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# THE FLOW COMPLETION OF A MANIFOLD WITH VECTOR FIELD

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Abstract. For a vector field $X$ on a smooth manifold $M$ there exists a smooth but not necessarily Hausdorff manifold $M_{\mathbb{R}}$ and a complete vector field $X_{\mathbb{R}}$ on it which is the universal completion of $(M, X)$.

**1. Theorem.** *Let $X \in \mathfrak{X}(M)$ be a smooth vector field on a (connected) smooth manifold $M$.*

*Then there exists a universal flow completion $j : (M, X) \to (M_{\mathbb{R}}, X_{\mathbb{R}})$ of $(M, X)$. Namely, there exists a (connected) smooth not necessarily Hausdorff manifold $M_{\mathbb{R}}$, a complete vector field $X_{\mathbb{R}} \in \mathfrak{X}(M_{\mathbb{R}})$, and an embedding $j : M \to M_{\mathbb{R}}$ onto an open submanifold such that $X$ and $X_{\mathbb{R}}$ are $j$-related: $Tj \circ X = X_{\mathbb{R}} \circ j$. Moreover, for any other equivariant morphism $f : (M, X) \to (N, Y)$ for a manifold $N$ and a complete vector field $Y \in X(N)$ there exists a unique equivariant morphism $f_{\mathbb{R}} : (M_{\mathbb{R}}, x_{\mathbb{R}}) \to (N, Y)$ with $f_{\mathbb{R}} \circ j = f$. The leaf spaces $M/X$ and $M_{\mathbb{R}}/X_{\mathbb{R}}$ are homeomorphic.*

*Proof.* Consider the manifold $\mathbb{R} \times M$ with coordinate function $s$ on $\mathbb{R}$, the vector field $\bar{X} := \partial_s \times X \in \mathfrak{X}(\mathbb{R} \times M)$, and let $M_{\mathbb{R}} := \mathbb{R} \times_{\bar{X}} M$ be the orbit space (or leaf space) of the vector field $\bar{X}$.

Consider the flow mapping $\mathrm{Fl}^{\bar{X}} : \mathcal{D}(\bar{X}) \to \mathbb{R} \times M$, given by $\mathrm{Fl}^{\bar{X}}_t(s, x) = (s + t, \mathrm{Fl}^X_t(x))$, where the domain of definition $\mathcal{D}(\bar{X}) \subset \mathbb{R} \times (\mathbb{R} \times M)$ is an open neighbourhood of $\{0\} \times (\mathbb{R} \times M)$ with the property that $\mathbb{R} \times \{x\} \cap \mathcal{D}(\bar{X})$ is an open interval times $\{x\}$.

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For each $s \in \mathbb{R}$ we consider the mapping

$$j_s : M \xrightarrow{\text{ins}_t} \{s\} \times M \subset \mathbb{R} \times M \xrightarrow{\pi} \mathbb{R} \times_{\bar{X}} M = M_{\mathbb{R}}.$$

Each mapping $j_s$ is injective: A trajectory of $\bar{X}$ can meet $\{s\} \times M$ at most once since it projects onto the unit speed flow on $\mathbb{R}$.

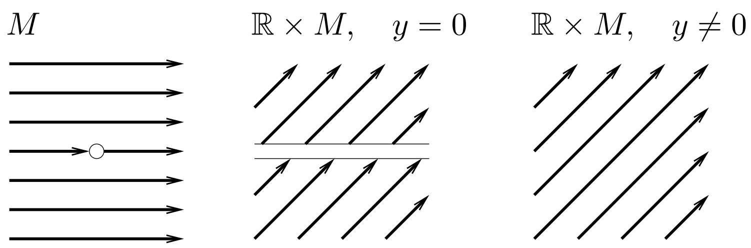
Obviously, the image $j_s(M)$ is open in $M_{\mathbb{R}}$ in the quotient topology: If a trajectory hits $\{s\} \times M$ in a point $(s, x)$, let $U$ be an open neighborhood of $x$ in $M$ such that $(-\varepsilon, \varepsilon) \times (s - \varepsilon, s + \varepsilon) \times U \subset \mathcal{D}(\bar{X})$. Then the trajectories hitting $(s - \varepsilon, s + \varepsilon) \times U$ fill a flow invariant open neighborhood which projects on an open neighborhood of $j_s(x)$ in $M_{\mathbb{R}}$ which lies in $j_s(M)$. This argument also shows that $j_s$ is a homeomorphism onto its image in $M_{\mathbb{R}}$.

Let us use the mappings $j_s : M \to M_{\mathbb{R}}$ as charts. The chart change then looks as follows: For $r < s$ the set $(j_s)^{-1}(j_r(M)) \subset M$ is just the open subset of all $x \in M$ such that $[0, s - r] \times \{(s, x)\} \subset \mathcal{D}(\bar{X})$, and $(j_s)^{-1} \circ j_r$ is given by $\text{Fl}^X_{s-r}$ on this set. Thus the chart changes are smooth.

