

THE RELATION BETWEEN SYSTEMS AND ASSOCIATED BUNDLES

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ABSTRACT. It is shown that a strong system of vector fields on a fiber bundle in the sense of [Mo] is induced from a principal fiber bundle if and only if each vertical vector field of the system is complete.

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1. INTRODUCTION

The notion of systems of vector fields and systems of connections for fibered manifolds were introduced by Marco Modugno as a generalisation of principal connections and as a means to give a lucid and easy construction of the universal connection on the bundle of connections, which for principal bundles is due to [Garcia, 1977]. This is a special

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case of the usual notion of a system as treated for example in [Gauthier, 1984]

In this paper we prove as the main result (theorem 3.6) that any such system for a fiber bundle which is strong in the sense of Modugno and has the further property that all vertical vector fields of the system are complete, is in fact an induced system on an associated bundle for a principal bundle. The structure group of the principal bundle is the holonomy group of the universal connection of the system. In 2.7 we show that the converse is true and we describe some simple examples of non-complete systems also.

We use heavily the concepts and techniques of [Mi]. These can be found also with more details and more complete proofs in [Michor, 1991].

I want to thank Marco Modugno for his hospitality, for asking the question answered in this paper, and for lots of discussions.

2. SYSTEMS

2.1. Let (E, p, M, S) be a smooth finite dimensional fibre bundle with base M and standard fibre S . By a *system of vector fields* on E we mean a pair (H, η) , where

- (1) $H = (H, q_H, M)$ is a vector bundle over M .
- (2) η is a mapping factoring as in the following diagram

$$\begin{array}{ccc} H \times_M E & \xrightarrow{\eta} & TE \\ \text{pr}_1 \downarrow & & \downarrow T_p \\ H & \xrightarrow{\underline{\eta}} & TM. \end{array}$$

Furthermore it is supposed to be fibre respecting and fiber linear over E . Then it turns out that $\underline{\eta} : H \rightarrow TM$ is fibre respecting and fiber linear over M . Also we suppose that $\underline{\eta}$ is fiberwise surjective.

In [Mo] this is called a linear, horizontally complete, and projectable system of vector fields.

2.2. A system of vector fields is called *monic* if the associated mapping $\tilde{\eta} : H \rightarrow \bigcup_{x \in M} C^\infty(TE|E_x)$ is injective.

It is called *involutive*, if the push forward of the associated mapping acting on sections $\tilde{\eta}_* : C^\infty(H) \rightarrow C^\infty(TE) = \mathfrak{X}(E)$ has as image a Lie sub algebra of the algebra of vector fields on E . Note that this is not involutivity of some sub bundle of TE , since all vector fields in the image of $\tilde{\eta}_*$ are "rigid" along the fibres of E .

A system is called *canonical* if there exist an open cover (U_α) of M , a fiber bundle atlas $(U_\alpha, \psi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times S)$ of (E, p, M) , and a vector bundle atlas $(U_\alpha, \varphi_\alpha : H|_{U_\alpha} \rightarrow TU_\alpha \times V)$ of (H, q_H, M) , such that

$$T\psi_\alpha \cdot \eta \cdot (\varphi_\alpha^{-1}(\xi_x, v), \psi_\alpha^{-1}(x, s)) = (\xi_x, \eta^\alpha(v)(s)),$$

where $\eta^\alpha : V \rightarrow \mathfrak{X}(S)$ is a linear mapping into the space of vector fields on the standard fiber S . So it is required that the mapping η^α does not depend on the foot point $x \in U_\alpha$. These data will be called *canonical atlases* for the system.

A system that is monic, involutive, and canonical is called a *strong system*, see [Mo].

2.3. Let (H, η) be a system of vector fields on the bundle E . Then the kernel of the vector bundle homomorphism $\underline{\eta} : H \rightarrow TM$ is a sub vector bundle A of H . Thus we have the following diagram

$$\begin{array}{ccccccc} & & H \times_M E & \xrightarrow{\eta} & TE & & \\ & & \text{\scriptsize } pr_1 \downarrow & & \downarrow \text{\scriptsize } Tp & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & H & \xrightarrow{\underline{\eta}} & TM \longrightarrow 0, \end{array}$$

where the lower line is an exact sequence of vector bundles. For $a_x \in A_x$ the vector field $\check{\eta}(a_x) \in \mathfrak{X}(E_x)$.

We say that the system (H, η) is *complete* if and only if each vector field $\check{\eta}(a_x) \in \mathfrak{X}(E_x)$ is a complete vector field on the fiber E_x ; so its flow should exist for all time.

2.4. Remark. The exact sequence $0 \rightarrow A \rightarrow H \rightarrow TM$ of a monic and involutive system is also called a *Lie algebroid*, see e. g. [Mackenzie, 1987, p. 100], or an *abstract Atiyah sequence*, see *Almeida-Molino, 1985*; one forgets the bundle E on which the sections of H induce projectable vector fields. In this paper we will concentrate on E .

2.5. Let (H, η) be an involutive monic system of vector fields on the bundle (E, p, M, S) . We consider the exact sequence $0 \rightarrow A \rightarrow H \rightarrow TM \rightarrow 0$ of vector bundles from 2.3 and the induced exact sequence of push forwards on the respective spaces of sections

$$0 \rightarrow C^\infty(A) \xrightarrow{i_*} C^\infty(H) \xrightarrow{\underline{\eta}_*} \mathfrak{X}(M) \rightarrow 0.$$

We have also the induced push forward mapping

$$\check{\eta}_* : C^\infty(H) \rightarrow \mathfrak{X}(E)$$

which is injective since the system is monic. The image of $\check{\eta}_*$ is closed under the Lie bracket, so there is an induced bracket

$$[\ , \]^H : C^\infty(H) \times C^\infty(H) \rightarrow C^\infty(H)$$

which is a bilinear differential operator of total degree 1.

