists a global lift $\bar{c} : \mathbb{R} \to \Sigma$, which as curve in V is orthogonal to each G-orbit it meets, by the properties of Σ. Note for further use that $\bar{c}$ is nowhere flat, since otherwise the curve c is not generic at some t.

If c is real analytic there are local lifts over $\sigma|\Sigma$ into Σ by Theorem 3.4. We claim that these local lifts are unique up to the action of a constant element in

W. Namely, let $\bar{c}_1$ and $\bar{c}_2$ be real analytic lifts defined on an interval I. Choose a convergent sequence $t_i \in I$ and elements $\alpha_i \in W$ with $\alpha_i.\bar{c}_1(t_i) = \bar{c}_2(t_i)$. Since W is finite, by passing to a subsequence we may assume that all $\alpha_i = \alpha \in W$. But then the real analytic curves $\bar{c}_2$ and $\alpha.\bar{c}_1$ coincide on a converging sequence, so they coincide on the whole interval. Thus we can glue the local lifts to a global real analytic lift $\bar{c}$ in Σ, which as curve in V is an orthogonal lift.

It remains to show that for two orthogonal lifts $\bar{c}_1, \bar{c}_2 : \mathbb{R} \to V$ of c there is a constant element $g \in G$ with $\bar{c}_1(t) = g.\bar{c}_2(t)$ for all t. We may assume that $\bar{c}_1$ lies in a section Σ, by the first assertion.

Since c is generic, $\bar{c}_1$ meets each stratum of V only in isolated points if it is not entirely contained in this stratum. Since the isotropy group of each point of Σ contains the isotropy groups of all sufficiently close points of Σ, it follows that for an open dense subset of $t \in \mathbb{R}$ the group $G_{\bar{c}_1(t)}$ is the same, say, H, and $H \subset G_{\bar{c}_1(t)}$ for any $t \in \mathbb{R}$.

From Lemma 3.8 we get that $\bar{c}_1(t) = g(t).\bar{c}_2(t)$ for some smooth or real analytic curve $g : I \to G$, locally near each t_0. We consider the right logarithmic derivative $X(t) := g'(t).g(t)^{-1} \in \mathfrak{g}$. Differentiating $\bar{c}_1(t) = g(t).\bar{c}_2(t)$ we get $\bar{c}_1'(t) - g(t).\bar{c}_2'(t) = X(t).g(t).\bar{c}_2(t) = X(t).\bar{c}_1(t)$, where the left hand side is orthogonal to the orbit through $\bar{c}_1(t)$, and the right hand side is tangential to it, so both sides are zero and $X(t)$ lies in the isotropy Lie algebra $\mathfrak{g}_{\bar{c}_1(t)}$ for each t, and hence $X(t)$ lies in the Lie algebra of H. But then $g(t)$ lies in a right coset of H. Obviously, this coset must be the same, say Hg, for all t_0 and hence $\bar{c}_1(t) = g\bar{c}_2(t)$ for all $t \in \mathbb{R}$. □

References

1. Alekseevky, D.; Kriegl, A.; Losik, M.; Michor; P. W., *Choosing roots of polynomials smoothly*, Israel J. Math. (1998), 203–233.
2. Bierstone, E., *Lifting isotopies from orbit space*, Topology **14** (1975), 245–252.
3. Dadok, J., *Polar coordinates induced by actions of compact Lie groups*, TAMS **288** (1985), 125–137.
4. Dadok, J.; Kac, V., *Polar representations*, J. of Algebra **92** (1985), 504–524.
5. Kolář, I.; Michor, Peter W.; Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin Heidelberg New York, 1993.
6. Łojasewicz, S., *Triangulation of semi-analytic sets*, Ann. Sc. Norm. Sup. Pisa, ser III **1834** (1964), 449–473.
7. Luna, D., *Sur certaines opérations differentiables des groupes de Lie*, Amer. J. Math. **97** (1975), 172–181.
8. Procesi, C.; Schwarz, G., *Inequalities defining orbit spaces*, Invent. Math. **81** (1985), 539–554.
9. Palais, R. S.; Terng, C. L., *A general theory of canonical forms*, Trans. AMS **300** (1987), 771–789.
10. Sartori, G., *A theorem on orbit structures (strata) of compact linear Lie groups*, J. Math. Phys. **24** (1983), 765–768.
11. Schwarz, G. W., *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68.
12. Schwarz, G. W., *Lifting smooth homotopies of orbit spaces*, Publ. Math. IHES **51** (1980), 37–136.
13. Terng, C. L., *Isoparametric submanifolds and their Coxeter groups*, J. Diff. Geom. **1985** (21), 79–107.

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