COADJOINT ORBITS FOR DOUBLE LOOP GROUPS

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Date: July 7, 2004.

Contents

1. Preface

In this paper we treat two articles concerning orbital theory for (twisted) affine Lie algebras. In [F] I.B. Frenkel introduced an infinite dimensional analogon of Kirillov's character formula for compact Lie groups. In particular, a classification of coadjoints orbits of the loop group, i.e. the group $C^\infty(S^1, G)$ where G denotes a compact Lie group, allowed the introduction of Wiener measure on the continouus closure of such a coadjoint orbit. So the character of an affine Kac Moody algebra, could be written as integral over a coadjoint orbit with respect to Wiener measure. In [W1] R. Wendt gave a generalization of Frenkels original program for twisted affine Lie algebras. Again, an analog of Kirillov's character formula could be deduced. In [W2] Wendt introduces a geometric approach to the orbital theory of affine Kac Moody algebras. There, an infinite dimensional analog of the Duistermaat Heckmann exact integration formula is presented and a formal integration with respect to the "Riemannian volume form" is introduced. This symplectic approach has the advantage, that integration is possible on spaces where no measure theory is yet developed. As an application several functions on coadjoint orbits of loop groups and double loop groups, i.e. groups of the form $C^\infty(\Sigma_\tau, G)$, where Σ_τ is an elliptic curve with modular parameter τ and G denotes a compact Lie group, are integrated. Furthermore it is shown, that the partition function of the gauged WZW model is a certain Hamiltonian function, so the formal integration is applicable. So one has a conceptual calculation of the functional integral representing this partition function. The contents of this paper is as follows: In the first chapter (see 2) we collect the most important facts concerning Kac Moody algebras and loop groups, which are needed in the sequal. For a detailed introduction to this subject the reader is refered to the books [K], [Wa]. Then we start treating [W1]. We decided to ommit the treatment of the Weyl character formula for non conncected Lie groups, since this was the master's thesis [W4] and the arguments are very similar to the ones employed in [BtD]. We compare the classification of coadjoint orbits in the loop group case with the one of double loop groups, introduced in [EF]. Then we treat the Duistermaat Heckmann formula as preperation for the infinite dimensional analog. There, a symplectic torus action on an infinite dimensional manifold is introduced. Then a functional on the set of Hamiltonian functions, corresponding to this action, is presented and a complete analog to the Duistermaat Heckmann exact integration formula is build. An important ingredient in Wendt's geometric approach are zeta reglarized products, we provide a more conceptual approach to his calculations. In particular we can deduce a stronger result on the modular invariance of the gauged WZW model. Usually one uses zeta regularized products to give sense to infinite products over sets of eigenvalues of a certain differential operator. In [W2] zeta regularization was only admitted for real numbers because of the analogy with the Wiener measure approach. So to calculate a zeta regularized product of the form

$$\prod_\zeta (n + \tau m + z) \tag{1}$$

with $m, n \in \mathbb{Z}$ the Epstein zeta function had to be employed. This is a Zeta function $\zeta(a, b)$ in two variables which works only for elements of $\mathbb{R}/\mathbb{Z}$. So in [W2] a new torus operation was introduced to overcome this difficulty. Unfortunately, to deduce the $SL(2\mathbb{Z})$ invariance of the partition function of the gauged WZW model, one is restricted to complex numbers. Calculating the zeta regularized product for the "original" eigenvalues we get the same result as in [W2], up to a constant factor, but now the calculations work arbitrary complex numbers and so the full modular invariance can be deduced. In the end, we present several open questions, two of them were stated in [W2] and an understanding of these problems was the starting point of this paper. We state the stochastic problem in a more concrete way then in [W2] and ask for further generalizations of the symplectic approach.

2. Kac Moody theory

This section follows mainly the introductory parts of [Wa]. As usual $\mathfrak{g}, \mathfrak{h}$ denote Lie Algebras, Δ is the corresponding root system. To distinguish the affine Kac Moody algebra from its finite counterpart we denote it by $\tilde{g}$. Let $A \in M_n\mathbb{Z}$ be a $n \times n$ matrix which satisfies the conditions

$$a_{ii} = 2 \quad (i = 1, \ldots, n) \tag{2}$$

$$a_{ij} \leq 0 \quad i \neq j \tag{3}$$

$$a_{ij} \neq 0 \Longleftrightarrow a_{ji} \neq 0 \tag{4}$$

then it is called a generalized Cartan matrix (GCM), one sees that the third condition of the usual Cartan matrix is ommited. We call a GCM indecomposable if it is not of the form

$$A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \tag{5}$$

A GCM is called symmetrizable if it is written as the product

$$A = DB$$

with a diagonal matrix D with all of its diagonal entries positive real numbers and a real symmetric matrix B. Indecomposable generalized Cartan matrices can be classified in three types, [K] Cor. 4.3

finite type: $Av \in (\mathbb{R}_{>0})^n$ for some $v \in (\mathbb{R}_>)^n$

affine type: $Av = 0$ for some $v \in (\mathbb{R}_>)^n$

indefinite type: $Av \in (\mathbb{R}_{<0})^n$ for some $v \in (\mathbb{R}_>)^n$

The transpose of a GCM is again a GCM and it can be shown that they are of same type. Like in the finite dimensional theory we can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The Lie algebra corresponding to a GCM is denoted by $\mathfrak{g}(A)$. Let $\mathfrak{h}^*$ denote the dual of $\mathfrak{h}$ as a vector space. We can choose a set of linearly independent elements $\alpha_i^\vee \quad (1 \leq i \leq n)$ from $\mathfrak{h}$ and $\alpha_i \quad (1 \leq i \leq n)$ from $\mathfrak{h}^*$ which satisfiy the condition

$$\langle \alpha_i^\vee, \alpha_i \rangle = a_{ij} \tag{6}$$

To a given GCM $A = (a_{ij})_{i,j=1,\ldots,n}$ one can associate a Dynkin diagram with n vertices and (oriented) multi lines as follows:

(D1) Put the label $\alpha_1, \ldots, \alpha_n$ on each of these vertices

(D2) Two vertices α_i and α_j with $i \neq j$ are connected by $\max\{|a_{ij}|, |a_{ji}|\}$ lines, and these lines are equipped with an arrow pointing towards i if $|a_{ij}| \geq 2$.

(D3) Tow vertices α_i and α_j for $i \neq j$ are not connected if $a_{ij} = 0$

It is known that there exists a non degenerate invariant symmetric bilinear form on $\mathfrak{g}(A)$ iff A is symmetrizable. The bilinear form $(.,.)$ is called invariant if

$$([x, y], z) = (x, [y, z]) \tag{7}$$

holds. The affine Dynkin diagrams fall in the three tables Aff1, Aff2, Aff3. See [K] chapter 4. One says a GCM $A = (a_{ij})_{0\le i,j\le l}$ is of type $X_N^{(r)}$, where r is called the tier number of A. For example A is of type $A_2^{(2)}$ or $A_{2l}^{(2)}$ and so on.
Given a GCM $A = (a_{ij})_{i,j=0,\dots,l}$ two $(l+1)$ tuples of positive integers, $(a_i^\vee)_{0\le i\le 0}$ and $(a_i)_{0\le i\le 0}$ are uniquely determined by the condition

$$(a_0^\vee,\dots,a_l^\vee)\cdot A = 0 \quad A\cdot(a_0,\dots,a_l)^t = 0 \tag{8}$$

and $gcd(a_0^\vee,\dots,a_l^\vee) = gcd(a_0,\dots,a_l) = 0$.
The numbers a_i and $a_i^\vee$ are called the label resp. the colabel. We define the two important *Coxeter* numbers

$$h := \sum_{i=1}^{l} a_i \quad h^\vee := \sum_{i=1}^{l} a_i^\vee \tag{9}$$

One can identify the Cartan subalgebra $\mathfrak{h}$ with its dual via the non degenerate bilinear form. One has

$$a_i^\vee \alpha_i^\vee = a_i\alpha_i \quad 0\le i\le l$$

The element

$$c := \sum_{i=0}^{l} \alpha_i^\vee a_i^\vee \tag{10}$$

is called the canonical central element. It satisfies

$$\langle c,\alpha_i\rangle = 0 \quad 0\le i\le l$$

so c commutes with all elements in $\mathfrak{g}(A)$ and $\mathbb{C}c$ is the center of $\mathfrak{g}(A)$. The dual element corresponding to c in $\mathfrak{h}^*$ is

$$\delta := \sum_{i=1}^{l} a_i\alpha_i \tag{11}$$

which is a positive root (i.e. an eigenvalue of the adjoint representation) which satisfies the following:

$$(\delta,\delta) = 0 \quad \{\alpha\in\Delta; (\alpha,\alpha) = 0\} = \{n\delta; n\in\mathbb{Z}-\{0\}\} \tag{12}$$

One can choose elements $\Lambda_0\in\mathfrak{h}^*$ and $d\in\mathfrak{h}$ such that the following relations are satisfied:

$$\Big\{\langle\alpha_i^\vee,\Lambda_0\rangle = \delta_{i,0} \quad (0\le i\le l) \quad \langle d,\Lambda_0\rangle = 0 \tag{13}$$

Under the natural isomorphism $\mathfrak{h}\cong\mathfrak{h}^*$, $a_0\Lambda_0$ is identified with d and δ is identified with the canonical central element c. We set

$$\bar{\mathfrak{h}} := \sum_{i=1}^{l}\mathbb{C}\alpha_i^\vee \quad\text{and}\quad \bar{\mathfrak{h}}^* := \sum_{i=1}^{l}\mathbb{C}\alpha_i \tag{14}$$

which are sometimes called finite part of $\mathfrak{h}$ and $\mathfrak{h}^*$ respectively. We use the following decompositions

$$\tilde{\mathfrak{h}} = \mathbb{C}d\oplus\bar{\mathfrak{h}}+\mathbb{C}c \quad\text{and}\quad \tilde{\mathfrak{h}}^* = \mathbb{C}\Lambda_0\oplus\bar{\mathfrak{h}}^*\oplus\mathbb{C}\delta \tag{15}$$

Next we introduce the notion of real and imaginariy roots of a Kac Moody algebra, the latter has no counterpart in the finite dimensional theory.

$$\Delta^{re} = \{\alpha\in\Delta; (\alpha,\alpha) > 0\} \quad\text{and}\quad \Delta^{im} = \{\alpha\in\Delta; (\alpha,\alpha)\le 0\} \tag{16}$$

In the case of an affine Kac Moody algebra the imaginary roots are described as follows:

$$\Delta^{im} = \{\alpha \in \Delta; (\alpha, \alpha) = 0\} = \{n\delta; n \in \mathbb{Z} - \{0\}\} \tag{17}$$

and the multiplicitiy $\text{mult}(n\delta) = \dim \mathfrak{g}_{n\delta}$ is given for a GCM $A \in X_N^{(r)}$ as follows:

$$\text{mult}(n\delta) = l \quad \text{if} r = 1 \tag{18}$$

and

$$\begin{cases} l \quad \text{if } n \notin r\mathbb{Z}, \qquad \text{and} \\ \frac{N-l}{r-1} \quad \text{if } n \in r\mathbb{Z}\,,\ n > 0\,. \end{cases} \tag{19}$$

when the tier number $r \geq 2$.

As in the finite dimensional theory one introduces the Weyl group, for each i, let τ_i be the element in $GL(\mathfrak{h}^*)$ defined by

$$\tau_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \qquad \lambda \in \mathfrak{h}^*$$

The subgroup of $GL(\mathfrak{h}^*)$ generated by τ_i $(0 \leq i \leq l)$ is called the Weyl group. An important set is the so called fundamental alcove

$$C_{af} := \{\lambda \in \bar{\mathfrak{h}}^*_{\mathbb{R}}; (\lambda, \alpha_i) \geq 0 \quad (1 \leq i \leq l) \quad (\lambda, \theta) \leq 1\}$$

where θ is the highest root in the finite dimensional positive roots $\bar{\Delta}_+$. We denote the root and coroot lattices by

$$Q = \sum_{i=0}^{l} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^\vee = \sum_{i=1}^{l} \mathbb{Z}\alpha_i^\vee$$

The finite root lattices are denoted with $\bar{Q}$ and $\bar{Q}^\vee$ respectively. Turning to the representation theory off affine Kac Moody algebras we introduce

$$\begin{gathered} \tilde{P} = \{\lambda \in \mathfrak{h}^*_{\mathbb{C}} | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 0, \dots, n\}, \\ \tilde{P}_+ = \{\lambda \in \tilde{P} | \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i = 0, \dots, n\} \text{ and} \\ \tilde{P}_{++} = \{\lambda \in \tilde{P}_+ | \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for all } i = 0, \dots, n\} \end{gathered}$$

where P is called the weight lattice, P_+, P_{++} are the integral resp, dominant integral weight lattices. Note, that there exists a bijection between the irreducible integrable highest weight modules of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and the dominant integral weights $\Lambda \in \tilde{P}_+$. Similarly co-weight lattices are denoted by

$$P^\vee = \{\lambda \in \mathfrak{h}^*; \langle \lambda, \alpha_i \rangle \in \mathbb{Z} \quad (0 \leq i \leq l)\}$$

We recall (13) and define elements $\Lambda_i \in \mathfrak{h}^*$ by

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \quad (0 \leq j \leq l)$$

$$\langle \Lambda_i, d \rangle = 0$$

and in the same manner for the cointegral forms:

$$\langle \Lambda_i^\vee, \alpha_j^\vee \rangle = \delta_{ij}$$

$$\langle \Lambda_i^\vee, d \rangle = 0$$

The finite weight lattices are denoted by

$$\bar{P} = \sum i = 1^l \mathbb{Z}\bar{\Lambda_i} \text{and} \qquad \bar{P^\vee} = \sum i = 1^l \mathbb{Z}\bar{\Lambda_i^\vee}$$

For $\lambda \in \mathfrak{h}^*$ we define the *level* of λ by (λ, δ). For a non-ngative integer m we define

$$P^m := \{\lambda \in P; \langle \lambda, c\rangle = m\}$$
$$P^m_+ := \{\lambda \in P_+; \langle \lambda, \delta\rangle = m\}$$
$$P^m_{++} := \{\lambda \in P^m_+; \langle \lambda, \alpha_i^\vee \in \mathbb{N}\rangle \quad (0 \le i \le l)\}$$

We now turn to the affine Weyl group.

Definition 2.1. $W := \bar{W} \ltimes M$

Now we turn to the Kac Weyl Character Formula. If $L(\Lambda)$ is an integrable irreducible highest weight module with highest weight $\Lambda \in \tilde{P}_+$ then the formal character of $L(\Lambda)$ is given by by the formal sum

$$chL(\Lambda) = \sum_{\lambda \in \tilde{\mathfrak{h}}^*} \dim L(\Lambda)_\lambda e(\lambda)$$

Here $L(\Lambda)_\lambda$ denotes the weight space corresponding to the weight λ and $e(\lambda)$ is a formal exponential. The Kac Weyl Character formula now reads:

$$chL(\Lambda) = \frac{\sum_{w \in \tilde{W}} \epsilon(w) e(w(\Lambda + \tilde{\rho}))}{e(\tilde{\rho}) \prod_{\alpha \in R}(1 - e(-\alpha))^{mult \alpha}}$$

Here $\tilde{\rho}$ is defined by $\langle \tilde{\rho}, \alpha_i^\vee\rangle = 1$ for $i = 0, \ldots, n$ and $\langle \tilde{\rho}, d\rangle = 0$ As usual we have set $\epsilon(w) = (-1)^{l(w)}$. So far the character was considered as a formal sum, involving the formal exponentials $e(\lambda)$. Now we set $e^\lambda(h) = e^{\langle \lambda, h\rangle}$ for $\mathfrak{h} \in \tilde{\mathfrak{g}}_\mathbb{C}$. In this way one can consider the character of a highest weight $\tilde{\mathfrak{g}}_\mathbb{C}$ module V as an infinite series. Let us set $Y(V) = \{h \in \tilde{\mathfrak{g}}_\mathbb{C} | ch_V(h)$ converges absolutely $\}$. Then ch_V defines a holomorphic function on $Y(V)$ and the following result holds (cf.[K]):

Proposition 2.2. *Let $V(\Lambda)$ be the irreducible highest weight module with highest weight $\Lambda \in P_+$. Then*

$$Y(V(\Lambda)) = \{h \in \tilde{\mathfrak{h}}_\mathbb{C} | Re\langle \delta, h\rangle > 0\}$$

where $\delta = \sum_{i=0}^n a_i \alpha_i$ and the a_i are the labels of the vertices of the affine Dynkin diagram. In this setting the Kac Weyl character formula gives an identity of holomorphic functions on $Y(V(\Lambda)))$.

Now we have $\tilde{\mathfrak{g}}_\mathbb{C} = \bar{\mathfrak{g}}_\mathbb{C} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with c and d as before. With this notion one gets:

$$Y(V(\Lambda)) = \{h + ac + bd | h \in \bar{\mathfrak{h}}_\mathbb{C}, a, b \in \mathbb{C}, Imb < 0\}$$

3. Classification of coadjoint orbits

Consider the following system of linear differential equations, or left logarithmic derivative

$$z'(t) = z(t)x(t) \tag{20}$$

where $z(t), x(t) \in M_n(\mathbb{C}) \forall t \ge 0$ and $x(t)$ is Lipschitz continuous in t. Now a fundamental result in the theory of differential equations states the existence of a unique solution of 20 for the initial condition $z(0) = I_n$ where I_n denotes the identity. This solution is usually called the fundamental solution. In [F] the same system is considered with periodic coefficients, that is $x(t+T) = x(t)$, but this will not allow us in the twisted case to classify affine adjoint orbits, Kleinfeld showed in [Kl] that then the monodromy map, which will be introduced below, is neither injective, nor surjective. So in [W1] twisted periodic coefficients are introduced.

Definition 3.1. *We call the coefficient $x(t)$ twisted periodic if $x(t+\frac{1}{r}) = \tau x \tau^{-1}$ for an invertible matrix τ with $\tau^r = I_n$ for all $t \geq 0$.*

If z is a fundamental solution of $z' = zx$ with the above definition it is obvious that $z_1(t) = \tau^{-1} z(t+\frac{1}{r})\tau$ is another solution. To see this, just plug into the definition 3.1 and one sees $z_1(t) = \tau^{-1}\tau z \tau^{-1}\tau = z$. This observation yields the following: There exists a matrix $\tilde{M}(x)$ such that $z_1(t) = \tilde{M}(x)z(t)$. But since we have chosen our initial condition $z(0) = I_n$ we get $\tilde{M}(x) = z_1(0) = \tau^{-1}z(0+\frac{1}{r})\tau$. Now

$$M(x) := z(\frac{1}{r}) \tag{21}$$

is called the "$\frac{1}{r}$-th monodromy" of the differential equation $z' = zx$. We then obtain

$$z(t+\frac{1}{r}) = M(x)\tau z(t)\tau^{-1} \tag{22}$$

for all $t \geq 0$. For a twisted periodic continuously differentiable g with $g(t) \in GL_n(\mathbb{C})$ for all $t \geq 0$ let us denote

$$z_g(t) = g(0)z(t)g^{-1}(t) \tag{23}$$

$$x_g(t) = g(t)x(t)g^{-1}(t) - g'(t)g^{-1}(t) \tag{24}$$

Then the following proposition holds:

Proposition 3.2. *Let x be a twisted periodic, continuous, matrixvalued function, and let z be the fundamental solution of $z' = zx$. Then*
(a) $z_g(t)$ is the fundamental solution of $z_g' = z_g x_g$
(b) $M(x_g) = g(0)M(x)\tau g^{-1}(0)\tau^{-1}$
(c) If x_1 is twisted periodic and there exists a g_0 such that

$$M(x_1) = g_0 M(x)\tau g_0^{-1}\tau^{-1},$$

then there exists a twisted periodic matrix $g(t)$ such that $g_0 = g(0)$ and $x_g(t) = x_1(t)$ for all $t \geq 0$.

Proof. (a) Let $z_g'(t) = (g(0)z(t)g^{-1}(t))' =$
$(z(t)g^{-1}(t))' =$

$$z'(t)g^{-1}(t) - z(t)g^{-1}(t)g'(t)g^{-1}(t).$$

But $z_g(t)x_g(t) = (z_g(t)g^{-1})(g(t)x(t)g^{-1} - g'(t)g^{-1}(t)) =$
$z(t)g^{-1}(t)g(t)x(t)g^{-1}(t) - z(t)g^{-1}g'(t)g^{-1}(t) =$
$z(t)x(t)g^{-1}(t) - z(t)g^{-1}g'(t)g^{-1}(t) =$

$$z'(t)g^{-1}(t) - z(t)g^{-1}(t)g'(t)g^{-1}(t)$$

This is the assertion.
(b) To show $M(x_g) = g(0)M(x)\tau g^{-1}(0)\tau^{-1})$. By (21) $M(x) = z(\frac{1}{r})$ so $M(x_g) = z_g(\frac{1}{r})$. But this is $g(0)z(\frac{1}{r})g^{-1}(\frac{1}{r})$ by (22). Now we use the definition of $M(x)$ and obtain

$$g(0)z(\tfrac{1}{r})g^{-1}(\tfrac{1}{r}) = g(0)M(x)g^{-1}(\tfrac{1}{r}) = g(0)M(x)\tau g^{-1}(0)\tau^{-1}$$

since $g^{-1}(0+\frac{1}{r}) = \tau g^{-1}(0)\tau^{-1}$. This proves (b).
(c) Put $g(t) = z_1(t)^{-1} g_0 z(t)$ where $z(t), z_1(t)$ are the fundamental solutions of our differential equation and $x(t), x_1(t)$ are the corresponding parameter matrices, then we obtain $g(0) = g_0$.
Now

$$x_g(t) = z_1^{-1} g_0 (zxz^{-1}) g_0^{-1} z_1 - z_1^{-1} g_0 (z'z^{-1}) g_0^{-1} z_1 - (z_1^{-1})' z_1 = \tag{25}$$

$$z_1^{-1} \underbrace{zx}_{z'} z^{-1} z_1 - z_1^{-1} z' z^{-1} z_1 - (z_1^{-1})' z_1 = \tag{26}$$

$$-(-z_1^{-1} z_1' z^{-1}) z_1 = \tag{27}$$

$$-(-z_1^{-1} z_1') = z_1^{-1} z_1' = x_1 \tag{28}$$

□

Remark 3.3. In the calculations above we used the general fact

$$\frac{d}{dt} c^{-1}(t) = -c(t)^{-1} c(t)' c(t)^{-1}$$

Before we can use this result to classify affine adjoint orbits, we recall a general result of differential geometry which is stated as follows in [F]:

Proposition 3.4. *Let $\mathfrak{g}_{\mathbb{C}} \subset M_n(\mathbb{C})$ be a matrix Lie algebra and $G_{\mathbb{C}} \subset GL_n(\mathbb{C})$ the corresponding Lie group. If z is a solution of the linear differential equation $z' = zx$ then $z(t) \in G_{\mathbb{C}}$ for all $t \geq 0$ if and only if $x(t) \in \mathfrak{g}_{\mathbb{C}}$ for all $t \geq 0$.*

Proof. A proof of this statement appears in [Nom] chapter two. □

Now let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an affine Lie algebra of type $X_n^{(r)}$ and let τ be the corresponding digaram automorphism of the underlying finite dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}$ used in the loop realization of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. In the case of an untwisted affine Lie algebra, τ is just the identity on $\mathfrak{g}_{\mathbb{C}}$. Let $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ denote the corresponding compact forms. Now in [W] an "affine shell" (in [F] standard paraboloid) is introduced. This is a submanifold of codimension 2 in $\tilde{\mathfrak{g}}_{\mathbb{C}}$.

$$\mathcal{P}_{\mathbb{C}}^{a,b} = \{x(\cdot) + a_1 C + b_1 D \in \tilde{\mathfrak{g}}_{\mathbb{C}} | 2a_1 b_1 + (x,x) = a, b_1 = b\}$$

where $a, b \in \mathbb{C}$ and $b \neq 0$. The zero hyperplane in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is defined to be the subspace

$$\hat{\mathfrak{g}}_{\mathbb{C}} = \{x(\cdot) + aC + bD \in \tilde{\mathfrak{g}}_{\mathbb{C}} | b = 0\}$$

For the compact case we denote the affine shell by $\mathcal{P}^{a,b}$ with $a, b \in \mathbb{R}$, $a \neq 0$ and the zero hyperplane with $\hat{\mathfrak{g}}$. Now let $\mathcal{O}_X$ be the $\mathcal{L}G_{(\mathbb{C})}$-orbit of X in $\tilde{\mathfrak{g}}_{(\mathbb{C})}$, and let $\mathcal{O}_{g\tau}$ be the $G_{(\mathbb{C})}$ orbit of $g\tau$ in $(G\tau)_{\mathbb{C}}$. Here $G\tau$ denotes the connected component of the principal extension $\tilde{G}$ of the compact group G constructed in the first section. $\tilde{G}_{\mathbb{C}}$ is the corresponding complexification. Now we can use our Proposition and the definition of the coadjoint action to obtain a classification of coadjoint orbits:

Theorem 3.5. *(a) Each $\mathcal{L}(G_{\mathbb{C}}, \tau)$ (resp.$\mathcal{L}(G, \tau)$) orbit in the complex (resp. compact) affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ (resp.$\tilde{\mathfrak{g}}$) is contained either in one of the affine shells $\mathcal{P}_{\mathbb{C}}^{a,b}$ resp. $\mathcal{P}^{a,b}$ or in the zero hyperplane.*
(b) For a fixed affine shell the monodromy map

$$\mathcal{O}_{x(\cdot)+aC+bD} \mapsto \mathcal{O}_{M(\frac{1}{b}x)\tau}$$

is well defined and injective.
(c) For a fixed affine shell the map defined in (b) gives a bijection between the $\mathcal{L}(G_{\mathbb{C}},\tau)$ (resp. $\mathcal{L}(G,\tau)$) orbits in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ (resp.$\tilde{\mathfrak{g}}$) which contain a constant loop and the $G_{\mathbb{C}}$ resp. G orbits in $G_{\mathbb{C}}\tau$ resp. $G\tau$ which contain an element which is invariant under conjugation with τ.

Proof. (a) The first statement follows from the definition of the coadjoint action given above. Just look at the formula for the coadjoint representation and the definition of $\mathcal{P}^{a,b}_{(\mathbb{C})}$.
(b) We look at the map $\mathcal{O}_{x(\cdot)+aC+bD} \mapsto \mathcal{O}_{M(\frac{1}{b})\tau}$ where, as above $M(\frac{1}{b}x) = z(\frac{1}{r})$ and z is the fundamental solution of $z' = z \cdot \frac{1}{b} \cdot x$. Now, we use the definition of the coadjoint action. That is

$$\widetilde{\mathrm{Ad}}(g)(aC + bD + x) = (\tilde{a}C + dD + gxg^{-1} - bg'g^{-1}).$$

where $\tilde{a} = a + (g^{-1}g', y) - \frac{b}{2}(g'g^{-1}, g'g^{-1})$ and $g' = \frac{dg'(t)}{dt}$.
But using 3.2(a) we see, that $\mathcal{O}_{\tilde{a}C+bD+gxg^{-1}-bg'g^{-1}}$ is being mapped to $\mathcal{O}_{z_g(\frac{1}{r}\tau)}$, but reffund(b) yields

$$z_g(\frac{1}{r}) = M(\frac{1}{b}x_g) = g(0)M(\frac{1}{b}x)\tau g(0)^{-1}\tau{-}1,$$

hence

$$z_g(\frac{1}{r})\tau = g(0)z(\frac{1}{r})\tau g(0)^{-1} \in \mathcal{O}_{z(\frac{1}{r})\tau}$$

Here we used again the definition of $M(x)$. So the map is well defined and injectivity follows from 3.2.
(c) If $s\tau \in G_{(\mathbb{C})}\tau$ is invariant under conjugation with τ then so is $r \cdot b \cdot log(s)$ and $\mathcal{O}_{a_1+bD+r\cdot b\cdot log(s)}$ is a preimage of $\mathcal{O}_{s\tau}$ whenever it belongs to $\mathcal{P}^{a,b}$. This proves the second direction. On the other hand, if the orbit $\mathcal{O}_{a_2C+bD+x(\cdot)}$ contains a constant loop $aC + bD + x_0$, then clearly x_0 has to be invariant under conjugation with τ. Now the fundamental solution of the differential equation $z' = z \cdot \frac{1}{b}x_0$ is given by $z(t) = \exp(t\frac{1}{b}x_0)$. Hence $z(\frac{1}{r})$ is invariant under conjugation with τ as well. □

Corollary 3.6. *If* $\mathrm{ord}(\tau) = 1$ *or G is compact, then the monodromy map defined in 2.27(b) is surjective and hence defines a bijection between the $\mathcal{L}G$ - orbits in a fixed affine shell $\mathcal{P}^{a,b}$ and the G orbits in $G\tau$.*

Proof. If $\tau = id$ the statement is trivial, if G is compact we use the fact, that every G orbit in $G\tau$ intersects $S\tau$, where S is a Cartansubgroup of $\tilde{G}$ which contains τ. □

This was the classification of $\mathcal{L}G$ orbits in a fixed affine shell, the fundamental observation is, the these orbits live in the affine Lie algebra $\tilde{\mathfrak{g}}$ and are in one-one correspondence with conjugacy classes in a connected component of the underlying finite dimensional Lie group. It is easy to see, that the image under the monodromy map are the $G_{(\mathbb{C})}$ orbits in $G_{(\mathbb{C})}\tau$ for which there exists a C^∞ path $z : [0, 1] \to G_{(\mathbb{C})}$ such that $z(0) = e$ and $z(t + \frac{1}{r}) = z(\frac{1}{r})\tau z(t)\tau^{-1}$, which could also be written as $M(x)\tau z(t)\tau^{-1}$ for all $t \geq 0$. This could be refereed to as "rolling up the loops". In the case of complex groups, the classification of $\mathcal{L}G_{\mathbb{C}}, \tau$ orbits in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ remains open. Here it is no longer true, that every $G_{\mathbb{C}}$ orbit in $G_{\mathbb{C}}\tau$ contains a $\tau-$ invariant element. (cf. [Mo] for an example), so different arguments may have to be applied. We have also not dealt with the $\mathcal{L}(G, \tau)$ orbits in the zero hyperplane, which are basically the orbits of the adjoint representation of $\mathcal{L}(G, \tau)$ on its Lie algebra. As

we shall see later, the orbits relevant for representation theory are these in a fixed affine shell with $b \neq 0$. Also the classification of $\mathcal{L}(G,\tau)$ orbits in $\tilde{\mathfrak{g}}$ is presumably not manageable, i.e it certainly yields an infinite dimensional "moduli space".
Later we will introduce appropriate measures on the path space we obtained now, through a classification of affine adjoint orbits, to interpret as final goal the numerator of the Kac Weyl character formula as integral over an appropriate closure of an affine adjoint orbit.

