GEODESICS ON SPACES
OF ALMOST HERMITIAN STRUCTURES

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ABSTRACT. A natural metric on the space of all almost hermitian structures on a given manifold is investigated.

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0. INTRODUCTION

If M is a (not necessarily compact) smooth finite dimensional manifold, the space $\mathcal{M} = \mathcal{M}(M)$ of all Riemannian metrics on M can be endowed with a structure of an infinite dimensional smooth manifold modeled on the space $C_c^\infty(S^2 T^* M)$ of symmetric $(0, 2)$-tensor fields with compact support. Analogously, the space $\Omega_{\mathrm{nd}}^2(M)$ of non degenerate 2-forms on M, is an infinite dimensional smooth manifold, modeled on the space $\Omega_c^2(M)$ of 2-forms with compact support. See [6] and [7].

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Here we consider the space of almost Hermitian structures on M, i.e. the subset $\mathcal{H}$ of $\mathcal{M}(M) \times \Omega^2_{\mathrm{nd}}(M)$ of those elements (g, ω) such that the $(1,1)$-tensor field $J = g^{-1}\omega$ is an almost complex structure on M. The aim of this paper is to study the geometry of $\mathcal{H}$. First we prove in section 1 that $\mathcal{H}$ is a splitting submanifold of the product $\mathcal{M}(M) \times \Omega^2_{\mathrm{nd}}(M)$. In section 2, after splitting the tangent space of the product in a form well adapted to our problem, we derive the equations that a curve in $\mathcal{H}$ should satisfy in order to be a geodesic for the metric on $\mathcal{H}$ induced by the product metric on $\mathcal{M}(M) \times \Omega^2_{\mathrm{nd}}(M)$. In section 3 we give an independent variational derivation of the geodesic equations, by parameterizing elements of $\mathcal{H}$ by automorphisms of TM, i.e. by sections of the bundle $GL(TM, TM)$. Finally, section 4 is devoted to the study of the geodesic equations found in section 2. We were not able to find an explicit solution, nevertheless we can give some explicit properties of the geodesics, see 4.1 and 4.2. The subspaces $\mathcal{H}_J$, $\mathcal{H}_\omega$, and $\mathcal{H}_g$ of all almost hermitian structures with fixed almost complex structure J, 2-form ω, or metric g, respectively, are splitting submanifolds of $\mathcal{H}$. This follows from splitting the bundle. Some other interesting subsets (hermitian structures, Kähler structures, with symplectic ω) are much more difficult to treat: we do not know whether they are submanifolds, since the differential operators describing them are complicated in the charts we use. Moreover $\mathcal{H}_J$ is totally geodesic in $\mathcal{M}(M)$, and the other two are totally geodesic in the manifold of all metrics with a fixed volume form. The geodesics are known explicitly in all three cases.

1. Almost hermitian structures

1.1. Almost hermitian structures. Let M be a smooth manifold of even dimension $n = 2m$. Let g be a Riemannian metric on M and let ω be a non degenerate 2-form on M; both of them will be regarded as fiber bilinear functionals on TM or as fiber linear isomorphisms $TM \to T^*M$ without any change of notation. The symmetry of g is then expressed $g^t = g$ where the transposed g^t is given by $TM \to T^{**}M \xrightarrow{g^*} T^*M$, and we also have $\omega^t = -\omega$

Then we consider the endomorphism $J := g^{-1}\omega\colon TM \to TM$ which satisfies $J^* = \omega^t(g^t)^{-1} = -\omega g^{-1}$. Then the following conditions are equivalent:

(1) J is an isometry for g, i.e. $g(JX, JY) = g(X, Y)$ for all $X, Y \in T_xM$.
(2) $g^{-1}\omega + \omega^{-1}g = 0$.
(3) J is an almost complex structure, i. e. $J^2 = -Id$ or equivalently $J^{-1} = -J$.

If these equivalent conditions are satisfied we say that (g, ω) is an **almost Hermitian structure**.

Remark: For a given almost complex structure J, condition (1) is equivalent to the fact that gJ is skew symmetric. So there is a bijective correspondence between the set of almost hermitian structures and the set of Riemannian almost complex structures.

1.2. The bundle of almost hermitian structures. We consider the subspace

$$\text{Herm} := \{(g, \omega): g^{-1}\omega + \omega^{-1}g = 0\} \subset S_+^2 T^*M \times \Lambda_{\text{nd}}^2 T^*M,$$

where $S_+^2 T^*M$ is the set of all positive definite symmetric 2-tensors on M, and we claim that it is a subbundle. For that we consider the following commutative diagram:

(1)
$$
\begin{array}{ccc}
\text{Herm} & \xrightarrow{\;pr_1\;} & S_+^2 T^*M \\
{\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \pi} \\
\Lambda_{\text{nd}}^2 T^*M & \xrightarrow[\pi]{} & M.
\end{array}
$$