Now for sections $a_1, a_2 \in C^\infty(A)$ and a function $f \in C^\infty(M, \mathbb{R})$ we have $[\check{\eta}_*(f.a_1), \check{\eta}_*(a_2)] = f.[\check{\eta}_*(f.a_1), \check{\eta}_*(a_2)]$, which is again vertical, since the vector fields $\check{\eta}_*(a_i) \in \mathfrak{X}(E)$ are vertical and f is constant along the fibres. Thus the induced bracket $[\ , \] : C^\infty(A) \times C^\infty(A) \rightarrow C^\infty(A)$ is of order 0 and is thus a push forward by a smooth fiberwise Lie bracket $[\ , \]^A : A \times_M A \rightarrow A$. Note that the isomorphism type of the Lie algebra $(A_x, [\ , \]_x^A)$ need not be locally constant, if the Lie algebra is not rigid, for example.

Let us now assume furthermore that the monic involutive system is also canonical (see 2.2) and let (U_α, ψ_α) and $(U_\alpha, \varphi_\alpha)$ be canonical atlases for this system as spelled out in 2.2. We want to express the bracket $[h_1, h_2]^H$ for $h_1, h_2 \in C^\infty(H)$ in terms of the canonical atlases. We have $\varphi_\alpha(h_i)(x) = (X_{h_i}(x), v_{h_i}(x)) \in TU_\alpha \times V$, and h_i has values in the sub bundle A if and only if the vector field X_{h_i} is zero. We have then

$$\begin{aligned} (T\psi_\alpha \circ \check{\eta}_*(h))(\psi_\alpha^{-1}(x, s)) &= (T\psi_\alpha \circ \eta)(\varphi_\alpha^{-1}(X_h(x), v_h(x)), \psi_\alpha^{-1}(x, s)) \\ &= (X_h(x), \eta^\alpha(v_h(x)))(s) \\ (T\psi_\alpha \circ [\check{\eta}_*(h_1), \check{\eta}_*(h_2)]^{\mathfrak{X}(E)})(\psi_\alpha^{-1}(x, s)) &= \\ &= [(X_{h_1}, \eta^\alpha \circ v_{h_1}), (X_{h_2}, \eta^\alpha \circ v_{h_2})]^{\mathfrak{X}(U_\alpha \times S)}(x, s) \\ &= \left([X_{h_1}, X_{h_2}]^{\mathfrak{X}(U_\alpha)}(x), [\eta^\alpha(v_{h_1}(x)), \eta^\alpha(v_{h_2}(x))]^{\mathfrak{X}(S)}(s) \right. \\ &\quad \left. + d(\eta^\alpha \circ v(h_2))(x)(X_{h_1}(x))(s) - d(\eta^\alpha \circ v(h_1))(x)(X_{h_2}(x))(s) \right). \end{aligned}$$

If $a_1, a_2 \in C^\infty(A)$ then we get

$$\begin{aligned} (T\psi_\alpha \circ [\check{\eta}_*(a_1), \check{\eta}_*(a_2)]^{\mathfrak{X}(E)})(\psi_\alpha^{-1}(x, s)) &= \\ &= \left(0, [\eta^\alpha(v_{a_1}(x)), \eta^\alpha(v_{a_2}(x))]^{\mathfrak{X}(S)}(s) \right) \\ &=: \eta^\alpha([v_{a_1}(x), v_{a_2}(x)]^V)(s). \end{aligned}$$

So the canonical atlases for a canonical system restrict to a vector bundle atlas for the Lie algebra bundle $(A, [\ , \]^A)$ in which the Lie algebra structure is locally trivial, thus constant along connected components of M . To simplify notation we assume that it is constant, isomorphic to $V, [\ , \]^V$.

2.6. Connections for a system. Let (H, η) be a system of vector fields on the bundle E . We consider a vector bundle homomorphism $\sigma : TM \rightarrow H$ which splits the exact sequence $0 \rightarrow A \rightarrow H \rightarrow TM \rightarrow 0$. Then σ defines a horizontal lifting $C_\sigma : TM \times_M E \rightarrow TE$ by the prescription

$$C_\sigma(\xi_x, u_x) := \eta(\sigma(\xi_x), u_x) \in TE.$$

So C_σ is linear over E and is a right inverse to $(Tp, \pi_E) : TE \rightarrow TM \times_M E$. By [Mi, 1.1] C_σ specifies a connection for the fiber bundle E . We call all connections obtained in this way *connections respecting the system H* or just *H -connections*.

Let us suppose now for the moment that the system H is canonical and let $(U_\alpha, \psi_\alpha), (U_\alpha, \varphi_\alpha)$ be canonical atlases for the system H as required in 2.2. The splitting $\sigma : TM \rightarrow H$ can then be written as $\varphi_\alpha(\sigma(\xi_x)) = (\xi_x, \sigma^\alpha(\xi_x))$, where $\sigma^\alpha \in \Omega^1(U_\alpha; V)$ is a one form on U_α with values in the vertical part V of the standard fiber of H .

The space of all vector bundle splittings of the exact sequence $0 \rightarrow A \rightarrow H \rightarrow TM \rightarrow 0$ parametrizes thus the space of all connections of the fiber bundle E which respect the system (H, η) . These splittings are exactly the sections of the affine bundle

$$C(H) := \{s_x \in L(T_x M, H_x) : \underline{\eta}_x \circ s_x = Id_{T_x M}, x \in M\}.$$

The modelling bundle of that affine bundle is $L(TM, A) = T^*M \otimes A$.

2.7. Associated systems. Let (P, M, p, G) be a principal fiber bundle with structure group G , and let $\ell : G \times S \rightarrow S$ be a smooth left action on a smooth manifold. Then we have the *associated fiber bundle* $P[S] = P[S, \ell] = (P \times S)/G$. On the principal bundle P there is the strong system of all projectable G -equivariant vector fields $(TP/G, \eta_P)$, whose exact sequence in the sense of 2.3 is given by

$$0 \rightarrow VP/G = P[\mathfrak{g}, Ad] \rightarrow TP/G \rightarrow TM \rightarrow 0.$$

The sections of TP/G correspond to the infinitesimal automorphisms of the principal bundle. The vertical sections correspond to the infinitesimal gauge transformations.