4. Double loop groups

This section follows [EF] and the recent lecture notes [KW]. Let Σ_τ denote the elliptic curve, that is a two dimensional torus with a complex structure. We can write this as $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and call $\tau = \tau_1 + i\tau_2$ with $\tau_2 > 0$ the modular parameter of the elliptic curve. Let dz denote the canonical $(1,0)$ form on Σ.

Definition 4.1. *Let $\mathfrak{g}$ be simple finite dimensional complex Lie algebra, then $\mathfrak{g}^\Sigma$ denotes the current algebra of $\mathfrak{g}$ and Σ. A one dimensional central extension of $\mathfrak{g}^\Sigma$ is given by*

$$\hat{\mathfrak{g}} = \mathfrak{g}^\Sigma \oplus \mathbb{C}$$

defined by the following commutator:

$$[(X(z,\bar{z}),a),(Y(z,\bar{z}),b)] = \left([X(z,\bar{z}),Y(z,\bar{z})], \int_\Sigma \langle X, dY\rangle \wedge dz\right).$$

We decompose the exterior derivative $d = \partial + \bar{\partial}$ and write the Lie algebra cocycle ω^Σ as

$$\omega^\Sigma(X,Y) = \int_\Sigma (X, \bar{\partial} Y) \wedge dz.$$

The cocycle can be seen as Lie algebra cocycle on $\mathfrak{g}^\Sigma$ with values in the dual space of holomorphic one forms on Σ. Lie group corresponding to the current algebra $\mathfrak{g}^\Sigma$ is given by G^Σ, where G denotes a simly conncected, complex Lie group, corresponding to $\mathfrak{g}$. As in the loop group case one can define a central extension for the current group G^Σ, which was introduced in [EF].

4.1. **Coadjoint orbits.**

Proposition 4.2. *The smooth part of the dual $(\mathfrak{g}^\Sigma)*$ of the Lie algebra $\hat{\mathfrak{g}}^\Sigma$ can be identified with the space $\mathfrak{g}^\Sigma \oplus \mathbb{C}$ via the pairing*

$$\langle (A,\lambda),(Y,\mu)\rangle = \int_\Sigma (A,Y) dz \wedge d\bar{z} + \lambda\mu$$

where $(.,.)$ is the Killing form on $\mathfrak{g}$. In the coadjoint representation of $\hat{G}^\Sigma$ an element $g \in G^\Sigma$ acts on the space $(\hat{g}^\Sigma)*$ via:

$$g : (A,\lambda) \mapsto (gAg^{-1} + \lambda \bar{\partial} g g^{-1}, \lambda)$$

The center acts trivially in the coadjoint representation, so the this describes the coadjoint action completely.

Definition 4.3. *Let $\mathbb{C}/\mathbb{Z}$ denote the cylinder and consider the holomorphic loop group*

$$\mathcal{L}_{hol} G = \{g : \mathbb{C}/\mathbb{Z} \to G | g \text{ is holomorphic}\}$$

Definition 4.4. *Fix an element $\tau \in \mathbb{C}$. Then the "τ twisted" conjugacy classes of the loop group $\mathcal{L}_{hol}G$ are the orbits of the following action:*

$$g(z) : h(z) \mapsto h^g = g(z-\tau)h(z)g(z)^{-1} \tag{29}$$

Proposition 4.5. *Let Σ be an elliptic curve, $\Sigma = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Fix some $\lambda \neq 0$. As in the loop group case there is a one-one correspondence between the set of coadjoint orbits of the group $\hat{G}^{\Sigma}$ in the hyperplane $\{(X, \lambda)|X \in \mathfrak{g}^{\Sigma}\} \subset (\mathfrak{g}^{\Sigma})*$ and the set of τ twisted conjugacy classes in the holomorphic loop group $\mathcal{L}_{hol}G$.*

Remark 4.6. In the loop group case we considered the left logarithmic derivative

$$z' = Az$$

for an $A \in Gl(n, \mathbb{C})$, now we consider a partial differential equation, the eigenvalue problem for the $\bar{\partial}$ operator

Proof. Associate to an element $(A, \lambda) \in \mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ the partial differential equation

$$\lambda\bar{\partial}\psi = A\psi \tag{30}$$

Let ψ be a solution of (30), defined on the cylinder $\mathbb{C}/\mathbb{Z}$. Because ψ is periodic in τ we have another solution $\psi(z-\tau)$. The function $\nu(z) = \psi(z-\tau)^{-1}\psi(z)$ is again periodic in $\mathbb{Z}$ and holomorphic. Therefore it defines an element in the holomorphic loop group $\mathcal{L}_{hol}G$. Now let $\psi_1(z)$ be another solution of (30), then it will have the form $\psi_1(z) = \psi(z)\mu(z)$ for a G valued function μ, defined on the cylinder $\mathbb{C}/\mathbb{Z}$. We see, that μ satisfies the Cauchy Riemann euquations, since

$$\bar{\partial}(\mu) = \bar{\partial}(\psi^{-1}(z)\psi(z)) = 0$$

So μ is a holomorphic, G valued function on the cylinder and therefore an element of $\mathcal{L}_{hol}G$. Finally this yields

$$\nu(z) = \psi(z-\tau)^{-1}\psi(z) = \mu(z-\tau)^{-1}\psi_1(z-\tau)^{-1}\psi_1(z)\mu(z)$$

so the τ twisted conjugacy class in $\mathcal{L}_{hol}G$ does not depend on the choice of a solution of (30). Let $g \in G^{\Sigma}$ and as before ψ a solution of (30). Then the map $g\psi$ is a solution of the equation

$$\bar{\partial}\psi = (gAg^{-1} + \lambda\bar{\partial}gg^{-1})\psi$$

Since g is periodic in τ we get

$$\psi(z-\tau)^{-1}g(z-\tau)^{-1}g(z)\psi(z) = \psi(z-\tau)^{-1}\psi(z)$$

which shows that the restricted conjugacy class associated to the element $(A, \lambda) \in \mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ is well defined on the level of coadjoint orbits of the central extension $\hat{G}^{\Sigma}$. Thus, we can define a map

$\{\hat{G}^{\Sigma}$ - orbits in the hyperplane $\{(A, \lambda)\} \subset \mathfrak{g}^{\Sigma} \oplus \mathbb{C}\} \rightarrow \{\tau$ twisted conjugacy classes in $\mathcal{L}_{hol}G\}$

Injectivity of the map is clear from the above, the map is surjective since for every element $g \in \mathcal{L}_{hol}G$ we can find a solution of (30), $\psi : \mathbb{C}/\mathbb{Z} \rightarrow G$ such that $g(z) = \psi(z-\tau)^{-1}\psi(z)$. Then we set $A = \frac{1}{\lambda}\bar{\partial}\psi\psi^{-1}$. Since $\psi(z-\tau)^{-1}\psi(z)$ is holomorphic, A is periodic in τ. So A is a smooth map from the elliptic curve Σ to the Lie algebra $\mathfrak{g}$. The conjugacy class corresponding to the coadjoint orbit through $(A, \lambda) \in \mathfrak{g}^{\Sigma} \oplus \mathbb{C}$ is the τ twisted conjugacy class through the element $g \in \mathcal{L}_{hol}G$. $\square$

5. Poisson transformation and the numerator of the character formula

In this section we will present the first steps to derive an analogue of Frenkel's character formula in the twisted case. First we introduce some notions. Let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an arbitrary affine Lie algebra of type $X_n^{(r)}$ and let $\tilde{\mathfrak{g}}'_{\mathbb{C}}$ be the untwisted affine Lie algebra of $X_n^{(1)}$ such that $\tilde{\mathfrak{g}}_{\mathbb{C}} \subset \tilde{\mathfrak{g}}'_{\mathbb{C}}$ and let $\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}'$ be the corresponding compact forms. If $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is untwisted we have $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}'$. Furthermore let R be the root system of the finite dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}$ used to construct $\mathcal{L}(\mathfrak{g}_{\mathbb{C}}, \tau)$ in 2.1, and for a diagram automorphism τ of $\mathfrak{g}_{\mathbb{C}}$ let R^τ be the folded root system introduced in the first section. If $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is a twisted affine Lie algebra with root system $\tilde{R}$, then $R^\tau = R^\circ$. Also, let $(.,.)$ denote the Killing form on $\tilde{\mathfrak{g}}'_{\mathbb{C}}$ and let $(.,.)_\tau$ denote its restriction to $\tilde{\mathfrak{g}}_{\mathbb{C}}$. Now we turn to the analytic Kac Weyl character formula from above. Let Λ be a highest weight of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. There is no essential loss of generality by assuming $\langle \Lambda, d\rangle = 0$. Then, after identifying $\tilde{\mathfrak{h}}_{\mathbb{C}} \cong \tilde{\mathfrak{h}}^*_{\mathbb{C}}$ via $(.,.)_\tau$ we can choose $a \in \mathbb{C}$ and $H \in \mathfrak{h}^\circ_{\mathbb{C}}$ such that $\Lambda + \tilde{\rho} = aD + H$. The condition $\Lambda \in \tilde{P}_+$ implies $a \in i\mathbb{R}$, $Im(a) < 0$ and $H \in i\mathfrak{h}^\circ$. The numerator of the Kac Weyl character formula evaluated at $bD + K$ now reads:

$$\sum_{w \in \tilde{W}} \epsilon(w) e^{(w(aD+H), bD+K)_\tau} \tag{31}$$

We now turn to a theorem of [F] which uses the Poisson transformation formula to give an interesting identity of the numerator of the Kac Weyl character formula.

Proposition 5.1. [F]*4.3.4 The following identity is valid*

$$\sum_{\lambda \in P_+} \chi_\lambda(e^h)\chi_\lambda(e^{-k})^{-\frac{t}{2}|\lambda+\rho|^2} = \left(\frac{2\pi}{t}\right)^{\frac{l}{2}} \mathrm{vol}(Q^\vee) \frac{e^{\frac{1}{2t}||h||^2 + \frac{1}{2t}||k||}}{\sigma(h)\sigma(-k)} \sum_{w \in W} \epsilon(w) e^{-\frac{1}{t}} \langle w(2\pi i d + h), 2\pi i d + k\rangle$$

Proof. Here χ_λ denotes the finite dimensional irreducible character. Taking the usual identity for normalized characters in the finite dimensional theoy one can rewrite the left hand side of the above equality. $\sigma(h)$ and $\sigma(-k)$ denotes the denominator of the Weyl character formula and $h, k \in \mathfrak{h}$. With these definitions we can rewrite the left hand side of the above equation:

$$\sum_{w \in W} \epsilon(w) e^{-\frac{1}{t}} \langle w(2\pi i d + h), 2\pi i d + k\rangle$$

$$= \frac{1}{\sigma(h)\sigma(-k)} \sum_{\lambda \in P_+} \left(\sum_{w_1 \in W} \epsilon(w_1) e^{\langle w_1(\lambda+\rho), h\rangle}\right)\left(\sum_{w_2 \in W} \epsilon(w_2) e^{\langle w_2(\lambda+\rho), -k\rangle}\right) e^{-\frac{t}{2}|\lambda+\rho|^2}$$

$$= \frac{1}{\sigma(h)\sigma(-k)} \sum_{\lambda \in P_+} \sum_{w_1 \in W} \sum_{w \in W} \epsilon(w_1)\epsilon(ww_1) e^{\langle w_1(\lambda+\rho), h - wk\rangle} e^{-\frac{t}{2}|\lambda+\rho|^2}$$

$$= \frac{1}{\sigma(h)\sigma(-k)} \sum_{p \in P} \sum_{w \in W} \epsilon(w) e^{\langle \mu, h - wk\rangle} e^{-\frac{t}{2}||\mu||^2}$$

As usual $\epsilon(w) = (-1)^{l(w)}$ where $l(w)$ denotes the length of Weyl group element. The above calculation is valid, since singular weights do not contribute to the sum. Next one observes that $\langle \tilde{w}(bd + h), bd + k\rangle$ can we rewritten. Here $\tilde{w} = w^{-1} \cdot \gamma, \gamma \in$

$Q^\vee, w \in W$ and the action of $Q^\vee$ on $\tilde{\mathfrak{h}}_\mathbb{C}$ is defined by

$$\gamma(h + ac + bd) = h + ac + bd + b\gamma - \left(\langle h, \gamma\rangle + \frac{b}{2}\langle \gamma, \gamma\rangle\right) c$$

with $\gamma \in Q^\vee, h \in \mathfrak{h}_\mathbb{C} \quad a, b \in \mathbb{C}$. So we first insert into this action and get

$$\langle bd + h + b\gamma - \langle h, \gamma\rangle c - \frac{b}{2}, bd + wk\rangle$$

Now we use the bilinearity and that the properties of $\langle, ., \rangle$ which are given by (2.8). This yields

$$\langle h + b\gamma, wk\rangle - \langle h, b\gamma\rangle - \frac{1}{2}\langle b\gamma, b\gamma\rangle$$

Applying the polarization identity yields

$$= -\frac{1}{2}||b\gamma + h - wk||^2 + \frac{1}{2}||h||^2 + \frac{1}{2}||k||^2$$

Now we insert these calculations into the statement of the theorem:

$$e^{\frac{1}{2t}||h||^2 + \frac{1}{2t}||k||} \sum_{w\in W} \epsilon(w) e^{-\frac{1}{t}}\langle w(2\pi i d + h), 2\pi i d + k\rangle = \sum_{\gamma\in 2\pi i Q^\vee} \sum_{w\in W} \epsilon(w) e^{\frac{1}{2t}||\gamma + h - wk||^2}$$

Now we apply the Poisson transformation formula stated below (a proof of the formula can be found in [Neu]).

Set $f(\mu) = e^{\langle\mu,x\rangle - \frac{t}{2}||\mu||^2}$ and insert into the definition of the Fourier Transformation

$$\hat{f}(\gamma) = \int e^{\langle\mu,\gamma\rangle} f(\mu) d\mu$$

$$\hat{f}(\gamma) = \int e^{\langle\mu,\gamma\rangle - \frac{t}{2}||\mu||^2} d\mu = e^{-\frac{t}{2}||x+y||^2} \int e^{-\frac{t}{2}||\mu - \frac{x+y}{t}||^2} d\mu$$

$$= e^{-\frac{t}{2}||x+y||^2} \left(\frac{2\pi}{t}\right)^{\frac{l}{2}}$$

Now we set $x = h - wk$ and obtain

$$\sum_{p\in P} \sum_{w\in W} \epsilon(w) e^{\langle\mu, h - wk\rangle} e^{-\frac{t}{2}||\mu||^2} = \left(\frac{2\pi}{t}\right)^{\frac{l}{2}} vol(Q^\vee) \sum_{\gamma\in 2\pi i Q^\vee} \sum_{w\in W} \epsilon(w) e^{\frac{1}{2t}||\gamma + h - wk||^2}$$

Putting this calculations together yields the assertion. □

Now we deduce a similar theorem for the case of twisted affine Lie algebras. First remember our identity 31. Let us assume in the following $b \in i\mathbb{R}$ and $K \in i\mathfrak{h}^\circ$. We then set $t = \frac{-1}{ab}$, $H = \frac{H}{a}$ and $k = \frac{K}{b}$ yielding $t \in \mathbb{R}_+$ and $h, k \in \mathfrak{h}$. With $2\pi i d = D$ the sum $\sum_{w\in\tilde{W}} \epsilon(w) e^{(w(aD+H), bD+K)_\tau}$ now reads

$$\sum_{w\in\tilde{W}} \epsilon(w) e^{\frac{-1}{t}(w(2\pi i D + h), 2\pi i D + k)_\tau} \tag{32}$$

Let us set $c = 2\pi i C$. Then the lattice M operates on $\tilde{h}_\mathbb{C}$ like in the theorem from [F] above. That is

$$\gamma(h + ac + bd) = h + ac + bd - ((h, \gamma)_\tau + \tfrac{ba_0}{2}||\gamma||_\tau^2)c$$

Now the same calculation as above shows for $w \in W^\circ$, $\gamma \in M$ and $w^{-1}\gamma \in \tilde{W}$:

$$(w^{-1}\gamma(2\pi id+h),2\pi id+k)_\tau = \tfrac{-1}{2}||2\pi ia_0\gamma+h-wk||^2_\tau + \tfrac{1}{2}||h||^2_\tau + \tfrac{1}{2}||k||^2_\tau.$$

Now we put our calculations together and this yields:

(33)

$$\sum_{w\in\tilde{W}}\epsilon(w)e^{(w(aD+H),bD+K)_\tau} = \sum_{w\in W^\circ}\epsilon(w)e^{\frac{-1}{t}(\frac{-1}{2}||2\pi ia_0\gamma+h-wk||^2_\tau+\frac{1}{2}||h||^2_\tau+\frac{1}{2}||k||^2_\tau)} =$$

$$\sum_{w\in W^\circ}\epsilon(w)e^{\frac{1}{2t}||2\pi ia_0\gamma+h-wk||^2_\tau}\cdot e^{\frac{-1}{t}\cdot(\frac{1}{2}||h||^2_\tau+\frac{1}{2}||k||^2_\tau)} =$$

$$\sum_{w\in W^\circ}\epsilon(w)e^{\frac{1}{2t}||2\pi ia_0\gamma+h-wk||^2_\tau}\cdot e^{\frac{-1}{2t}\cdot(||h||^2_\tau+||k||^2_\tau)} =$$

$$e^{\frac{-1}{2t}\cdot(||h||^2_\tau+||k||^2_\tau)}\cdot\sum_{w\in W^\circ}\epsilon(w)e^{\frac{1}{2t}||2\pi ia_0\gamma+h-wk||^2_\tau} =$$

$$e^{\frac{-1}{2t}\cdot(||h||^2_\tau+||k||^2_\tau)}\cdot\sum_{\gamma\in 2\pi ia_0M}\sum_{w\in W^\circ}\epsilon(w)e^{\frac{1}{2t}||\gamma+h-wk||^2_\tau} =$$

$$e^{\frac{-1}{2t}\cdot||h||^2_\tau-\frac{1}{2t}||k||^2_\tau}\cdot\sum_{\gamma\in 2\pi ia_0M}\sum_{w\in W^\circ}\epsilon(w)e^{\frac{1}{2t}||\gamma+h-wk||^2_\tau}$$

We now need to apply again the Poisson transformation formula. Recall, that for a euclidean vector space V, a lattice $Q\in V$ and a Schwartz function $f:V\to\mathbb{C}$ one has:

$$\sum_{\mu\in Q^\vee}\hat{f}(\mu) = \mathrm{vol}Q\sum_{\gamma\in Q}f(\gamma)$$

with

$$\hat{f}(\mu) = \int_V e^{2\pi i(\gamma,\mu)}f(\gamma)d\gamma.$$

For a fixed $x\in\mathfrak{h}^\circ$ set $f(\mu)=e^{(x,\mu)_\tau-\frac{t}{2}||\mu||^2_\tau}$. Then we get with the same calculation as in Frenkel's theorem:

$$\hat{f}(\gamma) = (\frac{2\pi}{t})^{\frac{l}{2}}e^{\frac{1}{2t}||x+2\pi i\gamma||^2_\tau}$$

with $l=\dim_{\mathbb{R}}\mathfrak{h}^\circ$. So for $x=h-wk$ with $h,k\in\mathfrak{h}^\circ$ and $w\in W^\circ$ we obtain the identity

$$(34)\qquad \sum_{\gamma\in a_0M}e^{\frac{1}{2t}||2\pi\gamma+h-wk||^2_\tau} = \mathrm{vol}(a_0M)^{-1}(\frac{2\pi}{t})^{\frac{l}{2}}\sum_{\mu\in(a_0M)^\vee}e^{(\mu,h-wk)_\tau-\frac{t}{2}}||\mu||^2_\tau$$

Here we just inserted $\hat{f}(\mu)$ in the last sum of the above expression. We will now go on and try to analyse which types of root systems can generate our lattice M. It will turn out, that we can exploit the root system R^1 introduced in the first section, and this will give us the possibility to use the characters of the underlying non connected Lie group.

If $\tilde{R}$ is of type $Aff1$ then θ is a long root in R°, and if $\tilde{R}$ is of type $Aff2$ or $Aff3$, but not of type $A^{(2)}_{2n}$, then θ is a short root in R°. In case $\tilde{R}$ is of type $A^{(2)}_{2n}$ then R° is of type BC_n, and θ is a root of medium length in R°. So if $\tilde{R}$ is of type $X^{(r)}_n$ with $r=2,3$ and $\tilde{R}\neq A^{(2)}_{(2n)}$ then θ° is a long root in $R^{\circ\vee}$ and hence M is the lattice which is generated by the long roots in $R^{\circ\vee}$. If $\tilde{R}$ is of type $A^{(2)}_{(2n)}$ then $\theta^\vee$ is of medium length in $R^{\circ\vee}$ and in this case we have $a_0=2$. Thus in all cases M is the lattice generated by the root system R^1 from the first section. Now for an arbitrary root system S let $P^\circ(S)$ denote the weight lattice of S. Then the above implies:

$$\sum_{\gamma\in a_0M} e^{\frac{1}{2t}||2\pi\gamma+h-wk||_\tau^2} = \mathrm{vol}(a_0M)^{-1}\left(\frac{2\pi}{t}\right)^{\frac{l}{2}} \sum_{\mu\in P^\circ(R^1)} e^{(\mu,h-wk)_\tau-\frac{t}{2}||\mu||_\tau^2} \tag{35}$$

Putting the above formulas together we get:

$$\sum_{w\in\tilde W}\epsilon(w)e^{(w(aD+H),bD+K)_\tau} = e^{\frac{-1}{2t}\cdot||h||_\tau^2-\frac{1}{2t}||k||_\tau^2}\cdot\sum_{\gamma\in 2\pi i a_0M}\sum_{w\in W^\circ}\epsilon(w)e^{\frac{1}{2t}||\gamma+h-wk||_\tau^2} \tag{36}$$

Then we get because of the Poisson resummation of the sum, which ranges over $\gamma\in a_0M$ and the above root system considerations:

$$\sum_{w\in\tilde W}\epsilon(w)e^{(w(aD+H),bD+K)_\tau} = \frac{e^{\frac{-1}{2t}\cdot||h||_\tau^2-\frac{1}{2t}||k||_\tau^2}}{\mathrm{vol}(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}}\sum_{w\in W^\circ}\sum_{\mu\in P^\circ(R^1)}\epsilon(w)e^{(\mu,h-wk)_\tau-\frac{t}{2}||\mu||_\tau^2} \tag{37}$$

Now let $W(R^1)$ denote the Weyl group of the root system R^1. It is a well known fact that after the choice of a basis of R^1 every weight $\lambda\in P^\circ(R^1)$ is conjugate under $W(R^1)$ to some dominant weight $\lambda'\in P^\circ_+(R^1)$. Since the root systems R^1 and R° are dual to each other, we have $W^\circ = W(R^1)$. So we apply this observation to the double some in the last expression above and get:

$$\sum_{w\in W^\circ}\sum_{\mu\in P^\circ(R^1)}\epsilon(w)e^{(\mu,h-wk)_\tau-\frac{t}{2}||\mu||_\tau^2} =$$

$$\sum_{\mu\in P^\circ_+(R^1)}\sum_{w\in W^\circ}\sum_{w'\in W^\circ}\epsilon(ww')\epsilon(w')e^{(w'\mu.h-wk)}e^{\frac{t}{2}||\mu||_\tau^2}$$

Here we have identified $\mathfrak{h}^\circ$ and $\mathfrak{h}^{\circ *}$ via $(.,.)$. In the above calculation singular weights cancel out, so it is enough to some over the strictly dominant weights, or equivalently to replace λ by $\lambda+\rho^r$ with $\rho^r = \frac{1}{2}\sum_{\bar\alpha\in R^1_+}\bar\alpha$ as in the introduction. Hence

$$\sum_{w\in W^\circ}\sum_{\mu\in P^\circ(R^1)}\epsilon(w)e^{(\mu,h-wk)_\tau-\frac{t}{2}||\mu||_\tau^2} =$$

$$\sum_{\lambda\in P^\circ_+(R^1)}\sum_{w\in W^\circ}\sum_{w'\in W^\circ}\epsilon(ww')\epsilon(w')e^{\langle w'(\lambda+\rho^\tau),h-wk\rangle}e^{-\frac{t}{2}||\lambda+\rho^\tau||_\tau^2}$$

Now the characters of the underlying non connected Lie group come into play. Remember the definition of χ^τ_λ, A^τ and δ^τ. We had $A^\tau(\mu) = \sum_{w\in W^\tau}\epsilon(w)\cdot e(w\mu)$ and ρ^τ where ρ^τ is the same as above. Furthermore we defined δ^τ as $\delta^\tau = e(\rho^\tau)\cdot\prod_{\bar\alpha\in R^1_+}(1-e(-\bar\alpha))$. And we defined χ^τ_λ to be $A^\tau(\lambda+\rho^\tau)/\delta^\tau$. So it is easy to see the our sum above can be written as

$$\sum_{\lambda\in P^\circ_+(R^1)}\delta^\tau(h)\delta\tau(-k)\chi^\tau_\lambda(h)\chi^\tau_\lambda(-k)e^{-\frac{t}{2}||\lambda+\rho^\tau||_\tau^2}$$

As before, let $\mathfrak{g}_\mathbb{C}$ be the finite dimensional complex Lie algebra used to construct $\mathcal{L}(\mathfrak{g}_\mathbb{C},\tau)$ with root system R and compact form $\mathfrak{g}$. Let G be the simply connected compact Lie group belonging to $\mathfrak{g}$ and let $G\tau$ denote the connected component belonging to the non connected Lie group $G \rtimes \langle\tau\rangle$ containing τ. In the first section we set $\chi^\tau_\lambda(h) = \tilde{\chi}_\lambda(e^h\tau)$ for $h \in \mathfrak{h}^\circ$. Here $\tilde{\chi}$ denotes the character of the $G \rtimes \langle\tau\rangle$ belonging to the highest weight λ. Observe that in the notion of the first section we have $\mathfrak{h}^\circ = LS_0$. So putting everything together, we have the following theorem, which is the analogue of the theorem proofed in [F]:

Theorem 5.2. *For $h, k \in \mathfrak{h}^\circ$ one has*

$$e^{\frac{1}{2t}||h||^2_\tau} e^{\frac{1}{2t}||k||^2_\tau} \sum_{w\in\tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi id+h),2\pi id+k)_\tau} =$$

$$\frac{\delta^\tau(h)\delta^\tau(-k)}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}} \sum_{\lambda\in P^\circ_+(R^1)} \chi_\lambda(e^h\tau)\chi_\lambda(e^{-k}\tau)e^{-\frac{t}{2}||\lambda+\rho^\tau||^2_\tau}$$

Proof. One has:

$$\sum_{w\in\tilde{W}} \epsilon(w)e^{(w(aD+H),bD+K)_\tau} = \frac{e^{\frac{-1}{2t}\cdot||h||^2_\tau - \frac{1}{2t}||k||^2_\tau}}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}} \sum_{w\in W^\circ} \sum_{\mu\in P^\circ(R^1)} \epsilon(w)e^{(\mu,h-wk)_\tau - \frac{t}{2}||\mu||^2_\tau}$$

But $\sum_{w\in W^\circ}\sum_{\mu\in P^\circ(R^1)} \epsilon(w)e^{(\mu,h-wk)_\tau-\frac{t}{2}||\mu||^2_\tau}$ was deduced to be

$$\sum_{\lambda\in P^\circ_+(R^1)} \delta^\tau(h)\delta^\tau(-k)\chi^\tau_\lambda(h)\chi^\tau_\lambda(-k)e^{-\frac{t}{2}||\lambda+\rho^\tau||^2_\tau}$$

and the characters in the above sum are exactly $\chi_\lambda(e^h\tau)$ resp. $\chi_\lambda(e^{-k}\tau)$. Inserting this into the above sum one get

$$\sum_{w\in\tilde{W}} \epsilon(w)e^{(w(aD+H),bD+K)_\tau} =$$

$$\frac{e^{\frac{-1}{2t}\cdot||h||^2_\tau - \frac{1}{2t}||k||^2_\tau}}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}} \sum_{\lambda\in P^\circ_+(R^1)} \delta^\tau(h)\delta^\tau(-k)\chi_\lambda(e^h\tau)\chi_\lambda(e^{-k}\tau)e^{-\frac{t}{2}||\lambda+\rho^\tau||^2_\tau}.$$

But this expression is clearly

$$e^{\frac{1}{2t}||h||^2_\tau} e^{\frac{1}{2t}||k||^2_\tau} \sum_{w\in\tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi id+h),2\pi id+k)_\tau} =$$

$$\frac{\delta^\tau(h)\delta^\tau(-k)}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}} \sum_{\lambda\in P^\circ_+(R^1)} \chi_\lambda(e^h\tau)\chi_\lambda(e^{-k}\tau)e^{-\frac{t}{2}||\lambda+\rho^\tau||^2_\tau}$$

□

6. The heat equation on a compact Lie group

Let Δ_G denote the Laplacian on the compact simply connected Lie group G with respect to the Riemannian metric on G induced by the negative of the Killing form on $\mathfrak{g}_{\mathbb{C}}$. We can pull back this metric to G_τ such that right multiplication with τ induces an isometry between the Riemannian manifolds G and $G\tau$. The Laplacian on $G\tau$ shall be denoted with $\Delta_{G\tau}$. Now for a fixed parameter $T > 0$ the heat equation on $G\tau$ reads

$$\frac{\partial f(g\tau, t)}{\partial t} = \frac{sT}{2}\Delta_{G\tau} f(g\tau, t)$$

with $g \in G$, $s \in \mathbb{R}$, $s > 0$ and $f : G\tau \to \mathbb{R}$ is continuous in both variables, C^2 in the first and C^1 in the second variable. The fundamental solution of the heat equation is defined by the initial data

$$f(g\tau, t)|_{t=+0} = \delta_\tau(g\tau)$$

where δ_τ is the Dirac delta distribution centered at $\tau \in G\tau$. For a highest weight $\lambda \in P_+^\circ(R)$ of G let $d(\lambda)$ denote the dimension of the corresponding irreducible representation of G and χ_λ its character. Then by results of [Fe] the fundamental solution of the heat equation is given by

$$u_s(g,t) = \sum_{\lambda \in P_+^\circ(r)} d(\lambda)\chi_\lambda(g) e^{-\frac{sT}{2}(||\lambda+\rho||^2 - ||\rho||^2)} \tag{38}$$

Now G and $G\tau$ are isometric as Riemannian manifolds, so the fundamental solutions of the corresponding heat equations coincide. That is, the fundamental solution of the heat equation on $G\tau$ is given by

$$v_s(g\tau,t) = \sum_{\lambda \in P_+^\circ(r)} d(\lambda)\chi_\lambda(g) e^{-\frac{sT}{2}(||\lambda+\rho||^2 - ||\rho||^2)} \tag{39}$$

There is a well known identity for the characters of a compact group which follow easily from the orthogonality relations

$$d(\lambda)\int_G \chi_\lambda(g_1 g g_2^{-1} g^{-1})dg = \chi_\lambda(g_1)\chi_\lambda(g_2^{-1}),$$

where dg denotes the normalized Haar measure on G. Now using a version of the orthogonality relations for non connected groups one can deduce an analogous formula for the characters of the outer component.

$$d(g)\int_G \chi_\lambda(g_1 \tau g \tau^{-1} g_2^{-1} g^{-1})dg = \chi_\lambda(g_1\tau)\chi_\lambda(\tau^{-1}g_2^{-1})$$

Hence we obtain by theorem 2.17 that $\chi_\lambda(g\tau) = 0$ if λ is not τ invariant. Furthermore, we have $||\lambda+\rho||^2 = ||\lambda+\rho^\tau||_\tau^2$ if λ is τ invariant Thus we can use theorem 5.2 exchanging the roles of s and t and fixing the parameter value $s = T$ we have proved the following proposition.