Lemma. *This is a double fiber bundle, where the standard fiber of pr_1 is the homogeneous space $O(2m, \mathbb{R})/U(m)$ and the standard fiber of pr_2 is the homogeneous space $Sp(m, \mathbb{R})/U(m)$.*

In fact all the fiber bundles of diagram (1) *are associated bundles for the linear frame bundle $GL(\mathbb{R}^{2m}, TM)$, where the structure group $GL(2m)$ acts from the left on the typical fibers given in the diagram*

(2)
$$
\begin{array}{ccc}
GL(2m, \mathbb{R})/U(m) & \longrightarrow & GL(2m, \mathbb{R})/O(2m, \mathbb{R}) \\
\downarrow & & \downarrow \\
GL(2m, \mathbb{R})/Sp(m, \mathbb{R}) & \longrightarrow & \{Id\}.
\end{array}
$$

Proof. If we fix a metric g, then in an orthonormal frame of $TM|U$ for an open subset $U \subset M$, there is of course a 2-form ω_0 such that (g, ω_0) is a local section of Herm. If (g, ω_1) is another local section with the same g then there is a g-isometric local isomorphism f of $TM|U$ with possibly smaller U such that $f^*\omega_1 = \omega_0$ which is fiberwise unique up to multiplication from the right by an element of $O(2m, \mathbb{R}) \cap Sp(m, \mathbb{R}) = U(m)$.

If we fix on the other hand a non degenerate 2-form ω, then there is a frame of $TM|U$ such that ω takes the usual standard form of a symplectic structure (the frame can be chosen holonomic if and only if $d\omega = 0$). Then obviously there is a metric g_0 (constant in that frame) such that (g_0, ω) is a local section of Herm. If (g_1, ω) is another local section then there is a fiberwise symplectic isomorphism f of $TM|U$ (with possibly smaller U) such that $f^*g_1 = g_0$, which is fiberwise unique up to right multiplication by an element of $Sp(m, \mathbb{R}) \cap O(2m, \mathbb{R}) = U(m)$.

To check the last statement we consider the diagram

$$
\begin{array}{ccc}
GL(\mathbb{R}^{2m}, TM) \times GL(2m, \mathbb{R})/O(2m, \mathbb{R}) & \xrightarrow{\;A\;} & S_+^2 T^*M \\
\downarrow & & \| \\
\dfrac{GL(\mathbb{R}^{2m}, TM) \times GL(2m, \mathbb{R})/O(2m, \mathbb{R})}{GL(2m, \mathbb{R})} & \xrightarrow{\;\cong\;} & S_+^2 T^*M
\end{array}
$$

where $A(u, g.O(2m, \mathbb{R}))(X, Y) = u^{-1}(X)^t g g^t u^{-1}(Y)$ for $u \in GL(\mathbb{R}^2 m, T_x M)$ and $X, Y \in T_x M$.

Likewise we have

$$
\begin{array}{ccc}
GL(\mathbb{R}^{2m}, TM) \times GL(2m, \mathbb{R})/Sp(m, \mathbb{R}) & \xrightarrow{\ B\ } & \Lambda^2_{\mathrm{nd}} T^* M \\
\downarrow & & \| \\
\frac{GL(\mathbb{R}^{2m}, TM) \times GL(2m, \mathbb{R})/Sp(m, \mathbb{R})}{GL(2m, \mathbb{R})} & \xrightarrow{\ \cong\ } & \Lambda^2_{\mathrm{nd}} T^* M
\end{array}
$$

where $B(u, g.Sp(m, \mathbb{R}))(X, Y) = u^{-1}(X)^t g J g^t u^{-1}(Y)$. $\quad\square$

1.3. The manifold of almost hermitian structures. The space of almost hermitian structures on M is just the space $\mathcal{H} := C^\infty(\mathrm{Herm})$ of smooth sections of the fiber bundle Herm. Since Herm is a subbundle of $S^2_+ T^* M \times \Lambda^2_{\mathrm{nd}} T^* M$ and since the latter is an open subbundle of the vector bundle $S^2 T^* M \times \Lambda^2 T^* M$ we see that $\mathcal{H}$ is a splitting submanifold of $\mathcal{M}(M) \times \Omega^2_{\mathrm{nd}}(M)$, by the following lemma. The splitting property will also follow directly in section 2.1.