The strong system $(TP/G, \eta_P)$ thus induces a system $(TP/G, \eta_{P[S]})$ on the associated bundle $P[S]$ which is monic if and only if the action ℓ is infinitesimally effective, i. e. the fundamental vector field mapping $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(S)$ is injective. By looking at a principal bundle atlas and the induced associated atlas (see [Mi, section 2]) one easily sees that these systems are canonical and complete. Also it is easily checked, that an

arbitrary system (H, η) on the associated bundle $P[S]$ is isomorphic to the induced system if and only if each H -connection is induced from a principal connection; by using [Mi, 2.5] one may recognize these induced connections.

If we take a suitable open subbundle E of the associated bundle $P[S]$ we obtain by restriction a (strong) system $(TP/G, \eta_E)$ which in general is not complete.

3. PROPERTIES OF COMPLETE STRONG SYSTEMS

3.1. Theorem. *Let (H, η) be a complete strong system of vector fields on the bundle (E, p, M, S) .*

Then each connection C_σ respecting the system H for any splitting of the exact sequence $0 \rightarrow A \rightarrow H \rightarrow TM \rightarrow 0$ is complete in the sense of [Mi, 1.6]: its parallel transport exists globally.

Proof. Let $c : [0, 1] \rightarrow M$ be a smooth curve. We have to show that for each $u_0 \in E_{c(0)}$ there exists a smooth curve $\text{Pt}(c, t, u_0)$ in E which covers $c(t)$, is horizontal, has initial value u_0 , and is defined for all $t \in [0, 1]$. We refer to [Mi, theorem 1.5] for the local existence and general properties of parallel transport.

Let $(U_\alpha, \psi_\alpha), (U_\alpha, \varphi_\alpha)$ be canonical atlases for the system H as required in 2.2. We choose a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that $c([t_i, t_{i+1}]) \subset U_{\alpha_i}$ for suitable α_i . It suffices to show that $\text{Pt}(c(t_i + \epsilon), t, u_{c(t_i)})$ exists for all $0 \leq t \leq t_{i+1} - t_i$ and all $u_{c(t_i)} \in E_{c(t_i)}$, for all i — then we may piece them together. So we may assume that $c : [0, 1] \rightarrow U_\alpha$ for some α .

By [Mi, third proof of 1.5] we have in $U_\alpha \times S$

$$\psi_\alpha(\text{Pt}(c, t, \psi_\alpha^{-1}(c(0), s))) = (c(t), \gamma(s, t)),$$

where $\gamma(s, t)$ is the evolution line (integral curve) of the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t))$ on S , where $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(S))$ is the Christoffel form for the connection C_σ in the fiber bundle chart (U_α, ψ_α) , see [Mi, 1.4], from where we use now the defining equation for Γ^α to compute as follows, where Φ_σ is the projection onto the vertical bundle VE along the horizontal bundle $C_\sigma(TM \times_M E)$:

$$\begin{aligned} (0_x, \Gamma^\alpha(\xi_x, s)) &= -T(\psi_\alpha) \cdot \Phi_\sigma \cdot T(\psi_\alpha)^{-1}(\xi_x, 0_s) \\ &= -T(\psi_\alpha) \cdot (T(\psi_\alpha)^{-1}(\xi_x, 0_s) - C_\sigma(\xi_x, \psi_\alpha^{-1}(x, s))) \\ &= -(\xi_x, 0_s) + T(\psi_\alpha) \cdot \eta \cdot (\sigma(\xi_x), \psi_\alpha^{-1}(x, s)) \\ &= -(\xi_x, 0_s) + T(\psi_\alpha) \cdot \eta \cdot (\varphi_\alpha^{-1}(\xi_x, \sigma^\alpha(\xi_x)), \psi_\alpha^{-1}(x, s)) \\ &= -(\xi_x, 0_s) + (\xi_x, \eta^\alpha(\sigma^\alpha(\xi_x))(s)) \\ &= (0_x, \eta^\alpha(\sigma^\alpha(\xi_x))(s)). \end{aligned}$$

Since the system H is complete by assumption we have

$$T(\psi_\alpha) \cdot \eta \cdot (\varphi_\alpha^{-1}(0_x, v), \psi_\alpha^{-1}(x, s)) = (0_x, \eta^\alpha(v)(s))$$

and $\eta^\alpha(v) \in \mathfrak{X}(S)$ is a complete vector field for each $v \in V$. So $\eta^\alpha : (V, [\cdot, \cdot]^V) \rightarrow \mathfrak{X}(S)$ is a homomorphism of Lie algebras whose image consists of complete vector fields. By the theorem of [Palais, 1957] there is a simply connected Lie group G_α with Lie algebra V and a right action $r_\alpha : S \times G_\alpha \rightarrow S$ of G_α on S such that η^α is the fundamental vector field mapping for this action: $\eta^\alpha(v)(s) = T_e(r_\alpha(s, \cdot))v$.

From the computation above we have $\Gamma^\alpha(\frac{d}{dt}c(t)) = \eta^\alpha(\sigma^\alpha(\frac{d}{dt}c(t)))$. Let us choose a left invariant Riemannian metric on the Lie group G_α . It is then a complete Riemannian metric, and the left invariant vector fields $L(v)$ generated by the $v \in V$ are all bounded with respect to this metric. Since $[0, 1]$ is compact, $L(\sigma^\alpha(\frac{d}{dt}c(t)))$ is a time dependent vector field which is bounded for the complete metric. Thus there exists the global evolution curve $t \mapsto g_\alpha(t) \in G_\alpha$ for $t \in [0, 1]$, uniquely given by

$$\begin{cases} \frac{d}{dt}g_\alpha(t) = T\lambda_{g_\alpha(t)} \cdot \sigma^\alpha(\frac{d}{dt}c(t)) \\ g_\alpha(0) = e, \end{cases}$$

where λ_g is left translation by $g \in G_\alpha$. But then we have

$$\begin{aligned} \frac{d}{dt}r_\alpha(s, g_\alpha(t)) &= T(r_\alpha(s, \cdot))\frac{d}{dt}g_\alpha(t) \\ &= T(r_\alpha(s, \cdot)) \cdot T_e(\lambda_{g_\alpha(t)}) \cdot \sigma^\alpha(\frac{d}{dt}c(t)) \\ &= T_e(r_\alpha(r_\alpha(s, g_\alpha(t)), \cdot)) \cdot \sigma^\alpha(\frac{d}{dt}c(t)) \\ &= \eta^\alpha(\sigma^\alpha(\frac{d}{dt}c(t)))(r_\alpha(s, g_\alpha(t))), \\ r_\alpha(s, g_\alpha(0)) &= r_\alpha(s, e) = s. \end{aligned}$$

