COADJOINT ORBITS IN INFINITE DIMENSIONS

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1. Introduction. Correspondence between coadjoint orbits and irreducible representations for Lie groups

A symmetry of a classical system is a group G acting on the symplectic manifold (G,ω) and preserving the symplectic form ω. An elementary classical system is then a G-orbit, hence a homogeneous symplectic G-space. A symmetry of a quantum system is a group G acting by projective unitary transformations on a Hilbert space. Then an elementary quantum system is a projective unitary irreducible representation of G.

By passing to a central extension of the group, the irreducible projective unitary representation corresponds to an irreducible unitary representation (see 3.4). By passing to the universal covering space of the symplectic G-space and to a central extension of the Lie group, the homogeneous symplectic G-space corresponds to a homogeneous Hamiltonian one (this means that the fundamental vector fields ζ_X, $X \in \mathfrak{g}$ are Hamiltonian and the action has a moment mapping $\mu : \mathfrak{g} \to C^\infty(M,\mathbb{R})$, $\zeta_X = H_{\mu(X)}$ which is a Lie algebra homomorphism).

The Kirillov-Kostant-Souriau Theorem assures that each homogeneous Hamiltonian G-space is locally isomorphic to a coadjoint orbit of G as Hamiltonian G-space (see 8.1 for the construction of the symplectic structure on a coadjoint orbit and the moment map of the coadjoint action). This suggests that there should be a connection between the coadjoint orbits of a Lie group and its irreducible unitary representations. Kirillov has found it.

Theorem [Kirillov, 1962]. *Let G be a simply connected nilpotent Lie group. Then:*

(1) *Each irreducible unitary representation is induced by a 1-dimensional unitary representation of some subgroup.*
(2) *Any $\alpha \in \mathfrak{g}^*$ defines an irreducible unitary representation $\operatorname{ind}_H^G \alpha$, where $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$ such that $\alpha|_{[\mathfrak{h},\mathfrak{h}]} = 0$, $H = \exp \mathfrak{h}$ and the 1-dimensional unitary representation on H is given by the character $\chi(\exp X) = e^{2\pi i \alpha(X)}$.*
(3) *The irreducible unitary representations $\operatorname{ind}_{H_1}^G \alpha_1$ and $\operatorname{ind}_{H_2}^G \alpha_2$ are unitarily equivalent if and only if α_1 and α_2 lie in the same coadjoint orbit.*

For compact simply connected Lie groups, the irreducible representations correspond to integral coadjoint orbits of maximal dimension. By definition, a coadjoint orbit $\mathcal{O}_\alpha$ is integral if the cohomology class of the symplectic form is integral; this is a necessary and sufficient condition for the existence of a character $\chi : G_\alpha \to S^1$ that integrates the restriction of α to $\mathfrak{g}_\alpha$, i.e. $\chi(\exp X) = e^{2\pi i \alpha(X)}$. (The theorem says that all the coadjoint orbits of a nilpotent Lie group are integral.)

An integral element $\alpha \in \mathfrak{g}$ for an arbitrary simply connected Lie group induces a family of irreducible unitary representations having l cyclic parameters, where $l = b_1(\mathcal{O}_\alpha)$=the number of connected components of $\mathcal{O}_\alpha$.

A theorem of Kostant-Auslander says that if G is a simply connected solvable Lie group with coadjoint orbit space T_0 and all coadjoint orbits are integral, then all irreducible unitary representations are obtained by the above construction from coadjoint orbits.

2. Infinite dimensional Lie groups

2.1. Calculus of smooth mappings.

The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces the main difficulty is that the composition of linear mappings stops being jointly continuous at the level of Banach spaces, for any compatible topology. The infinite dimensional calculus used in this work is the Frölicher-Kriegl calculus on convenient vector spaces: see the book [Frölicher-Kriegl, 1988], which is the general reference for this section, see also the forthcoming book [Kriegl-Michor] for sections 2.2 and 2.3.

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called smooth if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth curves. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).
(3) $\{E_B \to E : B$ bounded absolutely convex in $E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞*-topology* on E and is denoted $c^\infty E$. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Definition. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent (completeness) conditions is satisfied:

(1) Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges. This is also called c^∞-complete.
(2) If B is bounded closed absolutely convex, then E_B is a Banach space.
(3) Any Lipschitz curve in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1 = c_2'$ (existence of antiderivative).

Fréchet spaces are convenient.

Lemma. *Let E be a locally convex space. Then the following properties are equivalent:*

(1) *E is c^∞-complete.*
(2) *If $f : \mathbb{R}^n \to E$ is scalarwise $\mathcal{L}ip^k$, then f is $\mathcal{L}ip^k$, for $k > 1$.*
(3) *If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.*

Here a mapping $f : \mathbb{R}^n \to E$ is called $\mathcal{L}ip^k$ if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}^n$ and f scalarwise C^∞ means that $\lambda \circ f$ is C^∞ for all continuous linear functionals on E. This lemma says that on a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

Definition. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth*, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mapping from E to F.

For E and F finite dimensional this gives the usual notion of smooth mappings: this was first proved in [Boman, 1967]. Constant mappings are smooth. Multilinear mappings are smooth if and only if they are bounded. Therefore we denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$.

Lemma. *For locally convex spaces E and F we have:*

(1) *If F is convenient, then also $C^\infty(E, F)$ is convenient, for any E. The space $L(E, F)$ is a closed linear subspace of $C^\infty(E, F)$, so it is also convenient.*
(2) *If E is convenient, then a curve $c : \mathbb{R} \to L(E, F)$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in F for all $x \in E$.*

The category of convenient vector spaces and smooth mappings is cartesian closed: we have a natural bijection $C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G))$, which is even a smooth diffeomorphism. Other canonical mappings like evaluation, insertion and composition are also smooth.

Theorem. *Let E and F be convenient vector spaces. Then the differential operator*

$$d : C^\infty(E, F) \to C^\infty(E, L(E, F)),$$
$$df(x)v := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t},$$

exists and is linear and bounded (smooth). Also the chain rule holds:

$$d(f \circ g)(x)v = df(g(x))dg(x)v.$$

If one wants the cartesian closedness and assumes some other obvious properties, then the calculus of smooth functions is already uniquely determined. There are, however, smooth mappings which are not continuous. For example the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

2.2. Infinite dimensional manifolds.

Infinite dimensional smooth manifolds are defined by gluing c^∞-open sets in convenient vector spaces via smooth diffeomorphisms. Then we equip them with the identification topology with respect to the c^∞-topologies on the modeling spaces. We require this topology to be Hausdorff and regular.

A manifold is metrizable if and only if it is paracompact and modeled on Fréchet spaces. If the metrizable manifold is modeled on nuclear convenient spaces, then it admits smooth partitons of unity subordinated to locally finite open covers.

A subset N of a manifold M is called a *submanifold* if around each point of N there exists a chart (U, u) of M such that $u(U \cap N) = u(U) \cap F_U$, where F_U is a closed linear subspace of the convenient model space E_U. N is called a *splitting submanifold* if all F_U are complemented subspaces of E_U.

Definition. Let M be a manifold with a smooth atlas $(u_\alpha : U_\alpha \to E_\alpha)$. The space of equivalence classes $\bigsqcup_\alpha U_\alpha \times E_\alpha \times \{\alpha\}\big/_\sim$ where $(x, v, \alpha) \sim (y, w, \beta)$ if and only if $x = y$ and $d(u_{\alpha\beta})(u_\beta(x))w = v$ is called the *kinematic tangent bundle.*

This is the same as defining tangent vectors as equivalence classes of curves, but we do not obtain the same tangent space wenn taking derivations (then we get the operational tangent bundle).

Theorem. *Let M and N be smooth finite dimensional manifolds. Then the space $C^\infty(M, N)$ of all smooth mappings from M to N is a smooth manifold, modeled on spaces $C^\infty_c(f^*TN)$ of smooth vector fields with compact support on N along f.*

The construction of an atlas for $C^\infty(M, N)$: Choose a smooth Riemannian metric on N and let exp be the smooth exponential mapping of this Riemannian metric. There exists a smooth diffeomorphism onto an open neighborhood V of the diagonal $(\pi_N, \exp) : U \subset TN \to N \times N$. For $f \in C^\infty_c(M, N)$ let

$$U_f := \{g \in C^\infty(M, N) : (f(x), g(x)) \in V, \forall x \in M, f = g \text{ off some compact}\},$$
$$u_f : U_f \to C^\infty_c(f^*TN),$$
$$u_f(g)(x) = \exp^{-1}_{f(x)}(g(x)) = ((\pi_N, \exp)^{-1} \circ (f, g))(x).$$

Then u_f is a bijective mapping from U_f onto the set of vector fields along f with image in $U \subset TN$, whose inverse is $u_f^{-1}(s) = \exp \circ s$. The set $u_f(U_f)$ is open in $C^\infty_c(f^*TN)$ with the inductive limit topology of $C^\infty_K(f^*TN)$ where K runs through all compact sets in M and $C^\infty_K(f^*TN)$ has the topology of uniform convergence in all derivatives separately. This is a convenient vector space and the chart change mappings are smooth. This smooth structure does not depend on the choice of the Riemannian metric.

Lemma. *The smooth curves in $C^\infty(M, N)$ correspond exactely to the smooth mappings $\hat{c} \in C^\infty(\mathbb{R} \times M, N)$ satisfying the property: (*) for each compact interval $[a, b]$, there is a compact K in M such that $\hat{c}(t, x)$ is constant in $t \in [a, b]$ for all $x \in M - K$.*

Proof. Since $\mathbb{R}$ is locally compact, to show (*) for each compact interval is the same as to show (*) locally around each $t \in \mathbb{R}$. Hence it suffices to describe the smooth curves in the modeling space $C_c^\infty(E)$, where (E, p, M, V) is a vector bundle, $C_c^\infty(E)$ being equipped with the inductive limit topology $\operatorname{inj\,lim} C_K^\infty(E)$ over all compact subsets of M. This is a strict injective limit, so smooth curves locally factor to smooth maps in some $C_K^\infty(E)$. Now we apply the cartesian closedness property for each vector bundle chart (U_α, ψ_α) of E and get smooth maps $\mathbb{R} \times U_\alpha \to V$ which fit together to a smooth map $\mathbb{R} \times M \to E$ with support in $\mathbb{R} \times K$. □

2.3. Infinite dimensional Lie groups.

Definition. An infinite dimensional *Lie group* G is a smooth manifold and a group such that the multiplication $\mu : G \times G \to G$ and the inversion $\nu : G \to G$ are smooth.