Theorem 6.1.

$$\sum_{w\in\tilde{W}} \epsilon(w) e^{-\frac{1}{t}(w(2\pi i d+h), 2\pi i d+k)_\tau} =$$

$$\frac{e^{-\frac{1}{2t}||h||_\tau^2} e^{-\frac{1}{2t}||k||_\tau^2} e^{-\frac{t}{2}||\rho^\tau||_\tau^2}\delta^\tau(h)\delta^\tau(-k)}{\mathrm{vol}(\mathbb{Z}R^1)\left(\frac{2\pi}{t}\right)^{\frac{l}{2}}} \cdot \int_G v_{\frac{t}{T^2}}(ge^h\tau g^{-1}\tau^{-1}e^{-k}\tau, T)dg$$

7. Wiener measure

The Wiener measure of a euclidean vector space with variance $s > 0$ is a measure ω_V^s on the Banach space of paths

$$C_V = \{x : [0,T] \to V | x(0) = 0, x continuous\}$$

(the space has as norm the supremum norm) and is defined using the fundamental solution $w_s(x,t)$ of the heat equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{sT}{2} \Delta_V f(x,t)$$

on V as follows: First, one defines cylinder sets on C_V to be the following subsets of C_V

$$\{x \in C_V | x(t_1) \in A_1, \ldots, x(t_m) \in A_m\}$$

with $0 < t_1 \geq t_2, \ldots, \geq t_m \geq T$, $m \in \mathbb{N}$, and where $A_1, \ldots, A_m$ are Borel sets in V. Then the Wiener measure ω_V^s of variance $s > 0$ is defined on the cylinder sets of C_V via:

$$\omega_V^s(x(t_1) \in A_1, \ldots, x(t_m) \in A_m) = \int_{A_1} \cdots \int_{A_m} w_s(\Delta x_1, \Delta t_1) \ldots w_s(\Delta x_m, \Delta t_m) dx_1 \ldots dx_m$$

where dx is a Lebesgue measure on V and we have set $x_k = x(t_k), \Delta x_k = x_k - x_{k-1}, \Delta t_k = t_k - t_{k-1}$ and $x_0 = 0$. The conditional Wiener measure $\omega_{V,X}^s$ of variance $s > 0$ is defined on the closed subspace $C_{V,X} \subset C_V$ with fixed endpoints $x(t) = X$ on the cylinder sets via

$$\omega_{V,X}^s(x(t_1) \in A_1, \ldots, x(t_{m-1}) \in A_{m-1}) = \int_{A_1} \cdots \int_{A_m} w_s(\Delta x_1, \Delta t_1) \ldots w_s(\Delta x_m, \Delta t_m) dx_1 \ldots dx_{m-1}$$

where additionally $x_m = X$ and $t_m = T$. Now a classical result in the theory of Wiener measure states that the "measures" $\omega_V^s, \omega_{V,X}^s$ are σ additive on the σ algebra generated by the cylinder sets in C_V and $C_{V,X}$ respectively. This result is proved in [Kuo] chapter 4. Furthermore, the σ algebras generated by the cylinder sets are exactly the Borel σ algebras of the respective Banach spaces. As another result we have

$$\omega_V^s(C_V) = 1 \text{ and}$$
$$\omega_{V,X}^s(C_V, X) = w_s(X,T)$$

where $w_s(x,t)$ is the fundamental solution of the heat equation on V. Using the fundamental solution of the heat equation on the compact Lie group G we can define Wiener measure ω_G^s and the conditional Wiener measure $\omega_{G,Z}^s$ on the complete metric space

$$C_G = \{z : [0,T] \to T | z(0) = e, z continuous\}$$

and $C_{G,Z} = \{z \in C_G, Z(T) = Z\}$ in exactly the same fashion as the Wiener measure on V. The metric on C_G is given by $\varepsilon(z, z_1) = \sup_{t \in [0,T]} \varepsilon_0(z(t), z_1(t))$ where $\varepsilon_0(g, g_1)$ denotes the length of the shortest geodesic in G connecting two given points g, g_1 where the metric on G still being given by the negative of the Killing form on $\mathfrak{g}$. For the convenience of the reader we state the notion of Wiener measure on a compact Lie group, although it is exactly similar to the definition in

euclidean space.
As before we call cylinder sets the following subsets of C_G

$$\{z(\cdot) \in C_G : z(t_1) \in A_1, \ldots, z(t_m) \in A_m, 0 < t_1 < \ldots < t_m \leq T\}$$

where $A_1, \ldots, A_m$ are Borel subsets of G. Now the Wiener measure of variance $t > 0$ is defined on the cylinder sets of C_G by (a)

$$\omega^s_G(x(t_1) \in A_1, \ldots, x(t_m) \in A_m) = \int_{A_1} \cdots \int_{A_m} w_s(\Delta x_1, \Delta t_1) \ldots w_s(\Delta x_m, \Delta t_m) dx_1 \ldots dx_m$$

(b) The conditional Wiener measure reads:

$$\omega^s_{G,Z}(x(t_1) \in A_1, \ldots, x(t_{m-1}) \in A_{m-1}) = \int_{A_1} \cdots \int_{A_m} w_s(\Delta x_1, \Delta t_1) \ldots w_s(\Delta x_m, \Delta t_m) dx_1 \ldots dx_m$$

where dz is a Haar measure on G and $\Delta z_k = z_k z_{k-1}^{-1}, z_0 = e, \Delta t_k = t_k - t_{k-1}, t_0 = e$ and in case (b).
One of the fundamental properties of Wiener measure is its translation quasi invariance. This is a consequence of the Cameron Martin theorem, see e.g [Str] for a nice exposition on this theorem. First note, that we denote, following [F] and [W1] an integral with respect to Wiener measure by:

$$\int_{C_V} f(x) d\omega^s_V$$

if f is an integrable function and instead of the above expression by the symbolic expression:

$$\int_{C_V} f(x) e^{-\frac{1}{2t}(x',x')} d\omega^s_V$$

This notion is introduced in [F] and can be justified since the Wiener measure is viewed as pullback of the standard Gaussian measure on $\mathbb{R}^n$. Frenkel deduces the translation quasivariance of Wiener measure through the symbolic expression above:

$$\int_{C_V} f(x) e^{-\frac{1}{2t}(x',x')} d\omega^s_V = \int_{C_V} f(x+y) \{e^{-\frac{1}{2t} e^{(x'+y',x'+y')}}\} = \int_{C_V} f(x+y) e^{-\frac{1}{t}(x',y') - \frac{1}{2t}(y',y')} \{e^{-\frac{1}{2t}(x',x')} d\omega^s_V\}$$

Here $f : C_V \to \mathbb{R}$ is an integrable function an $y \in C_V$ is a C^∞ path. One can deduce the quasi-invariance for the conditional Wiener measure similar:
If $f : C_{V,X} \to \mathbb{R}$ is an integrable function the translation quasi-invariance now reads:

$$\int_{C_{V,X}} f(x) d\omega^s_{V,X}(x) = \int_{C_{V,X+Y}} f(x+y) e^{-\frac{1}{s}(x',y') - \frac{1}{2s}(x',y')} d\omega^s_{V,X+Y}(x)$$

with $Y = y(T)$ In the formulas above, (x', y') denotes the Stieltjes integral

$$\frac{1}{T}\int_0^T (y'(t), dx(t))$$

and $(.,.)$ denotes the scalar product on the vector space V. In Ito's construction of BM on a compact Lie group, an important connection between BM on the Lie algebra and BM on the Lie group appeared. The so called Ito map. Let $y \in C_{\mathfrak{g}}$ be a continuous path. For an $n \in \mathbb{N}$ and for $k = 0, \dots, 2^n - 1$ a path $z_n : [0,T] \to G$ is defined by $z_n(0) = e$ and

$$z_n(t) = z_n(\frac{k}{2^n}T)\exp(y(t) - y(\frac{k}{2^n}T))\text{for}\frac{k}{2^n}T < t \le \frac{k+1}{2^n}T$$

The goal of this construction is, as above, to deduce an isomorphism between the Wiener measures on the Lie algebra and the Lie group. If y is a differentiable path, then $\lim_{n\to\infty} z_n$ is the fundamental solution of the differential equation $z' = zy'$ and hence defines a path in G. Now a map i is defined, which according to [McKean] turns out to be an isomorphism. The map

$$i : C_{\mathfrak{g}} \to C_G$$

is defined via

$$y \to \lim_{n\to\infty} z_n \text{if the limit exists and} e \text{if the limit does not exist.}$$

Now the fundamental result of [McKean] is the following: the series z_n converges with $n \to \infty$ in the topology of C_G almost everywhere with respect to the Wiener measure on the Lie algebra, which is as usual denoted by $\omega^s_{\mathfrak{g}}$. Then as above the map i induces a measure on C_G, the Wiener measure ω^s_G. So i defines an isomorphism

$$I : L_1(C_{\mathfrak{g}}, \omega^s_{\mathfrak{g}}) \to L_1(C_G, \omega^s_G)$$

The proof of this appears in [McKean] page 117-123. We can state the analogous quasitransformation property of Wiener measure on a compact Lie group. Let $f : C_G \to \mathbb{R}$ be a integrable function on C_G, let $g \in C_G$ be a C^∞ path. Then the translation quasi-invariance of the Wiener measure with variance $s > 0$, respectively the conditional Wiener measure reads:

$$\int_{C_G} f(z)d\omega^s_G(z) = \int_{C_G} f(zg)e^{-\frac{1}{s}(z^{-1}z', g'g^{-1})_{\mathfrak{g}} - \frac{1}{2s}(g^{-1}g', g^{-1}g')_{\mathfrak{g}}}d\omega^s_G(z) \tag{40}$$

and for the conditional Wiener measure:

$$\int_{C_{G,Z}} f(z)d\omega^s_{G,Z}(z) = \int_{C_{G,zg(T)^{-1}}} f(zg)e^{-\frac{1}{s}(z^{-1}z', g'g^{-1})_{\mathfrak{g}}} \cdot e^{-\frac{1}{2s}(g^{-1}g', g^{-1}g')_{\mathfrak{g}}}d\omega^s_{G,Zg(T)^{-1}}(z)$$

Here the term $(z^{-1}z', g'g^{-1})_{\mathfrak{g}}$ should be interpreted as the Stieltjes integral

$$\frac{1}{T}\int_0^T (g'g^{-1}, d(i^{-1}(z)))_{\mathfrak{g}}$$

where $(.,.)$ denotes the negative of the Killing form on $\mathfrak{g}$. (The subscirpt $\mathfrak{g}$ is added in order to avoid confusions in the calculations below). Note, that i^{-1} is according to the results of [McKean] indicated above, a well defined map almost everywhere on C_G with respect to ω^s_G. In [F] Frenkel calculates the following path integral using the translation quasi-invariance of the Wiener measure.

Proposition 7.1. *([F]Prop.5.2.12) Let $K, Y \in \mathcal{L}(\mathfrak{g}, \tau)$, $g \in \mathcal{L}(G, \tau)$ be elements such that $K = gYg^{-1} - g'g^{-1}$ and $g(0) = e$ than one has*

$$e^{-\frac{||Y||^2_{\mathfrak{g}}}{2t}} \int_{C_{G,Z}} e^{\frac{1}{t}(z^{-1}z',Y)_{\mathfrak{g}}} d\omega^s_{G,Z}(z) = e^{-\frac{||K||^2_{\mathfrak{g}}}{2t}} \int_{C_{G,Z}} e^{\frac{1}{t}(z^{-1}z',K)_{\mathfrak{g}}} d\omega^s_{G,Z}(z) =$$

$v_s(Za_0^{-1}, T)$ *where $a_0 = a(T), Y = a^{-1}a', a(\cdot)$ is a C^∞ path in C_G*

Proof. We first make a change of variables in the first integral and then this reads:

$$\int_{C_{Gg_0^{-1},Z}} e^{\frac{1}{t}((zg)^{-1}(zg)',Y)_{\mathfrak{g}}} d\omega^s_{G,Zg_0^{-1}}(zg)$$

Let us first calculate the inner product, that is

$$((zg)^{-1}(zg'), Y)_{\mathfrak{g}} = ((g^{-1}z^{-1}(z'g + zg')), Y)_{\mathfrak{g}} =$$
$$(g^{-1}z^{-1}z'g, Y)_{\mathfrak{g}} + (g^{-1}z^{-1}zg', Y)_{\mathfrak{g}} = (g^{-1}z^{-1}z'g, Y)_{\mathfrak{g}} + (g^{-1}g', Y)_{\mathfrak{g}} =$$
$$(z^{-1}z', gYg^{-1})_{\mathfrak{g}} + (g^{-1}g', Y)_{\mathfrak{g}}$$

Inserting this in the integral above and considering the quasi-invariance of the integral we get:

$$\int_{C_{Gg_0^{-1},Z}} e^{\frac{1}{t}(g^{-1}g',Y)_{\mathfrak{g}} + \frac{1}{t}(z^{-1}z',gYg^{-1})_{\mathfrak{g}} - \frac{1}{t}(z^{-1}z',g'g^{-1})_{\mathfrak{g}} - \frac{1}{2t}(g^{-1}g',g^{-1}g')_{\mathfrak{g}}} d\omega^s_{G,Zg_0^{-1}}(z)$$

where $g(T) = g_0$. Let $g(\cdot) \in \mathcal{L}(G, \tau)$ i.e $g(T) = g(0) = e$ so we obtain the first part of the statement. For the second part let $g = a$, hence $g_0 = a_0$,$g^{-1}g' = Y$ since $Y = a^{-1}a'$ in the assumption of the proposition. So the inner product in our integral is now:
$-\frac{1}{2}(Y, Y) + (g^{-1}g', Y) + (z'z^{-1}, gYg^{-1}) - (z^{-1}z', g'g^{-1}) - \frac{1}{2}(g^{-1}g', g^{-1}g') = 0$
and we obtain the second part. □

In [W1] a slightly different notion is introduced, but the statement and the proof of the theorem stays the same. The theorem in [W1] reads:

$$e^{-\frac{||Y||^2_{\mathfrak{g}}}{2t}} \int_{C_{G,Z}} e^{\frac{1}{t}(z^{-1}z',Y)_{\mathfrak{g}}} d\omega^s_{G,Z}(z) = u_s(Zg(T)^{-1}, T),$$

where $g \in C_G$ is a C^∞ path and $g' = gY$. Now let $\mathcal{O}_{g\tau}$ denote the $G-$ orbit in $G\tau$ containing the element $g\tau$. Multiplying each element of $G\tau$ with τ^{-1} we can identify $\mathcal{O}_{g\tau}$ with a $G-$ orbit in G, where G acts on itself by twisted conjugation: $(h, g) \mapsto (hg\tau h^{-1}\tau^{-1})$. (In fact, this is twisted conjugation and the multiplication with τ^{-1}). This $G-$ orbit obtained by the above statement will be denoted by $\mathcal{O}_{g\tau}$ as well. We will rewrite Theorem 6.1 as integral over G. To do this we will again need a path space. So let us define $C_{G,\mathcal{O}_{g\tau}} \subset C_G$ to be the space

$$\mathcal{C} = \{C([0, T] \to G)\} \quad \text{with} \quad z(T) \in \mathcal{O}_{g\tau}$$

So we have a space of continuous path with endpoints in $\mathcal{O}_{g\tau}$. And on this path space we can introduce conditional Wiener measure $\omega^s_{G,\mathcal{O}g\tau}$. This is defined by

$$\int_{C_{G,\mathcal{O}g\tau}} f(z) d\omega^s_{G,\mathcal{O}g\tau}(z) = \int_G (\int_{C_{G,g_1g\tau g_1^{-1}\tau^{-1}}} f(z) d\omega^s_{C_{G,g_1g\tau g_1^{-1}\tau^{-1}}}(z)) dg_1$$

where f is integrable on $C_{G,g_1g\tau g_1^{-1}\tau^{-1}}$ for almost all $g_1 \in G$. Now we insert this definition in Theorem 7.1 and get:

Corollary 7.2. *Let $Y \in \mathcal{L}(g,\tau)$ and let $g \in C_G$ be a C^∞ path such that $g' = gY$ Then*

$$e^{-\frac{||Y||^2_{\mathfrak{g}}}{2t}} \int_{C_{G,Z}} e^{\frac{1}{t}(z^{-1}z',Y)_{\mathfrak{g}}} d\omega^s_{G,Z}(z) = \int_G u_s(g_1 Z\tau g_1^{-1}\tau^{-1}g(T)^{-1},T)dg_1$$

where as before $u_s(z,z)$ is the fundamental solution of the heat equation on G.

So let us introduce the path space

$$C_{G\tau} = \tilde{z} : [0,T] \to G\tau | \tilde{z}\tau^{-1} \in C_G$$

Now it is easy to introduce Wiener measure on this path space. This is exactly the same procedure as in the cases described before. Since the definition of Wiener measure depends on the fundamental solution of the heat equation and we observed, that G and $G\tau$ are isometric as Riemannian manifolds, so the fundamental solutions of the heat equations coincide. So Wiener measures on this spaces coincide as well. Due to this fact, we can define Wiener measure on $G\tau$ through Wiener measure on G.

$$\int_{C_{G_\tau}} f(\tilde{z})d\omega^s_{G\tau}(\tilde{z}) = \int_{C_G} \hat{f}(\tilde{z}\tau^{-1})d\omega^s_G(\tilde{z}^{-1}),$$

where $\hat{f}$ is a function on C_G which is given by $\hat{f}(z) = f(z\tau)$. The conditional Wiener measures $C_{G\tau,Z\tau}$ and $C_{G\tau,\mathcal{O}_{\mathfrak{g}\tau}}$ are defined analogously. Now recall 6.1 which reads:

Theorem 7.3.

$$\sum_{w\in\tilde{W}} \epsilon(w)e^{-\frac{1}{t}(w(2\pi id+h),2\pi id+k)_\tau} =$$

$$\frac{e^{-\frac{1}{2t}||h||^2_\tau}e^{-\frac{1}{2t}||k||^2_\tau}e^{-\frac{t}{2}||\rho^\tau||^2_\tau}\delta^\tau(h)\delta^\tau(-k)}{\mathrm{vol}(\mathbb{Z}R^1)\left(\frac{2\pi}{t}\right)^{\frac{l}{2}}} \cdot \int_G v_{\frac{t}{T^2}}(ge^h\tau g^{-1}\tau^{-1}e^{-k}\tau,T)dg$$

Looking at this expression we first observe, that we will have to change our paramter in the fundamental solution of the heat equation in Corollary 7.2. So let us set $s = \frac{t}{T^2}$. Now the Corollary reads, if we interpret it over $C_{G_\tau,\mathcal{O}_{Z\tau}}$

$$e^{-\frac{||Y||^2_{\mathfrak{g}}\tau^2}{2t}} \int_{C_{G_\tau,\mathcal{O}_{Z\tau}}} e^{\frac{T^2}{t}(z^{-1}z',Y)_{\mathfrak{g}}} d\omega^{\frac{t}{T^2}}_{G\tau,\mathcal{O}_{Z\tau}}(z\tau) = \int_G v_{\frac{t}{T^2}}(g_1Z\tau g_1^{-1}\tau^{-1}g(T)^{-1}\tau,T)dg_1$$

where we just inserted into the expression of Corollary 7.2 and used the new parameter, as well as the remark on the function $\hat{f}$. Of course, $v_s(g\tau,t)$ is the fundamental solution of the heat equation on $G\tau$. Let us fix a parameter value $T = \frac{1}{r}$. Observe, that for $Y \in \mathcal{L}(g,\tau)$ we then have $||Y||_r = -||Y||_{\mathfrak{g}}$. So for $h,k \in \mathfrak{h}^\circ$ we can set $Y = \frac{1}{T}k = rk$ and for $Z = e^h$. The motivation for this comes from the fact, that we want to insert this manipulations in the integral of Theorem 2.34, we then obtain the same integral as in the rewritten corollary 2.43 from above, and then we can use this, to rewrite Theorem 2.34 with respect to the Wiener measure. So from the above considerations we have $g(T)^{-1} = e^k$. So first we insert $Z = e^h$ into the integral of Theorem 2.34 and with $g(T)^{-1} = k$ we get:

$$\int_G v_{\frac{t}{T^2}}(gZ\tau g^{-1}g(T)^{-1}\tau, T)$$

but this is exactly the integral of the rewritten corollary, so we get for $h, k \in \bar{\mathfrak{h}}$:

Theorem 7.4.

$$\sum_{w\in\tilde{W}} \epsilon(w)e^{-\frac{1}{t}(w(2\pi id+h),2\pi id+k)_\tau} =$$

$$\frac{\delta^\tau(h)\delta^\tau(-k)e^{-\frac{1}{2t}||h||_\tau-\frac{t}{2}||\rho^\tau||_\tau}}{vol(\mathbb{Z}R^1)(\frac{2\pi}{t})^{\frac{l}{2}}}\int_{C_{G\tau,\mathcal{O}_{e^h}\tau}} e^{\frac{1}{tr^2}(z^{-1}z',k)_\mathfrak{g}}d\omega^{tr^2}_{G\tau,\mathcal{O}_{e^h}\tau}(z\tau)$$

7.1. **A generalization of Frenkel's character formula.** In this section we will end our exposition of [W1] and indicate, how the integral of Theorem 7.4 can be interpreted as an integral of a certain closure of a coadjoint orbit of $\mathcal{L}(G,\tau)$. All the necessary ingredients of a generalization of Frenkel's character formula, which was stated in the introduction, are present. The classification of affine adjoint orbits, the Poisson resumation of the numerator of the analytic Kac-Weyl character formula, the interpretation of the resulting integral as integral over a path space, on which a well understood measure could be introduced. To finish the analogue of Frenkel's program, Wendt introduces certain closures of the spaces so far discussed. Then using the natural mappings between this spaces, he obtains an analogue of Frenkel's character formula. Recall the notion of the affine shell

$$\mathcal{P}^{a,b}_\mathbb{C} = \{x(\cdot) + a_1C + b_1D \in \tilde{\mathfrak{g}}_\mathbb{C} | 2a_1b_1 + (x,x) = a, b_1 = b\}$$

So for $a, b \in \mathbb{C}$ and $b \neq 0$ let $\mathcal{P}^{a,b}$ be the affine shell just stated above. Since $\mathcal{P}^{a,b}$ is a submanifold of $\tilde{\mathfrak{g}}$ we can identify $\mathcal{P}^{a,b}$ with $\mathcal{L}(\mathfrak{g},\tau)$ via the projection $p : C \mapsto 0$ and $D \mapsto 0$. We now observe how the affine adjoint action transformes under this projections, the coadjoint action was given by

$$\widetilde{\mathrm{Ad}}g(aC + dD + y) = aC + bD + gyg^{-1} - bg'g^{-1} + (\langle g^{-1}g', y\rangle - \frac{b}{2})\langle g'g^{-1}, g'g^{-1}\rangle)C$$

Now the affine adjoint action responds under the projections above:

$$(g, y) \mapsto gyg^{-1} - bg'g^{-1}$$

we now have a series of maps

$$\mathcal{P}^{a,b} \xrightarrow{p} \mathcal{L}(\mathfrak{g},\tau) \xrightarrow{s} C^\infty_\mathfrak{g} \xrightarrow{i} C^\infty_G \xrightarrow{e_\tau} G\tau$$

where $s(x)(t) = \int_0^1 x(\kappa)d\kappa$ and where i maps a path $y \in C_{\mathfrak{g}^\infty}$ to the fundamental solution of the differential equation $z' = \frac{1}{b}zy$. The map e_τ is given by $e_\tau(z) = z(\frac{1}{r}\tau)$. From Ito's isomorphism we have a map $\tilde{i} : C_\mathfrak{g} \to C_G$ which is the extension of the map $i : C^\infty_\mathfrak{g} \to C^\infty_G$ above to the corresponding completions $C_\mathfrak{g}$ and C_G. Now every element $y \in C_\mathfrak{g}$ defines an element $dy \in \mathcal{L}(\mathfrak{g},\tau)^*$ via the Stieltjes integral

$$\langle x, dy\rangle = r\int_0^{\frac{1}{r}}(x(\kappa), dy(\kappa))$$

where $(.,.)$ denotes the Killing form on $\mathfrak{g}$. Now for $y \in \mathcal{L}(\mathfrak{g},\tau)$ we have $\langle x, d(s(y))\rangle = (x,y)_\tau$ where $(.,.)_\tau$ is the bilinear form on $\mathcal{L}(\mathfrak{g},\tau)$ as used throughout our exposition. Now we use the smooth part of $\mathcal{L}(\mathfrak{g},\tau)^*$, denoted by $\mathcal{L}(\mathfrak{g},\tau)*_0$, which is just the image of $C_\mathfrak{g}$ under this map with the topology induced from $C_\mathfrak{g}$. And let $\tilde{s} : \mathcal{L}(\mathfrak{g},\tau)*_0 \to C_\mathfrak{g}$ denote the inverse map. As seen in the diagram below, $\tilde{s}$ is the extension of the map s. Putting all together we get the following commutative diagram: Here e_τ denotes the extension of the map e_τ to C_G. In the classification of affine adjoint orbits we have seen that an $\mathcal{L}(G,\tau)$ Orbit $\mathcal{O}_{x(\cdot)+a_1C+b_1D}$, which lives in $\mathcal{P}^{a,b}$, is mapped to a $G\tau$ orbit $\mathcal{O}_{e_\tau \circ i \circ s(x)}$ When we discussed topologies on affine Lie algebras we introduced the C^∞ topology, as well as the Hilbert space $\mathcal{L}(\mathfrak{g},\tau)(L_2)$, on which we defined the norm

$$|x(\cdot)| = \sup_{t\in[0,\frac{1}{r}]} |\int_0^t x(\kappa)d\kappa|$$

The completion of $\mathcal{L}(\mathfrak{g},\tau)(L_2)$ with respect to this norm will be $\mathcal{L}(\mathfrak{g},\tau)_0^*$. So we obtained a series of completions:

$$\mathcal{L}(\mathfrak{g},\tau) \subset \mathcal{L}(\mathfrak{g},\tau)(L_2) \subset \mathcal{L}(\mathfrak{g},\tau)(L_2)_0^*$$

with respect to the L_2 topology on $\mathcal{L}(\mathfrak{g},\tau)$ and the norm on $\mathcal{L}(\mathfrak{g},\tau)(L_2)$ introduced above. Now we can introduce an appropriate closure of an affine adjoint orbit. If we look at the mappings in the diagram above, we see that the set $\tilde{s}^{-1} \circ \tilde{i}^{-1} \circ \tilde{e}_\tau^{-1}(\mathcal{O}_{e_\tau \circ i \circ s(x)}) \subset \mathcal{L}(\mathfrak{g},\tau)_0^*$ can be viewed as the closure of the affine adjoint orbit $\mathcal{O}_{x(\cdot)+a_1C+b_1D}$ in $\mathcal{L}(\mathfrak{g},\tau)_0^*$ and is mapped to $C_{G,\mathcal{O}_{i\circ s(x)}}$ under the map $\tilde{i} \circ \tilde{s}$. So the integral

$$\int_{C_{G\tau},\mathcal{O}_{Z\tau}} f(z)d\omega^s_{G\tau,\mathcal{O}_{Z\tau}}(z\tau)$$

can be viewed as integral over the closure of the affine adjoint orbit in $\mathcal{L}(\mathfrak{g},\tau)_0^*$. The observations above allow one to interpret the numerator of the Kac Weyl character formula in Theorem 7.4 as integral over the closure in $\mathcal{L}(\mathfrak{g},\tau)_0^*$ containing $aD+H = \Lambda+\tilde{\rho}$ where we use the notion introduced in section 2.2. To interpret the integral in proposition 2.44 as character formula, we have to use the formulations introduced in our first reformulation of the Kac- Weyl character formula, that is we have set the parameters $t = -\frac{1}{ab}, b = \frac{H}{a}$ and $K = \frac{k}{b}$ for $b \in i\mathbb{R}$ and $K \in i\mathfrak{h}^\circ$. Note the numerator of the Kac Weyl character formula is a function p not depending on the highest weight Λ of the corresponding representation, so it can be seen as the analogue of the universal function appearing in the original Kirillov character formula. With the repeated notions above the analogue of Frenkel's character formula in the twisted case now reads:

Theorem 7.5. *([W1] Theorem 4.9) Let $bD + K \in \tilde{\mathfrak{h}}$ with $K \in \mathfrak{h}^\circ$ and $b \in i\mathbb{R}$,$im(b) < 0$ Furthermore, for $\Lambda \in \tilde{P}_+$ let $\Lambda + \tilde{\rho} = aD + H$. Then the character of the highest weight representation corresponding to Λ evaluated at $bD + K$ is given by*

$$ch(L(\Lambda))(bD+K) = p^{-1}(bD+K) \cdot \frac{\delta^\tau(\frac{H}{a})\delta^\tau(-\frac{K}{b})e^{\frac{ab}{2}||\frac{H}{a}||_\tau - \frac{1}{2ab}||\rho^\tau||_\tau}}{vol(\mathbb{Z}R^1)(-2ab\pi)^{\frac{l}{2}}}$$
$$\cdot \int_{C_{G\tau},\mathcal{O}_{(e^{\frac{H}{a}})\tau}} e^{-\frac{ab}{r^2}(z^{-1}z',\frac{K}{b})_\mathfrak{g}} d\omega^{-\frac{r^2}{ab}}_{G\tau,\mathcal{O}_{(e^{\frac{H}{a}})\tau}}(z\tau).$$

Remark 7.6. It would be interesting to obtain a kind of ntegration, that allows one to integrate over the coadjoint orbit itself, rather the continouus closure. This would open the way to apply the orbit method in infinite dimensions, where no natural measure is available. This will be presented in the rest of the paper.