Lemma. *Let (E, p, M, S) and (E', p', M, S') be two fiber bundles over M and let $i \colon E \to E'$ be a fiber respecting embedding. Then the following embeddings of spaces of sections are splitting smooth submanifolds:*

$$
C^\infty(E) \xrightarrow{\ i_*\ } C^\infty(E') \hookrightarrow C^\infty(M, E').
$$

Proof. This is a variant of the results 10.6 and 10.10 in [10] and the proof is similar to the ones given there, by direct finite dimensional construction. $\quad\square$

1.4. Metrics on $\mathcal{H} = C^\infty(\mathrm{Herm})$. On the space $\mathcal{M}(M) \times \Omega^2_{\mathrm{nd}}(M)$ there are many pseudo Riemannian metrics which are invariant under the diffeomorphism group and which are of first order. We shall consider the product metric

$$
G_{(g,\omega)}((h_1, \varphi_1), (h_2, \varphi_2)) =
$$
$$
= \int_M \mathrm{tr}(g^{-1} h_1 g^{-1} h_2) \, \mathrm{vol}(g) + \int_M \mathrm{tr}(\omega^{-1} \varphi_1 \omega^{-1} \varphi_2) \, \mathrm{vol}(\omega),
$$

where $(h_1, \varphi_1), (h_2, \varphi_2) \in T_g \mathcal{M}(M) \times T_\omega \Omega^2_{\mathrm{nd}}(M) = C^\infty_c(S^2 T^* M) \times \Omega^2_c(M)$. Note that $\mathrm{vol}(g) = \mathrm{vol}(\omega)$ if $(g, \omega) \in \mathcal{H}$ since $Sp(m, \mathbb{R})$ and $O(2m, \mathbb{R})$ are both subgroups of $SL(2m, \mathbb{R})$. So the restriction of G to $\mathcal{H}$ is given by

$$
G_{(g,\omega)}((h_1, \varphi_1), (h_2, \varphi_2)) := \int_M \left(\mathrm{tr}(g^{-1} h_1 g^{-1} h_2) + \mathrm{tr}(\omega^{-1} \varphi_1 \omega^{-1} \varphi_2) \right) \mathrm{vol}(g).
$$

2. The geodesic equation in $\mathcal{H}$

2.1. Splitting the tangent space of the space of almost hermitian structures. Let $(g, \omega) \in \mathcal{H} = C^\infty(\text{Herm}) \subset \mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)$ so that $F(g, \omega) := g^{-1}\omega + \omega^{-1}g = 0$. Then for $(h, \alpha) \in C_c^\infty(S^2 T^* M \times \Lambda^2 T^* M)$ we put $H := g^{-1}h$, $A := \omega^{-1}\alpha$, and of course $J = g^{-1}\omega$. We have $(h, \alpha) \in T_{(g,\omega)}\mathcal{H}$ if and only if

$$dF(g, \omega)(h, \alpha) = -g^{-1}hg^{-1}\omega + g^{-1}\alpha + \omega^{-1}h - \omega^{-1}\alpha\omega^{-1}g = 0,$$

which is easily seen to be equivalent to $JH + HJ = JA + AJ$. Note that this implies $\text{tr}(H) = \text{tr}(A)$.

For $(g, \omega) \in \mathcal{H}$ we have the following G-orthogonal decomposition of the tangent space to $\mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)$ at (g, ω):

$$T_{(g,\omega)}(\mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)) = \mathcal{N}^1_{(g,\omega)} \oplus \mathcal{N}^2_{(g,\omega)} \oplus \mathcal{N}^3_{(g,\omega)} \oplus \mathcal{N}^4_{(g,\omega)}$$
$$\mathcal{N}^1_{(g,\omega)} = \{(H, 0) \colon JHJ = H\}$$
$$\mathcal{N}^2_{(g,\omega)} = \{(0, A) \colon JAJ = A\}$$
$$\mathcal{N}^3_{(g,\omega)} = \{(H, A) \colon JHJ = -H \text{ and } H = A\}$$
$$\mathcal{N}^4_{(g,\omega)} = \{(H, A) \colon JHJ = -H \text{ and } H = -A\}$$

The tangent space to $\mathcal{H}$ is given by $T_{(g,\omega)}\mathcal{H} = \mathcal{N}^1_{(g,\omega)} \oplus \mathcal{N}^2_{(g,\omega)} \oplus \mathcal{N}^3_{(g,\omega)}$ and its G-orthogonal complement is given by $(T_{(g,\omega)}\mathcal{H})^\perp = \mathcal{N}^4_{(g,\omega)}$. The restriction of the pseudometric to $\mathcal{H}$ is then non degenerate.

The projectors on these subspaces can easily be constructed and in particular we have the orthogonal projectors from $T_{(g,\omega)}\mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)$ to the tangent space $T_{(g,\omega)}\mathcal{H}$ and to the G-orthogonal complement

$$Pr^{\text{T}}_{(g,\omega)}(H, A) := \left(\frac{3H + JHJ + A - JAJ}{4}, \frac{3A + JAJ + H - JHJ}{4} \right)$$
$$Pr^{\perp}_{(g,\omega)}(H, A) := \left(\frac{H - JHJ - A + JAJ}{4}, \frac{A - JAJ - H + JHJ}{4} \right).$$

2.2. The geodesic equation. The space $\mathcal{H}$ is a splitting submanifold of $\mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)$ and the tangent space splits nicely in the direct sum of the tangential and the orthogonal part, by 2.1; note that the projection operators are algebraic. The tangential projection of the covariant derivative $\nabla_\xi \eta$ of smooth vector fields on $\mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)$ which along $\mathcal{H}$ are tangential to $\mathcal{H}$, is thus again a smooth vector field, and the usual proof of Gauß' formula involving the six term expression of the Levi-Civita covariant derivative shows that $Pr^T \nabla_\xi \eta$ is the smooth Levi-Civita covariant derivative of G on $\mathcal{H}$.