We shall use the following notation:
$\mu : G \times G \to G$, multiplication, $\mu(x, y) = x.y$.
$\mu_a : G \to G$, left translation, $\mu_a(x) = a.x$.
$\mu^a : G \to G$, right translation, $\mu^a(x) = x.a$.
$\nu : G \to G$, inversion, $\nu(x) = x^{-1}$.
$e \in G$, the unit element. The kinematic tangent mapping $T_{(a,b)}\mu : T_aG \times T_bG \to T_{ab}G$ is given by

$$T_{(a,b)}\mu.(X_a, Y_b) = T_a(\mu^b).X_a + T_b(\mu_a).Y_b.$$

and $T_a\nu : T_aG \to T_{a^{-1}}G$ is given by

$$T_a\nu = -T_e(\mu^{a^{-1}}) \circ T_a(\mu_{a^{-1}}) = -T_e(\mu_{a^{-1}}) \circ T_a(\mu^{a^{-1}}).$$

Invariant vector fields and Lie algebras. Let G be a Lie group. A (kinematic) vector field ξ on G is called left invariant, if $\mu_a^*\xi = \xi$ for all $a \in G$, where $\mu_a^*\xi = T(\mu_{a^{-1}}) \circ \xi \circ \mu_a$. Since we have $\mu_a^*[\xi, \eta] = [\mu_a^*\xi, \mu_a^*\eta]$, the space $\mathcal{X}_L(G)$ of all left invariant vector fields on G is closed under the Lie bracket, so it is a Lie subalgebra of $\mathcal{X}(G)$. Any left invariant vector field ξ is uniquely determined by $\xi(e) \in T_eG$, since $\xi(a) = T_e(\mu_a).\xi(e)$. Thus the Lie algebra $\mathcal{X}_L(G)$ of left invariant vector fields is linearly isomorphic to T_eG, and on T_eG the Lie bracket on $\mathcal{X}_L(G)$ induces a Lie algebra structure, whose bracket is again denoted by $[,]$. This Lie algebra will be denoted as usual by $\mathfrak{g}$.

We will also give a name to the isomorphism with the space of left invariant vector fields: $L : \mathfrak{g} \to \mathcal{X}_L(G)$, $X \mapsto L_X$, where $L_X(a) = T_e\mu_a.X$. Thus $[X, Y] = [L_X, L_Y](e)$.

A vector field η on G is called right invariant, if $(\mu^a)^*\eta = \eta$ for all $a \in G$. If ξ is left invariant, then $\nu^*\xi$ is right invariant, since $\nu \circ \mu^a = \mu_{a^{-1}} \circ \nu$ implies that $(\mu^a)^*\nu^*\xi = (\nu \circ \mu^a)^*\xi = (\mu_{a^{-1}} \circ \nu)^*\xi = \nu^*(\mu_{a^{-1}})^*\xi = \nu^*\xi$. The right invariant vector fields form a sub Lie algebra $\mathcal{X}_R(G)$ of $\mathcal{X}(G)$, which is again linearly isomorphic to T_eG and induces also a Lie algebra structure on T_eG. Since $\nu^* : \mathcal{X}_L(G) \to \mathcal{X}_R(G)$ is an isomorphism of Lie algebras, $T_e\nu = -Id : T_eG \to T_eG$ is an isomorphism between the two Lie algebra structures. We will denote by $R : \mathfrak{g} = T_eG \to \mathcal{X}_R(G)$ the isomorphism discussed, which is given by $R_X(a) = T_e(\mu^a).X$. We have $[L_X, R_Y] = 0$, thus the flows of L_X and R_Y commute.

Let $\varphi : G \to H$ be a smooth homomorphism of Lie groups. Then $\varphi' := T_e\varphi : \mathfrak{g} = T_eG \to \mathfrak{h} = T_eH$ is a Lie algebra homomorphism.

Definition. A *Lie subgroup* H of a group G is a subgroup of G which is also a submanifold. It follows that H is itself a Lie group.

A global *flow* for a vector field ξ on a manifold M is a smooth mapping $\mathrm{Fl}^\xi : \mathbb{R} \times M \to M$ with the following properties:

(1) $\frac{d}{dt}\mathrm{Fl}^\xi(t,x) = \xi(\mathrm{Fl}^\xi(t,x))$;
(2) $\mathrm{Fl}^\xi(0,x) = x$;
(3) $\mathrm{Fl}^\xi(t+s,x) = \mathrm{Fl}^\xi(t,\mathrm{Fl}^\xi(s,x))$.

For Banach manifolds the last relation follows from the first two. If the flow of a vector field exists, then it is unique.

Definition. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We say that G admits an *exponential mapping* if there exists a smooth mapping $\exp : \mathfrak{g} \to G$ such that $t \mapsto \exp(tX)$ is the 1-parameter subgroup with tangent vector X at 0. Then we have also:

(1) The exponential map is unique, since if α, β are 1-parameter subgroups with $\alpha'(0) = \beta'(0) = X$, then $\frac{d}{dt}\alpha(t)\beta(t)^{-1} = T\mu_{\alpha(t)}T\mu^{\beta(-t)}\frac{d}{ds}|_0\alpha(s)\beta(-s) = 0$.
(2) $\mathrm{Fl}^{L_X}(t,x) = x.\exp(tX)$.
(3) $\mathrm{Fl}^{R_X}(t,x) = \exp(tX).x$.
(4) $\exp(0) = e$ and $T_0\exp = Id : T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$ since $T_0\exp.X = \frac{d}{dt}|_0\exp(0+t.X) = \frac{d}{dt}|_0\mathrm{Fl}^{L_X}(t,e) = X$.
(5) Let $\varphi : G \to H$ be a smooth homomorphism of between Lie groups admitting exponential mappings. Then the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi'"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes, since $t \mapsto \varphi(\exp^G(tX))$ is a one parameter subgroup of H and $\frac{d}{dt}|_0\varphi(\exp^G tX) = \varphi'(X)$, so $\varphi(\exp^G tX) = \exp^H(t\varphi'(X))$.

The adjoint representation. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $a \in G$ we define $\mathrm{conj}_a : G \to G$ by $\mathrm{conj}_a(x) = axa^{-1}$. It is called the *conjugation* or the inner automorphism by $a \in G$. This defines a smooth action of G on itself by automorphisms.

The *adjoint representation* $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is given by $\mathrm{Ad}(a) = (\mathrm{conj}_a)' = T_e(\mathrm{conj}_a) : \mathfrak{g} \to \mathfrak{g}$ for $a \in G$. So $\mathrm{Ad}(a)$ is a Lie algebra homomorphism and $\mathrm{Ad}(a) = T_e(\mathrm{conj}_a) = T_a(\mu^{a^{-1}}).T_e(\mu_a) = T_{a^{-1}}(\mu_a).T_e(\mu^{a^{-1}})$. The *coadjoint representation* $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ is the dual of the adjoint representation: $\langle \mathrm{Ad}^*(a)\alpha, X\rangle = \langle \alpha, \mathrm{Ad}(a^{-1})X\rangle$ for for every $X \in \mathfrak{g}$.

Finally we define the (lower case) adjoint representation of the Lie algebra $\mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) := L(\mathfrak{g},\mathfrak{g})$, by $\mathrm{ad} := \mathrm{Ad}' = T_e\,\mathrm{Ad}$.

Like in the finite dimensional case we have:

(1) $L_X(a) = R_{\mathrm{Ad}(a)X}(a)$ for $X \in \mathfrak{g}$ and $a \in G$.
(2) $\mathrm{ad}(X)Y = [X,Y]$ for $X, Y \in \mathfrak{g}$.

Theorem. [Grabowski, 1993] *Let G be a Lie group with exponential mapping* $\exp : \mathfrak{g} \to G$. *Then for all $X, Y \in \mathfrak{g}$ we have*

$$\begin{aligned} T_X \exp .Y &= T_e\mu_{\exp X} . \int_0^1 \mathrm{Ad}(\exp(-tX))Y\,dt \\ &= T_e\mu^{\exp X} . \int_0^1 \mathrm{Ad}(\exp(tX))Y\,dt \end{aligned}$$

Remark. If G is a Banach Lie group then the series $\mathrm{Ad}(\exp(tX)) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \mathrm{ad}(X)^i$, so that we get the usual formula

$$T_X \exp = T_e\mu^{\exp X} . \sum_{i=0}^{\infty} \tfrac{1}{(i+1)!} \mathrm{ad}(X)^i .$$

It is not true in general that every convenient Lie agebra is the Lie algebra of a convenient Lie group. Also not every Lie subalgebra in the Lie algebra of a Lie group has a corresponding Lie subgroup.

3. Central extensions of Lie groups and the coadjoint action in the extended group

3.1. Lie algebra cohomology and central extensions.

Let $\mathfrak{a}$ be a linear representation of the Lie algebra $\mathfrak{g}$ in $\mathbb{R}^p$ and $C^k(\mathfrak{g}, \mathbb{R}^p)$ the space of k-multilinear skew-symmetric mappings from $\mathfrak{g} \times \cdots \times \mathfrak{g}$ (k times) to $\mathbb{R}^p$. We define $d_{\mathfrak{a}} : C^k(\mathfrak{g}, \mathbb{R}^p) \to C^{k+1}(\mathfrak{g}, \mathbb{R}^p)$ by

$$\begin{aligned} d_{\mathfrak{a}}\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i \mathfrak{a}(X_i)\omega(X_0, \dots, \hat{X}_i, \dots, X_k) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k). \end{aligned}$$

Because $d_{\mathfrak{a}} \circ d_{\mathfrak{a}} = 0$, we obtain by the usual procedure the cohomology groups $H^k_{\mathfrak{a}}(\mathfrak{g}, \mathbb{R}^p)$ of the Lie algebra $\mathfrak{g}$ with values in $\mathbb{R}^p$.

A *central extension* of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}^p$ is a central exact sequence of Lie algebras:

$$0 \longrightarrow \mathbb{R}^p \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

The word central means that $\mathbb{R}^p$ lies in the center of $\tilde{\mathfrak{g}}$, i.e. $[\mathbb{R}^p, \tilde{\mathfrak{g}}] = 0$.