8. The Duistermaat Heckmann formula

Standard references for this section are the books [McDS], [Aud] and the original articles [DH1], [DH2]. Also the lecture notes [Ci] are a good source for a detailed exposition of the Duistermaat Heckmann formula. Our exposition follows mainly [McDS] and [Ci].

Definition 8.1. *A symplectic manifold M is a $2n$ dimensional manifold with a closed, nondegenerate two form ω.*

Definition 8.2. *The volume form $\frac{\omega^{\wedge n}}{n!}$ on (M,ω) is called the Liouville volume form.*

Definition 8.3. *A complex structure on a vector space V is an endomorphism $\mathcal{J}: V \to V$, such that $\mathcal{J}^2 = -\mathbb{I}$. So V becomes a complex vector space with multiplication by $i = \sqrt{-1}$ corresponding to $\mathcal{J}$*

Definition 8.4. *A symplectic vector bundle (E,ω) over a manifold M is a real vector bundle*

$$\pi : E \to M$$

which is equipped with a symplectic bilinear form on each fiber E_q, which varies smoothly with $q \in M$. All these forms ω_q fit together to a smooth section of the natural bundle $\bigwedge^2 E^*$, where E^* denotes the dual bundle of E. This form ω is a non degenerate skew symmetric bilinear form, which will is called symplectic bilinear form.

Definition 8.5. *A complex structure $\mathcal{J}$ is called compatible with the symplectic form ω if $g_{\mathcal{J}} := \omega(.,\mathcal{J}\cdot)$ defines an inner product.*

Lemma 8.6. [Ci] *Let (V,ω) be a symplectic vector space. There exists a natural continouus map from the space of all inner products to the space of all compatible complex structures which maps each induced inner product $g_{\mathcal{J}}$ to $\mathcal{J}$. Thus the space of compatible complex structures is non empty and contractible.*

Proof. An inner product g defines an isomorphism $A : V \to V$ via $\omega(\cdot,\cdot) = g(A,\cdot)$. Skew symmetrie of ω implies $A^t = -A$. Recall from linear algebra that each positive definite operator P posesses a unique positive definite square root $\sqrt{P}$ and $\sqrt{P}$ commutes with any operator with which P commutes. So we can define:

$$J_g := (AA^t)^{-\frac{1}{2}}A$$

It follows that $J_g^2 = -\mathbb{I}$ and

$$\omega(\cdot,\mathcal{J}\cdot) = g(\sqrt{AA^t}\cdot,\cdot)$$

is an inner product. Continuity of the map follows from continuity of the square root. If $g = g_{\mathcal{J}}$ for some $\mathcal{J}$ then $A = \mathcal{J} = g_{\mathcal{J}}$. Contractability follows from convexity of the space of inner products and the following general fact: If $f : X \to Y$ and $g : Y \to X$ are continuous maps between topological spaces satisfying $f \circ g = -\mathbb{I}$, then a contraction h_t of X induces a contraction $f \circ h_t \circ g$ of Y. $\square$

The following proposition is a consequence of the above lemma, if we equipp the spaces of sectons with a resonable topology:

Proposition 8.7. *The space of compatible complex structures on a symplectic vector bundle* (E, ω) *is nonempty and contractible.*

Definition 8.8. *An almost complex structure on a manifold is a complex structure* $\mathcal{J}$ *of the vector bundle* $TM \to M$*. It is called compatible with a symplectic form* $\omega(\cdot,\cdot)$ *if* $\omega(\cdot, \mathcal{J}\cdot)$ *is a Riemannian metric.*

As a consequence of 8.7 there exist compatible almost complex structures on any symplectic manifold and they form a contractible space.
Now let (M, ω) be a finite dimensional symplectic manifold and assume that we have an action of some torus $T = \mathbb{R}^l/\mathbb{Z}^l$. That is a smooth map

$$g : T \times M \to M \quad (g, x) \mapsto g \cdot x$$

We denote the orbit of the action by

$$T \cdot x := \{t \cdot x | x \in M\}$$

and the stabilizer of x by

$$T_x := \{t \in T | t \cdot x = x\}$$

We have $T_{t \cdot x} = tT_x t^{-1}$ so all stabilizers are conjuagate along an obit. This conjuagcy class is called the type of the orbit. Let us assume that our torus action is symplectic, that is $g^*\omega = \omega$. where g denotes the action from above. We denote the Lie algebra of the torus by $\mathfrak{h}$ and every $H \in \mathfrak{h}$ gives us an $\mathbb{R}$ action and therefore a vector field $\tilde{H}$ on M. This follows from the well known correspondence (see e.g [Hs])

$$\{\text{one- parameter subgroups}\} \leftrightarrow \{\text{left invariant } \mathbb{R} \text{ actions}\} \leftrightarrow \{\text{left invariant vector fields}\}$$

Definition 8.9. *We call* $H \in \mathfrak{h}$ *generic if the subgroup generated by* $(\exp H)$ *is a dense subgroup of* T*.*

Furthermore let us assume that we have a Hamiltonian function J_H, which comes from the $\mathbb{R}$ action. Recall the notion of a hamiltonian vector field. If $f : M \to \mathbb{R}$ is a smooth function on a symplectic manifold M, then $\tilde{X}$ is called a hamiltonian vector field corresponding to f, if the identity $df = i_{\tilde{X}}\omega$ is valid. Here i denotes the usual insertion operator. The function f is that called the Hamiltonian corresponding to $\tilde{X}$. That is in our case the identity $dJ_H = i_{\tilde{H}}\omega$ is valid. Furthermore we assume that the torus acts effectively on M, that is $\bigcap_x T_x = 1$. We also assume that the fixed point set of our torus action consists of isolated points p, i.e. we can find a neighbourhood of p such that no other point is contained in this neighbourhood. So the torus action can also be viewed as a linear action on each tangent space with foot point a fixed point p. So we have a linearly torus action on T_pM. Now according to 8.7 we pick an almost complex structure on M which is compatible with ω. Furthermore we assume that ω commutes with the T action. Since representations can also be defined via group actions, the above gives us a decomposition of each $T_pM = \bigoplus_{j=1}^n V_j^p$ into complex one dimensional representations V_j^p of p. Here p is not a power, but an index for the fixed point. Here T acts on the one dimensional representation V_j^p via the complex character $t \mapsto \exp(2\pi i \alpha_j^p(H))$, where $\exp(H) =$

t. Again, p should be interpreted as an index. We now state the Duistermaat-Heckmann exact integration formula

$$\int_M e^{-tJ_H} \frac{\omega^{\wedge n}}{n!} = \sum_{p \in P} \frac{e^{-tJ_H(p)}}{\prod_{j=1}^n \alpha_j^p(H)} \tag{41}$$

where t can be any real or complex parameter. This theorem can be extended to the case when the fixed point set consists of submanifolds instead of isolated points. Now let us assume that there is not only a symplectic structure on our manifold, but we also chose a Riemannian metric σ. Let $d\sigma$ denote the Riemannian volume form. The symplectic and Riemannian volume form are related via the Pfaffian Pf. We call the skew symmetrix automorphism on the tangent bundle TM, associated to ω and the metric σ, B_σ. It is defined by

$$\omega_x(X,Y) = \sigma_x(B_\sigma(X),Y) \text{ for } X,Y \in T_xM.$$

Now the symplectic and Riemannian volume form are related by

$$\tfrac{\omega^{\wedge n}}{n!} = \mathrm{Pf}(B_\sigma)d\sigma$$

In the infinite dimensional case, the symplectic volume form (also called Liouville form) does not make sense. Ignoring this fact, physicists use the Duistermaat Heckmann formula to calculate certain integrals over infinite dimensional symplectic manifolds. The integration methods are merely justified by their analogy with the finite dimensional cases. Such infinite dimensional integrals are often called functional integrals, since they are by definition integrals of the form

$$\int_{C^\infty(\Sigma,G)} e^{\cdots}$$

Here Σ is a Riemann surface of genus $g \geq 1$ and G denotes a compact, simply connected Lie group. Such integrals play an important role in quantum field theory, but their exact meaning is at least today, still mysterious. More recently such functional integrals appeared also in economics, motivated by statistical mechanics, they are used in certain models for option pricing, see e.g [Vo]. Now let (M,ω) be an infinite dimensional, symplectic manifold, that is a Frechet manifold together with a closed, non degenerate two form ω. In our case, the non degeneracy of the two form means, that the tangent map $\phi : T_mM \to T_mM^*, X_m \mapsto \omega(X_m,.)$ is injective at each $m \in M$. Furthermore, we assume the T_mM to have a countable basis for all $m \in M$, in order to avoid not manageable calculations.

We now start to develop an infinite dimensional analogue of the Duistermaat Heckmann exact integration formula, to this end assume that there is an effective action of some torus T on M which preserves ω and I. Like in the finite dimensional case let us assume that the fixed point set of the torus action consist of isolated points p. But now there may be infinitely many p. Again, we have a torus action of T on the tangent spaces T_pM which decomposes into the direct sum of one dimensional complex representations, $T_pM = \bigoplus_{j=1}^n V_j^p$, since we are only concerned with current manifolds, in general this decomposition does not hold. As before we have the action of T on V_j^p via the complex character $t \mapsto \exp(2\pi i \alpha_j^p(H))$ for $H \in \mathfrak{h}$. Now we have $p \in \mathbb{R}$ for all $p \in P$ and $j \in \mathbb{N}$. This is the place, where an important tool for the development of the conceptual integration approach comes into play.

9. Zeta regularized products

The general theory of zeta regularization is developed in [QHS] and [Ill].

Definition 9.1. *Let λ_k be a sequence of nonzero complex numbers, then we define the zeta regularized product $\prod^{\zeta_k} \lambda_k$ to be* $\exp(-\zeta_k'(0))$*, where $\zeta_k(s)$ is the corresponding zeta function, that is $\sum_{k=0}^{\infty} \lambda_k^{-s}$.*

We assume, that $\zeta_k(s)$ has a meromorphic continuation with at most simple poles, to a half plane containing the origin and is analytic at the origin. Such a sequence is called zeta regularizable. Some basic calculatio rules are readly verified:

Lemma 9.2. *(a) Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ be a zeta multipliable series. If we denote the index $n \in \mathbb{N}$ for even natural numbers by n_{even} and for odd natural numbers by n_{odd} the following holds: If the series over all $n \in \mathbb{N}_{even}$ is zeta multipliable, and the series over all $n \in \mathbb{N}_{odd}$ is zeta multipliable, then the series over all $n \in \mathbb{N}$ is multipliable and equality holds:*

$$\prod_n^{\zeta} \lambda_n = \prod_{n_{even}}^{\zeta} \lambda_n \cdot \prod_{n_{odd}}^{\zeta} \lambda_n$$

(b) Let Λ be as above and let $a, b \in \mathbb{C}$. Then

$$\left(\prod_n^{\zeta} \lambda_n^a\right) = \left(\prod_n^{\zeta} \lambda_n\right)^a$$

and

$$\left(\prod_n^{\zeta} \lambda_n b\right) = \prod_n^{\zeta} \lambda_n \cdot b^{\zeta(0)}$$

Proof. (a) Let us denote the zeta functions for the corresponding indices by the following

$$\zeta_{\mathbb{N}}(s) = \sum_{n \in \mathbb{N}} e^{-s \log \lambda_n}$$
$$\zeta_{\mathbb{N}_{even}} = \sum_{n \in \mathbb{N}_{even}} e^{-s \log \lambda_n}$$
$$\zeta_{\mathbb{N}_{odd}} = \sum_{n \in \mathbb{N}_{odd}} e^{-s \log \lambda_n}$$

If $Re >> 0$ the even series is absolut convergence and the odd series is absolut convergent, then the series over all $n \in \mathbb{N}$ is absolut convergent. If the odd and even series are meromorphic, then the series over all $n \in \mathbb{N}$ is meromorphic and regular at 0.

$$\zeta_{\mathbb{N}}(s) = \zeta_{\mathbb{N}_{even}} + \zeta_{\mathbb{N}_{odd}}$$

$$(\zeta_{\mathbb{N}}'(0)) = \left(-\zeta_{\mathbb{N}_{odd}}'(0)\right) + \left(\zeta_{\mathbb{N}_{odd}}'(0)\right) \tag{42}$$

Let us now apply the exponential function to (3.1)

$$\exp\left(\zeta'_{\mathbb{N}}(0)\right) = \exp\left(-\zeta'_{\mathbb{N}_{odd}}(0) + \zeta'_{\mathbb{N}_{odd}}(0)\right) \tag{43}$$

and because of the functional equation of exp the assumption follows. Also the other properties of zeta regularized products hold, since meromorphic functions are a field.
(b) Let us denote the two zeta regularized products we consider here:

$$\left(\prod_n^{\zeta} \lambda_n^a\right), \prod_n^{\zeta} \lambda_n$$

again we denote the corresponding zeta functions by

$$\tilde{\zeta}(s) = \sum_{n\in\mathbb{N}} e^{-sa\log\lambda_n}, \zeta(s) = \sum_{n\in\mathbb{N}} e^{-s\log\lambda_n}$$

Now we can take derivatives of these zeta functions

$$\tilde{\zeta}'(s) = \sum_{n\in\mathbb{N}} -a\log\lambda_n e^{-sa\log\lambda_n} = a\zeta'(s),$$
$$\zeta'(s) = \sum_{n\in\mathbb{N}} -\log\lambda_n e^{-s\log\lambda_n}$$

Now for $s = 0$ this reads

$$-\tilde{\zeta}'(0) = -a\zeta'(0) \Rightarrow$$
$$\left(\prod_n^{\zeta} \lambda_n^a\right) = \left(\prod_n^{\zeta} \lambda_n\right)^a$$

since $e^{a\zeta'(0)} = \left(e^{\zeta'(0)}\right)^a$.
now we turn to the second part of (b). Let us denote

$$\zeta = \lambda_n$$
$$\tilde{\zeta} = \lambda_n b$$

Now we can write

$$\zeta(s) = \sum_{n\in\mathbb{N}} e^{-s\log\lambda_n}$$

$$\tilde{\zeta}(s) = \sum n \in \mathbb{N} e^{-s\log\lambda_n b} = \sum_{n\in\mathbb{N}} e^{-s(\log\lambda_n + \ln b)} =$$

$$\left(\sum_{n\in\mathbb{N}} e^{-s\log\lambda_n}\right) e^{-s\log b} =$$

$$\zeta(s)e^{-s\log b}$$

Now we can take derivatives

$$\tilde{\zeta}'(s) = \zeta'(s)e^{-s\log b} - \log\, b\,\zeta'(s)e^{-s\log b}$$

But

$$-\log\left(\prod_n^{\zeta}\lambda_n b\right) = \tilde{\zeta}'(0) = \zeta'(0)\cdot 1 - \log(b)\zeta(0)\cdot 1 =$$

$$= -\log\left(\prod_n^{\zeta}\lambda_n\right) - \log\left(b^{\zeta(0)}\right) =$$

$$= -\log\left(\left(\prod_n^{\zeta}\lambda_n\right)\left(b^{\zeta(0)}\right)\right)$$

It follows that

$$\left(\prod_n^{\zeta}\lambda_n\right) = \left(\prod_n^{\zeta}\lambda_n\right)\cdot b^{\zeta(0)}$$

Like in the first case, the meromorphically continuation follows. □

Definition 9.3. $\alpha = \inf\{s \in \mathbb{R} | \sum_{k=0}^{\infty} |\lambda_k|^{-s} < \infty\}$

For $\lambda \neq \lambda_k$ we consider the shifted sequence $\lambda_k - \lambda$. and the associated zeta function $\zeta(s, -\lambda) = \sum_{k=0}^{\infty}(\lambda_k - \lambda)^{-s}$ which converges for $Res > \alpha$. We use the following convention: for $|\lambda_k|$ large, $arg(\lambda_k - \lambda)$ is near $arg(\lambda_k)$.

Theorem 9.4. *If λ_k is zeta regularizable, then so is $\lambda_k - \lambda$*

Proof. This is proofed in [QHS] Theorem 1 □

Next we establish the relationship between zeta regularized products and the classical Weierstrass product. Let h be an integer such that $h + 1 > \alpha$. We define the Weierstrass product by

Definition 9.5. $W_h(\lambda) = \prod_{j=0}^{\infty}(1 - \frac{\lambda}{\lambda_j})\exp\left(P_h(\frac{\lambda}{\lambda_j})\right)$

where $P_h(x) = x + \frac{x^2}{2} + \ldots + \frac{x^h}{h}$. At s_0 we define the finite part of ζ at s_0, $FPZ(s_0)$, to be the constant term of the Laurent expansion for ζ at s_0. Let $Res\zeta(s_0)$ denote the residue of Z at s_0. $Res\zeta(s_0) = 0$ if s_0 if not a pole.

Theorem 9.6. *Suppose h is an integer with $h+1>\alpha$ and that F has poles of order at most 1 at integer points then*

$$\frac{\prod^{\zeta}(\lambda_k-\lambda)}{\prod^{\zeta}\lambda_k}=e^{-Q_h(\lambda)W_h(\lambda)}$$

where

$$Q_h(\lambda)=\sum_{j=1}^{h}FPZ(j)\frac{\lambda^j}{j}+\sum_{j=2}^{h}ResZ(j)\left(1+\frac{1}{2}+\ldots+\frac{1}{j-1}\right)\frac{\lambda^j}{j}$$

and $F(s)=\sum_{k=0}^{\infty}G_k(s)$ with

$$G_k(s)=(\lambda_k-\lambda)^{-s}-\sum_{j=0}^{k}\binom{s+j-1}{j}\left(\frac{\lambda}{\lambda_k}\right)^j$$

Proof. [QHS] Theorem 2 □

According to [QHS] a sequence is called admissible if it lies in the half plane $Re\lambda>0$, the function $\sum_{k=0}^{\infty}e^{-\lambda_k t}$ converges absolutely for $t>0$ and has a full asymptotic expansion $\sum_{\nu=0}^{\infty}c_{j_\nu}t^{j_\nu}, j_0<j_1<\ldots\to\infty$ as $t\to\infty$. Furhtermore $\lim_{t\to 0}\sum_{k=0}^{\infty}|e^{-\lambda_k t}|t^{\beta}=0$ for some $\beta>0$. The sequences that are studied in [QHS] are rotations of admissible sequences, that is of the form $a\lambda_k$ for an admissible sequence λ_k. With the knowledge of the function $\sum_{k=0}^{\infty}e^{-\lambda_k t}$ we can study the analytic continuation of $\zeta(s)$ for an admissible sequence. We first write $\zeta(s)$ as Mellin transform. Write

$$\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt,\quad Res>0$$

and make the change of variables $t\mapsto\lambda t$ ($\lambda\neq 0$) and we get

$$\Gamma(s)=\lambda^s\int_0^{\infty}e^{-\lambda t}t^{s-1}dt$$

For $Res>\beta$ (β from above), we sum over $\Lambda=\lambda_k$ getting

$$\sum_k\lambda_k^{-s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\sum_k e^{-\lambda_k t}t^{s-1}dt. \tag{44}$$

Now $\zeta(s)$ can be analytically continued using the following theorem:

Theorem 9.7. *Suppose $\phi(t)$ is a complex valued continuous function of a real variable t, with $|\phi(t)|\leq e^{-bt}$ for some $b>0$ and t large. If $\phi(t)$ has the asymptotic expansion*

$$\phi(t)\backsim\sum_{\nu=0}^{\infty}c_{j_{nu}}t^{j_\nu}\quad ast\to 0^+$$

$j_0<j_1<\ldots\infty$ then

$$F(s)=\int_0^{\infty}\phi(t)t^{s-1}dt$$

converges for $Res>-j_0$ to an analytic function and can be analytically continued to a function meromorphic in the complex plane with simple poles $-j_\nu$ and residues c_{j_ν}

Proof. For $N > 0$ let

$$\phi_N(t) = \phi(t) - \sum_{\nu=0}^{N} c_{j_\nu} t^{j_\nu}$$

and write

$$F(s) = \int_1^\infty \phi(t) t^{s-1} dt + \int_0^1 \sum_{\nu=0}^{N} c_{j_\nu} t^{j_\nu + s - 1} + \int_0^1 \phi_N(t) t^{s-1} dt = \sum_{\nu=0}^{N} c_{j_\nu} \frac{1}{j_\nu + s} + A(s)$$

where $A(s)$ is analytic for $Res > -j_{N+1}$. This holds for $Res > -j_0$ but provides an analytic continuation to a function meromorphic with simple poles in $Res > -j_{N+1}$. Heinz : Probability theory / Heinz Bauer. Transl. from the German by Robert B. Burckel. - Ber □

Applying 9.7 theorem to the zeta function $\zeta(s)$ for an admissible sequence λ_k, letting $\phi(t) = \sum_k e^{-\lambda_k t}$, and $\zeta(s) = F(s)/\Gamma(s)$, we see that for $j_\nu \neq 0, 1, \ldots$ then $\zeta(s)$ has a pole at $-j_\nu$ with residue $\frac{c_{j_\nu}}{\Gamma(-c_{j_\nu})}$. We also see that if $m = 0, 1, 2, \ldots$ then $\zeta(-m) = c_m (-1)^m m!$ where by convention $c_m = 0$ if $m \neq j_\nu$ for some ν. In particular $\zeta(0) = c_0$ To handle the double zeta function let $\phi(t) = e^{-at} \sum_n e^{-\lambda_n t}$, so we get

$$\zeta(0, a) = \sum_{m \geq 0} c_{-m} \frac{a^m}{m!}$$

The following theorem is a classicial result and its proof can be found in [SG] p.458

Theorem 9.8. *Let*

$$\theta(t) \backsim \sum_{\nu=0}^{\infty} c_{k_\nu t^{k_\nu}}$$

as $t \to 0$, *where* $-1 < k_0 < k_1, \ldots \to \infty$. *Assuming that the Laplace type integral*

$$\varphi(a) = \int_0^\infty e^{-at} \theta(t) dt$$

has a half plane of simple convergence, then $\varphi(a)$ *posseses an asymptotic expansion*

$$\varphi(a) \backsim \sum_{\nu=0}^{\infty} c_{k_\nu} \Gamma(k_\nu + 1) a^{-(k_\nu + 1)}$$

as $a \to \infty$ *in the angular sector* $|arg(a)| \leq \delta < \frac{\pi}{2}$

For admissible sequences the theorems 9.7 and 9.8 provide an asymptotic expansion for $\zeta(s, a)$ and $\prod^\zeta (\lambda_k + a)$ as $a \to \infty$ provided we have an asymptotic expansion 44 for $\phi(t) = \sum_k e^{\lambda_k t}$ as $t \to 0$.
Next we state several examples of zeta regularized products that where obtained in [QHS]. These examples will allow us to calculate the zeta regularized products in a more conceptual way then in [W2].

Lemma 9.9.

$$\prod^\zeta n = \sqrt{2\pi}$$

Proof. We insert into the definition of the zeta regularized product, $\exp(-\zeta'(0))$ and use the fact that $\zeta'(0) = -\log \sqrt{2\pi}$. □

Lemma 9.10.

$$ForMoreinformationonthisderivativeandonregularLiegroupssee[K-M] \prod_{n=1}^{\infty}{}^{\zeta} n\tau = \sqrt{2\pi}\tau^{-\frac{1}{2}}$$

Proof. According to the basic calculation rules we have

$$\prod_{n=1}^{\infty}{}^{\zeta} n\tau = \tau^{\zeta(0)} \prod_{n=1}^{\infty}{}^{\zeta} n = \tau^{-\frac{1}{2}}\sqrt{2\pi}$$

□

Lemma 9.11.

$$\prod_{n=1}^{\infty}{}^{\zeta} (n+a) = \frac{\sqrt{2\pi}}{\Gamma(a)}$$

Proof. Since the finite part of $\zeta(1) = \gamma$ theorem 9.6 together with 9.9 gives:

$$\prod_{n=1}^{\infty}{}^{\zeta} (n+a) = a\sqrt{2\pi}e^{\gamma a} \prod_{n=1}^{\infty} \left(1 + \frac{a}{n}\right) \exp\left(\frac{-a}{n}\right)$$

By we well known product formula for the Gamma function we have

$$\prod_{n=1}^{\infty}{}^{\zeta} (n+a) = \frac{\sqrt{(2\pi)}}{\Gamma(a)}$$

□

Lemma 9.12.

$$\prod_{m,n=-\infty}^{\infty}{}' |m+n\tau|^2 = (2\pi)^2 |\eta(\tau)|^4 \qquad Im\tau > 0$$

The evalutation of the product is Kronecker's limit formula, the ′ indicates, that in the product zero is ommitted. η is Dedekind's eta function.