So a curve $\sigma(t) = (g(t), \omega(t))$ is a geodesic for the induced metric if and only if the covariant derivative of its tangent vector σ_t in $\mathcal{M}(M) \times \Omega^2_{\text{nd}}(M)$ is everywhere orthogonal to $\mathcal{H}$; i. e. $\nabla_{\partial_t}\sigma_t \in \mathcal{N}^4_\sigma$, or equivalently $Pr^T_\sigma(\nabla_{\partial_t}\sigma_t) = 0$.

If we put $X := g^{-1}g_t$ and $W := \omega^{-1}\omega_t$ and use the Christoffel form from [7], (2.3 for $\alpha = 1/n$), these conditions become, respectively:

$$
\text{(1)} \quad \begin{cases} JX_t + \frac{1}{2}\operatorname{tr}(X)JX = X_tJ + \frac{1}{2}\operatorname{tr}(X)XJ \\ X_t + W_t = -\frac{1}{2}\operatorname{tr}(X)X - \frac{1}{2}\operatorname{tr}(W)W + \frac{1}{4}(\operatorname{tr}(X^2) + \operatorname{tr}(W^2))Id \end{cases}
$$

$$
\text{(2)} \quad \begin{cases} 3X_t + \frac{3}{2}\operatorname{tr}(X)X - \frac{1}{2}\operatorname{tr}(X^2)Id + JX_tJ + \frac{1}{2}\operatorname{tr}(X)JXJ+ \\ +W_t + \frac{1}{2}\operatorname{tr}(W)W - \frac{1}{2}\operatorname{tr}(W^2)Id - JW_tJ - \frac{1}{2}\operatorname{tr}(W)JWJ = 0 \\ 3W_t + \frac{3}{2}\operatorname{tr}(W)W - \frac{1}{2}\operatorname{tr}(W^2)Id + JW_tJ + \frac{1}{2}\operatorname{tr}(W)JWJ+ \\ +X_t + \frac{1}{2}\operatorname{tr}(X)X - \frac{1}{2}\operatorname{tr}(X^2)Id - JX_tJ - \frac{1}{2}\operatorname{tr}(X)JXJ = 0 \end{cases}
$$

2.3. The submanifold $\mathcal{H}_J$ of $\mathcal{H}$. For a fixed almost complex structure J on M let us consider $\mathcal{H}_J := \{(g,\omega) \in \mathcal{H} \colon g^{-1}\omega = J\}$, the space of almost hermitian structures with almost complex structure J. The tangent space is $T_{(g,\omega)}\mathcal{H}_J = \mathcal{N}^3_{(g,\omega)}$, see 2.1

Via the first projection the space $\mathcal{H}_J$ is diffeomorphic to the submanifold of $\mathcal{M}(M)$ consisting of all g making J an isometry. It is a totally geodesic submanifold in the Riemannian manifold $(\mathcal{M}(M), G)$. This follows from the general result:

Lemma. *Let $A\colon TM \to TM$ be a vector bundle isomorphism covering the identity. Then the space of all Riemannian metrics g on M such that $g(AX, AY) = g(X, Y)$ for all $X, Y \in T_xM$ is a geodesically closed submanifold of $(\mathcal{M}(M), G)$. This is also true for bilinear structures, or metrics with fixed signature.*

Proof. The space of these g is the space of sections of the open subbundle $S^2_+T^*M \cap \{g\colon A^* \circ g \circ A = g\}$ of the obvious vector subbundle, which is a smooth manifold. A tangent vector h at such g is a tensor field with compact support with $A^* \circ h \circ A = h$; for $H = g^{-1}h$ this is equivalent to $A^{-1} \circ H \circ A = H$. By [6], 3.2, the geodesic in $\mathcal{M}(M)$ starting at g in the direction h is the curve

$$g(t) = g \; e^{a(t)Id + b(t)H_0},$$

where H_0 is the traceless part of H and $a(t)$ and $b(t)$ are real valued functions. Then if g and h are A-invariant, so is the whole geodesic. $\quad\square$

2.4. The submanifold $\mathcal{H}_\omega$. For a fixed non degenerate 2-form ω we consider $\mathcal{H}_\omega := \mathcal{M}(M) \times \{\omega\} \cap \mathcal{H}$, the space of almost hermitian structures with fixed ω. It is the space of sections of the pullback bundle $\omega^*(\text{Herm}, pr_2, \Lambda^2_{\text{nd}}T^*M)$ in terms of 1.2, so a smooth manifold. Its tangent space is $T_g\mathcal{H}_\omega = \mathcal{N}^1_{(g,\omega)}$. The space $\mathcal{H}_\omega$ is a submanifold $\mathcal{M}_{\text{vol}(\omega)}$ of the space of all metrics with fixed volume equal to $\text{vol}(\omega)$, see 1.4.