Proposition. *There is a one-to-one correspondence between equivalence classes of central extensions of $\mathfrak{g}$ by $\mathbb{R}^p$ and the second cohomology group $H^2(\mathfrak{g}, \mathbb{R}^p)$ (here we consider the trivial representation of $\mathfrak{g}$ on $\mathbb{R}^p$).*

Proof. Let $\tilde{\mathfrak{g}}$ be such an extension and $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$ a linear splitting which permits the identification of $\tilde{\mathfrak{g}}$ with $\mathfrak{g} \times \mathbb{R}^p$ as vector spaces. Because $[\mathbb{R}^p, \tilde{\mathfrak{g}}] = 0$, we have

$$\left[\begin{pmatrix} \xi \\ \lambda \end{pmatrix}, \begin{pmatrix} \eta \\ \mu \end{pmatrix} \right] = \left[\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right] =: \begin{pmatrix} [\xi, \eta] \\ \omega(\xi, \eta) \end{pmatrix}, \quad \xi, \eta \in \mathfrak{g}, \quad \lambda, \mu \in \mathbb{R}^p.$$

Concretely: $\omega(\xi,\eta)=[s(\xi),s(\eta)]-s([\xi,\eta])$. This defines the bilinear symmetric map $\omega:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}^p$. The Jacobi identity in $\tilde{\mathfrak{g}}$ is equivalent with the cocycle condition:

$$\omega(\xi,[\eta,\zeta])+\omega(\eta,[\zeta,\xi])+\omega(\zeta,[\xi,\eta])=0.$$

Another splitting will define a cocycle cohomologous to ω, so to every central extension $\tilde{\mathfrak{g}}$, we can associate the class $[\omega]\in H^2(\mathfrak{g},\mathbb{R}^p)$. If two central extensions $\tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{g}}_2$ are equivalent, i.e. there exists a Lie algebra homomorphism $\Phi:\tilde{\mathfrak{g}}_1\to\tilde{\mathfrak{g}}_2$ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}^p \arrow[r] \arrow[dr] & \tilde{\mathfrak{g}}_1 \arrow[r] \arrow[d, "\Phi"'] & \mathfrak{g} \arrow[r] & 0, \\ & & \tilde{\mathfrak{g}}_2 \arrow[ur] & & \end{tikzcd}$$

then the two associated cocycles differ by a coboundary $\partial\tau$, where the linear map $\tau:\mathfrak{g}\to\mathbb{R}^p$ is defined by $\Phi\binom{\xi}{0}=:\binom{\xi}{\tau(\xi)}$. Indeed:

$$\begin{aligned}\begin{pmatrix}[\xi,\eta]\\ \omega_2(\xi,\eta)\end{pmatrix}=\left[\Phi\begin{pmatrix}\xi\\0\end{pmatrix},\Phi\begin{pmatrix}\eta\\0\end{pmatrix}\right]_2&=\Phi\left(\left[\begin{pmatrix}\xi\\0\end{pmatrix},\begin{pmatrix}\eta\\0\end{pmatrix}\right]_1\right)\\ &=\Phi\begin{pmatrix}[\xi,\eta]\\ \omega_1(\xi,\eta)\end{pmatrix}=\begin{pmatrix}[\xi,\eta]\\ \omega_1(\xi,\eta)+\tau([\xi,\eta])\end{pmatrix}.\end{aligned}$$

Hence $\omega_2=\omega_1+d\tau$. $\square$

If G is a compact connected Lie group with Lie algebra $\mathfrak{g}$, then the cohomology of G equals the cohomology of the Lie algebra $\mathfrak{g}$: $H^k(G)=H^k(\mathfrak{g},\mathbb{R})$. General extensions of $\mathfrak{g}$ with $\mathbb{R}^p$, i.e. $0\to\mathbb{R}^p\to\tilde{\mathfrak{g}}\to\mathfrak{g}\to 0$ where $\mathbb{R}^p$ is an ideal in $\tilde{\mathfrak{g}}$, are in correspondence with elements of $H^2_{\mathfrak{a}}(\mathfrak{g},\mathbb{R}^p)$, the representation $\mathfrak{a}$ being defined by $\mathfrak{a}(\xi)\lambda=[\lambda,s(\xi)]\in\mathbb{R}^p,\xi\in\mathfrak{g},\lambda\in\mathbb{R}^p$.

3.2. Lie group cohomology and central extensions.

Let G be a Lie group and E an abelian Lie group and G-module by some action A of G on E. We define the cohomology groups $H^k_A(G,E)$ as the cohomology of the following cochain complex

$$\begin{aligned}C^k(G,E)&:=\{c:G^k\to E: c \text{ smooth}\}\\ d_Ac(g_1,\dots,g_{k+1})&:=A(g_1)c(g_2,\dots,g_{k+1})\\ &\quad+\sum_{i=1}^k(-1)^ic(g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_{k+1})\\ &\quad+(-1)^{k+1}c(g_1,\dots,g_k).\end{aligned}$$

Another way to define the cohomology groups $H^k_A(G,E)$ is by using the smooth homogeneous k-cochains

$$\tilde{C}^k_A(G,E)=\{\tilde{c}:G^{k+1}\to E:\tilde{c}(gg_0,\dots,gg_k)=A(g)\tilde{c}(g_0,\dots,g_k)\}$$

and $\tilde{d}\tilde{c}(g_0, \dots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \tilde{c}(g_0, \dots, \hat{g}_i, \dots, g_{k+1})$. The isomorphism between C^k and $\tilde{C}^k$ is given by:

$$\begin{aligned} c(g_1, \dots, g_k) &= \tilde{c}(e, g_1, g_1 g_2, \dots, g_1 \dots g_k) \\ \tilde{c}(g_0, \dots, g_k) &= A(g_0) c(g_0^{-1} g_1, \dots, g_{k-1}^{-1} g_k). \end{aligned}$$

A *central extension* of the Lie group G by E is a central exact sequence of Lie groups

$$1 \longrightarrow E \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

i.e. E lies in the center of $\tilde{G}$.

Proposition. *There is a one-to-one correspondence between equivalence classes of central extensions of G by E which admit a global smooth section, and the second cohomology group $H^2(G, E)$, considering E as a trivial G-module.*

Proof. Let $\tilde{G}$ be a central extension of G by E, and $S : G \to \tilde{G}$ a smooth splitting of the exact sequence. We define the corresponding group cocycle by $c(g, h) = S(g)S(h)S(gh)^{-1}$, i.e.

$$\begin{pmatrix} g \\ e \end{pmatrix} \begin{pmatrix} h \\ e \end{pmatrix} = \begin{pmatrix} gh \\ c(g,h) \end{pmatrix}, \qquad c : G \times G \to E.$$

Using the property $E \subset Z(\tilde{G})$, we obtain that the group multiplication in $\tilde{G}$ is:

$$\begin{pmatrix} g \\ z \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} gh \\ zvc(g,h) \end{pmatrix}.$$

The associativity of this multiplication is equivalent with

$$c(g_2, g_3) c(g_1 g_2, g_3)^{-1} c(g_1, g_2 g_3) c(g_1, g_2)^{-1} = 1$$

which means c is a cocycle.

The cocycle c depends on the splitting, but not its cohomology class $[c] \in H^2(G, E)$. Indeed, let $S, S' : G \to \tilde{G}$ be two splittings. They differ by elements in E, so there exists a map $t : G \to E$ such that $S' = tS$. Then $c'(g, h) = c(g, h)t(g)t(h)t(gh)^{-1} = dt(g, h) = c(g, h)dt(g, h)$.

So, to every central extension $\tilde{G}$, we have associated a class $[c] \in H^2(G, E)$. Like in the Lie algebra case we can prove that equivalent extensions lead to the same cohomology class. □

General extensions

$$1 \longrightarrow E \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

with E a normal subgroup of G, are in correspondence with elements of $H^2_A(G, E)$, where the G-module structure of E is given by: $A(g)v := S(g)vS(g)^{-1} \in E, g \in G, v \in E$.

3.3. How to obtain the Lie algebra cocycle from the group cocycle.

Let $\mathfrak{g} = Lie(G), \mathbb{R}^p = Lie(E)$ and $\tilde{\mathfrak{g}} = Lie(\tilde{G})$. By differentiating at e the central extension of G by E:

$$1 \longrightarrow E \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

we obtain a central extension of $\mathfrak{g}$ by $\mathbb{R}^p$:

$$0 \longrightarrow \mathbb{R}^p \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Proposition. *The relation between the group cocycle c defined by the section S and the Lie algebra cocycle ω defined by the section $s = dS_e$ is $\omega(\xi,\eta) = \ddot{c}(\xi,\eta) - \ddot{c}(\eta,\xi)$, where $\ddot{c}(\xi,\eta) := \partial_1\partial_2 c(e,e)(\xi,\eta)$.*

There exists in general a map $f \in C^k(G,E) \mapsto [f] \in C^k(\mathfrak{g},\mathbb{R}^p)$ defined by $[f](\xi^1,\dots,\xi^k) = \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\partial_1\dots\partial_k f(e,\dots,e)(\xi^{\sigma(1)},\dots,\xi^{\sigma(k)})$, which induces a map between the cohomology groups.

Proof. If G is a compact Lie group it can be realised as a matrix group and the exponential map is $e^\xi = 1 + \xi + \frac{\xi^2}{2} + \cdots$.

The Taylor expansion of the cocycle c at $(1,1)$ is

$$c(1+t\xi, 1+t\eta) = 1 + t^2\partial_1\partial_2 c(1,1)(\xi,\eta) + o(t^2) = 1 + t^2\ddot{c}(\xi,\eta) + o(t^2)$$

because of the property: $c(g,1) = c(1,g) = 1$.