Thus $r_\alpha(s, g_\alpha(t)) = \gamma(s, t)$, the looked for global evolution curve for for the time dependent vector field $\Gamma^\alpha(\frac{d}{dt}c(t)) = \eta^\alpha(\sigma^\alpha(\frac{d}{dt}c(t)))$. \square

3.2. Curvature. Let (H, η) be a complete strong system of vector fields on the bundle (E, p, M, S) . We want to compute the curvature R of a H -respecting connection $C = C_\sigma$ for a splitting $\sigma : TM \rightarrow H$ in canonical coordinates.

From [Mi, 1.4] we have

$$((\psi_\alpha^{-1})^*R)((X_1, Y_1), (X_2, Y_2)) = d\Gamma^\alpha(X_1, X_2) + [\Gamma^\alpha(X_1), \Gamma^\alpha(X_2)]^{\mathfrak{X}(S)}.$$

From the proof of 3.1 we have $\Gamma^\alpha = \eta^\alpha \circ \sigma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(S))$, thus we may compute

$$\begin{aligned} ((\psi_\alpha^{-1})^* R)((X_1, Y_1), (X_2, Y_2)) &= d\Gamma^\alpha(X_1, X_2) + [\Gamma^\alpha(X_1), \Gamma^\alpha(X_2)]^{\mathfrak{X}(S)} \\ &= d(\eta^\alpha \circ \sigma^\alpha)(X_1, X_2) + [(\eta^\alpha \circ \sigma^\alpha)(X_1), (\eta^\alpha \circ \sigma^\alpha)(X_2)]^{\mathfrak{X}(S)} \\ &= \eta^\alpha \left(d\sigma^\alpha(X_1, X_2) + [\sigma^\alpha(X_1), \sigma^\alpha(X_2)]^V \right). \end{aligned}$$

3.3. The holonomy Lie algebra. Let (H, η) be a complete strong system. The holonomy Lie algebra of any (complete by 3.1) H -connection C_σ is given as follows (see [Mi], 3.2):

Let M be connected. Choose $x_0 \in M$, a base point, and identify the standard fiber S with E_{x_0} . For $x \in M$ and $X_x, Y_x \in T_x M$ we consider the horizontal lifts $C_\sigma(X_x)$ and $C_\sigma(Y_x)$ which are vector fields on E along E_x . Then the curvature applied to these fields is vertical, $R(C_\sigma(X_x), C_\sigma(Y_x)) \in \mathfrak{X}(E_x)$. Now we choose a piecewise smooth curve c in M from x_0 to x and consider the pullback under the parallel transport

$$\text{Pt}(c, 1, \quad)^* R(C_\sigma(X_x), C_\sigma(Y_x)) \in \mathfrak{X}(E_{x_0}) = \mathfrak{X}(S).$$

The closed linear span of all these vector fields in $\mathfrak{X}(S)$ with respect to the compact C^∞ -topology is called the *holonomy Lie algebra* $\text{hol}(C_\sigma, x_0)$ of the connection C_σ , centered at x_0 .

3.4. Lemma. *The holonomy Lie algebra $\text{hol}(C_\sigma, x_0)$ is a sub Lie algebra of $\eta_{x_0}(A_{x_0}) \subset \mathfrak{X}(E_{x_0})$ and is thus finite dimensional.*

Proof. Using 3.2 and the proof of 3.1 we get in turn

$$\begin{aligned} T(\psi_\alpha).R(C_\sigma(X_x, \psi_\alpha^{-1}(x, s)), C_\sigma(Y_x, \psi_\alpha^{-1}(x, s))) \\ &= ((\psi_\alpha^{-1})^* R)(T(\psi_\alpha).\eta.(\varphi_\alpha^{-1}(X_x, \sigma^\alpha(X_x)), \psi_\alpha^{-1}(x, s)), \dots) \\ &= ((\psi_\alpha^{-1})^* R)((X_x, (\eta^\alpha \circ \sigma^\alpha)(X_x)(s)), (Y_x, (\eta^\alpha \circ \sigma^\alpha)(Y_x)(s))) \\ &= \eta^\alpha \left(d\sigma^\alpha(X_x, Y_x) + [\sigma^\alpha(X_x), \sigma^\alpha(Y_x)]^V \right)(s). \end{aligned}$$

Thus $R(C_\sigma(X_x), C_\sigma(Y_x)) \in \eta_x(A_x)$. Next we prove that pull back via parallel transport does not move out of $\eta(A)$. From the proof of theorem 3.1 we know that for a smooth curve c in U_α we have

$$\psi_\alpha(\text{Pt}(c, t, \psi_\alpha^{-1}(c(0), s))) = (c(t), r_\alpha(s, g_\alpha(t))),$$

where $g_\alpha(t)$ is a globally defined curve in the Lie group G_α and where $r_\alpha : S \times G_\alpha \rightarrow S$ is a right action such that $\eta_\alpha : V \rightarrow \mathfrak{X}(S)$ is the