$\mathcal{H}_\omega$ is a geodesically closed submanifold of $\mathcal{M}_{\mathrm{vol}(\omega)}$, see Blair [1], [2]. The geodesics of $\mathcal{M}_{\mathrm{vol}(\omega)}$ have been determined by Ebin [3], see also [4]. The geodesic starting at g in the direction h is given by

$$g(t) = g \, e^{tH_0}.$$

Note that $\mathcal{M}_{\mathrm{vol}(\omega)}$ is not geodesically closed in $\mathcal{M}$. These results are given for compact M, but clearly they continue to hold for noncompact M in our setting.

2.5.   The submanifold $\mathcal{H}_g$. For a fixed metric g we may consider $\mathcal{H}_g := \{g\} \times \Omega^2_{\mathrm{nd}}(M) \cap \mathcal{H}$, the space of almost hermitian structures with fixed g. It is the space of all sections of the pullback bundle $g^*(H, pr_1, S^2_+ T^*M)$ in the notation of 1.2, so it is a smooth manifold. For the tangent space it easily follows that $T_\omega \mathcal{H}_g = \mathcal{N}^2_{(g,\omega)}$. Since the situation is symmetric with respect to g or ω, it follows from 2.4 that $\mathcal{H}_g$ is a geodesically closed submanifold of the space $\Omega^2_{\mathrm{nd}}(M)_{\mathrm{vol}(g)}$ of all almost symplectic structures with fixed volume equal to $\mathrm{vol}(g)$, and the geodesics in $\Omega^2_{\mathrm{nd}}(M)_{\mathrm{vol}(g)}$ are given by

$$\omega(t) = \omega \, e^{tA_0}.$$

3. The variational approach to the geodesic equation

3.1. It is not so easy to find an adapted chart for the subbundle Herm $\subset S^2_+ T^*M \times \Lambda^2_{\mathrm{nd}} T^*M$ which would allow us to parameterize curves and their variations in $\mathcal{H}$. In order to achieve this parameterization we will use the following scheme.

We consider the bundle $GL(TM, TM)$ of all isomorphisms of the tangent bundle. It is an open submanifold of the vector bundle $L(TM, TM)$. Any (fixed) almost hermitian structure $(g_0, \omega_0) \in \mathcal{H}$ induces a smooth mapping

$$\varphi = \varphi_{(g_0,\omega_0)} \colon GL(TM, TM) \to \mathrm{Herm} \subset S^2_+ T^*M \times \Lambda^2_{\mathrm{nd}} T^*M$$
$$\varphi(f) = \varphi_{(g_0,\omega_0)}(f) = (f^* g_0 f, f^* \omega_0 f).$$

The corresponding push forward mapping between the spaces of sections will be denoted by

$$\Phi \colon \mathcal{G} := C^\infty(GL(TM, TM)) \to \mathcal{H} = C^\infty(\mathrm{Herm}) \subset \mathcal{M} \times \Omega^2_{\mathrm{nd}}(M),$$
$$\Phi(f) = \Phi_{(g_0,\omega_0)}(f) = \varphi \circ f = (f^* g_0 f, f^* \omega_0 f).$$

Remark. The mapping $\Phi \colon \mathcal{G} = C^\infty(GL(TM, TM)) \to \mathcal{H}$ is **not** surjective in general. The mapping $\varphi \colon GL(TM, TM) \to \mathrm{Herm}$ is the projection of a fiber bundle with typical fiber $U(m)$. Since $U(m)$ has nontrivial homotopy trying to lift a section $s \colon M \to \mathrm{Herm}$ over φ will meet obstructions in general.

3.2. Lemma. *For every curve* $(g(t), \omega(t))$ *in* $\mathcal{H}$ *and also for every variation* $(g(t,s), \omega(t,s))$ *of such a curve in* $\mathcal{H}$ *with* $(g(0), \omega(0)) = (g_0, \omega_0)$ *there is a curve* $f(t)$ *or variation* $f(t,s)$ *in* $\mathcal{G} = C^\infty(GL(TM, TM))$ *with* $(g, \omega) = \Phi_{(g_0, \omega_0)}(f)$.

Proof. As noticed above $\varphi: GL(TM, TM) \to$ Herm is the projection of a smooth fiber bundle with compact fiber type $U(m)$. We choose a generalized connection for this bundle, see [11] or [9], section 9. Its parallel transport

$$\mathrm{Pt}(c, t): GL(TM, TM)_{c(0)} \to GL(TM, TM)_{c(t)}$$

is globally defined for each curve $c: \mathbb{R} \to$ Herm, and it is smooth in the choice of the curve, see loc. cit.