Consider the flow $(t, (s, x)) \mapsto (s + t, x)$ on $\mathbb{R} \times M$ which commutes with the flow of $\bar{X}$ and thus induces a flow on the leaf space $M_{\mathbb{R}} = \mathbb{R} \times_{\bar{X}} M$. Differentiating this flow we get a vector field $X_{\mathbb{R}}$ on $M_{\mathbb{R}}$.

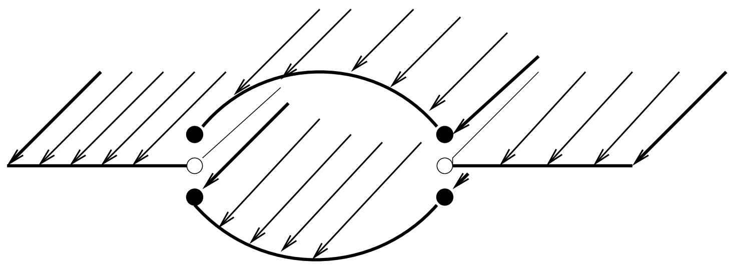
The construction $(M, X) \mapsto (M_{\mathbb{R}}, X_{\mathbb{R}})$ is a functor from the category of smooth Hausdorff manifolds with vector-fields and smooth mappings intertwining the vector fields into the category of possibly non-Hausdorff manifolds with complete smooth vector fields and smooth mappings intertwining these fields. For a pair $(M, X)$ with $X$ a complete vector field the flow completion $(M_{\mathbb{R}}, X_{\mathbb{R}})$ is equivariantly diffeomorphic to $(M, X)$ since then any of the charts $j_s : M \to M_{\mathbb{R}}$ is also surjective. From this the universal property follows. □

**2. Example.** Let $(M, X) = (\mathbb{R}^2 \setminus \{0\}, \partial_x)$. The trajectories of $X$ on $M$ and of $\bar{X}$ on $\mathbb{R} \times M$ in the slices $y =$ constant for $y = 0$ and $y \neq 0$ then look as follows:




The smooth manifold $M_{\mathbb{R}}$ then is $\mathbb{R}^2$ with the $x$-axis doubled: $(x, 0)_+$ and $(x, 0)_-$ cannot be separated for each $x \in \mathbb{R}$. The charts $j_s(M)$ all are diffeomorphic to $M = \mathbb{R}^2 \setminus \{0\}$ and contain $(x, 0)_-$ for $x < 0$ and $(x, 0)_+$ for $x > 0$. The charts $j_r(M)$ and $j_s(M)$ are glued together by the shift $x \mapsto x + s - r$. In this example $M_{\mathbb{R}}$ is not Hausdorff, but its Hausdorff quotient (given by the equivalence relation generated by identifying non-separable points) is again a smooth manifold and has the universal property described in (1).

**3. Example.** Let $(M, X) = (\mathbb{R}^2 \setminus \{0\} \times [-1, 1], \partial_x)$. The trajectories of $\bar{X}$ on $\mathbb{R} \times M$ in the slices $y =$ constant for $|y| \leq 1$ and $|y| \geq 1$ then look as in the second and third illustration above. The flow completion $M_{\mathbb{R}}$ then becomes $\mathbb{R}^2$ with the part $\mathbb{R} \times [-1, 1]$ doubled and the topology such that the points $(x, -1)_-$ and $(x, -1)_+$ cannot be separated as well as the points $(x, 1)_-$ and $(x, 1)_+$. The flow is just $(x, y) \to (x + t, y)$:



In this example $M_{\mathbb{R}}$ is not Hausdorff, and its Hausdorff quotient is not a smooth manifold any more. There are two obvious quotient manifolds which are Hausdorff, the cylinder and the plane. Thus none of these two has the universal property of (1).

**4. Non-Hausdorff smooth manifolds.** We met second countable smooth manifolds which need not be Hausdorff. Let us discuss a little their properties. They are $T_1$, since all points are closed; they are closed in a chart. The construction of the tangent bundle is by glueing the local tangent bundles. Smooth mappings and vector fields are defined as usual: Non separable pairs of points are mapped to non separable pairs. Vector fields admit flows as usual: These are given locally in the charts and are then glued together. If $x$ and $y$ are non separable points and if $X$ is a vector field on the manifold, then for each $t$ the points $\mathrm{Fl}^X_t(x)$ and $\mathrm{Fl}^X_t(y)$ are non separable. Theorem (1) can be extended to the category of not necessarily Hausdorff smooth manifolds and vector fields, without any change in the proof.

**5. Remark.** The ideas in this paper generalize to the setting of $\mathfrak{g}$-manifolds, where $\mathfrak{g}$ is a finite dimensional Lie group. Let $G$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then one may construct the $G$-completion of a non-complete $\mathfrak{g}$-manifold. There are difficulties with the property $T_1$, not only with Hausdorff. This was our original road which was inspired by [1]. We treat the full theory in [2]. We thought that the special case of a vector field is interesting in its own.

## References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