We use

$$\begin{aligned} e^{t\xi}e^{t\eta} - e^{t\eta}e^{t\xi} &= t^2(\xi\eta - \eta\xi) + o(t^2) \\ &= t^2[\xi,\eta] + o(t^2), \xi,\eta \in \mathfrak{g} \end{aligned}$$

to compute the bracket in the extended Lie algebra:

$$\begin{aligned} e^{t\binom{\xi}{0}}e^{t\binom{\eta}{0}} - e^{t\binom{\eta}{0}}e^{t\binom{\xi}{0}} &= \begin{pmatrix} e^{t\xi} \\ 1 \end{pmatrix}\begin{pmatrix} e^{t\eta} \\ 1 \end{pmatrix} - \begin{pmatrix} e^{t\eta} \\ 1 \end{pmatrix}\begin{pmatrix} e^{t\xi} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{t\xi}e^{t\eta} - e^{t\eta}e^{t\xi} \\ c(e^{t\xi},e^{t\eta}) - c(e^{t\eta},e^{t\xi}) \end{pmatrix} \\ &= t^2\begin{pmatrix} [\xi,\eta] \\ \ddot{c}(\xi,\eta) - \ddot{c}(\eta,\xi) \end{pmatrix} + o(t^2), \end{aligned}$$

hence

$$[\begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} \eta \\ 0 \end{pmatrix}] = \begin{pmatrix} [\xi,\eta] \\ \ddot{c}(\xi,\eta) - \ddot{c}(\eta,\xi) \end{pmatrix},$$

which was to be proved. $\square$

3.4. Projective representations.

A *projective representation* of the group G is a representation of G into the projective unitary group $\mathrm{PU}(H) = \mathrm{U}(H)/S^1$, where H denotes a Hilbert space. This means a map $T : G \to \mathrm{U}(H)$ such that

$$T(g_1g_2) = c(g_1g_2)^{-1}T(g_1)T(g_2), \quad g_1, g_2 \in G$$

with $c(g_1, g_2) \in S^1$. It corresponds to a unitary representation $\tilde{T}$ of the central extension $\tilde{G}$ of G by S^1, defined by the cocycle c. The representation is defined by $\tilde{T}\binom{g}{z} := zT(g)$, where $z \in S^1, g \in G$. Then

$$\tilde{T}\left(\begin{pmatrix} g_1 \\ z_1 \end{pmatrix}\begin{pmatrix} g_2 \\ z_2 \end{pmatrix}\right) = z_1 z_2 c(g_1, g_2) T(g_1 g_2) = z_1 z_2 T(g_1) T(g_2) = \tilde{T}\begin{pmatrix} g_1 \\ z_1 \end{pmatrix}\tilde{T}\begin{pmatrix} g_2 \\ z_2 \end{pmatrix}.$$

Conversely, every unitary representation $\tilde{T}$ of $\tilde{G}$ such that $\tilde{T}(z) = z\,\mathrm{Id}$ for every $z \in S^1$, defines a projective representation of G. Let $S : G \to \tilde{G}$ be a splitting such that $c(g_1, g_2) = S(g_1)S(g_2)S(g_1 g_2)^{-1}$. We define $T(g) = \tilde{T}(S(g))$. Then

$$T(g_1 g_2) = \tilde{T}(S(g_1 g_2)) = \tilde{T}(c(g_1 g_2)^{-1} S(g_1) S(g_2)) = c(g_1, g_2)^{-1} T(g_1) T(g_2).$$

3.5. Coadjoint action in the extended group.

Let $\tilde{G}$ be the central extension of G by S^1 with the cocycle $c : G \times G \to S^1$. Setting $g_3 = e$ in the cocycle condition (2.4) we get $c(g_1 g_2, e) = c(g_2, e) = c(e, e) = 1$, and setting $g_1 = g_3 = g_2^{-1}$ we get $c(g, g^{-1}) = c(g^{-1}, g)$. The inverse in $\tilde{G}$ is

$$\begin{pmatrix} g \\ z \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} \\ z^{-1} c(g, g^{-1})^{-1} \end{pmatrix}, \quad g \in G, z \in S^1.$$

The conjugation in $\tilde{G}$ is

$$\operatorname{conj}\begin{pmatrix} g \\ z \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} g h g^{-1} \\ v c(g, h) c(g, g^{-1})^{-1} c(gh, g^{-1}) \end{pmatrix} \quad h \in G, v \in S^1.$$

The adjoint action in $\tilde{G}$ is

$$\begin{aligned} \operatorname{Ad}\begin{pmatrix} g \\ z \end{pmatrix}\begin{pmatrix} \xi \\ \lambda \end{pmatrix} &= T_e\left(\operatorname{Conj}\begin{pmatrix} g \\ z \end{pmatrix}\right)\begin{pmatrix} \xi \\ \lambda \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{Ad}(g)\xi \\ \lambda c(g,e) c(g,g^{-1})^{-1} c(g,g^{-1}) + [d_2 c(g,e) c(g,g^{-1})^{-1} c(g,g^{-1})]\xi + \\ +[c(g,e) c(g,g^{-1})^{-1} d_1 c(g,g^{-1})] T_e\lambda_g.\xi \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{Ad}(g)\xi \\ \lambda + [d_2 c(g,e) + c(g,g^{-1})^{-1} d_1 c(g,g^{-1}) T_e \lambda_g].\xi \end{pmatrix}, \text{ for } \xi \in \mathfrak{g},\ \lambda \in \mathbb{R}. \end{aligned}$$

We define $h : G \to \mathfrak{g}^*$ by

$$h(g^{-1}) := d_2 c(g, e) + c(g, g^{-1})^{-1} d_1 c(g, g^{-1}) T_e \lambda_g.$$

Because the adjoint action of $\tilde{G}$ is really an action of G, we write

$$\operatorname{Ad}(g)\begin{pmatrix} \xi \\ \lambda \end{pmatrix} = \begin{pmatrix} \operatorname{Ad}(g)\xi \\ \lambda + \langle h(g)^{-1}, \xi \rangle \end{pmatrix}$$

Here $\langle , \rangle$ denotes the pairing between the Lie algebra and its dual.

The coadjoint action is:

$$\begin{aligned}
\langle \mathrm{Ad}^*(g)(p,c), \begin{pmatrix}\xi\\ \lambda\end{pmatrix}\rangle &= \langle (p,c), \mathrm{Ad}(g^{-1})\begin{pmatrix}\xi\\ \lambda\end{pmatrix}\rangle \\
&= \langle (p,c), \begin{pmatrix}\mathrm{Ad}(g^{-1})\xi\\ \lambda + \langle h(g),\xi\rangle\end{pmatrix}\rangle \\
&= \langle p, \mathrm{Ad}(g^{-1})\xi\rangle + c\lambda + c\langle h(g),\xi\rangle \\
&= \langle (\mathrm{Ad}^*(g)p + ch(g), c), \begin{pmatrix}\xi\\ \lambda\end{pmatrix}\rangle, p\in\mathfrak{g}^*, c\in\mathbb{R}.
\end{aligned}$$

Hence

$$\mathrm{Ad}^*(g)(p,c) = (\mathrm{Ad}^*(g)p + ch(g), c).$$

Using the fact that Ad^* is a group action on $\tilde{\mathfrak{g}}^*$, we get that h is a 1-cocycle of G with values in the G-module $\mathfrak{g}^*$ via the coadjoint action.

$$h(g_1g_2) = \mathrm{Ad}^*(g_1)h(g_2) + h(g_1), \text{ so } [h]\in H^1_{\mathrm{Ad}^*}(G,\mathfrak{g}^*).$$

3.6. Another way to compute the coadjoint action in $\tilde{G}$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\omega : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ a fixed Lie algebra cocycle, this means $\omega\in Z^2(\mathfrak{g})\subset\wedge^2\mathfrak{g}^*$. The coadjoint action Ad^* induces an action of G on $\wedge^2\mathfrak{g}^*$ denoted also by Ad^* and then an action on $Z^2(\mathfrak{g})$, because every $\mathrm{Ad}(g)$ is a Lie algebra homomorphism.

Proposition [Kirillov, 1982]**.** *The mapping $\alpha : \tilde{\mathfrak{g}}^*\to Z^2(\mathfrak{g})$ defined by $\alpha(p,c) := dp - c\omega, p\in\mathfrak{g}^*, c\in\mathbb{R}$ is G-equivariant. This is equivalent to*

$$dh(g) = \omega - \mathrm{Ad}^*(g)\omega,$$

where h is the map defined in 3.5.

Proof. Because G is connected, it suffices to verify that α is a $\mathfrak{g}$-equivariant mapping. The ad^*-action on $\tilde{\mathfrak{g}}^*$ is $\mathrm{ad}^*(\xi)(p,c) = (\mathcal{L}_\xi p - ci_\xi\omega, 0)$. Indeed

$$\begin{aligned}
\langle \mathrm{ad}^*(\xi)(p,c), \begin{pmatrix}\eta\\ \mu\end{pmatrix}\rangle &= \langle (p,c), -\begin{pmatrix}[\xi,\eta]\\ \omega(\xi,\eta)\end{pmatrix}\rangle \\
&= -\langle p,[\xi,\eta]\rangle - c\omega(\xi,\eta) = \langle \mathrm{ad}^*(\xi)p - ci_\xi\omega, \eta\rangle.
\end{aligned}$$

The $\mathfrak{g}$-equivariance means

$$\begin{aligned}
&\alpha(\mathrm{ad}^*(\xi)(p,c)) = \mathcal{L}_\xi(\alpha(p,c)) \\
\Leftrightarrow\ &d(\mathcal{L}_\xi p - ci_\xi\omega) = \mathcal{L}_\xi(dp - c\omega)
\end{aligned}$$

and this is satisfied because d commutes with the Lie derivative and $(d\circ i_\xi)\omega = \mathcal{L}_\xi\omega$, ω being a cocycle.

So we get the G-equivariance, which means

$$\begin{aligned}
&\alpha(\mathrm{Ad}^*(g)(p,c)) = \mathrm{Ad}^*(g)(\alpha(p,c)) \\
&\Leftrightarrow \alpha(\mathrm{Ad}^*(g)p + ch(g), c) = \mathrm{Ad}^*(g)(dp - c\omega) \\
&\Leftrightarrow d(\mathrm{Ad}^*(g)p) + cdh(g) - c\omega = d\,\mathrm{Ad}^*(g)p - c\,\mathrm{Ad}^*(g)\omega \\
&\Leftrightarrow dh(g) = \omega - \mathrm{Ad}^*(g)\omega. \quad \square
\end{aligned}$$

It follows in particular that the cocycles ω and $\mathrm{Ad}^*(g)\omega$ belong to the same cohomology class.

Remark. If the Lie algebra $\mathfrak{g}$ obeys to the condition $H^1(\mathfrak{g}) = 0$, then $h(g) \in \mathfrak{g}^*$ is uniquely determined by $dh(g)$. The proposition and the results in 3.5 give the coadjoint action in $\tilde{G}$. This condition is also equivalent to the fact that $\mathfrak{g}$ is *perfect*, i.e. the commutator algebra is the whole algebra:

$$\begin{aligned}
f \in Z^1(\mathfrak{g}) &\Leftrightarrow f \in \mathfrak{g}^* \text{ and } \langle f, [\xi,\eta]\rangle = 0, \forall \xi,\eta \in \mathfrak{g} \\
&\Leftrightarrow f \in \mathfrak{g}^* \text{ and } f|[\mathfrak{g},\mathfrak{g}] \equiv 0
\end{aligned}$$

Hence $H^1(\mathfrak{g}) = Z^1(\mathfrak{g}) = 0 \Leftrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Examples of perfect Lie algebras:

–the semisimple algebras;

–$C_c^\infty(M,\mathfrak{h})$ with M a smooth manifold and $\mathfrak{h}$ a finite dimensional Lie algebra such that $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$ (see chapter 4);

–$\mathrm{Vect}_c(M)$, the Lie algebra of compactly supported vector fields on a smooth manifold M (see chapter 5);

–certain subalgebras of $\mathrm{Vect}(M)$ for a smooth manifold M (see chapter 9).