Then we just define $f(t) := \mathrm{Pt}((g(x, \quad), \omega(x, \quad)), t)\,\mathrm{Id}_{T_xM}$ and $f(\quad, t)$ will be a curve in $\mathcal{G} = C^\infty(GL(TM, TM))$ with $(g(t), \omega(t)) = \Phi_{(g_0, \omega_0)}(f(t))$, and $f(x, t)$ varies in t only for those x where also $g(x, t)$ or $\omega(x, t)$ varies. So the compact support condition required of smooth curves of sections is automatically satisfied.

For a variation of a curve we first define in turn

$$f(x, t, 0) := \mathrm{Pt}((g(x, \quad, 0), \omega(x, \quad, 0)), t)\,\mathrm{Id}_{T_xM},$$
$$f(x, t, s) := \mathrm{Pt}((g(x, t, \quad), \omega(x, t, \quad)), s)f(x, t, 0). \quad \square$$

3.3. Let $(g(t), \omega(t))$ be a smooth curve in $\mathcal{H} = C^\infty(\text{Herm})$, so it is smooth $M \times \mathbb{R} \to$ Herm and for each compact $[a, b] \subset \mathbb{R}$ there is a compact set $K \subset M$ such that $(g(x, t), \omega(x, t))$ is constant in $t \in [a, b]$ for each $x \in M \backslash K$, see ([11], 6.2, a slight mistake there). Then its energy with respect to the metric G of 1.4 is given by

$$(1) \qquad E_a^b(g, \omega) = \int_a^b \int_M \left(\mathrm{tr}(g^{-1}g_t g^{-1}g_t) + \mathrm{tr}(\omega^{-1}\omega_t \omega^{-1}\omega_t) \right) \mathrm{vol}(g)dt.$$

Since we cannot parameterize curves and their variations in $\mathcal{H}$ explicitly we will parameterize them with the help of $\Phi = \Phi_{(g_0, \omega_0)}: \mathcal{G} \to \mathcal{H}$, where $(g_0, \omega_0) = (g(0), \omega(0))$. So let $f(t)$ be a smooth curve in $\mathcal{G} = C^\infty(GL(TM, TM))$. Then we have

$$\Phi(f) = \Phi_{(g_0, \omega_0)}(f) = (f^* g_0 f, f^* \omega_0 f),$$
$$T_f\Phi(f_t) = (f_t^* g_0 f + f^* g_0 f_t, f_t^* \omega_0 f + f^* \omega_0 f_t),$$

so the energy of the curve $\Phi(f(t))$ in $\mathcal{H}$ is given by

$$(2) \qquad E_a^b(\Phi(f)) =$$

$$= \int_a^b \int_M \left(\mathrm{tr}(f^{-1}g_0^{-1}(f^*)^{-1}(f_t^* g_0 f + f^* g_0 f_t)f^{-1}g_0^{-1}(f^*)^{-1}(f_t^* g_0 f + f^* g_0 f_t)) \right.$$

$$\left. + \mathrm{tr}(f^{-1}\omega_0^{-1}(f^*)^{-1}(f_t^* \omega_0 f + f^* \omega_0 f_t)f^{-1}\omega_0^{-1}(f^*)^{-1}(f_t^* \omega_0 f + f^* g_0 f_t)) \right)$$

$$\det(f)\,\mathrm{vol}(g_0)dt.$$

Lemma. *A curve $f(t)$ in $\mathcal{G}$ is a critical point of the functional (2) if and only if $\Phi(f(t))$ is a critical point of the functional (1), i. e. $t \mapsto \Phi(f(t))$ is a geodesic.*

Proof. An (infinitesimal) variation in $\mathcal{G}$ can (with the help of a connection for $\varphi \colon GL(TM, TM) \to \mathrm{Herm}$) be written as a sum of two variations: the horizontal one corresponds exactly to a variation of $\Phi(f)$ in $\mathcal{H}$, and along the vertical one the functional is stationery anyhow. $\square$

3.4. Lemma. *In the setting of 3.3, for a variation $f(t, s) \in \mathcal{G}$ with fixed endpoints we have the first 'variation formula'*

$$\left. \frac{\partial}{\partial s} \right|_0 E_a^b(\Phi_{(g_0,\omega_0)}(f(\quad , s))) =$$

$$= \int_a^b \int_M \mathrm{tr}\Big(E(g_0, \omega_0, f; t) f_s f^{-1} \Big) \det(f) \, \mathrm{vol}(g) \, dt.$$