fundamental vector field mapping. But then we have

$$\begin{aligned}
 & (\psi_\alpha^{-1})^* \text{Pt}(c, t)^* \check{\eta}(\varphi_\alpha^{-1}(0_{c(t)}, v)) = \\
 & = T\psi_\alpha \circ T\text{Pt}(c, t)^{-1} \circ \check{\eta}(\varphi_\alpha^{-1}(0_{c(t)}, v)) \circ \text{Pt}(c, t) \circ \psi_\alpha^{-1}(c(0), \quad) \\
 & = T(\psi_\alpha \circ \text{Pt}(c, t) \circ \psi_\alpha^{-1})(0_{c(t)}, \eta^\alpha(v) \circ \text{pr}_2 \circ \psi_\alpha \circ \text{Pt}(c, t) \circ \psi_\alpha^{-1}(c(0), \quad)) \\
 & = (0_{c(0)}, T(r_\alpha^{g_\alpha(t)^{-1}}) \circ \eta^\alpha(v) \circ r_\alpha^{g_\alpha(t)}) = \\
 & = (0_{c(0)}, (r_\alpha^{g_\alpha(t)})^* \eta^\alpha(v)) = \\
 & = (0_{c(0)}, \eta^\alpha(\text{Ad}(g_\alpha(t))v)),
 \end{aligned}$$

by well known properties of right Lie group actions. This implies the desired result. \square

3.5. Holonomy groups. Let (E, p, M, S) be a fibre bundle with a complete connection Φ , and let us assume that M is connected. We choose a fixed base point $x_0 \in M$ and we identify E_{x_0} with the standard fiber S . For each closed piecewise smooth curve $c : [0, 1] \rightarrow M$ through x_0 the parallel transport $\text{Pt}(c, \quad, 1) =: \text{Pt}(c, 1)$ (pieced together over the smooth parts of c) is a diffeomorphism of S . All these diffeomorphisms form together the group $\text{Hol}(\Phi, x_0)$, the *holonomy group* of Φ at x_0 , a subgroup of the diffeomorphism group $\text{Diff}(S)$. If we consider only those piecewise smooth curves which are homotopic to zero, we get a subgroup $\text{Hol}_0(\Phi, x_0)$, called the *restricted holonomy group* of the connection Φ at x_0 .

3.6. Theorem. *Let (H, η) be a complete strong system of vector fields on the bundle (E, p, M, S) . Let M be connected.*

Then there is a principal bundle (P, p, M, G) with finite dimensional structure group G and a smooth action of G on S such that the following statements hold.

- (1) *The Lie algebra \mathfrak{g} of G is isomorphic to the standard Lie algebra $(V, [\quad, \quad])$ of the Lie algebra bundle (A, p, M) .*
- (2) *The fibre bundle E is isomorphic to the associated bundle $P[S]$.*
- (3) *The system (H, p, M) is induced from the system TP/G of G -invariant projectable vector fields on P .*
- (4) *Any connection respecting H is induced from a principal connection on P .*

Proof. Let us again identify E_{x_0} and S , and also A_{x_0} and the standard Lie algebra of the Lie algebra bundle (A, p, M) . Then $\eta_{x_0} : A_{x_0} \rightarrow \mathfrak{X}(E_{x_0})$ is a Lie algebra homomorphism whose image consists of complete vector fields, since the system is complete. There exists a Lie group G_0

with Lie algebra A_{x_0} and a effective smooth left action $\ell : G_0 \times S \rightarrow S$ such that $-\eta_{x_0}$ is the fundamental vector field mapping for it (which is a Lie algebra anti homomorphism for left actions). We call \mathfrak{g} the image of η_{x_0} . This is then a finite dimensional sub Lie algebra of $\mathfrak{X}(S)$ which is anti isomorphic to the Lie algebra of G_0 . We view G_0 as a finite dimensional subgroup of the group of all diffeomorphisms of $S = E_{x_0}$. For the rest of the proof we choose an H -connection C , given by some splitting $\sigma : TM \rightarrow H$, which we fix from now on.

Claim 1. G_0 contains $\text{Hol}_0(\Phi, x_0)$, the restricted holonomy group. Let $f \in \text{Hol}_0(\Phi, x_0)$, then $f = \text{Pt}(c, 1)$ for a piecewise smooth closed curve c through x_0 , which is nullhomotopic. Since the parallel transport is essentially invariant under reparametrisation, see [Mi, 1.5.3], we can replace c by $c \circ g$, where g is smooth and flat at each corner of c . So we may assume that c itself is smooth. Since c is homotopic to zero, by approximation we may assume that there is a smooth homotopy $H : \mathbb{R}^2 \rightarrow M$ with $H_1|_{[0, 1]} = c$ and $H_0|_{[0, 1]} = x_0$. Then $f_t := \text{Pt}(H_t, 1)$ is a curve in $\text{Hol}_0(\Phi, x_0)$ which is smooth as a mapping $\mathbb{R} \times S \rightarrow S$.

Claim 2. $(\frac{d}{dt}f_t) \circ f_t^{-1} =: Z_t$ is in \mathfrak{g} for all t .

To prove claim 2 we consider the pullback bundle $H^*E \rightarrow \mathbb{R}^2$ with the induced connection H^Φ . It is sufficient to prove claim 2 there. Let $X = \frac{d}{ds}$ and $Y = \frac{d}{dt}$ be the constant vector fields on \mathbb{R}^2 , so $[X, Y] = 0$. Then $\text{Pt}(c, s) = \text{Fl}_s^{CX} |_{(H^*E)_{(1,0)}}$ and so on. We put

$$f_{t,s} = \text{Fl}_{-s}^{CX} \circ \text{Fl}_{-t}^{CY} \circ \text{Fl}_s^{CX} \circ \text{Fl}_t^{CY} : S \rightarrow S,$$

so $f_{t,1} = f_t$. Then we have in the vector space $\mathfrak{X}(S)$

$$\begin{aligned} (\frac{d}{dt}f_{t,s}) \circ f_{t,s}^{-1} &= -(\text{Fl}_s^{CX})^*CY + (\text{Fl}_s^{CX})^*(\text{Fl}_t^{CY})^*(\text{Fl}_{-s}^{CX})^*CY, \\ (\frac{d}{dt}f_{t,s}) \circ f_{t,s}^{-1} &= \int_0^s \frac{d}{ds} \left((\frac{d}{dt}f_{t,s}) \circ f_{t,s}^{-1} \right) ds \\ &= \int_0^s \left(-(\text{Fl}_s^{CX})^*[CX, CY] + (\text{Fl}_s^{CX})^*[CX, (\text{Fl}_t^{CY})^*(\text{Fl}_{-s}^{CX})^*CY] \right. \\ &\quad \left. - (\text{Fl}_s^{CX})^*(\text{Fl}_t^{CY})^*(\text{Fl}_{-s}^{CX})^*[CX, CY] \right) ds. \end{aligned}$$