Proof. Let $\zeta(s) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} (|m+n\tau|^2)^{-s}$ and $\zeta_1(c,s) = \sum_{m=-\infty}^{\infty} (|m+a|^2)^{-}s$ where $a = b + ic$, b fixed. Using Theorem 9.8 we get an asymptotic expansion for ζ_1 as $c \to \infty$ if we have an expansion for $\sum_{m=-\infty}^{\infty} e^{-(m+b)^2 t}$ as $t \to 0^+$. The corresponding integral reads

$$\int_{-\infty}^{\infty} e^{-(m+b)^2 t} dx = \sqrt{(\pi)} t^{-\frac{1}{2}}$$

and as $t \to 0$ the difference between the sum and the integral is shown to be $O(t^n)$ for all $n > 0$. Thus we have the full asymptotic expansion

$$\sum_{m=-\infty}^{\infty} e^{-(m+b)^2 t} \backsim \sqrt{(\pi)} t^{-\frac{1}{2}} \tag{45}$$

as $t \to 0^+$. Now 9.8 and 45 give

$$\zeta_1(s,c) - \sqrt{\pi} \frac{\Gamma(-\frac{1}{2}+s)}{\Gamma(s)} c^{-2s+1} = O(c^{-2s-2}) \tag{46}$$

as $c \to \infty$ for $Res > -1$. Letting $a = n(x+iy)$ and summing up over n gives
(47)

$$\sum_{n=1}^{\infty}\left(\zeta_1(s,n^2y^2) - \sqrt{\pi}\frac{\Gamma(-\frac{1}{2}+s)}{\Gamma(s)}(ny)^{-2s+1}\right) = \zeta(s) - \sqrt{\pi}\frac{\Gamma(-\frac{1}{2}+s)}{\Gamma(s)}\zeta_R(2s-1)y^{1-2s}$$

where ζ_R denotes the Riemann zeta function, and we have $Res > -1$. Now differentiating 47 at $s = 0$ and using $\prod_{n=-\infty}^{\infty}{}^{\zeta}(n+b)^2 = 4\sin^2 \pi b$ as well as $\zeta_R(-1) = -\frac{1}{12}$ and $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ we get

$$\sum_{n=1}^{\infty}(-\log 4|sinn\pi\tau|^2 + 2\pi ny) = \zeta'(0) - \frac{\pi y}{6} \tag{48}$$

Exponentiating both sides gives

$$\prod_{m=-\infty}^{\infty}|m+n\tau|^2 = |\eta(\tau)|^2, \tag{49}$$

where in the product above $n = 1$. Now we can use 9.9, which gives

$$\prod{}^{\zeta} m^2 = (2\pi)^2$$

and the proof is complete. ☐

Lemma 9.13.

$$\prod_{m=-\infty}^{\infty}{}^{\zeta}(m+a) = \begin{cases} 1 - e^{2\pi i a} & Im\ a > 0 \\ 1 - e^{-2\pi i a} & Ima < 0 \end{cases} \tag{50}$$

where we have $Im\ a \neq 0$ *and* $-\pi < arg(m+a) < \pi$

Proof. If Im $a > 0$ we have $-(m+a) = e^{2\pi}(m+a)$ now we can use 9.11, the basic calculation rules for zeta regularized products and get

$$\prod_{m=-\infty}^{\infty}{}^{\zeta}(m+a) = a\prod_{m=1}^{\infty}{}^{\zeta}(m+a)\prod_{m=1}^{\infty}{}^{\zeta}e^{\pi i}(m+a) =$$
$$a^{-1}\frac{\sqrt{2\pi}}{\Gamma(a)}e^{\pi i(-\frac{1}{2}+a)}\frac{\sqrt{2\pi}}{\Gamma(-a)}$$

Now we use the identity $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$ and get the final result

$$1 - e^{2\pi i a}$$

We do a simiar calculation for Im $a < 0$ and get the desired result. ☐

Lemma 9.14.

$$\prod_{m,n=-\infty}^{\infty}{}^{\zeta'}(m+n\tau)) = 2\pi i\eta^2(\tau)e^{\frac{2\pi i}{6}}$$

where $Im\tau > 0$ *and* $-\pi \leq arg(m+n\tau) < \pi$

Proof. Let $\zeta_1(s,a)$ denote the analytic continuation of $\sum_{m=1}^{\infty}(m+n)^{-s}$. We have seen from 9.8 that

$$\zeta_1(s,a) = \frac{a^{1-s}}{s-1} - \frac{1}{2}a^{-1} + \frac{s}{12}a^{-1-s} + O(|a|^{-3-s})$$

for Re$s > -3$, $a \to \infty$, $|arg a| < \pi - \delta$. Likewise $\zeta_2(s,a) = e^{\pi i s}\zeta_1)(s, e^{\pi i}a)$ is the analytic continuation of $\sum_{m=1}^{\infty}(-m+a)^{-s}$ where $arg(-m) = -\pi i$ and substituting $\zeta_1(s,a)$ yields:

$$\zeta_1(s,a) = -\frac{a^{1-s}}{s-1} - \frac{1}{2}a^{-1} + \frac{s}{12}a^{-1-s} + O(|a|^{-3-s})$$

Now letting $\zeta(s,a)$ being the analytic continuation of the whole series $\sum_{m=-\infty}^{\infty}(m+a)^{-s}$ we have

$$\zeta(s,a) = \zeta_1(s,a) + a^{-s} + \zeta_2(s,a)$$

and adding the two above asymptotic expansions we have

$$\zeta(s,a) = O(|a|^- 3 - s) \tag{51}$$

for Re$s > -3$, as $a \to \infty$, $-\pi + \delta < arg a < \pi - \delta$. Checking the residues we see that 51 is an entire function of s. Also by 51 we see that

$$\sum_{n \neq 0}(m + n\tau)^{-s} = \sum_{n=-\infty}^{\infty}{}' \zeta(s, n\tau) \tag{52}$$

Furthermore the sum over n on the right converges absolutely for Re$s > -2$ and the analytic continuation of 52 is also an entire function of s. Now we can take the iterated product

$$\prod_{n \neq 0}^{\zeta}(m+n\tau) = \prod_{n=-\infty}^{\infty}{}^{\zeta'} \prod_{m=-\infty}^{\infty}{}^{\zeta}(m+n\tau) =$$

$$= \prod_{n=1}^{\infty}\prod_{m=-\infty}^{\infty}{}^{\zeta}(m+n\tau)\prod_{n=1}^{\infty}\prod_{m=-\infty}^{\infty}{}^{\zeta}(m-n\tau) =$$

$$\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})^2 = \eta^2(\tau)e^{\frac{-\pi i\tau}{6}}$$

where we used the basic calculation rules and 9.13.
Now 9.9 and the rules yield

$$\prod_{m=-\infty}^{\infty}{}^{\zeta} m = \pi i$$

FInally for Im$\tau > 0$ we get

$$\prod_{m,n=-\infty}^{\infty}{}^{\zeta'}(m+n\tau) = 2\pi i\eta^2(\tau)e^{-\frac{\pi i\tau}{6}} = 2\pi i\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})^2$$

□

Lemma 9.15.

$$\prod_{m,n=-\infty}^{\infty}{}^{\zeta}(m+n\tau+z) = i\eta^{-1}(\tau)\exp\left(-\frac{\pi i\tau}{6} - \pi i z\right)\vartheta_1(z)$$

Here Im$\tau > 0$, $-\pi < arg(m+n\tau+z) < \pi$ and $\vartheta_1(z)$ denotes a Jacobi theta function as in [SG], *that is*

$$\vartheta_1(z) = i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-\frac{1}{2})^2}\exp(2n-1)\pi i z$$

Here $q = e^{\pi i \tau}$.

Proof. Let Ω denote the lattice with basis $1, \tau$ and $-\pi < arg\gamma y\pi$. The consider the corresponding zeta function $\zeta(s) = \sum_{\gamma\in\Omega} \gamma^{-s}$. We have

$$\zeta(s) = \zeta_R(s) + (e^{-\pi i})^{-s}\zeta_R(s) + \sum_{n\neq 0}(m+n\tau)^{-s}. \tag{53}$$

We can convert the sum as in 52 into an iterated sum, since ζ is an entire function of s we can apply theorem 9.6 and get

$$\prod_{\gamma\in\Omega}^{\zeta}(\gamma+z) = \left(\prod_{\gamma\in\Omega}{}'^{\zeta}\gamma\right)\exp\left(-\frac{\zeta(2)}{2}z^2 + \zeta(1)z\right)\sigma(z)$$

where $\sigma(z)$ denotes the Weierstrass product associated with the sequence γ. By 53 and 52 we can compute $\zeta(2)$ and $\zeta(1)$ by iterated summation. For $\zeta(2)$ this is the Eisenstein summation formula, applying this formula yields:

$$\zeta(2) = -4\pi i \frac{\eta'(\tau)}{\eta(\tau)} \tag{54}$$

We can directly compute from 53 that

$$\zeta(1) = -\pi i, \zeta(0) = -1 \tag{55}$$

Finally we use 53, 54, 55 and 9.14 to get

$$\prod^{\zeta}(\gamma+z) = 2\pi i\eta^2(\tau)\exp\left(-\frac{\pi i\tau}{6} + 2\pi i\frac{\eta'(\tau)}{\eta(\tau)}z^2 - \pi i\tau\right)\sigma(z)$$

Now using the Jacobi theta function $\vartheta_1(z)$ we can deduce the following relationship

$$\sigma(z) = \vartheta(z)(2\pi)^{-1}\eta^{-3}(\tau)\exp\left(-2\pi i\frac{\eta'(\tau)}{\eta(\tau)}z^2\right)$$

Now the final result follows

$$\prod^{\zeta}(\gamma+z) = i\eta^{-1}(\tau)\exp\left(-\frac{\pi i\tau}{6} - \pi i z\right)\vartheta_1(z)$$

□

10. The Liouville functional

With these definitions we can define an analogue of the denominator of the Duistermaat Heckmann formula for the infinite dimensional setup stated above. Let $\alpha_j^p(H)$ be the weights from above and let us assume, that the $\alpha_j^p(H)$ are zeta multipliable, then we introduce

$$Z_p(H) = \left(\prod_{j\in\mathbb{N}}^{\zeta}|\alpha_j^p(H)|\right)$$

Up to sign this is the exact analogue of the denominator of the Duistermaat Heckmann formula for finite dimensional compact manifolds. To take care of the some we consider the number p of rotation planes V_j^p for which $\alpha_j^p(H) < 0$. We have to make the assumption that p is finite for all $p \in P$. Then $(-1)^{\sharp p}$ will be the desired sign. Now all the necessary structures for the definition of the Liouville functional are collected. This yields

Definition 10.1. *Let (M,ω) be an infinite dimensional symplectic manifold with a torus action for which all the assumptions above are satisfied. For $H \in \mathfrak{h}$ as above , let J_H be a Hamiltonian function of the $\mathbb{R}$ action on M defined by H. Then for $t \in \mathbb{R}_{>0}$ we define the Liouville functional $L_t(J_H)$ via*

$$L_t(J_H) = \sum_{p \in P} (-1)^{\sharp p} \frac{e^{-tJ_H(p)}}{t^{\zeta(0)} Z_p(H)}$$

whenever this sum makes sense.

Now we consider, like in the finite dimensional case, a Riemannian metric σ on the manifold M. That is, σ is a non degenerate, symmetric bilinear form $\sigma_m = \langle .,. \rangle$ on each tangent space T_mM which varies smoothly with m. As before, the non degeneracy means, that the induced map $T_mM \to T_m^*M$ is injective. Now an analogue to the integration with respect to the Riemannian volume form in the finite dimensional case is defined. Remember, that in the finite dimensional case the symplectic form ω and the Riemannian metric σ where related by a skew symmetric automorphism B_σ of the tangent bundle. So in the infinite dimensional case the compatibility looks like the following: Assume that there is a skew symmetric automorphism $B_{\sigma,x}$ of the tangent space T_xM for each $x \in M$ such that

$$\omega_x(X,Y) = \sigma_x(B_{\sigma,x}(X), Y)$$

Furthermore, let us assume that the zeta regularized determinant (i.e the zeta regularized product of the eigenvalues of $B_{\sigma,x}$) exists. Now, if the zeta regularized determinant defines a nowhere vanishing positive function on M, we can define the zeta regularized Pfaffian $Pf_\zeta : M \to \mathbb{R}$ of $B_{\sigma,x}$ by $Pf_\zeta(x)^2 = \det_\zeta(B_{\sigma,x})$. We will call the symplectic form ω and the Riemannian metric σ zeta compatible, or just compatible, if $Pf_\zeta(B_\sigma)$ exists. (This means, that the zeta regularized product of the eigenvalues of B_σ exists). So we can define another functional which will be refereed to as "integration" with respect to the Riemannian volume form:

$$\int_M e^{-tJ_H} Pf_\zeta(B_\sigma) d\sigma = L_t(J_H)$$

In the cases considered in [W2] the manifold M is always a homogenous space G/G' and σ and ω can be chosen invariant under the canonical G action. So if $Pf_\zeta(B_\sigma)$ exists, it will be constant. Therefore, such an M is always orientable and we can rewrite the above expression

$$\int_M e^{-tJ_H} d\sigma = \frac{L_t(J_H)}{Pf_\zeta(B_\sigma(x_0))}$$

10.1. **The Liouville functional on a complex manifold.** In this section we will carry over the formalism of the Liouville functional to the case where M is a complex manifold. There are no major changes, except that we have to take care of the definition of the number of rotation planes. The zeta regularization works also for complex series, so there arises no problem in calculating the zeta regularized determinants. We first recall some basic definitions of complex geometry. For an introduction to this vast branch of differential geometry see [F-G] or the forthcoming book [Huy]. First consider a Haussdorf space with countable basis, equipped with a complex coordinate system. Like in the real case one has charts (U, ϕ), but now the homeomorphisms ϕ map the open subset $U \subset M$ into an open ball $B \subset M$. Also the coordinate changes are now biholomorphic maps and a covering of M with pairwise compatible complex coordinate systems is called a complex atlas. An equivalence class of a complex atlas is called a complex structure and one defines

Definition 10.2. *A complex manifold M is a Haussdorf space M with countable basis, equipped with complex structure.*

Note, that there exist also other versions of this definition, in particular one which we learned from [Joy].
Let M be a real manifold of dimension $2m$. An almost complex structure J is defined as a tensor J_a^b which satisfies the cocycle condition $J_a^b J_b^c = -\delta_a^c$. For each vector field v on M define Jv by $J(v)^b = J_a^b v^a$. So J satisfies $J^2 = -1$ and gives each tangent space T_pM the structure of a complex vector space. Now one associates the so called *Nijenhuis Tensor* to J. That is a tensor $N = N_{bc}^a$ which satisfies

$$N_{bc}^a v^b w^c = ([v,w] + J([Jv,w] + [v,Jw]) - [Jv,Jw])^a$$

for all vector fields v, w on M and where $[.,.]$ denotes the usual Lie bracket for vector fields. One calls J a complex structure if $N \equiv 0$. Then again M is called a complex manifold if it is equipped with a complex structure J. It turns out that this definition is equivalent to the above, but is more convenient for the treatment of Calabi-Yau manifolds. For more information we refer to [Joy]. In our case we consider the infinite dimensional manifold M which is equipped with a closed, non degenerate, complex valued $\mathbb{C}$ linear two form $\omega = \omega_1 + i\omega_2$. In this case ω will not be compatible with the natural complex structure I on TM which is given by multiplication by i, since we have $\omega(IX, IY) = -\omega(X,Y)$. To handle this problem we assume that TM has a second complex structure J which anticommutes with I and is compatible with ω, that is $J * \omega = \omega$, $\omega(., J(.))$ is "positive definite" in the sense, that for all $m \in M$, $X_m \in T_m$ we have either $\omega_2(X_m, J(X_m)) > 0$ or $\omega_2(X_m, J(X_m)) = 0$ and $\omega_1(X_m, J(X_m)) > 0$. Like in the real case we assume that there is an action of some torus T which preserves I and J and leaves ω invariant. Furthermore we assume the $T-$ action to have a discrete fixed point set P. Following [BtD] J gives the tangent spaces T_pM the structure of quaternionic representations of T. Since we are working in a complex setting, we can give T_pM a decomposition according to the $\pm i$ eigenspaces of J. That is, we have a decomposition $T_pM = T_pM^+ \oplus T_pM^-$, where T_pM^+ denotes the $+i$ eigenspace of J and T_pM^- denotes the $-i$ eigenspace of J. Viewed as vector spaces T_pM^- and T_pM^+ are isomorphic. Now we can decompose T_pM^+ like in the real case into ints direct sum of complex one dimensional representations, that is $T_pM^+ = \bigoplus_{j \in \mathbb{N}} V_j^p$(with respect to the complex structure J), such that T acts via the complex character $t \mapsto \exp(2\pi i \alpha_j^p(H))$. With theses choices made we can define $Z_p(H)$, p and the Liouville functional $L_t(J_H)$ of some Hamiltonian function J_H exactly the way we did in the real case. Of course it is natural in the complex setting to consider the H which give rise to the $\mathbb{R}$ action to live in $\mathfrak{h}_{\mathbb{C}}$, rather then in the real Lie algebra $\mathfrak{h}$. Since M is a complex manifold, such an H defines a vector field $\tilde{H}$ and we call the function $J_H : M \to \mathbb{C}$ Hamiltonian if the identity $dJ_H = i_{\tilde{H}}\omega$ is valid. The formalism of the Liouville functional can be generalized to this setting without major changes. Again we get a decomposition of tangent spaces $T_pM = \bigoplus_{j \in \mathbb{N}} V_j^p$

into complex (with respect to I) one dimensional representations V_j^p on which the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ acts via the character $2\pi\alpha_j^p$. This causes only problems with the definition of p, since the zeta regularization works also in the complex case. To generalize the notion of p to the complex case we observe what happened in the case $H \in \mathfrak{h}$. We have decompositions of the tangent spaces of M at p into 4-dimensional real representations $T_pM = \bigoplus_j (V_{\alpha_j}^p \oplus V_{-\alpha_j}^p)$ of T (where T is the torus), such that if one diagonalizes the $T-$action with respect to the complex structure I the torus acts on V_j^p via the complex character $\exp(2\pi i\alpha_j^p)$. In this setting the other complex structure J defines an $\mathbb{R}$ linear map $J : V_\alpha^p \to V_{-\alpha}^p$. Restricting the T action to the $+i$ eigenspace of J amounts to picking one character out of each pair $\pm\alpha_j^p$ appearing in the decomposition of T_pM. Now the choice of some regular $H \in \mathfrak{h}$ (i.e $\beta(H) \notin \mathbb{Z}$ for all characters $\exp(2\pi i\beta)$ of T) gives a decomposition of the character lattice Q into positive and negative characters if we declare $\beta \in Q$ to be positive if $\beta(H) > 0$. In this picture our number p is exactly the number of negative characters appearing in the series $\{\alpha_j^p\}_{j\in\mathbb{N}}$. Now it is straightforward to generalize the definition of p to the complex case. The element $H \in \mathfrak{h}_{\mathbb{C}}$ comes from the action of the Lie algebra of the complexified torus $T_{\mathbb{C}}$ on M. So we pick a decomposition of the character lattice $Q = Q_- \cup Q_+$ of $T_{\mathbb{C}}$ into positive and negative characters. In analogy with the real case the decomposition should come from an element $H \in \mathfrak{h}_{\mathbb{C}}$. We define $\alpha \in Q$ to be positive if $Im(\alpha(H)) > 0$ or $Im(\alpha(H)) = 0$ and $Re(\alpha(H)) > 0$. This choice of decomposition of the character lattice Q into positive and negative characters agrees with the definition of positive definiteness of the $\mathbb{C}$ valued symmetric bilinear form $\omega(., J(.))$ above. Then as before we can set p to be the number of negative characters appearing in the series $\{\alpha_j^p\}_{j\in\mathbb{N}}$

11. A partition function for a compact Lie group

In this section a Liouville functional is calculated which gives rise to an interesting function. We first consider the infinite dimensional manifold $\mathcal{L}G/T$ where G is a compact, semi simple, simply connected Lie group, T denotes its maximal torus and as usual $\mathcal{L}G$ is th1e loop group, i.e $C^\infty(S^1, G)$. Let $\mathfrak{g}$ denote the Lie algebra of G and so $\mathcal{L}\mathfrak{g}$ is the loop algebra. Let $\langle ., . \rangle$ denote the negative of the Killing form on $\mathfrak{g} \otimes \mathbb{C}$, this gives rise to a $G-$ invariant bilinear form on $\mathfrak{g}$. If we want to use the Liouville functional approach for any calculation, we should ask if it is possible to give $\mathcal{L}G/T$ a symplectic structure. So let us first establish a skew symmetric bilinear form on $\mathcal{L}\mathfrak{g}$. The latter is defined via $\langle ., . \rangle$:

$$\omega(X, Y) = \int_0^1 \langle X'(t), Y(t)\rangle dt,$$

Lemma 11.1. *ω gives rise to a symplectic form on $\mathcal{L}G/G$.*

Proof. We note that $\langle \cdot, \cdot \rangle$ is an invariant form and $X'(t)$ denotes the usual derivative. We have to show, that ω is a antisymetric, closed, non degenerate billinear form on $\mathcal{L}G/G$. Obviously, it is non degenerate on the space of constant loops. Partial integration yields the skew symmetry:
Let

$$c(t) \mapsto \langle X(t), Y(t)\rangle$$

Then $c'(t)$ is

$$X'(t)Y(t) + X(t)Y'(t)$$

Now the fundamental theorem of calculus yields

$$\int_0^1 c'(t) = c(1) - c(0) = 0$$

since our loops are closed and we have $\omega(X,Y) = -\omega(Y,X)$ as desired.
To show that ω is closed, we first note that the map

$$LG \quad (56)$$
$$\downarrow$$
$$LG/G$$

Now consider the map is a principal fiber bundle. Consider the map

$$\sigma : S^1 \times (-\epsilon, \epsilon) \to G$$
$$\sigma(z,t) = \sigma_t(z)$$

$$\frac{d}{dt}|_0 \sigma_t : S^1 \times (-\epsilon, \epsilon) \to TG$$

Now we can calculate

$$d\omega(\zeta, \eta, \xi) = \zeta \cdot d\omega(\eta, \xi) \pm \ldots - \omega([\zeta, \eta], \xi) \pm \ldots$$

and we define two maps

$$\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \qquad \tilde{\omega} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

$\tilde{\omega}(\zeta, \eta) = \omega(T\lambda_{g^{-1}} \cdot \zeta, T\lambda_{g^{-1}} \cdot \eta)$. Now

$$\tilde{\omega}(L_X, L_Y) = \omega(X,Y)$$

and so

$$d\omega(L_X, L_Y, L_Z) = -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X)$$

which shows that ω is closed by the Jacobi identity. □

It is a well known fact, that G/T is a generic coadjoint orbit of G and hence possesses a canonic symplectic structure, the Kirillov-Kostant-Souriau form ω_0^H. On the tangent space eT this form is given by

$$\omega_0^H(X,Y) = \langle H, [Y,X] \rangle$$

As a manifold, $\mathcal{L}G/T$ is isomorphic to $\mathcal{L}G/G \times G/T$. So for generic $H \in \mathfrak{h}$(i.e $\exp(H)$is dense in T)we have a symplectic form on $\mathcal{L}G/T$, which is given by

$$\omega^H = pr_1^*\omega + pr_2^*\omega_0^H$$

It was shown in [PS] that the symplectic manifold $\mathcal{L}G/G$ admits a complex structure, which makes it into a Kaehler manifold. According to the formalism of the Liouville functional we will choose a Riemannian metric σ on $\mathcal{L}G/G$ and show that σ and ω are compatible. First we define a Riemannian metric σ by

$$\sigma_{eT}(X,Y) = \int_0^1 \langle X(t), Y(t) \rangle$$

where again $\langle ., . \rangle$ denotes the negative of the Killing form. According to the above definition the symplectic form σ is given by

$$\sigma_{eT} = \int_0^1 \langle X'(t), Y(t) \rangle + \int_0^1 \langle H, [Y(t), X(t)] \rangle dt = \; = \sigma_{eT}(X'(t), Y(t)) - \int_0^1 \langle [H, X(t)], Y(t) \rangle dt$$

So the skew symmetric endomorphism relating the two structures is given by

$$B_{\sigma,eT} = \frac{\partial}{\partial t} - ad(H)$$

Since the structures σ and ω^H on $\mathcal{L}G/T$ are both defined by left translation the zeta regularized Pfaffian PfB_σ - if it exists - will be constant. So the next goal is to show, that PfB_σ indeed exists. To this end, we identify the root system of $\mathfrak{g}\otimes\mathbb{C}$ with a subset of the character lattice Q of T used above in the complex case.

Lemma 11.2. *The zeta regularized Pfaffian $PfB_\sigma(eT)$ exists and is given by*

$$\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$$

Remark 11.3. The zeta regularized Pfaffian was defined as $\sqrt{\det_\zeta B_\sigma(eT)}$, so it is sufficient to show the existence of $\det_\zeta B_\sigma(eT)$. The zeta regularized determinant was defined as the product of the eigenvalues of the endomorphism in question. We will provide another calculation than that one employed in [W2], since we have 9.11 at hand.

Proof. Let us first consider the root space decomposition of $C^\infty(S^1, \mathfrak{g})$. Remember that we identified the root system with a subset of the character lattice, the the root space decomposition reads

$$\begin{aligned}\bigoplus C^\infty(S^1, \mathfrak{h}_\mathbb{C} \oplus_{\alpha\in\Delta} \mathbb{C}X_\alpha) = \\ = \bigoplus_{\alpha\in\Delta\cup\mathcal{B}} \mathbb{C}e^{2\pi int} X_\alpha\end{aligned}$$

Here $\mathcal{B}$ denotes a basis of $\mathfrak{h}$. Since we identified Δ with a subset of Q and the torus acts exactly by the character $\exp(2\pi i\alpha_j^p)$, with p a fixed point of the torus action, the eigenvalues of $B_\sigma(eT)$ are

$$\{\pm 2\pi in)\}_{n\in\mathbb{N}} \cup \{\pm 2\pi i\alpha(H) \pm 2\pi in\}_{n\in\mathbb{N}_0}$$

The multiplicities of the eigenvalues are 1 if $\alpha \neq 0$ and $l = \dim T$ if $\alpha = 0$.
So we calculate

$$\begin{aligned}\prod_{n\in\mathbb{Z}}^{\infty\ \zeta} 2\pi ia(n+\alpha) &= 2\pi ia\alpha \left(\prod_{n\in\mathbb{N}}^{\infty\ \zeta} 2\pi ia(n+\alpha)\right)\cdot\left(\prod_{n\in\mathbb{N}}^{\infty\ \zeta} 2\pi ia(-n+\alpha)\right) \\ &= 2\pi ia\alpha\,(2\pi ia)^{\zeta_\alpha(0)}\left(\prod_{n\in\mathbb{N}}^{\infty\ \zeta}(n+\alpha)\right)(-2\pi ia)^{\zeta_{-\alpha}(0)}\left(\prod_{n\in\mathbb{N}}^{\infty\ \zeta}(n-\alpha)\right) \\ &= 2\pi ia\alpha\,(2\pi a)^{\zeta_\alpha(0)+\zeta_{-\alpha}(0)}\,\frac{\sqrt{2\pi}}{\alpha\Gamma(\alpha)}\cdot\frac{\sqrt{2\pi}}{-\alpha\Gamma(-\alpha)} \\ &= 2\pi ia\alpha(2\pi a)^{-1}\cdot\frac{2\pi\sin(\pi\alpha)}{\pi\alpha} \\ &= \frac{2\pi i\alpha 2\pi}{2\pi a\pi\alpha}\cdot\sin(\pi\alpha) \\ &= 2i\sin(\pi\alpha)\end{aligned}$$

Since we want to range the product over all positive roots, we use the same calculation again, which gives the result

$$-2i\sin\pi\alpha$$

Putting these calculations together we get

$$\prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha)$$

Taking we square root yields the desired result, now we calculate the term involving only $2\pi ian$

$$\prod_{n\in\mathbb{Z}/\{0\}}^{\zeta} 2\pi ian = \left(\prod_{n\in\mathbb{N}}^{\zeta} 2\pi ian\right)\cdot\left(\prod_{n\in\mathbb{N}}^{\zeta} -2\pi ian\right) =$$

$$= (2\pi ia)^{\zeta(0)}\prod_{n\in\mathbb{N}}^{\zeta} n\cdot(-2\pi ian)^{\zeta(0)}\prod_{n\in\mathbb{N}}^{\zeta} n = (2\pi a)^{2\zeta(0)}\cdot\sqrt{2\pi}\cdot\sqrt{2\pi} = \frac{1}{a}$$

□

Remark 11.4. Note, that $\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$ is the denominator in the Weyl character formula for the compact Lie group.

12. A torus action on $\mathcal{L}G/T$

We will now introduce a torus action on our homogenous space, this will be done in conceptual analogy with the introduction of the symplectic form. We first consider a torus action on $\mathcal{L}G/G$ and on G/T, together this yields a torus action on $\mathcal{L}G/G$. So let us identify $\mathcal{L}G/G$ with the space of based loops $\Omega G = \{\gamma\in\mathcal{L}G|\gamma(0)=e\}$. S^1 acts on ΩG by rotation, that is

$$R_t(\gamma)(u) = \gamma(u+t)\gamma(t)^{-1}$$

One sees immediately that this is indeed an action.
But there is a further action on $\mathcal{L}G/G$. The maximal torus $T\subset G$ acts on $\mathcal{L}G/G$ by conjugation. We can put the two actions together and get the following $S^1\times T$ action:
Note, that we write elements of the torus as $\exp(H)$ where $H\in\mathfrak{h}$ and the latter denotes the Lie algebra of the torus. So an element $(t,\exp(H))\in S^1\times T$ acts by

$$(t,\exp(H)) : \gamma\mapsto \exp(tH)R_t(\gamma)\exp(-H)$$

Clearly, the fixed point set of this action are the homomorphisms $\gamma : S^1\to T$. There is also an action on G/T, here the torus acts by left multiplication. So the fixed point set of this action is $N(T)/T$ where $N(T)$ denotes the normalizer of T in G. It is well known, that $N(T)/T$ is the Weyl group. Letting S^1 act trivially on G/T we get an $S^1\times T$ action on G/T with fixed point set $Q^\vee\times W$, where $Q^\vee$ denotes the lattice of homomorphisms $\gamma : S^1\to T$ and W is the Weyl group of G. It is easy to see, that this torus action preserves the symplectic form as well as the complex structure I. Now take a generic $H\in\mathfrak{h}$, this defines an $\mathbb{R}$ action on $\mathcal{L}G/T$ by the construction above. Now we can compute the denominator $Z_p(H)$ of the Liouville functional. The tangent space of $\mathcal{L}G/T$ at eT is isomorphic to $\mathcal{L}\mathfrak{g}/\mathfrak{h}$. Its decomposition into rotation planes is exactly the decomposition of $\mathcal{L}\mathfrak{g}/\mathfrak{h}$ into eigenspaces of the endomorphism $B_\sigma(eT)$ used in the lemma above. In the proof of 3.9 that the eigenvalues of the torus action are given by the two series $\{2\pi i(\pm\alpha(H)\pm n)\}_{\alpha\in\Delta_+,n\geq 0}$ and $\{\pm 2\pi in\}_{n>0}$ again with the multiplicities 1 for

$\alpha \neq 0$ and l for $\alpha = 0$. Note also, that the eigenvalues of the torus must be multiplied by $\frac{1}{2\pi}$ according to the definition of Z_{eT}. Therefore we have to calculate the zeta regularized product

$$Z_{eT} = \prod_{\alpha\in\Delta_+} \left((\alpha(H))^2 \prod_{n=1}^{\infty} (n^2 - \alpha(H)^2)^2 \right)_\zeta \cdot \left(\prod_{n=1}^{\infty} n^2 \right)_\zeta^l$$

So exactly the same calculation as in 11.2 leads to the result

Lemma 12.1. *The denominator of the Liouville functional is given by*

$$Z_{eT} = (\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$$

So the series defining Z_{eT} is zeta multipliable and all the necessary premises to calculate the Liouville functional of a Hamiltonian of our $\mathbb{R}$ action are checked. Now we have to calculate the number of fixed points $(\beta, w) \in Q^\vee \times W$. To do this, we identify the fixed point set $Q^\vee \times W$ with the affine Weyl group $\tilde{W}$ of the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$ (see Prop 2.30) corresponding to the Lie algebra $\mathfrak{g}$. Furthermore the set $\{\pm\alpha \pm n | \alpha \in \Delta_+, n \geq 0\} \cup \{\pm n | n > 0\}$ can be identified with the root system of $\tilde{\Delta}$ of $\tilde{\mathfrak{g}}$ (cf.[K]). Let us assume that $H \in \mathfrak{h}$ lies in a fundamental alcove of the action of the affine Weyl group on $\mathfrak{h}$. Then the set $\tilde{\Delta}_+ = \{\alpha \in \tilde{\Delta} | \alpha(H) > 0\}$ defines a decomposition of $\tilde{\Delta}$ into positive and negative roots. Now we can see by identifying the spaces $T_{\beta,w}\mathcal{L}G/T$ with $T_e\mathcal{L}G/T$ via left multiplication of a representative of $(\beta, w)^{-1}$ in $\mathcal{L}G$ that the number of fixed points (β, w) is exactly the number of positive roots of $\tilde{\mathfrak{g}}_\mathbb{C}$ which are mapped to negative roots by the action of (β, w) by the action of (β, w) on the root system of $\tilde{\mathfrak{g}}_\mathbb{C}$. By definition, this is the length $l(w)$ of (β, w) in $\tilde{W}$.