where

$$
\begin{aligned}
E(g_0, \omega_0, f; t) = {}& -2 f_{tt} f^{-1} - 2 g_0^{-1}(f^*)^{-1} f_{tt}^* g_0 + 2 f_t f^{-1} f_t f^{-1} \\
& - 2\,\mathrm{tr}(f_t f^{-1}) f_t f^{-1} + \mathrm{tr}(f_t f^{-1} f_t f^{-1}) Id - 2 g_0^{-1}(f^*)^{-1} f_t^* g_0 f_t f^{-1} \\
& + 2 g_0^{-1}(f^*)^{-1} f_t^* (f^*)^{-1} f_t^* g_0 + 2 f_t f^{-1} g_0^{-1}(f^*)^{-1} f_t^* g_0 \\
& - 2\,\mathrm{tr}(f_t f^{-1}) g_0^{-1}(f^*)^{-1} f_t^* g_0 + \mathrm{tr}(g_0^{-1}(f^*)^{-1} f_t^* g_0 f_t f^{-1}) Id \\
& - 2 f_{tt} f^{-1} - 2 \omega_0^{-1}(f^*)^{-1} f_{tt}^* \omega_0 + 2 f_t f^{-1} f_t f^{-1} \\
& - 2\,\mathrm{tr}(f_t f^{-1}) f_t f^{-1} + \mathrm{tr}(f_t f^{-1} f_t f^{-1}) Id \\
& - 2 \omega_0^{-1}(f^*)^{-1} f_t^* \omega_0 f_t f^{-1} + 2 \omega_0^{-1}(f^*)^{-1} f_t^* (f^*)^{-1} f_t^* \omega_0 \\
& + 2 f_t f^{-1} \omega_0^{-1}(f^*)^{-1} f_t^* \omega_0 - 2\,\mathrm{tr}(f_t f^{-1}) \omega_0^{-1}(f^*)^{-1} f_t^* \omega_0 \\
& + \mathrm{tr}(\omega_0^{-1}(f^*)^{-1} f_t^* \omega_0 f_t f^{-1}) Id
\end{aligned}
$$

Proof. This is a long but straightforward computation. We may interchange $\frac{\partial}{\partial s}|_0$ with the first integral since this is finite dimensional analysis, and we may interchange it with the second one, since $\int_M$ is a continuous linear functional on the space of all smooth densities with compact support on M, by the chain rule. Then we use that tr_* is linear and continuous, $d(\mathrm{vol})(g)h = \frac{1}{2}\,\mathrm{tr}(g^{-1}h)\,\mathrm{vol}(g)$, and that $d((\quad)^{-1})_*(g)h = -g^{-1}hg^{-1}$ and partial integration; there are no boundary terms since we assumed the variation to have fixed endpoints. $\square$

3.5. Lemma. *For curve $f(t)$ in $\mathcal{G}$ the curve $\Phi_{(g_0,\omega_0)}(f(t))$ is a geodesic in $(\mathcal{H}, G)$ if and only if $f(t)$ satisfies the following equation:*

$$E(g_0, \omega_0, f, t) = 0.$$

Proof. This follows from 3.4 since the integral in (3) describes a nondegenerate inner product on $\mathcal{G}$, given by

$$G_f(h, k) = \int_M \mathrm{tr}(h f^{-1} k f^{-1}) \det(f) \, \mathrm{vol}(g_0). \quad \square$$

3.6. Comparison with section 2. Let $\Phi_{(g_0,\omega_0)}(f(t)) = (f^*g_0 f, f^*\omega_0 f) =:$
$(g(t), \omega(t))$. Then the expressions used in section 2 become

$$X = g^{-1}g_t = f^{-1}g_0^{-1}(f^*)^{-1}f_t^* g_0 f + f^{-1}f_t,$$
$$W = \omega^{-1}\omega_t = f^{-1}\omega_0^{-1}(f^*)^{-1}f_t^* \omega_0 f + f^{-1}f_t,$$
$$J = g^{-1}\omega = f^{-1}g_0^{-1}\omega_0 f.$$

If we compute X_t, W_t and insert this into the second equation of 2.2.(1), we get
exactly $f^{-1}E(g_0, \omega_0, f, t)f = 0$. So we get the same geodesic equation as in 2.2

4. SOME PROPERTIES OF THE GEODESICS

We are not able to give the explicit solution of the geodesic equation on $\mathcal{H}$.
But we can give some explicit formulas of the time evolution of some functions
of the structures.