Since $[X, Y] = 0$ we have $[CX, CY] = \Phi[CX, CY] = R(CX, CY)$ and

$$\begin{aligned} (\text{Fl}_t^{CX})^*CY &= C \left((\text{Fl}_t^X)^*Y \right) + \Phi \left((\text{Fl}_t^{CX})^*CY \right) \\ &= CY + \int_0^t \frac{d}{dt} \Phi(\text{Fl}_t^{CX})^*CY dt = CY + \int_0^t \Phi(\text{Fl}_t^{CX})^*[CX, CY] dt \\ &= CY + \int_0^t \Phi(\text{Fl}_t^{CX})^*R(CX, CY) dt. \end{aligned}$$

Thus all parts of the integrand above are in \mathfrak{g} and so $(\frac{d}{dt}f_{t,s}) \circ f_{t,s}^{-1}$ is in \mathfrak{g} for all t and claim 2 follows.

Now claim 1 can be shown as follows. There is a unique smooth curve $g(t)$ in G_0 satisfying $T_e(\rho_{g(t)})Z_t = Z_t \cdot g(t) = \frac{d}{dt}g(t)$ and $g(0) = e$ where ρ_g denotes right translation by g in G . Via the action of G_0 on S the curve $g(t)$ is a curve of diffeomorphisms on S , generated by the time dependent vector field Z_t , so $g(t) = f_t$ and $f = f_1$ is in G_0 . So we get $\text{Hol}_0(\Phi, x_0) \subseteq G_0$.

Step 3. Now let G be the subgroup of the group of all diffeomorphisms of S which is generated by the full holonomy group $\text{Hol}(\Phi, x_0)$ and by G_0 . We make G into a Lie group by taking G_0 as its connected component of the identity. This is possible: G/G_0 is a countable group, since the fundamental group $\pi_1(M)$ is countable (by Morse Theory M is homotopy equivalent to a countable CW-complex).

Step 4. Construction of a cocycle of transition functions with values in G . Let $(U_\alpha, u_\alpha : U_\alpha \rightarrow \mathbb{R}^m)$ be a locally finite smooth atlas for M such that each $u_\alpha : U_\alpha \rightarrow \mathbb{R}^m$ is surjective. Put $x_\alpha := u_\alpha^{-1}(0)$ and choose smooth curves $c_\alpha : [0, 1] \rightarrow M$ with $c_\alpha(0) = x_0$ and $c_\alpha(1) = x_\alpha$. For each $x \in U_\alpha$ let $c_\alpha^x : [0, 1] \rightarrow M$ be the smooth curve $t \mapsto u_\alpha^{-1}(t \cdot u_\alpha(x))$, then c_α^x connects x_α and x and the mapping $(x, t) \mapsto c_\alpha^x(t)$ is smooth $U_\alpha \times [0, 1] \rightarrow M$. Now we define a fibre bundle atlas $(U_\alpha, \psi_\alpha : E|U_\alpha \rightarrow U_\alpha \times S)$ by $\psi_\alpha^{-1}(x, s) = \text{Pt}(c_\alpha^x, 1) \text{Pt}(c_\alpha, 1) s$. Then ψ_α is smooth since $\text{Pt}(c_\alpha^x, 1) = \text{Fl}_1^{C X_x}$ for a local vector field X_x depending smoothly on x . Let us investigate the transition functions.

$$\begin{aligned} \psi_\beta \psi_\alpha^{-1}(x, s) &= (x, \text{Pt}(c_\alpha, 1)^{-1} \text{Pt}(c_\alpha^x, 1)^{-1} \text{Pt}(c_\beta^x, 1) \text{Pt}(c_\beta, 1) s) \\ &= (x, \text{Pt}(c_\beta \cdot c_\beta^x \cdot (c_\alpha^x)^{-1} \cdot (c_\alpha)^{-1}, 4) s) \\ &=: (x, \psi_{\beta\alpha}(x) s), \text{ where } \psi_{\beta\alpha} : U_{\beta\alpha} \rightarrow G. \end{aligned}$$

Clearly $\psi_{\beta\alpha} : U_{\beta\alpha} \times S \rightarrow S$ is smooth which implies that $\psi_{\beta\alpha} : U_{\beta\alpha} \rightarrow G$ is also smooth. $(\psi_{\alpha\beta})$ is a cocycle of transition functions and we use it to glue a principal bundle with structure group G over M which we call (P, p, M, G) . From its construction it is clear that the associated bundle $P[S] = P \times_G S$ equals (E, p, M, S) .

We have thus shown assertions (1) and (2) of the theorem. The two remaining assertions will be shown later in 4.7. \square

4. THE UNIVERSAL CONNECTION

4.1. The extension of the system. Let (H, η) be a complete strong system of vector fields on the bundle E , and let $(C = C(H), p_C, M)$ be

the bundle of H -connections on E described in 2.6. We consider the following fibered products

$$\begin{array}{ccc} TC \times_{TM} H & \xrightarrow{pr_2} & H & & TC \times_M A & \xrightarrow{pr_2} & A \\ pr_1 \downarrow & & \downarrow \eta & & pr_1 \downarrow & & \downarrow p_A \\ TC & \xrightarrow{T(p_C)} & TM, & & TC & \xrightarrow{\pi_M \circ T(p_C)} & M. \end{array}$$

There is a smooth diffeomorphism, fibered over TC , between these two fibered products, whose description involves the properties of $C = C(H)$:

$$\begin{aligned} \kappa : TC \times_{TM} H &\rightarrow TC \times_M A \\ (X, h) &\mapsto (X, h - \pi_C(X) \cdot \eta(h)) \\ (X, a + \pi_C(X) \cdot Tp_C \cdot X) &\leftarrow (X, a). \end{aligned}$$

Since $TC \times_M A = TC \times_C p_C^* A \rightarrow C$ is a vector bundle, we may regard also $TC \times_{TM} H \rightarrow C$ as a vector bundle via κ .