13. Calculation of a Liouville functional

Let $H \in \mathfrak{g}$ and $\gamma \in \mathcal{L}G$ be given. Recall the notion of a vector field along a curve: [Michor]

Definition 13.1. *Let γ be a curve from an interval J into a manifold M. Let $\pi_M : TM \to M$ and call $\gamma'(t) : J \to TM$ a vector field along γ if $\pi_M \circ \gamma' = \gamma$.*

We define the vector field ad$H(\gamma)$ along γ by

$$\mathrm{ad}H(\gamma) = \frac{\partial}{\partial s}|_0 \exp(sH)\gamma\exp(-sH).$$

With the above definition and the well known fact that the tangent bundle for a Lie group is trivial, it is immediately seen that ad$H(\gamma)$ is a vector field along γ. Now let $H \in \mathfrak{h}$ be a generic element (i.e the group generated by $\exp(H)$ is dense in T). Let us define a function $J_H : \mathcal{L}G \to \mathbb{R}$ via

$$\gamma \mapsto \frac{1}{2}\int_0^1 ||(\frac{\partial}{\partial t}\gamma(t) - \mathrm{ad}H(\gamma(t)))\gamma^{-1}(t)||^2 dt$$

Since the scalar product $\langle .,. \rangle$ is $G-$ invariant we have $J_H(\gamma) = J_H(\gamma h)$ for all $h \in T$. So J_H defines a function on $\mathcal{L}G/T$ which will be denoted by the same symbol. Our next goal is to show that J_H is a Hamiltonian. This is done in analogy to the proof 8.9.3 in [PS]. In order to view the γ's as matrix valued functions we fix a faithful representation of the group G (such a representation can be found for every compact

Lie group G). So we can write $J_H(\gamma) = \frac{1}{2}\int_0^1 ||\gamma'(t)\gamma^{-1}(t) + \gamma(t)H\gamma^{-1} - H||^2 dt$. We show

Lemma 13.2. *The Hamiltonian vector field on $\mathcal{L}G/T$ corresponding to $J_H(\gamma)$ is exactly the vector field coming from the $\mathbb{R}$ action defined by H.*

Proof. Choose a representative γ for $\gamma T \in \mathcal{L}G/T$ and let $\delta\gamma$ be the variational derivative of γ. The vector field generated by the $\mathbb{R}$ action on $\mathcal{L}G/T$ is given at the point γT by $\gamma' + adH(\gamma)(\text{mod}\mathfrak{h})$. So according to the definition of a Hamiltonian function we have to show that

$$(dJ_H)_\gamma(\delta\gamma) = \omega_\gamma^H(\delta\gamma, \gamma' + adH(\gamma))$$

But $dJ_{H_\gamma}(\delta\gamma)$ is given by

$$\begin{aligned} dJ_{H_\gamma}(\delta\gamma) = \int_0^1 \langle \delta(\gamma'\gamma^{-1} + \gamma H\gamma^{-1} - H), \gamma'\gamma^{-1} + \gamma H \gamma^{-1} - H)\rangle dt = \\ = \int_0^1 (\langle \delta(\gamma'\gamma^{-1}), \gamma'\gamma^{-1}\rangle + \langle \delta(\gamma'\gamma^{-1}), \gamma H\gamma^{-1}\rangle - \\ -\langle \delta(\gamma'\gamma^{-1}), H\rangle + \langle \delta(\gamma^{-1}H\gamma), \gamma'\gamma^{-1}\rangle + \\ +\langle \delta(\gamma^{-1}H\gamma), \gamma H\gamma^{-1}\rangle - \langle \delta(\gamma^{-1}H\gamma), H\rangle)dt \end{aligned}$$

where we just used the definition of J_H and the bilinearity of the scalar product. Now a short calculation shows

$$\delta(\gamma^{-1}\gamma') = (\delta\gamma\gamma^{-1})' + [\delta\gamma\gamma^{-1}, \gamma^{-1}\gamma']$$

and

$$\delta(\gamma H\gamma^{-1}) = [\delta\gamma\gamma^{-1}, \gamma H\gamma^{-1}]$$

Furthermore the pointwise G invariance of the scalar product $\langle ., . \rangle$ implies

$$\langle [\delta\gamma\gamma^{-1}, \gamma^{-1}\gamma'], \gamma^{-1}\gamma'\rangle = 0 = \langle [\delta\gamma\gamma^{-1}, \gamma H\gamma^{-1}], H\rangle$$

Partial integration yields

$$\int_0^1 \langle (\delta\gamma\gamma^{-1})', H\rangle = 0$$

So we get

$$dJ_{H_\gamma(\delta\gamma)} = \int_0^1 \langle (\delta\gamma\gamma^{-1})', \gamma'\gamma^{-1}, \gamma H\gamma^{-1}\rangle dt + \int_0^1 \langle H, [\gamma'\gamma^{-1} + \gamma H\gamma^{-1}, \delta\gamma\gamma^{-1}]\rangle$$

Comparing with the definition of ω_γ^H we see that this is the assertion. □

Next we will according to the definition, calculate J_H in the fixed points. Since the fixed point set is exactly $Q^\vee \times W$ we choose representatives $g_w \in N(T)$ for $w \in W$ and let β be in $Q^\vee$. Set $\gamma(t) = g_w \exp(t\beta)$ and recall the notion of J_H, that is

$$J_H(\gamma) = \frac{1}{2}\int_0^1 ||\gamma'(t)y^{-1}(t) + \gamma(t)H\gamma^{-1} - H||^2 dt$$

but $y'(t) = g_w \exp(t\beta)\beta$ and $\gamma^{-1}(t) = (\exp(t\beta))^{-1} g_w^{-1}$. So $\gamma'(t)\gamma^{-1}(t) = \beta$. And $\gamma(t)H\gamma^{-1} = w(H)$. So the function J_H reads

$$J_H(\gamma) = \frac{1}{2}\int_0^1 ||\beta + w(H) - H||^2 = \frac{1}{2}||\beta + w(H) - H||^2$$

Now we plug into the definition of the Liouville functional. First recall the definition

$$L_1(J_H) = \sum_{p\in P} (-1)^{\sharp p} \frac{e^{-J_H}}{Z_p(H)}$$

So $L_1(J_H)$ reads

$$L_1(J_H) = \frac{1}{(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))} \sum_{w\in W} \sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{1}{2}||\beta+w(H)-H||^2}$$

Let $\rho = \sum_{\alpha\in\Delta_+} \alpha$ denote the half sum of positive roots and recall a theorem due to I.Frenkel which was stated in **??**:
There $\sigma(h)\sigma(-k)$ denotes the denominator of the Weyl character formula for a compact Lie group. We saw that $\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$, i.e the denominator of the Weyl character formula for a compact Lie group, appeared in the calculation of the zeta regularized Pfaffian. Furthermore in our case $t = 1$ and $h = k$ so we can rewrite the double sum in $L_1(J_H)$

$$\frac{\sum_{w\in W} \sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{1}{2}||\beta+w(H)-H||^2}}{\prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha(H))} e^{\frac{1}{2}||h||^2+\frac{1}{2}||-h||^2} \cdot (2\pi)^{\frac{l}{2}} \cdot volQ^\vee = \\ = \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2}$$

So the double sum in $L_1(J_H)$ reads

$$\sum_{w\in W} \sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{1}{2}||\beta+w(H)-H||^2} = \\ \frac{\prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha(H))}{(2\pi)^{\frac{l}{2}} \cdot volQ^\vee} \cdot \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2}$$

Now we insert this into $L_1(J_H)$ and get

$$L_1(J_H) = \frac{1}{(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))} \cdot \frac{\prod_{\alpha\in\Delta_+} 4\sin^2(\pi\alpha(H))}{(2\pi)^{\frac{l}{2}} \cdot volQ^\vee} \cdot \\ \cdot \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2} = \\ \frac{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))}{(\sqrt{2\pi})^{2l} vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2}$$

Now we recall the definition of the formal integration with respect to the Riemannian volume form, which was

$$\int_M e^{-tJ_H} d\sigma = \frac{L_t(J_H)}{Pf(B_\sigma eT)}$$

The zeta regularized Pfaffian was calculated in 11.2 and is given by $\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$ Now inserting these identities in the definition of $\int_M e^{-tJ_H} d\sigma$ yields:

$$\int_{\mathcal{L}G/T} e^{-J_H} d\sigma = \frac{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))}{(2\pi)^l vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2} \cdot$$
$$\cdot \left(\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))\right)^{-1} =$$
$$= \frac{1}{(2\pi)^l vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2}$$

According to [W2] we will also formulate this result as a theorem:

Theorem 13.3. *The following identity is valid:*

$$\int_{\mathcal{L}G/T} e^{-J_H} d\sigma = \frac{1}{(2\pi)^l vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho||^2}$$

Remark 13.4. One can give a physical interpretation of the calculations leading to theorem 3.13. Consider a quantum mechanical particle moving on a compact Lie group G with classical action J_H. Then according to Feynman's path integral formulation of quantum mechanics, the trace or partition function of this quantum mechanical system is formally given by $\int_{C^\infty(S^1,G)} e^{-J_H(\gamma)} d\gamma$, where the integration is over all closed loops in G. But this is basically the same as the "integral" $\int_M e^{-J_H} d\sigma$ we calculated above. Indeed, the definition of $\int_M e^{-J_H} d\sigma$ for a symplectic manifold M is merely a formalization of the heuristic techniques employed in the physics literature in calculating such integrals.

As we saw in the statement of Frenkel's theorem above, our $H \in \mathfrak{h}$ defining the symplectic structure on $\mathcal{L}G/T$ was chosen to be the same as the element $K \in \mathfrak{h}$ which defined the $\mathbb{R}$ action. This is in fact not necessary and was only done to emphase the similarities with the calculations leading to the partition function of the WZW model. We will see, that for certain choices of H and K there is a natural interpretation of the integral $\int_{\mathcal{L}G/T} e^{-J_H} d\sigma$ as characters of the affine Lie algebra $\tilde{\mathfrak{g}}_\mathbb{C}$.
Choose $a \in \mathbb{R}_{>0}$ and $H \in \mathfrak{h}$ such that aH is generic in $\mathfrak{h}$. Then we define like in the above discussion a non degenerate closed two form $\omega^{H,a}$ on $\mathcal{L}G/T$ via $\omega^{H,a} = pr_1^*\omega^a + pr_2^*\omega_0^{aH}$, where as before $\mathcal{L}G/T$ is identified with $\mathcal{L}G/G \times G/T$ and we have $\omega_{eT}(X,Y) = \int_0^1 \langle aX'(t), y(t)\rangle dt$. Furthermore let us choose $b \in \mathbb{R}_{>0}$ and $K \in \mathfrak{h}$. As in the case above we define an $\mathbb{R}$ action on $\mathcal{L}G/T$ via

$$u : \gamma \mapsto \exp(buK) R_{bu}(\gamma) \exp(-buK)$$

If bK is generic, the fixed point set of this action $Q^\vee \times W$ as before. The vector field defined by this $\mathbb{R}$ action at a point $\gamma T \in \mathcal{L}G/T$ is by $b\gamma' + ad(bK)(\gamma)$, and as in the proof of Lemma 3.12 one deduces that this vector field is exactly the Hamiltonian vector field on $\mathcal{L}G/T$ corresponding to the function $J_{H,K,a,b}$ which is defined by

$$J_{H,K,a,b} = \frac{ab}{2}\int_0^1 ||\gamma'(t)\gamma^{-1}(t) + \gamma(t)K\gamma^{-1}(t)||^2 dt$$

The skew symmetric endomorphism $B_\sigma(eT)$ relating the Riemannian metric and the new symplectic form is now given by $a\frac{\partial}{\partial t} + \mathrm{ad}(aH)$ and the zeta regularized

Pfaffian is given, by exactly the same calculations as in 11.2, by

$$Pf_\zeta(B_\sigma)(eT) = \frac{1}{a^l} \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))$$

and

$$Z_{eT}(K) = \frac{1}{b^l}(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K))$$

Like in the case above we calculate

$$L_1(J_{H,K,a,b}) = \frac{b^l}{(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K))} \cdot \\ \cdot \sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{ab}{2}||\beta+w(K)-H||^2}$$

Set $c = \frac{1}{ab}$ and again apply Frenkels theorem stated above, now the reformulation of our double sum above yields:

$$\sum_{w\in W}\sum_{\beta\in Q^\vee} (-1)^{l(w)} e^{-\frac{1}{2c}||\beta+w(K)-H||^2} = \\ = \frac{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)) \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K))}{(\sqrt{2\pi})^l vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K)) e^{-\frac{c}{2}}$$

Now we calculate $L_1(J_{H,K,a,b})$

$$L_1(J_{H,K,a,b}) = \frac{b^l}{(\sqrt{2\pi})^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K))} \cdot \\ \cdot \frac{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H)) \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(K))}{(\sqrt{2\pi})^l vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K)) e^{-\frac{c}{2}} = \\ = \frac{b^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))}{(\sqrt{2\pi})^{2l} vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K)) e^{-\frac{c}{2}}$$

Now we calculate $\int_M e^{-J_{H,K,a,b}}$ which is just $L_1(J_{H,K,a,b})$ divided by $Pf_\zeta(B_\sigma)(eT)$, that is

$$\int_M e^{-J_{H,K,a,b}} = \frac{a^l}{\prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))} \cdot \\ \cdot \frac{b^l \prod_{\alpha\in\Delta_+} 2\sin(\pi\alpha(H))}{(\sqrt{2\pi})^{2l} vol(Q^\vee)(2\pi)^{\dim\mathfrak{g}}} \cdot \sum_{\lambda\in P_+} \chi_\lambda(\exp(-h))\chi_\lambda(\exp(K)) e^{-\frac{c}{2}} = \\ = \frac{(ab)^l}{(2\pi)^l vol(Q^\vee)} \cdot \sum_{\lambda\in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K)) e^{-\frac{c}{2}}$$

Theorem 13.5. *Let a, b, H, K as above. Then the following identity is valid:*

$$\int_M e^{-J_{H,K,a,b}} =$$

$$\frac{(ab^l)}{(2\pi)^l vol(Q^\vee)} \cdot \sum_{\lambda \in P_+} \chi_\lambda(\exp(-H))\chi_\lambda(\exp(K))e^{-\frac{c}{2}}$$

It was shown in [F] how for certain H and b the numerator of the right hand side can be interpreted as numerator of the Kac Weyl character where t can be any real or complex parameter. This theorem can be extended to the case when the fixed point set consists of submanifolds instead of isolated points. Now let us assume that there is not only a symplectic structure on our manifold, but we also chose a Riemannian metric σ. Let $d\sigma$ denote the Riemannian volume form. The symplectic and Riemannian volume form are related via the Pfaffian Pf. We call the skew symmetrix automorphism on the tangent bundle TM, associated to ω and the metric σ, B_σ. It is defined by

$$\omega_x(X, Y) = \sigma_x(B_\sigma(X), Y) \text{ for } X, Y \in T_xM.$$

Now the symplectic and Riemannian volume form are related by

$$\frac{\omega^{\wedge n}}{n!} = \mathrm{Pf}(B_\sigma)d\sigma$$

formula for highest weight representations of the untwisted affine Lie algebra corresponding to $\mathfrak{g}$ evaluated at K, b. (The generalization of Frenkels program for twisted affine Lie algebras was discussed in chapter two). So theorem 3.16 gives a realization of the affine characters as integral of a coadjoint orbit. This is one of the main features of Kirillov's "method of orbits" in the representation theory of Lie groups. In the following section this approach will be compared with the one from chapter two. Recall that the main difference between the Liouville functional approach and the analytic one applied in chapter two is, that we can apply the Liouville functional in situations, where no natural measure is available. So in the calculations above we integrated over a coadjoint orbit, while in chapter two we were forced to take an appropriate closure of the affine adjoint orbit, on which the Wiener measure exists.

14. Comparison with Wiener measure

We already mentioned that physicists use the Duistermaat Heckmann formula for calculations on infinite dimensional manifolds. One example appears in [A] where some interesting ideas are discussed. In his heuristic deduction of the index theorem for the Dirac operator on a Riemannian manifold, Witten suggested that that the Wiener measure on a Riemannian manifold M should closely be related to the "Riemannian measure" on the loop space of M. (Of course the loop space of a Riemannian manifold is not a symplectic manifold in our sense, but one can extend the definition of $d\sigma$ to this case.) In the case of the homogenous space $\mathcal{L}G/T$ we consider, one can make this connection between the Riemannian volume form $d\sigma$ and the Wiener measure explicit. In fact, we can embed $\mathcal{L}G/T$ into a space of continuous maps $[0, 1] \to G$ on which the Wiener measure is defined (cf.chapter 2). So the first guess would be that possibly after some identifications, the Riemannian volume form and the Wiener measure coincide. But the Riemannian volume form is translation invariant, where the Wiener measure is only quasi invariant; Set let

$$C_G = \{z : [0, 1] \to G | z(0) = e, z\text{continuous}\}$$

and let $f : C_G \to \mathbb{R}$ be integrable with respect to the Wiener measure $\bar{\omega}$ on C_G. Then

$$\int_{C_G} f(z)d\bar{\omega}(z) = \int_{C_G} f(gz)e^{\langle z'z^{-1},g^{-1}g'\rangle - \frac{1}{2}\langle g'g^{-1},g'g^{-1}\rangle}d\bar{\omega}(z)$$

where $g \in C_G$ and $\langle X, Y\rangle = \int_0^1 \langle X(t), Y(t)\rangle dt$ for $X, Y \in C([0,1], \mathfrak{g})$. To get rid of this defect, we will replace $d\bar{\omega}(z)$ with $d\tilde{\bar{\omega}} = e^{\frac{1}{2}||z'z^{-1}||^2}d\bar{\omega}(z)$. This "new" measure $d\tilde{\bar{\omega}}$ is indeed invariant under left translations and we will formally have $d\sigma = d\tilde{\bar{\omega}}$ as desired. To be more concrete remember the classification of the $\mathcal{L}G$ orbits on $\mathcal{L}\mathfrak{g} \times \{1\}$ from 3: Let $\mathcal{O}_g$ denote the conjugacy class of G containing the element g and set

$$C_{G,\mathcal{O}_g} = \{z \in C_G \, s.t. \, z(1) \in \mathcal{O}_g\}$$

Let us identify $\mathcal{L}\mathfrak{g} \times \{1\}$ with $\mathcal{L}\mathfrak{g}$. Then $\mathcal{L}G$ acts via $\gamma : X \mapsto \gamma X \gamma^{-1} + \gamma'\gamma^{-1}$. After identifying $\mathcal{L}G/T$ with the $\mathcal{L}G$ orbit through H, we then define a map

$$\phi : \mathcal{L}G/T \to C_{G,\mathcal{O}_{\exp(H)}}$$

via

$$\gamma H \gamma^{-1} + \gamma'\gamma^{-1} \mapsto z_{\gamma H\gamma^{-1}+\gamma'\gamma^{-1}}$$

where z_X denotes the fundamental solution of the differential equation $z' = -Xz$. One can identify C_G with a subspace $(L\mathfrak{g})^*_0$ of $L\mathfrak{g}*$ and in this identification $C_{G,\mathcal{O}_{\exp(H)}}$ can be viewed as the closure in $(L\mathfrak{g})^*_0$ of the coadjoint orbit containing H. As we have seen in 7, the most natural measure on $C_{G,\mathcal{O}_{\exp(H)}}$ is the conditional Wiener measure introduced in [F]. Let $C_{G,\mathcal{O}_g} \to \mathbb{R}$ be an integrable function with respect to this measure. The integral over f will be denoted by

$$\int_{C_{G,\mathcal{O}_g}} f(z)d\bar{\omega}_{G,\mathcal{O}_g}(z)$$

This integral has the quasiinvariance properties stated above. As outlined before we replace $d\bar{\omega}_{G,\mathcal{O}_g}$ by $d\tilde{\bar{\omega}}_{G,\mathcal{O}_g}(z) = e^{\frac{1}{2}||z'z^{-1}||^2}d\bar{\omega}_{G,\mathcal{O}_g}(z)$ such that we can write

$$\int_{C_{G,\mathcal{O}_g}} f(\gamma z)d\tilde{\bar{\omega}}_{G,\mathcal{O}_{\gamma(2\pi)g}}(z) = \int_{C_{G,\mathcal{O}_g}} f(z)d\tilde{\bar{\omega}}_{G,\mathcal{O}_g}(z)$$

for all $\gamma \in C_G$.
Now let us define a function $\tilde{J}_H : C_{G,\mathcal{O}_{\exp(H)}} \to \mathbb{R}$ via $\tilde{J}_H(z) = \frac{1}{2}||z'z^{-1} + H||^2$. Now one checks easily that $\phi^*\tilde{J}_H = J_H$ with $\phi : \mathcal{L}G/T \to C_{G,\mathcal{O}_{\exp(H)}}$ as before. The main result of this section is the following:

Proposition 14.1.

$$\int_{\mathcal{L}G/T} e^{-J_H(\gamma)}d\sigma(\gamma) = c \cdot \int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\tilde{J}_H(z)}d\tilde{\bar{\omega}}_{G,\mathcal{O}_{\exp(H)}}$$

where $c = e^{\frac{1}{2}||\rho||^2}(2\pi)^l\mathrm{vol}(Q^\vee)$.
So up to a constant, which does not depend on H, the Wiener measure on $C_{G,\mathcal{O}_{\exp(H)}}$ and the integration with respect to the Riemannian volume form on $\mathcal{L}G/T$ are equal.

Proof. For $z \in C_{G,\mathcal{O}_{\exp(H)}}$ we have

$$\tilde{J}_H(z) = \frac{1}{2}||z'z^{-1}||^2 + \langle H, z'z^{-1}\rangle + \frac{1}{2}||H||^2,$$

so we get

$$\int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\tilde{J}_H(z)} d\tilde{\bar{\omega}}_{G,\mathcal{O}_{\exp(H)}}(z) = e^{-\frac{1}{2}||H||^2} \int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\langle H, z'z^{-1}\rangle} d\bar{\omega}_{G,\mathcal{O}_{\exp(H)}}(z)$$

But the last integral was computed in **??**

$$e^{-\frac{1}{2}||H||^2} \int_{C_{G,\mathcal{O}_{\exp(H)}}} e^{-\langle H, z'z^{-1}\rangle} d\bar{\omega}_{G,\mathcal{O}_{\exp(H)}}(z) = \sum_{\lambda\in P_+} |\chi_\lambda(H)|^2 e^{-\frac{1}{2}||\lambda+\rho||^2 - ||\rho||^2}$$

Now compare with Theorem 13.3 and we see that indeed

$$\int_{\mathcal{L}G/T} e^{-J_H(\gamma)} d\sigma(\gamma) = e^{\frac{1}{2}||\rho||^2} (2\pi)^l vol(Q^\vee) \cdot \sum_{\lambda\in P_+} |\chi_\lambda(H)|^2 e^{-\frac{1}{2}||\lambda+\rho||^2 - ||\rho||^2}$$

□

15. The twisted partition function

In this section functions on the coadjoint orbits of twisted loop groups will be integrated using the Liouville functional approach. Let ψ a an outer automorphism of order $ord(\psi) = r$ of the simply connected compact semi simple Lie group G such that ψ acts as an automorphism of the Dynkin diagram on the root system of the complexified Lie algebra $\mathfrak{g}\otimes\mathbb{C}$. As in the previous sections denote the twisted loop group by

$$\mathcal{L}(G,\psi) = \{\gamma \in \mathcal{L}G | \psi(\gamma(t)) = \gamma(t + \frac{1}{r})\}$$

for all $t \in [0,1]$. The Lie algebra of the twisted loop group will be denoted by $\mathcal{L}(\mathfrak{g},\psi)$. By restriction the symmetric form $\langle .,,\rangle$ and the antisymmetric form ω on $\mathcal{L}\mathfrak{g}$ give a symmetric and antisymmetric form on $\mathcal{L}(\mathfrak{g},\psi)$ by restriction and will be denoted by the same symbols. The form $\langle .,,\rangle$ is non degenerate on ω on $\mathcal{L}\mathfrak{g}$ and defines a Riemannian metric on $\mathcal{L}(\mathfrak{g},\psi)$ by left translation. The form ω is degenerate exactly in the subspace of constant loops so that it defines a symplectic form on $\mathcal{L}(\mathfrak{g},\psi)/G^\psi$ where G^ψ denotes the group of fixed points under the automorphism ψ. Since we chose G to be compact and semisimple, so will be G^ψ with maximal torus T^ψ. Like in the untwisted case the manifold G^ψ/T^ψ can be viewed as coadjoint orbit of G^ψ through a generic $H \in \mathfrak{h}^\psi$. As before, the Kirillov form of such an orbit is denoted by ω_0^H. After identifying $\mathcal{L}G^\psi/T^\psi$ with $\mathcal{L}G^\psi/G^\psi \times G^\psi/T^\psi$ we can define a symplectic structure ω_H on $L(G,\psi)/T^\psi$ via $\omega^H = pr_1^*\omega + pr_2^*\omega_0^H$. As in the untwisted case the skew symmetric endomorphism of the tangent space at eT^ψ relating the Riemannian metric and the symplectic structure, reads

$$B_{\sigma,eT^\psi} : X \mapsto X' + adH(X)$$

The calculation of the zeta regularized Pfaffian of B_{σ,eT^ψ} is essentially the same as in the untwisted case, but now we have to take care of the multiplicities of the eigenvalues. Let Δ denote the root system of $\mathfrak{g}\otimes\mathbb{C}$ and let Δ^ψ denote the "folded" root system, that is $\Delta^\psi = \{\bar{\alpha}|\alpha\in\Delta\}$ where $\bar{\alpha} = \frac{1}{ord(\psi)}\sum_{i=1}^{ord(\psi)} \psi^i(\alpha)$. Now let us assume for the moment that Δ is an irreducible root system of type ADE but not of type A_{2n}. In this case Δ^ψ is a root system of type $BCFG$. Let Δ_l^ψ and Δ_s^ψ denote the corresponding subsets of long and short roots in Δ^ψ respectively. Now one uses [K] Prop. 6.3 to see that the eigenvalues of $B_{\sigma,eT^{\psi}}$ are given by

$$\{\pm 2\pi i n | n \in \mathbb{N}_{>0}\} \cup \{2\pi i(\pm\alpha(H)+n) | \alpha \in \Delta_s^{\psi}, n \in \mathbb{Z}\} \cup \{2\pi i(\pm\alpha(H)+nr) | \alpha \in \Delta_l^{\psi}, n \in \mathbb{Z}\}$$

Furthermore if Δ is of type X_N in the notion of [K] then for an arbitrary eigenvalue $2\pi i\lambda$ of $B_{\sigma,eT^{\psi}}$, we have $mult(2\pi i\lambda) = 1$ if $\lambda = \alpha(H)+n$, with $\alpha \in \Delta^{\psi}$. And in the case $\lambda = n$ we have $mult(2\pi i\lambda) = l = \dim(T^{\psi})$ if r divides n and $mult(2\pi i\lambda) = \frac{(N-l)}{r-1}$ if r does not divide n. Now we can calculate the zeta regularized determinant in complete analogy with Lemma 11.2, but now we take the different multiplicities into account. First we consider the terms with multiplicity 1. This product was already calculated in Lemma 11.2

$$\prod_{n\in\mathbb{Z}}^{\zeta} 2\pi i n = 1$$

Now we calculate the zeta regularized product for the terms which have values in the short roots:

$$\prod_{n\in\mathbb{Z}}^{\zeta} 2\pi i(\alpha+n) = \prod_{\alpha\in\Delta_{s+}^{\psi}} 4\sin^2(\pi\alpha(H))$$

Now we calculate the zeta regularized product for the terms which take values in the long roots, here we have to take the term $(\pm\alpha+nr)$ into account:

$$\begin{aligned}\prod_{n\in\mathbb{Z}}^{\zeta}(2\pi i r)(\frac{\alpha}{r}+n) &= 2\pi i r\frac{\alpha}{r}\prod_{n\in\mathbb{N}}^{\zeta}2\pi i r(\frac{\alpha}{r}+n)\prod_{n\in\mathbb{N}}^{\zeta}2\pi i r(\frac{\alpha}{r}-n) = \\ &= 2\pi i\alpha(2\pi r)^{\zeta_{\frac{\alpha}{r}}(0)+\zeta_{\frac{-\alpha}{r}}(0)}\cdot\prod_{n\in\mathbb{N}}^{\zeta}(\frac{\alpha}{r}+n)\prod_{n\in\mathbb{N}}^{\zeta}(-\frac{\alpha}{r}+n) = \\ &= \frac{2\pi i\alpha}{2\pi r}\cdot\frac{\sqrt{2\pi}\sqrt{2\pi}}{(\frac{\alpha}{r})\Gamma(\frac{\alpha}{r})(-\frac{\alpha}{r})\Gamma(\frac{-\alpha}{r})} = \\ &= \frac{2\pi i\alpha\cdot 2\pi}{2\pi r\cdot\pi\frac{\alpha}{r}}\cdot\sin(\pi\frac{\alpha}{r}) = 2i\sin(\pi\frac{\alpha}{r})\end{aligned}$$