4. Current groups, loop groups

4.1. Some facts.

The *current group* $C_c^\infty(M,H)$ is the space of smooth mappings with compact support from a finite dimensional smooth manifold M into a finite dimensional Lie group H, i.e. mappings which equal constantly the unit element in H outside a compact set of M, with pointwise multiplication. This is an infinite dimensional Lie group: like in 2.2 we endow $C_c^\infty(M,H)$ with a smooth manifold structure and it remains only to show that multiplication and inversion are smooth

$$\begin{aligned}
\mu : (\varphi,\psi) \in C_c^\infty(M,H) \times C_c^\infty(M,H) &\mapsto \mu_H \circ (\varphi,\psi) \in C_c^\infty(M,H) \\
\nu : \varphi \in C_c^\infty(M,H) &\mapsto \nu_H \circ \varphi \in C_c^\infty(M,H).
\end{aligned}$$

We do this by showing that they map smooth curves into smooth curves. Let (c_1,c_2) be a smooth curve in $C_c^\infty(M,H) \times C_c^\infty(M,H)$. Then $(\mu \circ (c_1,c_2))\hat{} = \mu_H \circ (\hat{c}_1,\hat{c}_2) : \mathbb{R} \times M \to H$ is smooth and because of the compact support it has automatically property (*) from 2.2, hence $\mu \circ (c_1,c_2)$ is a smooth curve. The same is true for ν.

The Lie algebra is $C_c^\infty(M,\mathfrak{h})$ with the inductive limit topology of $C_K^\infty(M,\mathfrak{h})$, where K runs through all compact sets in M and each $C_K^\infty(M,\mathfrak{h})$ has the topology of uniform convergence on K in all derivatives separately. This is a convenient vector space and if M is compact it is a Fréchet space.

The *loop group* of H is by definition $LH := C^\infty(S^1,H)$ and its Lie algebra is denoted by $L\mathfrak{h}$. This is a Fréchet Lie group.

The exponential map $\exp : C_c^\infty(M,\mathfrak{h}) \to C_c^\infty(M,H)$ is just composition with the exponential map on H. Therefore it is smooth and it is a local diffeomorphism near the identity. Even if the exponential map of H is surjective, the exponential map on LH is not always surjective. An example is provided by $z \in S^1 \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix} \in$ SU(2). For if $g = \exp\xi$ for some $\xi \in L\mathfrak{h}$, then ξ must commute with g and hence ξ must be diagonal, i.e. $z \in S^1 \mapsto \begin{pmatrix} i\vartheta(z) & 0 \\ 0 & -i\vartheta(z)\end{pmatrix} \in \mathfrak{su}(2)$ but there is no smooth function $\vartheta : S^1 \to \mathbb{R}$ such that $e^{i\vartheta(z)} = z$. But if H is compact, the exponential image of $L\mathfrak{h}$ is dense in the identity component of LH.

In the same way one can show that $C_c^\infty(E)$, the space of smooth sections with compact support of a locally trivial bundle E with a Lie group G as fiber, is an infinite dimensional Lie group. An example is the Gauge group: Let (P,p,M,G) be a principal bundle with compact base space M; then G acts on P by the principal right action and on G by conjugation. The group of principal bundle automorphisms of P over the identity on M is called the Gauge group and it is isomorphic to $C^\infty(P,G)^G = C^\infty(P[G,\mathrm{conj}])$. Its Lie algebra is $C^\infty(P,\mathfrak{g})^G = C^\infty(P[\mathfrak{g},\mathrm{Ad}])$.

4.2. Central extensions of $L\mathfrak{h}$.

Let H be a compact connected Lie group and $\mathfrak{h}$ its Lie algebra. One can always find an H-invariant scalar product $\langle,\rangle_\mathfrak{h}$ on $\mathfrak{h}$ by choosing any scalar product on $\mathfrak{h}$ and integrating over H

$$\langle\xi,\eta\rangle_\mathfrak{h} = \int_H \langle \mathrm{Ad}^*(h)\xi, \mathrm{Ad}^*(h)\eta\rangle dh.$$

On the Lie algebra level the invariance of the scalar product is written

$$\langle[\xi,\eta],\zeta\rangle_\mathfrak{h} = \langle\xi,[\eta,\zeta]\rangle_\mathfrak{h}, \quad \forall \xi,\eta,\zeta \in \mathfrak{h}.$$

Proposition. *The form $\omega(\xi,\eta) := \int_{S^1}\langle\xi(t),\eta'(t)\rangle_\mathfrak{h} dt$ on $L\mathfrak{h}$ is a Lie algebra cocycle invariant under conjugation with constant loops and under the action of* $\mathrm{Diff}_+(S^1)$.

Proof.

1.The cocycle identity is $\omega([\xi,\eta],\zeta)+\omega([\eta,\zeta],\xi)+\omega([\zeta,\xi],\eta) = 0$. Using the bilinearity of the Lie bracket and then the invariance of the scalar product, we get

$$\begin{aligned}\omega([\xi,\eta],\zeta) &= -\int_{S^1}\langle\zeta,[\xi,\eta]'\rangle \\ &= -\int_{S^1}\langle\zeta,[\xi',\eta]\rangle_\mathfrak{h} - \int_{S^1}\langle\zeta,[\xi,\eta']\rangle_\mathfrak{h} \\ &= -\int_{S^1}\langle[\eta,\zeta],\xi'\rangle_\mathfrak{h} - \int_{S^1}\langle[\zeta,\xi],\eta'\rangle_\mathfrak{h} \\ &= -\omega([\eta,\zeta],\xi) - \omega([\zeta,\xi],\eta).\end{aligned}$$

2.The H-invariance of ω follows from the H-invariance of $\langle , \rangle_{\mathfrak{h}}$.

3.The $\mathrm{Diff}_+(S^1)$-invariance. The action of $\mathrm{Diff}_+(S^1)$ on $L\mathfrak{h}$ is by composition on the right. Then

$$\begin{aligned}\omega(\varphi\cdot\xi,\varphi\cdot\eta) &= \int_{S^1}\langle\xi(\varphi(t)),\eta'(\varphi(t))\varphi'(t)\rangle_{\mathfrak{h}}dt\\ &= \int_{S^1}\langle\xi(\vartheta),\eta'(\vartheta)\rangle_{\mathfrak{h}}\varphi'(\varphi^{-1}(\vartheta))|(\varphi^{-1})'(\vartheta)|d\vartheta = \omega(\xi,\eta),\end{aligned}$$

where $\varphi\in\mathrm{Diff}_+(S^1)$ and $\varphi' : S^1\to\mathbb{R}_+$ is defined by $T_t\varphi.\frac{d}{dt} = \varphi'(t).\frac{d}{dt}\big|_{\varphi(t)}$. $\square$

Theorem [Pressley-Segal, 1986]**.** *The only continuous H-invariant cocycles ω on $L\mathfrak{h}$, where H is compact and semisimple, are of the form*

$$\omega(\xi,\eta) = \int_{S^1}\langle\xi,\eta'\rangle_{\mathfrak{h}},\quad \xi,\eta\in\mathcal{L}\mathfrak{h}$$

with $\langle , \rangle_{\mathfrak{h}}$ an invariant scalar product on $\mathfrak{h}$.

First we need a

Lemma. *If H is a compact and semisimple Lie group, then every H-invariant bilinear map $\alpha : \mathfrak{h}_{\mathbb{C}}\times\mathfrak{h}_{\mathbb{C}}\to\mathbb{C}$ is symmetric.*

Proof. Let $\mathfrak{h} = \oplus\mathfrak{h}_i$ be the decomposition of the semisimple Lie algebra $\mathfrak{h}$ into a direct sum of simple Lie algebras. The H-equivariant linear map induced by α, denoted $\hat\alpha : \mathfrak{h}_{\mathbb{C}} = \oplus\mathfrak{h}_{i,\mathbb{C}}\to\mathfrak{h}^*_{\mathbb{C}} = \oplus\mathfrak{h}^*_{i,\mathbb{C}}$ decomposes, by Schur's lemma, (because the H-modules $\mathfrak{h}_i$ are non-isomorphic), into a sum of H-equivariant maps : $\hat\alpha = \oplus\hat\alpha_i$, $\hat\alpha_i : \mathfrak{h}_{i,\mathbb{C}}\to\mathfrak{h}^*_{i,\mathbb{C}}$.

We have reduced the problem to the case of a compact and simple group. Here the adjoint action of H on $\mathfrak{h}_{\mathbb{C}}$ is irreducible because the fact that $\mathfrak{h}_{\mathbb{C}}$ has no ideals implies that the ad-action, which is the differential of the Ad-action, is irreducible. Then the coadjoint action is also irreducible. By the Schur lemma all the H-equivariant linear maps : $\mathfrak{h}_{\mathbb{C}}\to\mathfrak{h}^*_{\mathbb{C}}$ differ by a complex factor. On the other hand there is a choice which corresponds to a symmetric map : $\mathfrak{h}_{\mathbb{C}}\times\mathfrak{h}_{\mathbb{C}}\to\mathbb{C}$. So any choice of α is symmetric. $\square$

Proof of the theorem. Let $\omega : L\mathfrak{h}\times L\mathfrak{h}\to\mathbb{R}$ be a cocycle and $\omega : L\mathfrak{h}_{\mathbb{C}}\times L\mathfrak{h}_{\mathbb{C}}\to\mathbb{C}$ its complex extension. Every $\xi\in L\mathfrak{h}$ can be expanded in a Fourier series $\xi(t) = \sum_{n\in\mathbb{Z}}\xi_n e^{int}$, $\xi_n\in\mathfrak{h}_{\mathbb{C}}$. The H-invariant, complex, bilinear maps $\omega_{p,q}$ on $\mathfrak{h}_{\mathbb{C}}$ defined by $\omega_{p,q}(\xi,\eta) = \omega(\xi e^{ipt},\eta e^{iqt})$ completely determine the cocycle ω. By the lemma, $\omega_{p,q}$ is symmetric. Together with the antisymmetry of ω this implies $\omega_{p,q} = -\omega_{q,p}$.