We consider now the fiber bundle $C(H) \times_M E = C \times_M E \rightarrow C$ with standard fiber S , and the extended system of vector fields on it which is given by the following diagram:

$$\begin{array}{ccc} (TC \times_{TM} H) \times_C (C \times_M E) & \xrightarrow{\tilde{\eta}} & T(C \times_M E) \\ \parallel & & \parallel \\ TC \times_{TM} (H \times_M E) & \xrightarrow{TC \times_{TM} \eta} & TC \times_{TM} TE. \end{array}$$

Since the vertical part of $\tilde{\eta}$ is the same as that of η we see that the system $(TC \times_{TM} H, \tilde{\eta})$ is again strong and complete if the system H is it. Canonical atlases for the extended system are given by base extensions of the canonical atlases for (H, η) . The exact sequence in the sense of 2.3 is here an exact sequence of vector bundles over C :

$$0 \rightarrow C \times_M A \rightarrow TC \times_M A \xrightarrow{pr_1} TC \rightarrow 0.$$

4.2. The universal connection. We now consider the bundle of connections $C(TC \times_{TM} H) \rightarrow C$ for the extended system of vector fields, in the sense of 2.6. It is just the affine bundle of all splittings of the exact sequence of vector bundles over C in the bottom line of the big diagram of 4.1. Thus we have

$$\begin{aligned} C(TC \times_{TM} H) &= \{\sigma \in L(TC, TC \times_M A) : pr_1 \circ \sigma = Id_{TC}\} \\ &\cong L(TC, C \times_M A), \end{aligned}$$

since this affine bundle has a canonical section, namely $(Id_{TC}, 0)$. This canonical section gives rise to a distinguished connection on the bundle $C \times_M E \rightarrow C$ which is called the *universal connection* since it has the universal property described in lemma 4.3 below. Its horizontal lift will be called

$$C^{\text{univ}} : TC \times_C (C \times_M E) = TC \times_M E \rightarrow T(C \times_M E) = TC \times_{TM} TE.$$

By the general formula of 2.6 we have (taking into account all isomorphisms):

$$\begin{aligned} C^{\text{univ}}(X, e) &= \tilde{\eta}(\kappa^{-1}(X, 0), e) \\ &= (TC \times_{TM} \eta)(X, (\pi_C(X).Tp_C.X), e) \\ &= (X, \eta(\pi_C(X).Tp_C.X, e)). \end{aligned}$$

This coincides with the coordinate formula of [Mo]. The universal connection itself is then given by

$$\begin{aligned} \Phi^{\text{univ}} : T(C \times_M E) &= TC \times_{TM} TE \rightarrow V(C \times_M E) = C \times_M TE \\ \Phi^{\text{univ}}(X, Y) &= (\pi_C(X), Y - \eta(\pi_C(X).Tp_C.X, \pi_E(Y))). \end{aligned}$$

4.3. Lemma. *The universal connection has the following universal property: Let $\sigma \in C^\infty(C(H))$ be a section describing a horizontal lift C_σ of a H -connection on E . Consider the extended section $\sigma \times_M E : E = M \times_M E \rightarrow C \times_M E$. Then the universal connection C^{univ} on $C \times_M E$ and the connection C_σ are $(\sigma \times_M E)$ -related, i. e. the following diagram commutes:*

$$\begin{array}{ccc} TM \times_M E & \xrightarrow{C_\sigma} & TE \\ T\sigma \times_M E \downarrow & & \downarrow T(\sigma \times_M E) \\ TC \times_M E & \xrightarrow{C^{\text{univ}}} & T(C \times_M E). \end{array}$$

Likewise the vertical projection $\Phi_\sigma : TE \rightarrow VE$ and the vertical projection Φ^{univ} of the universal connection are $(\sigma \times_M E)$ -related, i. e. the following diagram commutes:

$$\begin{array}{ccc} TE = T(M \times_M E) & \xrightarrow{\Phi_\sigma} & VE = V(M \times_M E) \\ T\sigma \times_{TM} TE \downarrow & & \downarrow \sigma \times_M \text{ins} \\ T(C \times_M E) = TC \times_{TM} TE & \xrightarrow{\Phi^{\text{univ}}} & V(C \times_M E) = C \times_M TE. \end{array}$$

Proof. Check from the definitions that the diagrams commute. \square

4.4. The universal holonomy group. Since the universal connection C^{univ} respects the system $TC \times_{TM} H$ on $C \times_M E \rightarrow C$, and since this system is complete as noted in 4.1, C^{univ} is a complete connection by theorem 3.1.

Now we choose $c_0 \in C$ with $p_C(c_0) = x_0 \in M$ and we identify again the standard fiber S with $(C \times_M E)_{c_0} \cong E_{x_0}$. Then we can consider the holonomy group $\text{Hol}^{\text{univ}}(c_0) := \text{Hol}(C^{\text{univ}}, c_0)$ within the group of all diffeomorphisms of the standard fiber S . We may now apply the first half of the proof of theorem 3.6 to the universal connection C^{univ} on the bundle $C \times_M E \rightarrow C$. From step 3 of that proof it follows that the universal holonomy group $\text{Hol}^{\text{univ}}(c_0)$ is a subgroup of the Lie group G constructed there. The groups coincide, but we will not need this fact.