So again taking the product over all positive long roots we get

$$\prod_{\alpha\in\Delta_{l+}^{\psi}} 4sin^2(\alpha(\pi\frac{\alpha}{r}))$$

As in Lemma 11.2 we have used the results of [QHS]. Now we come to the last term, which involves the $2\pi i r n$

$$\prod_{n\in\mathbb{Z}}^{\zeta}(2\pi i r n) = (\frac{1}{r})$$

according to our calculations in Lemma 11.2. Now we take the multiplicities into accout and this yields:

$$\frac{1}{r^{l-\frac{(N-l)}{r-1}+2|\Delta_{l+}^{\psi}|}}$$

Finally we get for the zeta regularized Pfaffian:

$$Pf_{\zeta}(B_{\sigma})(eT^{\psi}) = \frac{1}{\sqrt{r}^{l-\frac{(N-l)}{r-1}+2|\Delta_{l+}^{\psi}|}}\prod_{\alpha\in\Delta^{\psi^{vee}}} 2\sin(\pi\alpha(H))$$

Note, that like in the untwisted case the zeta regularized Pfaffian is exactely the denominator of the Weyl character formula for the compact Lie group with root system $\Delta^{\psi\vee}$ or equivialently, the denominator of the characters of the outer component of the principal extension of the Lie group with root sytem Δ, discussed in chapter 2. In any case we see, that like in the untwisted case the Riemannian metric and the symplectic form are compatible. So let us consider a $S^1 \times T^\psi$ action on $\mathcal{L}G/T^\psi$ by restriction of the $S^1 \times T$ action of the untwisted case. That is, $S^1 \times T^\psi$ acts on $\mathcal{L}(G,\psi)/G^\psi$ by "twisted rotation" and T^ψ acts on G^ψ/T^ψ by conjugation. The same arguments as in the untwisted case show now that the fixed point set of this action is $W^\psi \times M$ where $M \subset \mathfrak{h}$ is the lattice generated by the long roots of Δ^ψ (where we identified $\mathfrak{h}$ with $\mathfrak{h}^*$ via the negative of the Killing form). As we saw before, and in Prop.2.30 we can identify $W^\psi \times M$ with the affine Weyl group $\tilde{W}$ belonging to the twisted affine Lie algebra $\mathcal{L}(G,\psi)$. The number of fixed points $\sharp(\beta, w)$ for $W^\psi \times M$ is again given by $\sharp(\beta, w) = l((w,\beta))$, where $l(w,\beta)$ denotes the length of (β, w) in $W^\psi \times M$. Like in the untwisted case, the same calculation as for the Pfaffian, but now the eigenvalues are divided by 2π yields:

$$Z_{eT^\psi}(H) = c \cdot \prod_{\alpha\in\Delta^1_+} 2\sin(\pi\alpha(H))$$

where $c = (\sqrt{2\pi})^l((\sqrt{r})^{l-\frac{(N-l)}{r-1}+2|\Delta^\psi_{l+}|})^{-1}$. In exact analogy with the untwisted case we introduce a function $\tilde{J}_H$ which will be the Hamiltonian function for the calculation of the Liouville functional. We get this function just by restriction to the function considered in the untwisted case. That is our $\tilde{J}_H$ is $J_H|\mathcal{L}(G,\psi)/T^\psi$: $\mathcal{L}(G,\psi)/T^\psi \to \mathbb{R}$, where J_H is just the function considered in the untwisted case. So we will denote $\tilde{J}_H$ by the same symbol and this yields:

$$J_H(\gamma) = \frac{1}{2}\int_0^1 ||\gamma'(t)\gamma^{-1}(t) + \gamma(t)H\gamma^{-1}(t) - h||dt$$

Since for generic $H \in \mathfrak{h}^\psi$, the corrseponding $\mathbb{R}$ action on $\mathcal{L}(G,\psi)/T^\psi$ is just the restriction of the corresponding $\mathbb{R}$ action on $\mathcal{L}G/T$, it follows from lemma 3.12 that the Hamiltonian vector field on $\{L(G,\psi)/T^\psi$ corresponding to J_H is the vector field generated by the $\mathbb{R}$ action. Therefore one can calculate the Liouville functional. This calculation is in exact analogy with the calculations in the untwisted case, remember that we used there a theorem of I.Frenkel [F]Prop. 4.3.4, now we consider the twisted case of this theorem, which was obtained in [W2] and presented in our Theorem 5.1. In rest of the calculation is exactly the same as in the untwisted case. So we first take J_H in the fixed points, which was $J_H(\gamma) = \frac{1}{2}||\beta + w(H) - H||^2$. Now we insert into the definition of the Liouville functional and get

$$L_1(J_H) = \frac{1}{c \cdot \prod_{\alpha\in\Delta^1_+} 2\sin(\pi\alpha(H))} \cdot \sum_{w\in W^\psi}\sum_M (-1)^{l(w)} e^{-\frac{1}{2}||\beta+wH-H||}$$

According to the calculations leading to Theorem 5.1 we can rewrite the double sum in the equation above:

$$\sum_{w\in W^\psi}\sum_M (-1)^{l(w)} e^{-\frac{1}{2}||\beta+wH-H||} =$$

$$= \frac{\prod_{\alpha\in\Delta^1_+} 4\sin^2(\pi\alpha(H))}{(2\pi)^{\frac{l}{2}} vol(a_0M)} \sum_{\lambda\in P_+(\Delta^1)} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho^\psi||^2}$$

where $P(\Delta^{\psi\vee})$ denotes the weight lattice of the corresponding root system, $P_+(\Delta^{\psi\vee})$ denotes the cone of dominant weights, χ_λ denotes the irreducible character of the compact simply connected semi simple Lie group G of the same type as the root system $\Delta^{\psi\vee}$, and as usual ρ^ψ is the half sum of positive roots of corresponding to the root system $\Delta^{\psi\vee}$. So let us put the above calculations together:

$$L_1(J_H) = \frac{\prod_{\alpha\in\Delta^1_+} 2\sin(\pi\alpha(H))}{c(2\pi)^{\frac{l}{2}} vol(a_0 M)} \sum_{\lambda\in P_+(\Delta^1)} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho^\psi||^2},$$

where c is the constant of above.
Now we calculate the integral with respect to the Riemannian volume form, that is by definition

$$\int_M e^{-tJ_H} = \frac{L_1}{Pf(B_\sigma(eT))}$$

The Pfaffian was calculated above and we just insert into the definition

$$\int_{\mathcal{L}(G,\psi)/T^\psi} e^{-J_H}(\gamma)d\sigma(\gamma) = \frac{1}{((\sqrt{r})^{l-\frac{(N-l)}{r-1}+2|\Delta^\psi_{l+}|})^{-1}\prod_{\alpha\in\Delta^{\psi\vee}} 2\sin(\pi\alpha(H))}\cdot$$

$$\cdot\frac{\prod_{\alpha\in\Delta^1_+} 2\sin(\pi\alpha(H))}{c(2\pi)^{\frac{l}{2}} vol(a_0 M)} \sum_{\lambda\in P_+(\Delta^1)} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho^\psi||^2}$$

This yields

Proposition 15.1.

$$\int_{\mathcal{L}(G,\psi)/T^\psi} e^{-J_H}(\gamma)d\sigma(\gamma) = c\cdot\sum_{\lambda\in P_+(\Delta^{\psi\vee})} |\chi_\lambda(\exp(H))|^2 e^{-\frac{1}{2}||\lambda+\rho^\psi||^2}$$

Where $c = (2\pi)^{\frac{l}{2}} vol(M)^{-1}\cdot((\sqrt{r})^{l-\frac{N-l}{r-1}+2|\Delta^\psi_{l+}|})^{-2}$

As in the non twisted case one can compare the Liouville functional with the Wiener measure, and again the Wiener measure is up to a constant the same as the formal integration with respect to the Riemannian volume form.

16. The WZW model

We can now use the Liouville functional approach to integrate functions on spaces, where no measure theory is yet developed. A particular example of such a situation is the partition function of the so called Wess-Zumino-Witten (WZW) model, which is a quantum field theory on a Riemann surface Σ with values in a simply connected semi simple Lie group G, or more generally in its complexification $G_\mathbb{C}$. In our discussion the Riemann surface will be an elliptic curve, i.e the torus Σ_τ with modular parameter $\tau = \tau_1 + i\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$, $\tau_2 > 0$. The torus $S^1\times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ is equipped with a complex structure which is defined by $f : \Sigma_\tau \to \mathbb{C}$. The complex structure is holomorphic if $\bar\partial f := (\partial_s + \tau\partial_t)f = 0$. We now come to the definition of the WZW action functional. For more information on the physical background see e.g [G]. Let $\langle .,.\rangle$ denote the Killing form on $\mathfrak{g}_\mathbb{C}$ normalized in such a way that the long roots have square length 2 and set $\partial = \partial_s + \bar\tau\partial_t$. In this normalization the action functional of the WZW model reads:

$$S_{G,\kappa}(g) = -\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle g^{-1}\partial g, g^{-1}\bar\partial g\rangle dsdt + \frac{i\kappa\pi}{3}\int_B tr((\tilde g^{-1}d\tilde g)^{\wedge 3})$$

Here B is a three dimensional manifold with boundary $\partial B = \Sigma$, and $\tilde{g} : B \to G$ is a map such that $\tilde{g}|_{\partial B} = g$ and trace denotes the negative of the normalized Killing form as well. The second term in the action is the so called Wess-Zumino-Witten term. Up to the factor $i\kappa$, it is the integral over the pull back of the generator of $H^3(G, \mathbb{Z})$ to B via the map $\tilde{g}$. There is some need for an explantation of the 3-form $tr((\tilde{g}^{-1}d\tilde{g}))^{\wedge 3}$. The term inside the bracket is a Lie algebra valued one form. That is we have a map

$$\tilde{g}^{-1}d\tilde{g} = \delta^l \tilde{g} : TM \to \mathfrak{g} \subset \mathfrak{g}l(n)$$

That is the left logarithmic derivative. Note, that the Lie algebra is really viewed as matrix Lie algebra. So the the operation inside $tr((\tilde{g}^{-1}d\tilde{g}))^{\wedge 3}$ is matrix multiplication. If one would view the Wess Zumino term as real valued three form, this would not make any sence, because of the alternating propertie of differential forms this term would vanish. Viewed as Lie algebra valued one form, the anti commutativity of the Lie bracket kills the alternating property. So the term $tr((\tilde{g}^{-1}d\tilde{g}))^{\wedge 3}$ can be viewed as three form, which makes sense on a three dimensional ball.
According to [Hi1] the Wess Zumino term is an invariant with values in $\mathbb{R}/2\pi\mathbb{Z}$ associated to a map $g : \Sigma \to G$. The Wess Zumino term can be viewed as the holonomy around Σ of the canonical gerbe on G(see. [Hi1], [Hi2], [Hi3] for more information on this point of view). The classical interpretation of the Wess Zumino term is that of the integral on G over an extension of g to a three manifold M with with boundary Σ. If G is a compact, simple, simply connected Lie group and $B(X, Y)$ denotes the Killing form of the corresponding Lie algebra, then $B(X, [Y, Z])$ defines a bi-invariant closed three form on G whose cohomology class generates $H^3(G, \mathbb{R})$. A particular multiple of this form gives a form Ω such that $[\Omega]$ generates $H^3(G, 2\pi\mathbb{Z}) \cong \mathbb{Z}$. As an example in [Hi1] the case $G = SU(2)$ is mentioned, in this case the three form is given by

$$\Omega = \frac{1}{12} tr(g^{-1}dg)^{\wedge 3}$$

(In our notion g is the map $\tilde{g}$). Now for a **general** Lie group $g^{-1}dg$ is replaced by the Maurer Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$. If $\tilde{g}_1$ and $\tilde{g}_2$ are two different extensions of g they differ by a map $\tilde{h} : B \to G$ such that $\tilde{h}|_{\partial B} = e$. But for such $\tilde{h}$ we have $\frac{\pi}{3}\int_B tr(\tilde{h}^{-1}d\tilde{h})^{\wedge 3} \in 2\pi\mathbb{Z}$ such that the action $e^{S_{G,\kappa,(g)}}$ is well defined for all $\kappa \in \mathbb{Z}$. An important formula for calculations with the Wess Zumino term is the Polyakov-Wiegmann formula.

Proposition 16.1. *Let $g, h : \Sigma \to G$. Then the following identity is valid:*

$$S_{G,\kappa}(gh) = S_{G,\kappa}(g) + S_{G,\kappa}(h) - \frac{\kappa\pi}{\tau_2} \int_\Sigma \langle g^{-1}\partial g, \bar{\partial} h h^{-1}\rangle dsdt$$

Note that the imaginary part of the term $\frac{\kappa\pi}{\tau_2}\langle g^{-1}\partial g, \bar{\partial} h h^{-1}\rangle$ is exactly the cocycle in the explicit construction of the central extension $\hat{G}$ of the loop group $\mathcal{L}G$ as a quotient see [KW] and also [FKh].

Proof. One checks directly by inserting into the definitions:

$$-\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle h^{-1}g^{-1}\partial(gh), h^{-1}g^{-1}\bar\partial(gh)\rangle dsdt =$$

$$-\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle g^{-1}\partial g, g^{-1}\bar\partial g\rangle dsdt - \frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle h^{-1}\partial h, h^{-1}\bar\partial h\rangle dsdt$$

$$-\frac{\kappa\pi}{2\tau_2}\int_\Sigma \langle (g^{-1}\partial_s g, \partial_s hh^{-1})\rangle + \tau_1\langle g^{-1}\partial_s g, \partial_t hh^{-1}\rangle$$

$$+\tau_1\langle g^{-1}\partial_t g, \partial_s hh^{-1}\rangle + \tau\bar\tau\langle g^{-1}\partial_t g, \partial_t hh^{-1}\rangle dsdt$$

Now we insert into the Wess Zumino term and this yields

$$\frac{i\kappa\pi}{3}\int_B tr((gh)^{-1}d(gh))^{\wedge 3} = \frac{i\kappa\pi}{3}\int_B tr(g^{-1}dg)^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B tr(h^{-1}dh)^{\wedge 3}$$

$$+i\kappa\pi\int_B tr((g^{-1}dg)^{\wedge 2}\wedge dhh^{-1} + g^{-1}dg\wedge (dhh^{-1})^{\wedge 2})$$

$$= \frac{i\kappa\pi}{3}\int_B tr(g^{-1}dg)^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B tr(h^{-1}dh)^{\wedge 3}$$

$$+i\kappa\pi\int_B dtr(g^{-1}dg\wedge dhh^{-1})$$

Since B is a manifold with boundary $\partial B = \Sigma$ we can apply Stoke's theorem and this yields

$$i\kappa\pi\int_B dtr(g^{-1}dg\wedge dhh^{-1}) = i\kappa\pi\int_\Sigma tr(g^{-1}dg\wedge dhh^{-1}).$$

Now we write $dg = \partial_s gds + \partial_t gdt$ and $dh = \partial_s hds + \partial_t hdt$, this yields the assertion. □

Now we come to the definition of the gauged WZW model, in this generalization of the WZW model one adds a particular term. So let $H \in \mathfrak{h}$ be a generic element. We will extend the WZW model by adding an $H-$ dependent term. Set

$$S_{G,H,\kappa}(g) = S_{G,\kappa}(g) + \frac{\kappa\pi}{\tau_2}\int_{\Sigma_\tau} (\langle g^{-1}\partial g, H\rangle - \langle\bar\partial gg^{-1}, H\rangle - -\langle H, g^{-1}Hg\rangle + \langle H, H\rangle)dsdt$$

$S_{G,H,\kappa}(g)$ is the action functional of the gauged WZW model, studied in [GK]. The partition funtion of this model at level κ is formally given by the integral

$$\int_{C^\infty(\Sigma_\tau, G_\mathbb{C})} e^{S_{G,H,\kappa}(g)}\mathcal{D}(g,)$$

The measure theoretic meaning of this integral is at least at this time unclear, $\mathcal{D}(g)$ is interpreted as "formal" measure, in the sequel we will use the Liouville functional approach to calculate the partition function of the gauged WZW model.

16.1. **Double loop groups and a torus action.** We will now apply the results of 4 to the Liouville functional approach Now, we use slightly different notion and denote the double loop group by $\mathcal{LL}G$.

Let $H \in \mathfrak{h}$ be generic and choose a modular parameter $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ such that $\tau_2 > 0$. In analogy with the loop case, we define a non degenerate closed two form and and $\mathbb{R}$ action on $\widetilde{\mathcal{LL}G_C}/T_\mathbb{C}$. Like in the loop group case we have $\mathcal{LL}G_C/T_\mathbb{C} \cong \Omega\Omega G_\mathbb{C} \times G_\mathbb{C}/T_\mathbb{C}$ where $\Omega\Omega G_\mathbb{C}$ denotes the set of based double loops, that is the set of all maps $g \in \mathcal{L}\mathcal{L}G_C$ such that $g(1,1) = e$. Let $X, Y : S^1 \times S^1 \to \mathfrak{g}_{\mathbb{C}}$ be elements of the corresponding Lie algebra. Then

$$\omega_e(X,Y) = \frac{\pi}{\tau_2} \int_{S^1 \times S^1} \langle \bar{\partial} X(s,t), Y(s,t) \rangle ds dt$$

defines a $\mathbb{C}$ valued skew symmetric bilinear form on $\mathcal{L}\mathcal{L}\mathfrak{g}_C$ which is degenerate on the set of holomorphic maps. Since our elliptic curve Σ_τ is compact any holomorphic map has to be constant, so we can use the form above to define a non-degenerate $\mathbb{C}$ valued two form on $\mathcal{L}\mathcal{L}G_C/G_{\mathbb{C}} \cong \Omega\Omega G_{\mathbb{C}}$. Like in the loop case we can choose a $\mathbb{C}$ valued two form ω_0^H on $G_{\mathbb{C}}/T_{\mathbb{C}}$ which is defined via $\omega^H_{0,eT_{\mathbb{C}}}(A,B) = \frac{\pi}{\tau_2}\langle H, [A,B]\rangle$ for $A, B \in T_{eT_{\mathbb{C}}}$ and extended to $G_{\mathbb{C}}/T_{\mathbb{C}}$ via left translation. Again, we put this forms together and get a symplectic form $\omega^H = pr_1^*\omega + pr_2^*\omega_0^H$. The formalism of the Liouville functional asks as a next prerequisite for an almost complex structure which is compatible with the symplectic form. We are now working in the complex setting described in 3.3. So the almost complex structure J must satisfy $J^*\omega = \omega$ and $\omega(\cdot, J(.))$ is positive definite in the sense that for all $m \in M$, $X_m \in T_xM$ we have either $\omega_2(X_m, J_m(X_m)) > 0$ or $\omega_2(X_m, J_m(X_m)) = 0$ and $\omega_1(X_m, J_m(X_m)) > 0$, where the symplectic form $\omega = \omega_1 + i\omega_2$.
Consider the root space decomposition of $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} = \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ into one dimensional root spaces. Here Δ denotes the root system of $\mathfrak{g}_{\mathbb{C}}$. For each $\alpha \in \Delta$ we choose a root vector $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha}\rangle = 1$. Furthermore we choose an orthonormal basis of the Cartan subalgebra $\mathfrak{h}$. Denote the basis vectors with $H_1, \ldots, H_l$. Then we can write any $X \in \mathcal{L}\mathcal{L}\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ as

$$X(s,t) = \sum_{(n,m)\in\mathbb{Z}^2} \sum_{\alpha\in\Delta} c_{n,m,\alpha} X_\alpha e^{2\pi i(ns+mt)} + \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ (n,m)\neq(0,0)}} \sum_{j=1}^{l} c_{n,m,j} H_j e^{2\pi i(ns+mt)}$$

with $c_{n,m,\alpha}, c_{n,m,j} \in \mathbb{C}$.
As always we have $\tau = \tau_1 + i\tau_2$, the modular parameter of the elliptic curve Σ_τ. Now let Δ_+ be the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to a basis of Δ. Let us decompose the set $\tilde{\tilde{\Delta}} = \{(\alpha, n, m) | \alpha \in \Delta \cup 0, (n,m) \in \mathbb{Z}^2, (\alpha, n, m) \neq (0,0,0)\}$ into $\tilde{\tilde{\Delta}}_+ \cup \tilde{\tilde{\Delta}}_-$ via defining (α, n, m) to be positive if either $n + \tau_1 m > 0$ or $n + \tau_1 m = 0$ and $m < 0$ or $n = m = 0$ and $\alpha \in \Delta_+$. Now we can define an $\mathbb{R}$ linear anti-involution J of $\mathcal{L}\mathcal{L}\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ which anticommutes with the natural complex structure, which is given by multiplication with i: For $c \in \mathbb{C}$ set

$$\begin{aligned} J(c_{n,m,\alpha} X_\alpha e^{2\pi i(ns+mt)}) = & \bar{c}_{n,m,\alpha} X_{-\alpha} e^{-2\pi i(ns+mt)} if (\alpha, n, m) \in \tilde{\tilde{\Delta}}_+ and \\ & -\bar{c}_{n,m,\alpha} X_{-\alpha} e^{-2\pi i(ns+mt)} if (\alpha, n, m) \in \tilde{\tilde{\Delta}}_- \end{aligned}$$

analogously set

$$\begin{aligned} J(c_{n,m,\nu} H_\nu e^{2\pi i(ns+mt)}) = & \bar{c}_{n,m,\nu} H_\nu e^{-2\pi i(ns+mt)} if (0, n, m) \in \tilde{\tilde{\Delta}}_+ \\ & -\bar{c}_{n,m,\nu} H_\nu e^{-2\pi i(ns+mt)} if (0, n, m) \in \tilde{\tilde{\Delta}}_- \end{aligned}$$

Now if the set of positive roots Δ_+ is chosen in such a way that H lies in the fundamental chamber of the Weyl group with respect to Δ_+, one checks by inserting into the appropriate formula of the pullback, that the complex structure is indeed compatible with the symplectic form ω^H. Furthermore J commutes with the natural $T_{\mathbb{C}}$ action on $\mathcal{L}\mathcal{L}\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$. So J defines an automorphism of the tangent bundle

and thus an almost complex structure on $\mathcal{LL}G_{\mathbb{C}}/T_{\mathbb{C}}$ by left translation.
A bilinear form on $\mathcal{LL}\mathfrak{g}_{\mathbb{C}}$ is given by

$$\sigma(X,Y) = \pi \int_{S^1\times S^1} \langle X(s,t), Y(s,t)\rangle dsdt$$

Hence the skew symmetric endomorphism of the tangent bundle relating ω^H and σ reads $B_{\sigma,eT_{\mathbb{C}}} = \frac{1}{\tau_2}(\bar{\partial} + ad(H))$. As before let Δ denote the root system of G Then the eigenvalues of $B_{\sigma,eT_{\mathbb{C}}}$ are

$$\begin{cases} \frac{2\pi i}{\tau_2}(n+\tau m + \alpha(H)), & \alpha \in \Delta, \quad n,m \in \mathbb{Z} \\ \frac{2\pi i}{\tau_2}(n+\tau m), & n,m \in \mathbb{Z}, n \neq 0 \, or \, m \neq 0 \end{cases} \tag{57}$$

The multiplicity of the eigenvalues in the first series is 1 and the multiplicities of the second series is $l = \dim_{\mathbb{R}} \mathfrak{h}$. This can be seen by using the root space decomposition of the semi simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. With Lemma 9.15 at hand we can easily calculate the zeta regularized product $Pf_\zeta \cdot Z(H)$ where $Z(H)$ denotes the nominator of the Liouville functional with respect to the Riemannian volume form.

Proposition 16.2. *The zeta regularized product $Pf_\zeta \cdot Z(H)$ for the above eigenvalues is given by*

$$C\cdot(2\pi\cdot\sqrt{\tau_2}|\eta(\tau)|^2)^l\cdot \prod_{\alpha\in\Delta_+} |q^{12}(e(\frac{1}{2}\alpha(H))-e(-\frac{1}{2}\alpha(H)))\times\prod_{n=1}^{\infty}(1-q^n e(\alpha(H)))(1-q^n e(-\alpha(H)))|^2$$

Proof. According to 9.15 we calculate the product once for Pf_ζ than for $Z(H)$, which was just the zeta regularized product of the eigenvalues, since we are taking the product over all positive roots we will get the desired square in the product. In the following $q = e2\pi i\tau$ and $e(z) = e^{2\pi i z}$

$$\prod_{m,n}^{\zeta} \frac{2\pi i}{\tau_2}(m+\tau n+\alpha(H)) = \left(\frac{2\pi i}{\tau_2}\right)^{-1} \prod_{\alpha\in\Delta} i\eta^{-1}(\tau)\exp\left(\frac{-\pi i\tau}{6} - \pi i\alpha(H)\right)\cdot\vartheta_1\alpha(H)$$

Now we use the product expansion of the Jacobi theta function, which can be found in [SG] p.266 Inserting this one yields

$$\left(\frac{2\pi i}{\tau_2}\right)^{-1} \prod_{\alpha\in\Delta} i\eta^{-1}(\tau)\exp\left(\frac{-\pi i\tau}{6} - \pi i\alpha(H)\right)\cdot$$
$$\prod_{n=1}^{\infty}(1-q^n)q^{\frac{1}{2}}\cdot(-i)\prod_{n=1}^{\infty}((1-q^n\ e(\alpha(H)))(1-q^n\ e(-\alpha(H))))\cdot$$
$$(1-e(-\alpha(H)))e(\frac{1}{2}\alpha(H)) =$$
$$= \left(\frac{2\pi i}{\tau_2}\right)\prod_{\alpha_\Delta} e^{\frac{\pi i\tau}{12}}\exp\left(-\frac{\pi i\tau}{6} - \pi i\alpha(H)\right)\exp\left(\frac{\pi i\tau}{4}\right)\cdot$$
$$\prod_{n=1}^{\infty}((1-q^n\ e(\alpha(H)))(1-q^n\ e(-\alpha(H))))\cdot\left(e(\frac{1}{2}\alpha(H)) - e(-\frac{1}{2}\alpha(H))\right)$$

Now we calculate

$$e^{-\frac{\pi i\tau}{12}}\cdot\exp(-\frac{\pi i\tau}{6})\exp(-\pi i\alpha(H))\exp(\frac{\pi i\tau}{4}) =$$
$$\exp(-\frac{\pi i\tau}{6})\cdot\exp(-\frac{\pi i\tau}{6})\cdot\exp(\frac{\pi i\tau}{4})\cdot\exp(-\pi i\alpha(H)) =$$

$$\exp(-\frac{21\pi i\tau}{12}) = q^{-\frac{1}{12}}$$

Putting these calculations together we get

(58)

$$\left(\frac{2\pi i}{\tau_2}\right)^{-1} \prod_{\alpha\in\Delta} q^{-\frac{1}{12}}(e(\frac{1}{2}\alpha(H) - e(-\frac{1}{2}\alpha(H))) \times \prod_{n=1}^{\infty} ((1-q^n\ e(\alpha(H)))(1-q^n\ e(-\alpha(H))))$$

Doing the same calculatio for $Z(H)$, but now the eigenvalues are divided by 2π, and taking into account that $Z(H)$ is the zeta regularized product of the absolut value of the eigenvalues we get as final result for the eigenvalues with multiplicity 1:

(59)

$$C\cdot \prod_{\alpha\in\Delta} |q^{\frac{1}{12}}(e(\frac{1}{2}\alpha(H) - e(-\frac{1}{2}\alpha(H))) \times \prod_{n=1}^{\infty} ((1-q^n\ e(\alpha(H)))(1-q^n\ e(-\alpha(H))))\,|^2$$

and $C = \frac{\tau_2}{2\pi}$. Now we calculate the zeta regularized product for the eigenvalues with multiplicity l, using 9.12.

$$\prod_{m,n=-\infty}^{\infty}{}^{\zeta'} \frac{2\pi i}{\tau_2}\cdot\frac{i}{\tau_2}|m+n\tau|^2 = \sqrt{\tau_2}(2\pi)^{-\frac{1}{4}}(2\pi^2)|\eta(\tau)|^2 \tag{60}$$

since the Pfaffian is defined as the square root of the zeta regularized determinant. Puttng the eigenvalues with both multiplicities together yields the final result:

$$Pf_\zeta\cdot Z(H) = C\cdot((2\pi)^2|\eta(\tau)|^2)^l\cdot$$

$$\prod_{\alpha\in\Delta} |q^{\frac{1}{12}}(e(\frac{1}{2}\alpha(H) - e(-\frac{1}{2}\alpha(H))) \times \prod_{n=1}^{\infty} ((1-q^n\ e(\alpha(H)))(1-q^n\ e(-\alpha(H))))\,|^2$$

where $C = (2\pi)^{-\frac{l}{4}}\cdot\frac{\tau_2}{2\pi}$ □

Finally, we have to calculate the number of fixed points $\sharp p$ in order to obtain our sign $(-1)^{\sharp p}$ for the fixed points p of the torus action on $\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$. Again, this is done exactly the way we did it in the loop case: Let $(\beta_1,\beta_2,w)\in Q^\vee\times Q^\vee\times W$ be a fixed point of the $S^1\times S^1\times T$ action. We choose a representative $g_w\in G$ for each $w\in W$, so (β_1,β_2,g_w) can be viewed as an element of $\mathcal{L}\mathcal{L}G_{\mathbb{C}}$. But $\mathcal{L}\mathcal{L}G_{\mathbb{C}}$ acts transitively on $\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$, so we can use left translation by (β_1,β_2,g_w) to identify the tangent spaces of $T_{eT_{\mathbb{C}}}\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$ and $T_{(\beta_1,\beta_2,g_w)}\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$. But (β_1,β_2,g_w) is an element of the normalizer of $T_{\mathbb{C}}$, so the identification is well defined. Now with this identification the infinitesimal actionof $S^1\times S^1\times T$ is given on the tangent space $T_{(\beta_1,\beta_2,g_w)}\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$ by $(\beta_1,\beta_2,g_w)^{-1}(\partial_s+\partial_t+H)(\beta_1,\beta_2,g_w)$ for $(\partial_s+\partial_t+H)\in Lie(S^1\times S^1\times T)$. (And ∂ was defined as $\partial=\partial_s+\bar\tau\partial_t$). But this defines an action of $Q^\vee\times Q^\vee\times W$ on the above defined set $\tilde{\tilde{\Delta}}$. According to the Definition in 2.2, $\sharp(\beta_1,\beta_2,w)$ is the number of elements of $\tilde{\tilde{\Delta}}_+$ mapped to $\tilde{\tilde{\Delta}}_-$ under (β_1,β_2,w). As in the case of affine roots systems and the affine Weyl group one can see by using the fact that the cardinality of the finite root system Δ, which was used in the definition of $\tilde{\tilde{\Delta}}$, that again the number of fixed points $\sharp(\beta_1,\beta_2,w)=(-1)^{l(w)}$ where $l(w)$ denotes the lenght of the Weyl group element w.