The cocycle identity gives

$$\omega_{p+q,r}([\xi,\eta],\zeta)+\omega_{q+r,p}([\eta,\zeta],\xi)+\omega_{r+p,q}([\zeta,\xi],\eta) = 0.$$

Applying the H-invariance of $\omega_{p,q}$, we get

$$(\omega_{p+q,r}+\omega_{q+r,p}+\omega_{r+p,q})([\xi,\eta],\zeta) = 0.$$

But the semisimplicity of $\mathfrak{h}$ assures that $[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$. Hence

$$\omega_{p+q,r}+\omega_{q+r,p}+\omega_{r+p,q}=0.$$

Setting $q=r=0$ we find $\omega_{p,0}=0$. Setting $r=-p-q$ we find $\omega_{p,-p}=p\omega_{1,-1}$. Setting $r=n-p-q$ we find $\omega_{n-k,k}=k\omega_{n-1,1}$. This implies $0=\omega_{0,n}=n\omega_{n-1,1}=\frac{n}{k}\omega_{n-k,k}$, so $\omega_{p,q}=0$ if $p+q\neq 0$. Now

$$\begin{aligned}\omega(\xi,\eta)&=\sum_{p,q}\omega_{p,q}(\xi_p,\eta_q)=\sum_p p\omega_{1,-1}(\xi_p,\eta_{-p})\\&=\frac{1}{i}\sum_{p,q}\delta(p+q)iq\omega_{1,-1}(\xi_p,\eta_q)=\frac{i}{2\pi}\int_0^{2\pi}\omega_{1,-1}(\xi(t),\eta'(t))dt\end{aligned}$$

is of the required form. □

In 3.6 we saw that for any Lie algebra cocycle ω, $[\mathrm{Ad}^*(g)\omega]=[\omega], g\in G$. Then by averaging ω over H, the group of constant loops, we get an H-invariant cocycle $\int_H \mathrm{Ad}^*(h)\omega dh$ in the same cohomology class. Hence the H-invariance is not really a restriction.

4.3. The coadjoint action of LH on $\widetilde{L\mathfrak{h}}$.

Let H be a connected compact Lie group. An invariant scalar product $\langle,\rangle_{\mathfrak{h}}$ on $\mathfrak{h}$ defines an LH-invariant form on $L\mathfrak{h}$: $\langle\xi,\eta\rangle=\int_o^{2\pi}\langle\xi(t),\eta(t)\rangle_{\mathfrak{h}}dt$ and a cocycle $\omega(\xi,\eta)=\langle\xi,\eta'\rangle$.

We compute the coadjoint action in the extended Lie algebra using the result from 3.6 and that $[L\mathfrak{h},L\mathfrak{h}]=L\mathfrak{h}$, which follows from the following proposition.

Proposition. *If $\mathfrak{h}$ is a finite dimensional Lie algebra with $[\mathfrak{h},\mathfrak{h}]=\mathfrak{h}$, then also $C_c^\infty(M,\mathfrak{h})=[C_c^\infty(M,\mathfrak{h}),C_c^\infty(M,\mathfrak{h})]$.*

Proof. Let $\xi_1,\dots,\xi_n$ be a basis of $\mathfrak{h}$ and $\xi\in C_c^\infty(M,\mathfrak{h})$. Every ξ_i is a linear combination $\sum_{j,k}a_i^{j,k}[\xi_j,\xi_k]$. Then

$$\begin{aligned}\xi(x)&=\sum_i c^i(x)\xi_i=\sum_{i,j,k}a_i^{jk}c^i(x)[\xi_j,\xi_k]\\&=\sum_{i,j,k}a_i^{jk}[c^i(x)\xi_j,\xi_k]=:\sum_{i,j,k}a_i^{jk}[\eta_j^i(x),\zeta_k(x)]\end{aligned}$$

hence ξ is a linear combination of brackets $[\eta_j^i,\zeta_k]$ in $C_c^\infty(M,\mathfrak{h})$. □

Remark. An analogous statement is true for compact semisimple Lie groups H: the identity component of $C_c^\infty(M,H)$ is a perfect Lie group.

We can realize H as a matrix group, since it is compact. Then the adjoint action of H is $\mathrm{Ad}(h)\xi = h\xi h^{-1}$ and we can write:

$$\begin{aligned}
\mathrm{Ad}^*(g)\omega(\xi,\eta) &= \omega(g^{-1}\xi g, g^{-1}\eta g) \\
&= \langle g^{-1}\xi g, (g^{-1}\eta g)'\rangle \\
&= \langle g^{-1}\xi g, g^{-1}\eta' g + g^{-1}\eta g' - g^{-1}g'g^{-1}\eta g\rangle \\
&= \langle \xi, \eta' + \eta g' g^{-1} - g'g^{-1}\eta\rangle \\
&= \omega(\xi,\eta) + \langle \xi, [\eta, g'g^{-1}]\rangle \\
&= \omega(\xi,\eta) + \langle g'g^{-1}, [\xi,\eta]\rangle.
\end{aligned}$$

We can identify $(\widetilde{L\mathfrak{h}})^*$ with $(L\mathfrak{h})^* \oplus \mathbb{R}$. The subspace of regular elements of $(L\mathfrak{h})^*$ is $L\mathfrak{h}^*$ and can be identified with $L\mathfrak{h}$ under $\langle , \rangle$. Then the preceding calculation gives $\mathrm{Ad}^*(g)\omega - \omega = d(g'g^{-1})$ and the result in 3.6 gives the coadjoint action of LH on the regular subspace of $(\widetilde{L\mathfrak{h}})^*$:

$$\mathrm{Ad}^*(g)(p,c) = (gpg^{-1} + cg'g^{-1}, c), \quad p \in L\mathfrak{h} = (\widetilde{L\mathfrak{h}})^*_{reg}, c \in \mathbb{R}.$$

In this case $h(g)$ is the right logarithmic derivative of g. We used that the coadjoint action on the regular dual equals the adjoint action because $\langle , \rangle$ is invariant.

4.4. About the smooth coadjoint orbits.

This paragraph follows the book [Pressley-Segal, 1986].

To a regular element $(p,c) \in L\mathfrak{h} \oplus \mathbb{R}$ in the dual of the extended algebra we associate an ordinary differential equation in H:

$$cu' = pu.$$

Let $u_0 : \mathbb{R} \to H$ be the solution with initial condition $u_0(0) = 1$. Then the solution with initial condition $u(0) = a$ is $u = u_0 a$, because $cu'(t) = cu_0'(t)a = p(t)u_0(t)a = p(t)u(t)$. The map $t \mapsto u(t+2\pi)$ is also a solution, so there exist $M_u \in H$, called the *monodromy*, such that $u(t+2\pi) = u(t)M_u$. Then $M_u = u(0)^{-1}u(2\pi)$. The monodromy matrix of another solution $v = ua$ is conjugated to M_u. The conjugacy class of the monodromy is an invariant of the differential equation $cu' = pu$.

We fix a nonzero real number c. There is a bijection Φ between $L\mathfrak{h}$ and the set $\{u : \mathbb{R} \to H : u(t+2\pi) = u(t)M, u(0) = 1, M \in H\}$ (this is the set of multivalued mappings on S^1 with values in H such that $u(0) = 1$). To every $p \in L\mathfrak{h}$ we assign the solution u_0 and conversely, to every map u we assign $cu'u^{-1}$, which is 2π-periodic, hence an element in $L\mathfrak{h}$.

$$p(t+2\pi) = cu'(t+2\pi)u^{-1}(t+2\pi) = cu'(t)M(u(t)M)^{-1} = cu'(t)u(t)^{-1} = p(t)$$

We recall that the coadjoint action of LH on $(\widetilde{L\mathfrak{h}})^*_{reg} = L\mathfrak{h} \oplus \mathbb{R}$ is

$$\mathrm{Ad}^*(g)(p,c) = (gpg^{-1} + cg'g^{-1}, c)$$

Under the bijection Φ, the element $\bar{p} = gpg^{-1} + cg'g^{-1} \in L\mathfrak{h}$ corresponds to the function $\bar{u}(t) = g(t)u(t)g(0)^{-1}$. Thus the monodromy matrix for $(\bar{p}, c)$ is $M_{\bar{u}} = \bar{u}(2\pi) = g(2\pi)u(2\pi)g(0)^{-1} = g(0)M_u g(0)^{-1}$, conjugate to the monodromy for (p,c).

Proposition. *The map π which assigns to every element $(p,c) \in L\mathfrak{h} \times \{c\}$ the monodromy conjugacy class of the associated differential equation*

$$\pi : L\mathfrak{h} \times \{c\} \to [H]$$

is the projection onto the space of coadjoint orbits.

Proof. 1.Claim: π is surjective. For every $M \in H$ we can find a path $u : [0,2\pi] \to H$ from 1 to M and then extend it to (a multivalued function on S^1) $u : \mathbb{R} \to H$ by $u(t+2\pi) = u(t)M$. Let $p \in L\mathfrak{h}$ be the element corresponding to u by the bijection Φ. Then $\pi(p,c) = [M]$.