4.5. Lemma. *The parallel transport Pt^{univ} of the universal connection has the following universal property:*

Let $\sigma \in C^\infty(C(H))$ be a section describing a horizontal lift C_σ of a H -connection on E . Let $c : [0, 1] \rightarrow M$ be a (piecewise) smooth curve in M . Then the universal parallel transport Pt^{univ} and the parallel transport Pt^σ of the connection C_σ are related by the following formulas:

$$\begin{aligned} \text{Pt}^{\text{univ}}(\sigma \circ c, t) \circ (\sigma \times_M E) &= (\sigma \times_M E) \text{Pt}^\sigma(c, t) \\ pr_2 \circ \text{Pt}^{\text{univ}}(\sigma \circ c, t) &= \text{Pt}^\sigma(c, t) \circ pr_2. \end{aligned}$$

Proof. We only have to show that for $u \in E_{c(0)}$ the following formula holds:

$$\text{Pt}^{\text{univ}}(\sigma \circ c, t, (\sigma(c(0)), u)) = (\sigma \times_M E) \text{Pt}^\sigma(c, t, u)$$

Both curves cover the curve $\sigma \circ c$ in C and have the same initial value $(\sigma(c(0)), u) \in C \times_M E$. Moreover by lemma 4.3 we have

$$\begin{aligned} \Phi^{\text{univ}} \frac{d}{dt} (\sigma \times_M E) \text{Pt}^\sigma(c, t, u) &= \Phi^{\text{univ}} (T\sigma \times_{TM} TE) \frac{d}{dt} \text{Pt}^\sigma(c, t, u) \\ &= (\sigma \times_M TE) \Phi_\sigma \frac{d}{dt} \text{Pt}^\sigma(c, t, u) = 0. \quad \square \end{aligned}$$

4.6. Lemma. *Let $b : [0, 1] \rightarrow C_x$ be a vertical (piecewise) smooth curve in C . Then the universal parallel transport along b is just given by the affine structure of $C \rightarrow M$, i. e. we have $\text{Pt}^{\text{univ}}(b, t, (b(0), e)) = ((b(t), e))$ for each $e \in E_x$*

Proof. By the formula for Φ^{univ} in 4.2 we have

$$\begin{aligned} \Phi^{\text{univ}} \frac{d}{dt} (b(t), 0_e) &= \Phi^{\text{univ}}(b'(t), 0_e) \\ &= (b(t), 0_e - \eta(b(t).0_x, e)) = 0_{(b(t), e)}. \quad \square \end{aligned}$$

4.7. Rest of the proof of theorem 3.6. We assume that we are again in the situation at the end of the proof.

Step 5. Lifting each H -connection to P .

For this we have to compute the Christoffel symbols of C_τ for an arbitrary section $\tau \in C^\infty(C(H))$ with respect to the atlas of step 4. To do this directly is quite difficult since we have to differentiate the parallel transport with respect to the curve. Fortunately there is another way using the universal parallel transport. Let again Pt^τ denote the parallel transport of C_τ and as above $\text{Pt} = \text{Pt}^\sigma$ that one of $C = C_\sigma$. Let us identify $S \cong E_{x_0} \cong (C \times_m E)_{\sigma(x_0)} = \{\sigma(x_0)\} \times S$. Let $c : [0, 1] \rightarrow U_\alpha$ be a smooth curve. Then we have

$$\begin{aligned} \psi_\alpha(\text{Pt}^\tau(c, t)\psi_\alpha^{-1}(c(0), s)) &= \\ &= \left(c(t), \text{Pt}(c_\alpha^{-1}, 1) \text{Pt}((c_\alpha^{c(t)})^{-1}, 1) \text{Pt}^\tau(c, t) \text{Pt}(c_\alpha^{c(t)}, 1) \text{Pt}(c_\alpha, 1)s \right). \end{aligned}$$

Let now $b_0 : [0, 1] \rightarrow C_{c(0)}$ be a vertical smooth curve from $\sigma(c(0))$ to $\tau(c(0))$, and let $b_t : [0, 1] \rightarrow C_{c(t)}$ be one from $\sigma(c(t))$ to $\tau(c(t))$. Using lemmas 4.5 and 4.6 the last expression then gives

$$\begin{aligned} \psi_\alpha(\text{Pt}^\tau(c, t)\psi_\alpha^{-1}(c(0), s)) &= \\ &= \left(c(t), \text{pr}_2 \text{Pt}^{\text{univ}}(\sigma \circ (c_\alpha^{c(t)}.c_\alpha)^{-1}, 2) \text{Pt}^{\text{univ}}(b_t^{-1}, 1) \text{Pt}^{\text{univ}}(\tau \circ c, t) \right. \\ &\quad \left. \text{Pt}^{\text{univ}}(b_0, 1) \text{Pt}^{\text{univ}}(\sigma \circ (c_\alpha^{c(t)}.c_\alpha), 2)(\sigma(x_0), s) \right) \\ &= (c(t), \gamma(t).s), \end{aligned}$$

where $\gamma(t)$ is a smooth curve in the holonomy group G since we have $\text{Hol}^{\text{univ}}(\sigma(x_0)) \subset G$ as remarked in 4.4. Now let $\Gamma_\tau^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(S))$ be the Christoffel symbol of the connection Φ_τ with respect to the chart (U_α, ψ_α) . From the third proof of theorem [Mi, 1.5] we have

$$\psi_\alpha(\text{Pt}^\tau(c, t)\psi_\alpha^{-1}(c(0), s)) = (c(t), \bar{\gamma}(t, s)),$$

where $\bar{\gamma}(t, s)$ is the integral curve through s of the time dependent vector field $\Gamma_\tau^\alpha(\frac{d}{dt}c(t))$ on S . But then we get

$$\Gamma_\tau^\alpha(\frac{d}{dt}c(t))(\bar{\gamma}(t, s)) = \frac{d}{dt}\bar{\gamma}(t, s) = \frac{d}{dt}(\gamma(t).s) = (\frac{d}{dt}\gamma(t)).s,$$

where $\frac{d}{dt}\gamma(t) \circ \gamma(t)^{-1} \in \mathfrak{g}$. So Γ_τ^α takes values in the Lie sub algebra of fundamental vector fields for the action of G on S . Theorem [Mi, 2.5] shows that the connection Φ_τ is induced from a principal connection ω_τ on P .

Thus any H -connection on $E = P[S]$ is induced by a principal connection on P . By 2.7 this also implies that the system (H, η) is induced from the system TP/G of G -invariant projectable vector fields on P . \square

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