16.2. **Calculation of the partition function.** In this section the partition function of the gauged WZW model will be calculated, so like in the case of a partition function for the compact Lie group G, we will show, that the action functional of the WZW model is the Hamiltonian of the $\mathbb{R}-$ action defined above. We first note, that the action functional $S_{G,H,\kappa}(g)$ does not depend on the representative of g modulo the complexified torus $T_{\mathbb{C}}$. So by left translation, $S_{G,H,\kappa}(g)$ defines a function on $\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$. To apply the Liouville functional approach we have to check that $S_{G,H,\kappa}(g)$ is the Hamiltonian of the $S^1 \times S^1 \times T$ action defined above. This is done in analogy with Lemma 8.9.3 in [PS] and the calculations in the loop case. We have

Lemma 16.3. *The Hamiltonian vector field on $\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$ corresponding to $-S_{G,H,1}$ is exactly the vector field on $\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$ defined by the element $(\partial, H) \in Lie(S^1 \times S^1 \times T)$ considered above.*

Proof. We choose a representative g of $gT_{\mathbb{C}}$ in $\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}$ and let δg be the cariational derivative of g (i.e a vector field along g). The vector field generated by the element $(\partial, H) \in Lie(S^1 \times S^1 \times G)$ is given at the point $gT_{\mathbb{C}}$ by $\partial g + adH(g)$. So we have to show that

$$-dS_{G,H,1}(\delta g) = \omega^H_{gT}(\delta g, \partial g + adH(g)).$$

In order to handle the action functional more conveniently we denote the H dependent term in $S_{G,H,1}(g)$ by $\tilde{S}_H(g)$ so that we have

$$S_{G,H,1}(g) = S_{G,1} + \tilde{S}_H(g)$$

Now we apply the Polyakov Wiegmann Formula (prop. 4.1) and see that

$$-dS_{G,1}(\delta g) = \frac{\pi}{2}\int_\Sigma \langle g^{-1}\partial g, \bar{\partial}(g^{-1}\delta g)\rangle dsdt$$

(This is just inserted into the Polyakov Wiegmann Formula) Now we apply the same for the H dependent term, so we just insert into the last term in $S_{G,H,\kappa}(g)$

$$-d\tilde{S}_H(\delta g) = \frac{\pi}{2}\int_\Sigma (\langle \delta(g^{-1}dg), H\rangle - \langle \delta(\bar{\partial} g g^{-1}), H\rangle - \langle H, \delta(g^{-1}Hg)\rangle)dsdt$$

We already know that $\delta(g^{-1}\partial g) = \partial(g^{-1}\delta g) + [g^{-1}\partial g, g^{-1}\delta g]$. Thus partial integration yields $\int_\Sigma \langle \delta(g^{-1}\partial g, H)\rangle = \int_\Sigma \langle [g^{-1}\partial g, g^{-1}\delta g]\rangle dsdt$. Furthermore $\langle \delta(\bar{\partial} g g^{-1}), H\rangle = \langle \bar{\partial}(g^{-1}\delta g), g^{-1}Hg\rangle$ and as in the proof of Lemma 11.2 we have $\langle H, \delta(g^{-1}Hg), H\rangle = \langle H, [g^{-1}Hg, g^{-1}\delta g]\rangle$. So we put all the terms together and this yields

$$\begin{aligned}-dS_{G,H,1} = &\frac{\pi}{\tau_2}\int_\Sigma \langle \partial(g^{-1}\delta g), g^{-1}\partial g + g^{-1}Hg\rangle dsdt\\ &+\frac{\pi}{\tau_2}\int_\Sigma \langle H, [g^{-1}\delta g, g^{-1}\partial g - g^{-1}Hg]\rangle dsdt\end{aligned}$$

Which is the assertion, by the definition of ω^H_{gT}. □

Since $S_{G,H,\kappa} = \kappa S_{G,H,1}$ (which is immedeately seen from the definition) this lemma allows us to calculate the formal integral $\int_{\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}}$ via the Liouville functional approach. before we start with the calculations, we recall some facts from the representaton theory of Kac Moody algebras. According to [K] chapter 12, one can give the Kac Weyl character formula another meaning, using theta functions. So let $\tilde{\mathfrak{g}}_{\mathbb{C}} = \mathcal{L}\mathfrak{g}_{\mathbb{C}} \oplus +\mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Lie algebra corresponding to the semi simple Lie algbra $\mathfrak{g}_C$ and let A denote the generalized Cartan matrix introduced in chapter 1. As usual $\tilde{\Delta}$ denotes the root system of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and we choose a root basis $\alpha_0, \dots, \alpha_l$ of simple roots. Denote by $\alpha_0^\vee, \dots, \alpha_l^\vee$ the dual simple roots, that is $\alpha_i \in \tilde{\mathfrak{h}}$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = (A)_{ij}$. Let a_i be the minimal integers such that $A(a_0, \dots, a_n) = 0$ and set $\delta = \sum_{i=0}^n a_i \alpha_i$. Following [K] chapter 12.4 one introduces the so called canonical central element of $\tilde{\mathfrak{g}}_{\mathbb{C}}$

$$K = \sum_{i=0}^{l} a_i \alpha_i^\vee$$

We repeat the notions of chapter 2

$$\begin{aligned} \tilde{P} &= \{\lambda \in \mathfrak{h}_{\mathbb{C}}^* | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 0, \dots, n\}, \\ \tilde{P}_+ &= \{\lambda \in \tilde{P} | \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i = 0, \dots, n\} \text{ and} \\ \tilde{P}_{++} &= \{\lambda \in \tilde{P}_+ | \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for all } i = 0, \dots, n\} \end{aligned}$$

Remember that there exists a bijection between the irreducible integrable highest weight models of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and the dominant integral weights $\lambda \in \tilde{P}_+$. For $\lambda \in P_+$ let $L(\lambda)$ denote the corresponding irreducible highest weight module of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. There is no loss of generality for the representation theory of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ if we assume $\langle \lambda, d \rangle = 0$ for highest weight λ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$, so from now on, we restrict $\tilde{P}$ to $\{\lambda | \langle \lambda, d \rangle = 0\}$. Since the highest weight modules $L(\lambda)$ are irreducible, K operates as a scalar on $L(\lambda)$. We define the level k of λ to be the non negative integer $k = \langle \lambda, K \rangle$ and set

$$\tilde{P}_+^k = \{\lambda \in P_+ | \quad \text{level}(\lambda) = k\}$$

Now for $\lambda \in \tilde{P}_+$ let $ch(\lambda)$ denote the character and χ_λ denotes the normalized character of the $\tilde{\mathfrak{g}}_{\mathbb{C}}$ module $L(\lambda)$. That is for $\lambda \in \mathfrak{h}^*$ we set

$$\chi_\lambda = \frac{\sum_{w \in W^\circ} (-1)^{l(w)} e(w(\lambda + \bar{\rho}) - \bar{\rho})}{\prod_{\alpha \in \Delta^\circ} (1 - e(-\alpha))}$$

if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+$ for $i = 1, \dots, l$ then χ_λ is nothing else but the formal character of the $\mathfrak{g}_{\mathbb{C}}$ module $L(\lambda)^\circ$. Choose an element $\Lambda_0 \in \tilde{\mathfrak{h}}$ such that $\langle \Lambda_0, \alpha_0 \rangle = 1$ and $\langle \Lambda_0, \Lambda_0 \rangle = \langle \Lambda_0, \alpha_i^\vee \rangle = 0$ for all $i = 1, \dots, l$. If we choose orthonormal coordinates of $v_1, \dots, v_l$ of $\mathfrak{h}$ with respect to the negative of the Killing form on $\mathfrak{g}$, we can coordinatize $\tilde{\mathfrak{h}}_{\mathbb{C}}$ via

$$v = 2\pi i (\sum_{\nu=1}^{l} z_\nu v_\nu - \tau \Lambda_0 + u\delta),$$

and identify $v \in \tilde{\mathfrak{h}}_{\mathbb{C}}$ with the vector (τ, H, u) with $H = \sum z_\nu v_\nu \in \mathfrak{h}_{\mathbb{C}}$ and $\tau, u \in \mathbb{C}$. It is known that for any $\lambda \in \tilde{P}_+$ the character $ch(\lambda)$ and the normalized character χ_λ converge absolutely on the domain

$$Y = \{(\tau, H, u) | H \in \mathfrak{h}_{\mathbb{C}}; \tau, \, u \in \mathbb{C}, \, Im(\tau) > 0\}$$

Therefore $ch(\lambda)$ and χ_λ define holomorphic functions on Y. Since the center of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ acts on $L(\lambda)$ by scalar multiplication, we can view $ch(\lambda)$ and χ_λ as functions $ch(\lambda)(\tau, H)$ and $\chi_\lambda(\tau, H)$ of τ and H and forget about the central u coordinate without loss of generality. Note, that there is also a geometric interpretation of these characters as sections of certain line bundles over abelian varieties, see e.g [EFK] and [Lo].

An explicit formula for the normalized character is given by the Kac Weyl characte formula. As before let $Q^\vee \subset \mathfrak{h}$ be the dual root lattice of $\mathfrak{g}_{\mathbb{C}}$(with the appropriate identifications) and let Δ_+ and $\tilde{\Delta}_+$ be set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ and $\tilde{\mathfrak{g}}_{\mathbb{C}}$ respectively(w.r.t the simple roots $\alpha_0, \ldots, \alpha_l$). Set $\rho = \sum_{\Delta_+} \alpha$. For $\mu \in \tilde{P}$ define $\bar{\mu} \in \mathfrak{h}$ to be the projection of μ to $\mathfrak{h}$ and for $x, \tau \in \mathbb{C}$ set $e(x) = e^{2\pi i x}$ and $q = e^{2\pi i \tau}$. Then for $\lambda \in \tilde{P}^k_+$define

$$\Theta_\lambda(\tau, H) = \sum_{\gamma \in Q^\vee + k^{-1}\bar{\lambda}} e(\frac{1}{2} k\tau \langle \gamma, \gamma \rangle + k\langle \gamma, H \rangle)$$

With these definitions the Kac Weyl character formula (cf.[K]ch.10,12) reads

$$\chi_\lambda(\tau, H) = \frac{\sum_{w \in W} (-1)^{l(w)} \Theta_{w(\lambda + \tilde{\rho})}(\tau, H)}{q^{\frac{\dim \mathfrak{g}}{24}} e(\langle \rho, H \rangle) \prod_{\alpha \in \tilde{\Delta}_+} (1 - e(-\alpha(\tau, H)))^{mult \alpha}}$$

where $mult\, \alpha = \dim\, \mathfrak{g}_\alpha$ denotes the dimension of the root space corresponding to $\alpha \in \tilde{\Delta}$ and $\tilde{\rho} \in \mathfrak{h}$ is defined via $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$ for $i = 0, \ldots, l$ and $\langle \tilde{\rho}, d \rangle = 0$. As already said, the squared absolute value of the denominator of this formula is, up to the coefficient C the product $Pf_\zeta(B_\sigma) \cdot Z(H)$. This can be seen using the root space decomposition of $\tilde{\Delta}_+$ into real and imaginary roots

$$\tilde{\Delta}_+ = \Delta_+ \cup \bigcup_{n=1}^{\infty} \{\Delta + n\delta\} \cup \bigcup_{n=1}^{\infty} n\delta$$

Here δ is the same as in 11 $\Delta_+^{im} = \{n\delta | n \in \mathbb{N}\}$ is called the set of positive imaginary roots. The multiplicities of the roots are given by $mult\, \alpha = 1$ for $\alpha \in \Delta_+ - \Delta_+^{im}$ and $mult\, \alpha = l$ for Δ_+^{im}. We can now state the theorem which gives a conceptual calculation of the partition function of the gauged WZW model:

Theorem 16.4. *Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}_{\mathbb{C}}$, that is $h^\vee = \sum a_i^\vee$ and let k be a positive level of $\tilde{\mathfrak{g}}$. Set $\kappa = h^\vee + k$(This is motivated by the physical meaning of the action functional, see* [G]*). Then the following identity is valid:*

$$\int_{\mathcal{L}\mathcal{L}G_{\mathbb{C}}/T_{\mathbb{C}}} = \frac{C_0}{C_1 C} \sum_{\lambda \in \tilde{P}^k_+} |\chi_\lambda(\tau, H)|^2$$

with $C_0 = \kappa^l$, $C_1 = \frac{(\sqrt{2\kappa\tau_2})^l}{vol\, \kappa Q^\vee}$ and $C = (2\pi)^{-\frac{l}{2}}$.

Proof. The equality of the denominators is clear because of the definition of the integral w.r.t the Riemannian volume form (that was $\frac{L_1(J_H)}{Pf_\zeta(B_\sigma)}$). Above we saw that the product of the zeta regularized Pfaffian with the denominator of the Liouville functional is equals the squared absolute value of the Kac Weyl character formula, so the equality of the denominators follow. Next we calculate $S_{G,H,\kappa}(g)$ in the fixed points of the $S^1 \times S^1 \times T$ action which are given by $(s, t) \mapsto g_w \cdot \exp(s\beta) \cdot \exp(t\mu)$, with $\beta, \mu \in Q^\vee$ and where as before g_w is a representative of $w \in W$.The action functional can be written as

$$S_{G,H,\kappa}(g) = -\frac{\kappa}{\pi} \int_\Sigma (\langle g^{-1}\partial g + g^{-1}Hg - H, g^{-1}\bar{\partial} g + g^{-1}Hg - H\rangle$$
$$-2i\langle g^{-1}\partial_t g, g^{-1}Hg + H\rangle) ds dt + \frac{i\kappa\pi}{3} \int_B tr(\tilde{g}^{-1} d\tilde{g})^{\wedge 3}$$

For $g(s,t) = g_w \cdot \exp(s\beta) \cdot \exp(t\mu)$, the Wess Zumino term can be calculated using the Polyakov Wiegmann formula and the calculation of Lemma 4.1

$$\frac{i\kappa\pi}{3}\int_B tr((gh)^{-1}d(gh))^{\wedge 3} = \frac{i\kappa\pi}{3}\int_B tr(g^{-1}dg)^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B tr(h^{-1}dh)^{\wedge 3} + i\kappa\pi\int_B dtr(g^{-1}dg \wedge dhh^{-1})$$

Writing $g(s,t) = g_w \cdot (\exp(s\beta)\cdot\exp(t\mu))$ we see that the constant term does not contribute to the integral. Furthermore we have

$$\frac{i\kappa\pi}{3}\int_B tr((\exp(s\beta)\,\exp(t\mu))^{-1}d(\exp(s\beta)\cdot\exp(t\mu)))^{\wedge 3} = \frac{i\kappa\pi}{3}\int_B tr(\exp(-s\beta)d(\exp(s\beta)))^{\wedge 3} + \frac{i\kappa\pi}{3}\int_B tr(\exp(-s\mu)d(\exp(s\mu)))^{\wedge 3} + i\kappa\pi\int_\Sigma tr(\exp(-s\beta)d\exp(s\beta))\wedge d(\exp(t\mu)\exp(-t\mu))$$

The first two terms of the right hand side of the equation vanish, and the third term is calculated to be $\kappa\pi i\langle\beta,\mu\rangle$ So in the fixed points of the torus action the action functional reads

$$S_{G,H,\kappa}(g_w\,\exp(s\beta)\exp(t\mu)) = -\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+w^{-1}H-H, \beta+\bar\tau\mu+w^{-1}H-H\rangle + \pi i\kappa\langle\mu, w^{-1}H+H\rangle - \pi i\kappa\langle\beta,\mu\rangle.$$

Now after rearranging the order of the terms the theorem follow with Lemma 16.5, since the denominators have already been observed to be equal. ☐

Lemma 16.5. *The following identity is valid:*

$$\frac{vol\kappa Q^\vee}{(\sqrt{2\kappa\tau_2})^l}\sum_{\beta,\mu\in Q^\vee}\sum_{w\in W}(-1)^{l(w)}e^{-\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+H-wH,\beta+\bar\tau\mu+H-wH\rangle} \times e^{\pi i\kappa\langle\beta,\mu\rangle+\pi i\kappa\langle\mu,H+wH\rangle} = \sum_{\lambda\in\tilde P^k_+}\sum_{w_1\in W}(-1)^{l(w_1)}\Theta_{w_1(\lambda+\tilde\rho)}(\tau,H) \times \sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}.$$

Proof. Let us denote the right hand side of the equation with $N_k(\tau,H)$. Note that we have $\bar{\tilde\rho} = \rho$ and level $\tilde\rho = h^\vee$. (see [K] ch. 12). Therefore

$$N_k(\tau,H) = \sum_{\lambda\in\tilde P^k_+}\sum_{w_1\in W}(-1)^{l(w_1)}\Theta_{w_1(\lambda+\tilde\rho)}(\tau,H)\sum_{w_2\in W}(-1)^{l(w_2)}\overline{\Theta_{w_2(\lambda+\tilde\rho)}(\tau,H)}$$

Now we use the definition of Θ and note that $\kappa = h^\vee + k$. So the above reads:

$$= \sum_{\lambda\in\tilde{P}^{\kappa}_{++}} \sum_{w_1\in W} \sum_{\gamma\in Q^\vee+\frac{1}{\kappa}w_1\bar{\lambda}} (-1)^{l(w_1)} e(\frac{1}{2}\kappa\tau\langle\gamma,\gamma\rangle + \kappa k\langle\gamma,H\rangle)$$
$$\times \sum_{w_2\in W} (-1)^{l(w_2)} \overline{\Theta_{w_2(\lambda+\tilde{\rho})}(\tau,H)}$$
$$= \sum_{\lambda\in\tilde{P}^{\kappa}_{++}} \sum_{\alpha\in Q^\vee} \sum_{w,w'\in W} (-1)^{l(w')}(-1)^{l(ww')} e(\frac{1}{2}\kappa\tau\langle\frac{1}{\kappa}w'\bar{\lambda}+\alpha, \frac{1}{\kappa}w'\bar{\lambda}+\alpha\rangle)$$
$$\times e(\kappa\langle\frac{1}{\kappa}w'\bar{\lambda}+\alpha, H\rangle)\overline{\Theta_{w'\bar{\lambda}+\alpha}(\tau,wH)}$$

The set $\{\frac{1}{\kappa}\bar{\lambda}|\lambda\in\tilde{P}^{\kappa}_{++}\}$ lies in a fundamental alcove of the affine Weyl group and the singular weights do not contribute to the sum below. Since Θ_γ only depends on the class of γ modulo $(Q^\vee)$ we can sum over $\alpha\in Q^\vee$ and $w'\in W$ and get

$$= \sum_{\gamma\in\frac{1}{\kappa}P} \sum_{w\in W} (-1)^{l(w)} e(\frac{1}{2}\tau\kappa\langle\gamma,\gamma\rangle + \kappa\langle\gamma,H\rangle)\cdot\overline{\Theta_\tau(\tau.wH)}$$
$$= \sum_{\gamma\in\frac{1}{\kappa}P} \sum_{w\in W} \sum_{\mu\in Q^\vee} (-1)^{l(w)} e(\frac{1}{2}\tau\kappa\langle\gamma,\gamma\rangle + \kappa\langle\gamma,H\rangle)$$
$$\times e(-\frac{1}{2}\bar{\tau}\kappa\langle\gamma+\mu,\gamma+\mu\rangle - \kappa\langle\gamma+\mu,wH\rangle)$$
$$= \sum_{\gamma\in\frac{1}{\kappa}P} \sum_{w\in W} \sum_{\mu\in Q^\vee} (-1)^{l(w)} e^{-2\pi\kappa\tau_2\langle\gamma,\gamma\rangle - 2\pi i\kappa\bar{\tau}\langle\gamma,\mu\rangle - \pi i\kappa\bar{\tau}\langle\mu,\mu\rangle}$$
$$\times e^{2\pi i\kappa\langle\gamma,H\rangle - 2\pi i\kappa\langle\gamma+\mu,wH\rangle}$$
$$= \sum_{\gamma\in\frac{1}{\kappa}P} \sum_{w\in W} \sum_{\mu\in Q^\vee} (-1)^{l(w)} e^{-2\pi\kappa\tau_2(\langle\gamma,\gamma\rangle+\langle\gamma,\mu\rangle) - 2\pi i\kappa\tau_1\langle\gamma,\mu\rangle + 2\pi i\kappa\langle\gamma,H-wH\rangle}$$
$$e^{-\pi i\bar{\tau}\kappa\langle\mu,\mu\rangle - 2\pi i\kappa\langle\mu,wH\rangle}$$

In fact this calculations can be seen as technically analogous to the calculations leading to theorem 5.1. Also here we will apply the Poisson transformation formula, that is for an euclidean vector space V, a Schwartz function $f: V\to\mathbb{C}$,and a lattice $M\subset V$ we have

$$\sum_{\beta\in M^\vee} \hat{f}(\beta) = volM \sum_{\gamma\in M} f(\gamma)$$

with

$$\hat{f}(\beta) = \int_V e^{2\pi i\langle\gamma,\beta\rangle} f(\gamma)d\gamma$$

and where $M^\vee$ denotes the lattice dual to M with respect to the scalar product on V. if we choose

$$f(\gamma) = e^{-2\pi\kappa\tau_2(\langle\gamma,\gamma\rangle+\langle\gamma,\mu\rangle) - 2\pi i\kappa\tau_1\langle\gamma,\mu\rangle + 2\pi i\kappa\langle\gamma,H-wH\rangle - \pi i\bar{\tau}\kappa\langle\mu,\mu\rangle - 2\pi i\kappa\langle\mu,wH\rangle}$$

inserting into the definition yields

$$\hat{f}(\beta) = \frac{1}{(\sqrt{2\kappa\tau_2})^l} e^{-\pi i\langle\mu,\beta\rangle - \frac{\pi}{2\tau_2\kappa}\langle\beta,\beta\rangle + \frac{\pi\tau_1}{\tau_2}\langle\beta,\mu\rangle - \frac{\pi}{\tau_2}\langle\beta,H-wH\rangle - \frac{\pi\cdot(\tau_1^2+\tau_2^2)\kappa}{2\tau_2}\langle\mu,\mu\rangle}$$

$$\times e^{\frac{\pi\tau_1\kappa}{\tau_2}\langle\mu,H-wH\rangle - \frac{\pi\kappa}{2\tau_2}\langle H-wH,H-wH\rangle - \pi i\kappa\langle\mu,H+wH\rangle}.$$

So by the Poisson summation formula, we get

$$N_\kappa(\tau,H) = \frac{1}{vol\frac{1}{\kappa}P\cdot(\sqrt{2\kappa\tau_2})^l} \sum_{\beta\in\kappa Q^\vee}\sum_{w\in W}\sum_{\mu\in Q^\vee}(-1)^{l(w)} e^{-\pi i\langle\mu,\beta\rangle - \frac{\pi}{2\tau_2\kappa}\langle\beta,\beta\rangle}$$

$$\times e^{\frac{\pi\tau_1}{\tau_2}\langle\beta,\mu\rangle - \frac{\pi}{\tau_2}\langle\beta,H-wH\rangle - \frac{\pi\cdot(\tau_1^2+\tau_2^2)\kappa}{2\tau_2}\langle\mu,\mu\rangle + \frac{\pi\tau_1\kappa}{\tau_2}\langle\mu,H-wH\rangle}$$

$$\times e^{-\frac{\pi\kappa}{2\tau_2}\langle H-wH,H-wH\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$

$$= \frac{vol\kappa Q^\vee}{(\sqrt{2\kappa\tau_2})^l}\sum_{\beta,\mu Q^\vee}\sum_{w\in W}(-1)^{l(w)} e^{-\pi i\kappa\langle\mu,\beta\rangle - \frac{\pi\kappa}{2\tau_2}\langle\beta,\beta\rangle}$$

$$\times e^{\frac{\pi\tau_1\kappa}{\tau_2}\langle\beta,\mu\rangle - \frac{\pi\kappa}{\tau_2}\langle\beta,H-wH\rangle - \frac{\pi\cdot(\tau_1^2+\tau_2^2)\kappa}{2\tau_2}\langle\mu,\mu\rangle + \frac{\pi\tau_1\kappa}{\tau_2}\langle\mu,H-wH\rangle}$$

$$\times e^{-\frac{\pi\kappa}{2\tau_2}\langle H-wH,H-wH\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$

$$= \frac{vol\kappa Q^\vee}{(\sqrt{2\kappa\tau_2})^l}\sum_{\beta,\mu Q^\vee}\sum_{w\in W}(-1)^{l(w)} e^{-\frac{\pi\kappa}{2\tau_2}\langle\beta-\tau\mu+H-wH,\beta-\bar{\tau}\mu+H-wH\rangle}$$

$$\times e^{-\pi i\kappa\langle\beta,\mu\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$

$$= \frac{vol\kappa Q^\vee}{(\sqrt{2\kappa\tau_2})^l}\sum_{\beta,\mu Q^\vee}\sum_{w\in W}(-1)^{l(w)} e^{-\frac{\pi\kappa}{2\tau_2}\langle\beta+\tau\mu+H-wH,\beta+\bar{\tau}\mu+H-wH\rangle}$$

$$\times e^{\pi i\kappa\langle\beta,\mu\rangle - \pi i\kappa\langle\mu,H+wH\rangle}$$

□

An important fact of conformal field theories is their invariance under a certain $SL(2,\mathbb{Z})$ action. The modular group $SL(2,\mathbb{Z})$ acts on the torus $S^1\times S^1$. Under this action the modular parameter τ of the elliptic curve Σ_τ is transformed via

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

This $SL(2,\mathbb{Z})$ action can be extended to the domain Y via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau,H,u) \mapsto (\frac{a\tau+b}{c\tau+d}, \frac{H}{c\tau+d}, u - \frac{c\langle H,H\rangle}{2(c\tau+d)})$$

It was shown in [KP] that for each $k\in\mathbb{N}$ the $SL(2,\mathbb{Z})$ action on Y defined above gives rise to an $SL(2,\mathbb{Z})$ action on the set of normalized characters of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ at level k.

In particular it follows from the explicit transformation properties of the characters under the $SL(2,\mathbb{Z})$ action that the sum $\sum_{\lambda\in\tilde{P}^k_+}|\chi_\lambda, H|^2$ is $SL(2,\mathbb{Z})$ invariant. Since the $SL(2,\mathbb{Z})$ action on the modular parameter arises naturally in the functional integral setup in [W2] one can deduce the modular invariance of the partition function of the WZW model, since zeta regularization works also for complex numbers.

17. Open Questions

The results presented in this paper lead to several open questions, two of them where stated in [W2]. But we will indicate, that there are much more directions for further extensions.

17.1. **Stochastic point of view.** According to the classification of coadjoint orbits for double loop groups, one can find a path space, which is defined on the unit square, the exact determination of this space is work in progress. Surprinsingly, this space will again be a G valued path space, although the coadjoint orbits are classified in terms of conjugacy classes, which live in the holomorphic loop group. It would be interesting to build a measure on this space, which will certainly arise as probabhility measure of a multiparameter process, e.g. the Brownian sheet ([D], [Nu]). Some kind of Brownian sheet would be especially usefull because of its simple covariance structure. This could open the way, to build an $SL(2,\mathbb{Z})$ invariant measure. Such a measure on double loop groups could perhaps lead to a measure theoretic interpretatin for the gauged WZW model, certainly one could compare a possible measure with the formal Liouville functional approach. It would also be interesting to investigate, which well esthablished results from stochastic analysis so far developed on Loop groups, can be generalized to the case of double loop groups. For example Driver and Lohrenz showed in [BL] the existence of logarithmic Sobolov inequalities on loop groups, perhaps similar results could be obtained in the double loop case. Another problem stated in [W2] would be to obtain a classification of infinite dimensional symplectic manifolds on which a measure theory can be developed and which contains the Duistermaat Heckmann formula as a theorem.

17.2. **Geometric point of view.** The conceptual approach to the partition function of the gauged WZW model leads to the question, which other Quantum field theories are suitable for such a symplectic approach. It would interesting to obtain a similar approach to the Boundary WZW model.
We want to underline the importance of the mathematical methods presented in this paper and certain branches of conformal field theory in Condensed matter phyiscs and nanotechnology, see e.g. the papers [Aff], [Al-D'All], [D'All1], [D'All2], [Sch], [Tv], [Tv1].

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