2.Claim: $\pi^{-1}([M])$ is a coadjoint orbit. We saw that $\pi(\mathrm{Ad}^*(g)(p,c)) = \pi(\bar{p},c) = [M_{\bar{u}}] = [g(0)M_u g(0)^{-1}] = [M_u] = \pi(p,c)$. It remains to prove that $\pi^{-1}([M])$ contains only one coadjoint orbit. Let $M_1 = aMa^{-1}$ and $u_1, u_2 : \mathbb{R} \to H$ be multivalued functions on S^1, $u_1(0) = u_2(0) = 1$, with monodromy M_1, M and let $p_1, p \in L\mathfrak{h}$ the elements corresponding by Φ to u_1, u. We show that $\mathrm{Ad}^*(g)(p_1,c) = (p,c)$ for $g \in LH$ defined by $g(t) = u(t)a^{-1}u_1(t)^{-1}$. (The map g is really 2π-periodic: $g(t+2\pi) = u(t+2\pi)a^{-1}u_1(t+2\pi)^{-1} = u(t)Ma^{-1}M_1^{-1}u_1(t)^{-1} = u(t)a^{-1}u_1(t)^{-1} = g(t)$). Indeed:

$$\begin{aligned}
\mathrm{Ad}^*(g)(p_1,c) &= (gp_1g^{-1} + cg'g^{-1}, c) \\
&= (ua^{-1}u_1^{-1}cu_1'u_1^{-1}u_1au^{-1} + c(ua^{-1}u_1^{-1})'(ua^{-1}u_1^{-1})^{-1}, c) \\
&= (cu'u^{-1}, c) = (p,c).
\end{aligned}$$

Hence the space $[H]$ of conjugacy classes in H can be considered as the orbit space. $\square$

Corollary. *Every coadjoint orbit in the regular part of $(\tilde{L\mathfrak{h}})^*$ contains a constant element.*

Proof. The Lie group H is compact, hence exponential. Let $M_u = \exp\xi$, $\xi \in \mathfrak{h}$ be the monodromy of $u \leftrightarrow (p,c)$. We claim that (p_0,c) where p_0 is constant equal to $\frac{c}{2\pi}\xi$ is in the same coadjoint orbit as (p,c). By the proposition, we need only to verify that $\pi(p,c) = \pi(p_0,c)$. The solution u_0 of $cu_0' = p_0u_0 \Leftrightarrow u_0' = \frac{\xi}{2\pi}u_0$ with condition $u_0(0) = 1$ is $u_0(t) = \exp(t\frac{\xi}{2\pi})$. The monodromy $M_{u_0} = \exp\xi = M_u$. $\square$

Proposition. *The isotropy group $G_{(p,c)}$ for $(p,c) \in \tilde{\mathfrak{g}}^*_{reg}$ is isomorphic to the centralizer H_{M_u} of the monodromy matrix.*

Proof. The isomorphism is the evaluation at zero. The fact that g stabilizes (p,c) is equivalent to $g(t) = u(t)g(0)u(t)^{-1}$ and g is uniquely determined by its value at zero. Indeed $\mathrm{Ad}^*(g)(p,c) = (p,c) \Leftrightarrow \bar{u} = u \Leftrightarrow g(t)u(t)g(0)^{-1} = u(t)$. Then $g(0)$ is really in the centralizer of M_u.

The converse: If $g(0) \in H_{M_u}$, then g given by $g = ug(0)u^{-1}$ is 2π-periodic: $g(t+2\pi) = u(t+2\pi)g(0)u(t+2\pi)^{-1} = u(t)M_ug(0)M_u^{-1}u(t)^{-1} = u(t)g(0)u(t)^{-1} = g(t)$ and centralizes (p,c). $\square$

4.5. Finite dimensional coadjoint orbits in $C_c^\infty(M,H)$.

Let M be a smooth manifold and H a Lie group with Lie algebra $\mathfrak{h}$ being a perfect Lie algebra. The adjoint action of $G = C_c^\infty(M,H)$ on $\mathfrak{g} = C_c^\infty(M,\mathfrak{h})$ is pointwise. The dual of $\mathfrak{g}$ is $\mathfrak{g}^* = C^{-\infty}(M,\mathfrak{h}^*) = \mathcal{D}'(M) \otimes \mathfrak{h}^*$ the space of distributions on M with values in $\mathfrak{h}^*$. From 4.3 we get $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$.

Theorem [Kirillov, 1974]**.** *The coadjoint orbit $\mathcal{O}_\delta$ is finite dimensional if and only if the distribution $\delta \in \mathfrak{g}^*$ has finite support.*

Proof. Let $\delta \in \mathfrak{g}^*$ with support $\{x_1,\dots,x_N\}$. Then for every $g \in G$, $\mathrm{Ad}^*(g)\delta$ has as support a subset of $\{x_1,\dots,x_N\}$ and at every point the order of $\mathrm{Ad}^*(g)\delta$ is less or equal the order of δ. Hence

$$\dim \mathcal{O}_\delta \le \dim\{\delta \in C^{-\infty}(M,\mathfrak{h}^*) : \operatorname{supp}\delta \subset \{x_1,\dots,x_N\}, \operatorname{ord}_{x_i}\delta \le s\} =: d$$

where $s = \max\{\operatorname{ord}_{x_i}\delta : i = 1,\dots,N\} < \infty$. But d is the dimension of the space of Nn-tuples of polynomials of degree s in m-variables, where $m = \dim M$ and $n = \dim H$, hence finite. Namely $d = N\dim(\oplus_{j=0}^s L^j_{sym}(\mathbb{R}^m,\mathbb{R}^n)) = Nn\binom{m+s}{s}$. For the converse suppose that δ has infinite support. We show that for any natural number N, there exist N linearly independent vectors in the tangent space at δ to $\mathcal{O}_\delta$, which is $\mathrm{ad}^*(\mathfrak{g})\delta \subset \mathfrak{g}^*$. For this let $x_1,\dots,x_N \in \operatorname{supp}\delta$ and $U_1,\dots,U_N$ be disjoint open neighbourhoods of $x_1,\dots,x_N$. Then there exist $\xi_1,\dots,\xi_N \in \mathfrak{g}$ with $\operatorname{supp}\xi_i \subset U_i$ such that $\langle\delta,\xi_i\rangle \ne 0$. Because $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$, we can find $\eta_i^{(l)}, \zeta_i^{(l)} \in \mathfrak{g}$ with support in U_i, such that $\xi_i = \sum_{l=1}^{k_i}[\eta_i^{(l)},\zeta_i^{(l)}]$. For some $l_i \in \{1,\dots,k_i\}$ is

$$\langle \mathrm{ad}^*(\zeta_i^{(l_i)}).\delta, \eta_i^{(l_i)}\rangle = \langle\delta, [\eta_i^{(l_i)},\zeta_i^{(l_i)}]\rangle \ne 0$$

and so $\mathrm{ad}^*(\zeta_i^{(l_i)}).\delta$ are N linearly independent (having disjoint supports) vectors in $T_\delta\mathcal{O}_\delta$ for every natural N. This contradicts $\dim\mathcal{O}_\delta < \infty$. $\square$

5. The group of diffeomorphisms

5.1. The Lie group $\mathrm{Diff}(M)$.

Theorem [Kriegl-Michor]**.** *Let M be a smooth connected finite dimensional manifold. The group $G = \mathrm{Diff}(M)$ of all smooth diffeomorphisms of M is an infinite dimensional Lie group.*

Proof. $\mathrm{Diff}(M)$ is an open submanifold of $C^\infty(M,M)$, because it is an open subset of $C^\infty(M,M)$ for the Whitney C^∞-topology and the topology on $C^\infty(M,M)$ as smooth manifold is finer.

The composition μ and inversion ν are smooth, because they map smooth curves into smooth curves:

Let (c_1,c_2) be a smooth curve in $\mathrm{Diff}(M)\times\mathrm{Diff}(M)$. Then $(\mu\circ(c_1,c_2))\hat{}(t,x) = \hat{c}_1(t,\hat{c}_2(t,x))$ is smooth as composition of smooth maps. It also has property (*) from 2.2: Let $[a,b] \subset \mathbb{R}$ and K_1, K_2 compact subsets of M such that $\hat{c}_i(t,x) = f_i(x)$, $i = 1,2$, doesn't depend on t for $x \in M - K_i$ by property (*) of $\hat{c}_1,\hat{c}_2$. Because

$c_2(t)$ is a diffeomorphism, $f_2(M-K_1)\cap(M-K_2)$ is the complement of a compact set K and $\hat{c}_1(t,\hat{c}_2(t,x)) = f_1\circ f_2(x)$ for $t\in[a,b]$, $x\in M-K$. So $\mu\circ(c_1,c_2)$ is again a smooth curve.

Let c be a smooth curve in $\operatorname{Diff}(M)$. Then $(\nu\circ c)\hat{}$ fulfills the implicit equation $\hat{c}(t,(\nu\circ c)\hat{}(t,x)) = x$. By the finite dimensional implicit function theorem $(\nu\circ c)\hat{}$ is smooth. It also satisfies (*): Let $[a,b]\subset\mathbb{R}$ and a compact K in M such that $\hat{c}(t,x) = f(x)$ for $t\in[a,b]$, $x\in M-K$. Because $c(t)$ is a diffeomorphism, $f(M-K)$ is again the complement of a compact set in M and for $t\in[a,b]$ and $x\in f(M-K)$ we have $(\nu\circ c)\hat{}(t,x) = f^{-1}(x)$ doesn't depend on t. So $\nu\circ c$ is again a smooth curve.

Hence $\operatorname{Diff}(M)$ is a Lie group having $\operatorname{Vect}_c(M)$ as tangent space at the identity. $\square$

The same is true for the space $\operatorname{Diff}_c(M)$ of diffeomorphisms of M which equal to the identity outside a compact set. $\operatorname{Diff}(M)$ only has more connected components.

Proposition. *The exponential map associates to a vector field with compact support its flow at time 1 and is smooth. The image of the exponential map lies in* $\operatorname{Diff}_c(M)$.

Proof. Let $\xi\in T_{\mathrm{Id}_M}\operatorname{Diff}(M) = \operatorname{Vect}_c(M)$. Because $\operatorname{supp}\xi$ is compact, the vector field ξ is complete, hence its flow exists $\mathrm{Fl}^\xi_t\in\operatorname{Diff}_c(M)$ at every time $t\in\mathbb{R}$. The defining differential equation for $\exp t\xi$ is

$$\begin{cases} \frac{d}{dt}\exp t\xi = T(\mu^{\exp t\xi}).\xi \\ \exp 0\xi = \mathrm{Id}_M\,. \end{cases}$$

Here μ^φ is the right translation in $\operatorname{Diff}_c(M)$ by φ, which is linear, so $T(\mu^\varphi).\xi = \xi\circ\varphi$. Evaluating the differential equation on $x\in M$, we obtain the defining equation for $\mathrm{Fl}^\xi_t(x)$, hence $\exp t\xi = \mathrm{Fl}^\xi_t$.