4.1. Proposition. *Let $(g(t), \omega(t))$ be the geodesic of $\mathcal{H}$ starting at (g_0, ω_0) in the
direction (h, α), let $H = g_0^{-1}h$ and $A = \omega_0^{-1}\alpha$, and let $(X, W) = (g^{-1}g_t, \omega^{-1}\omega_t)$
as in 2.2. Then we have*

$$\mathrm{tr}(X^2) + \mathrm{tr}(W^2) = (\det(g_0^{-1}g))^{-1/2}(\mathrm{tr}(H^2) + \mathrm{tr}(A^2)).$$

Proof. Since (g, ω) is in $\mathcal{H}$ its tangent vector (g_t, ω_t) satisfies

(1) $$JXJ - X = JWJ - W$$
(2) $$\mathrm{tr}(X) = \mathrm{tr}(W)$$

so that 2.2.(2) becomes

$$\begin{cases} 3X_t + JX_tJ + W_t - JW_tJ + 2\,\mathrm{tr}(X)X - \tfrac{1}{2}(\mathrm{tr}(X^2) + \mathrm{tr}(W^2))Id = 0 \\ 3W_t + JW_tJ + X_t - JX_tJ + 2\,\mathrm{tr}(W)W - \tfrac{1}{2}(\mathrm{tr}(W^2) + \mathrm{tr}(X^2))Id = 0 \end{cases}$$

We multiply now the first equation by X, the second equation by W and add
them to obtain

$$2(\mathrm{tr}(X_tX) + \mathrm{tr}(W_tW)) + \tfrac{1}{2}(\mathrm{tr}(X^2) + \mathrm{tr}(W^2))\,\mathrm{tr}(X) = 0.$$

From that it is easy to see that the derivative of $(\mathrm{tr}(X^2) + \mathrm{tr}(W^2))(\det(g_0^{-1}g))^{1/2}$
is zero, where we also use $((\det(g_0^{-1}g))^{1/2})_t = \tfrac{1}{2}(\det(g_0^{-1}g))^{1/2}\,\mathrm{tr}(X)$. $\square$

4.2. Proposition. *Let $(g(t), \omega(t))$ be the geodesic of $\mathcal{H}$ starting at (g_0, ω_0) in the direction (h, α), let $H = g_0^{-1}h$ and $A = \omega_0^{-1}\alpha$, and let $(X, W) = (g^{-1}g_t, \omega^{-1}\omega_t)$. We put $p(t) := \frac{1}{2}(\det(g_0^{-1}g))^{1/2}$. Then we have*

(1) $$p(t) = \tfrac{n}{32}(\operatorname{tr}(H^2) + \operatorname{tr}(A^2))t^2 + \tfrac{1}{2}\operatorname{tr}(H)t + 1,$$

(2) $$\operatorname{tr}(X) = \operatorname{tr}(W) = 2\frac{p'(t)}{p(t)}$$

$$= 4\frac{n(\operatorname{tr}(H^2) + \operatorname{tr}(A^2))t + 8\operatorname{tr}(H)}{n(\operatorname{tr}(H^2) + \operatorname{tr}(A^2))t^2 + 8\operatorname{tr}(H)t + 32},$$

(3) $$X + W = \frac{1}{p(t)}\left(\frac{1}{4}(\operatorname{tr}(H^2) + \operatorname{tr}(A^2))tId + H + A\right)$$

Proof. We take the trace in the second expression of 2.2.(1) and use (2) from the proof of 4.1 to obtain

$$2\operatorname{tr}(X)' = -\operatorname{tr}(X)^2 + \tfrac{n}{4}(\operatorname{tr}(X^2) + \operatorname{tr}(W^2)).$$

Inserting 4.1 we get

$$2\operatorname{tr}(X)' = -\operatorname{tr}(X)^2 + \tfrac{n}{4}\frac{C}{p(t)},$$

where $C = \operatorname{tr}(H^2) + \operatorname{tr}(A^2)$. From the proof of 4.1 we have in turn

$$p'(t) = \tfrac{1}{2}p(t)\operatorname{tr}(X)$$

$$p''(t) = \tfrac{1}{4}p(t)\operatorname{tr}(X)^2 + \tfrac{1}{2}p(t)\operatorname{tr}(X)' = \frac{nC}{16}.$$

For the initial conditions $p(0) = 1$ and $p'(0) = \frac{1}{2}\operatorname{tr}(H)$ this gives

$$p(t) = \tfrac{nC}{32}t^2 + \tfrac{1}{2}\operatorname{tr}(H)t + 1$$

and consequently assertions (1) and (2).

Now we take the tracefree part of the second expression in 2.2.(1)

$$(X + W)_0' = -\tfrac{1}{4}\operatorname{tr}(X + W)(X + W)_0,$$

and by an argument similar to that used in the proof of [7], 2.5 we get

$$X + W = a(t)Id + b(t)(H_0 + A_0), \quad \text{where } a(t) = \frac{2\operatorname{tr}(X)}{n} = \frac{4}{n}\frac{p'(t)}{p(t)},$$

and it just remains to find $b(t)$ when $(H_0 + A_0) \neq 0$.

We have $(X + W)_0 = b(t)(H_0 + A_0)$, thus $(X + W)_0' = b'(t)(H_0 + A_0)$. But we also know that $(X + W)_0' = -\frac{n}{4}a(t)b(t)(H_0 + A_0)$ and so we get

$$\frac{b'}{b} = -\frac{na}{4} = -\frac{p'}{p}$$

with $p(0) = 1$ and $b(0) = 1$, so $b = 1/p$ and we get assertion (3). $\square$

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