The exponential mapping $\exp:\operatorname{Vect}_c(M)\to\operatorname{Diff}(M)$ is smooth, because it maps smooth curves into smooth curves. Indeed, a smooth curve $\xi:\mathbb{R}\to\operatorname{Vect}_c(M)$ is a time dependent vector field with compact support K. The map $(\exp\circ\xi)\hat{}(t,x) = \mathrm{Fl}^{\xi(t)}_1(x)$ is smooth because $s\mapsto\mathrm{Fl}^{\xi(t)}_s(x)$ is the solution of an ordinary differential equation with smooth parameter t and initial condition x. Moreover $(\exp\circ\xi)\hat{}(t,x) = x$ for $x\in M-K$, hence property (*) is satisfied. Hence $\exp\circ\xi$ is smooth. $\square$

Proposition. *The Lie bracket in* $\operatorname{Vect}_c(M)$ *is just the negative of the usual bracket of vector fields.*

Proof. The adjoint action of $\operatorname{Diff}_c(M)$ on $\operatorname{Vect}_c(M)$ is

$$\operatorname{Ad}(\varphi)\xi = \frac{d}{dt}\Big|_0 \varphi\circ\mathrm{Fl}^\xi_t\circ\varphi^{-1} = T\varphi\circ\xi\circ\varphi^{-1} = (\varphi^{-1})^*\xi.$$

By the theorem in 2.3 due to Grabowski

$$\operatorname{ad}(\eta)\xi = \frac{d}{dt}\Big|_0 \operatorname{Ad}(\mathrm{Fl}^\eta_t)\xi = \frac{d}{dt}\Big|_0 (\mathrm{Fl}^\eta_{-t})^*\xi = -[\eta,\xi]$$

is the Lie bracket in $\operatorname{Vect}_c(M)$. $\square$

5.2. Some properties of $\mathrm{Diff}_c(M)$**.**

The dual of $\mathfrak{g} = \mathrm{Vect}_c(M)$ is $\mathfrak{g}^* = \Omega^1(M) \underset{C^\infty(M)}{\otimes} \mathcal{D}'(M)$, where $\mathcal{D}'(M)$ denotes the space of distributions on M, and is also called the *space of moments*. The coadjoint action is $\mathrm{Ad}^*(\varphi) = (\varphi^{-1})^*$. Indeed, for $\alpha \in \Omega^1(M)$ and $\delta \in \mathcal{D}'(M)$ we have

$$\begin{aligned}
\langle \mathrm{Ad}^*(\varphi)(\alpha \otimes \delta), \xi\rangle &= \langle \alpha \otimes \delta, \varphi^*\xi\rangle \\
&= \langle \delta, \alpha(\varphi^*\xi)\rangle \\
&= \langle \delta, \varphi^*(((\varphi^{-1})^*\alpha)(\xi))\rangle \\
&\overset{def}{=} \langle (\varphi^{-1})^*\delta, ((\varphi^{-1})^*\alpha)(\xi)\rangle \\
&= \langle (\varphi^{-1})^*\alpha \otimes (\varphi^{-1})^*\delta, \xi\rangle \\
&= \langle (\varphi^{-1})^*(\alpha \otimes \delta), \xi\rangle, \forall \xi \in \mathfrak{g}.
\end{aligned}$$

We define the regular dual $\mathfrak{g}^*_{reg} = \Omega^1(M) \underset{C^\infty(M)}{\otimes} \Omega^n(M)$ and the space of moments with finite support $\mathfrak{g}^*_f = \Omega^1(M) \underset{C^\infty(M)}{\otimes} \mathcal{D}'_f(M)$, where $\mathcal{D}'_f(M)$ denotes the space of distributions with finite support. Both are G-invariant subspaces of $\mathfrak{g}^*$.

Proposition [Kirillov, 1982]**.** *Every coadjoint orbit of* $\mathrm{Diff}_c(M)$ *has infinite codimension in* $\mathfrak{g}^*$.

Proof. The spaces $\mathfrak{g}_{reg^*}$ and $\mathfrak{g}^*_f$ are transversal in $\mathfrak{g}^*$ and their intersection is 0. Both are G-invariant, infinite dimensional and dense in $\mathfrak{g}^*$ (in the weak topology).

If the orbit intersects $\mathfrak{g}^*_{reg}$, then it is contained in $\mathfrak{g}^*_{reg}$ and its complement contains $\mathfrak{g}^*_f$, hence the complement is infinite dimensional. Analogously for orbits in $\mathfrak{g}^*_f$, if the orbit doesn't meet $\mathfrak{g}^*_{reg} \cup \mathfrak{g}^*_f$, then it is contained in the complement of the orbit, and so the orbit again has infinite codimension. □

Proposition. *Let M be a connected smooth manifold of dimension $\dim M \geq 2$. Then the group $\mathrm{Diff}_c(M)$ of all smooth diffeomorphisms with compact support acts n-transitively on M, for each finite n. I.e. for any two ordered sets of n different points $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in M there is a smooth diffeomorphism φ with compact support such that $\varphi(x_i) = y_i$ for each i.*

Proof. Let $M^{(n)}$ denote the open submanifold of all n-tuples $(x_1, \dots, x_n) \in M^n$ of pairwise distinct points. $\mathrm{Diff}_c(M)$ acts on $M^{(n)}$ by the diagonal action, and we have to show, that this action is transitive.

Let us first assume that $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are pairwise disjoint. For some $\varepsilon > 0$ let $c_i : (-\varepsilon, 1 + \varepsilon) \to M$ be smooth curves with $c_i(0) = x_i$ and $c_i(1) = y_i$ which are embeddings and do not intersect each other. From a drawing it can be seen that this exists if $\dim M \geq 2$, since $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are disjoint. We choose pairwise disjoint tubular neighborhoods of $c_i(-\varepsilon, 1+\varepsilon)$, extend the velocity vector fields of the curves to them, and use a smooth bump function to obtain a vector field ξ on M with compact support which coincides with the velocity vector field $c' \circ c^{-1}$ along each curve c_i. Then the flow mapping Fl^ξ_1 maps each x_i to y_i.

This argument shows that each $\mathrm{Diff}_c(M)$-orbit in $M^{(n)}$ is dense. We may replace in the argument the points y_i by points z_i in small open pairwise disjoint neighborhoods U_i of y_i, not meeting $\{x_1, \dots, x_n\}$. Then the argument shows that each orbit contains an open set in $M^{(n)}$, thus is open. Since the dimension of M is at least 2, $M^{(n)}$ is connected, so there is only one orbit and the result follows. □

Proposition. *For every smooth manifold M, $\mathrm{Vect}(M)$ and $\mathrm{Vect}_c(M)$ are perfect Lie algebras.*

Proof. By dimension theory it follows that there exists a number $p \le \dim M + 1$ and an open cover with coordinate domains $\mathcal{O} = \{U_\nu : \nu \in J = J_1 \cup \cdots \cup J_p \text{ partition}\}$ such that the open sets $(U_\nu)_{\nu \in J_k}$ are pairwise disjoint. Let $(\rho_\nu)_{\nu \in J}$ be a partition of unity subordinated to $\mathcal{O}$ and $(\varphi_\nu)_{\nu \in J}$ smooth functions on M with support in U_ν and identically 1 in a neighbourhood of $\operatorname{supp} \rho_\nu$.

Let ξ be an arbitrary vector field on M and $\xi|U_\nu = \sum_{k=1}^n \xi^k \frac{\partial}{\partial x^k}$, where all $\xi^k \in C^\infty(U_\nu)$. Setting

$$\eta_\nu^k := \varphi_\nu \frac{\partial}{\partial x^k}, \zeta_\nu^k := \varphi_\nu \big(\int_0^{x^k} \rho_\nu \xi^k \frac{1}{\varphi_\nu^2} dx^k \big) \frac{\partial}{\partial x^k}$$

we get

$$[\eta_k^\nu, \zeta_k^\nu] = [\varphi_\nu \frac{\partial}{\partial x^k}, \varphi_\nu (\int_0^{x^k} \rho_\nu \xi^k \frac{1}{\varphi_\nu^2} dx^k) \frac{\partial}{\partial x^k}] = \varphi_\nu^2 (\rho_\nu \xi^k \frac{1}{\varphi_\nu^2}) \frac{\partial}{\partial x^k} = \rho_\nu \xi^k \frac{\partial}{\partial x^k}.$$

The integral is well defined because $\operatorname{supp} \rho_\nu \subset \operatorname{supp} \varphi_\nu$. Then

$$\rho_\nu \xi = \sum_{k=1}^n [\eta_\nu^k, \zeta_\nu^k]$$

and

$$\xi = \sum_{\nu \in J} \rho_\nu \xi = \sum_{i=1}^p \sum_{\nu \in J_i} \sum_{k=1}^n [\eta_\nu^k, \zeta_\nu^k] = \sum_{i=1}^p \sum_{k=1}^n [\eta_i^k, \zeta_i^k]$$

where $\eta_i^k = \sum_{\nu \in J_i} \eta_\nu^k$ and $\zeta_i^k = \sum_{\nu \in J_i} \zeta_\nu^k$. This shows that ξ is in the commutator algebra of $\mathrm{Vect}(M)$. □

There is a similar result on the group level: the connected component of the identity in $\mathrm{Diff}_c(M)$, i.e. the group of those diffeomorphisms with compact support isotopic to the identity by an isotopy with compact support, is a simple group.

5.3. Special cases: the circle and the torus.

Let $p : t \in \mathbb{R} \mapsto e^{it} \in S^1$ be the universal covering group of S^1.

Proposition. *The space*

$$\widetilde{\mathrm{Diff}}(S^1) := \{\tilde\varphi : \mathbb{R} \to \mathbb{R} : p \circ \tilde\varphi = \varphi \circ p, \varphi \in \mathrm{Diff}(S^1)\}$$

coincides with

$$\{\tilde\varphi \in \mathrm{Diff}(\mathbb{R}) : \tilde\varphi(t + 2\pi) = \tilde\varphi(t) \pm 2\pi, t \in \mathbb{R}\}$$

and defines with the projection $P : \tilde{\varphi} \mapsto \varphi$ the universal covering group of $\mathrm{Diff}(S^1)$.

Proof. The fiber over $\varphi \in \mathrm{Diff}(S^1)$ consists of all lifts to $\mathbb{R}$ of the function $\varphi \circ p : \mathbb{R} \to S^1$. These lifts are also smooth functions and the difference between any two of them is an integer multiple of 2π. Each fiber is then isomorphic to $\mathbb{Z}$. In general $\widetilde{\varphi^{-1}} \circ \tilde{\varphi} = \widetilde{\mathrm{Id}}$ is a translation by an integral multiple of 2π. But for every $\tilde{\varphi}$ over φ we can find a $\widetilde{\varphi^{-1}}$ such that $\widetilde{\varphi^{-1}} = \tilde{\varphi}^{-1}$. Hence $\tilde{\varphi} \in \mathrm{Diff}(\mathbb{R})$.

We have $p \circ \tilde{\varphi}(t+2\pi) = \varphi(p(t+2\pi)) = \varphi(p(t)) = p \circ \tilde{\varphi}(t)$ and using the continuity of $\tilde{\varphi}$ we get that there exists a $k \in \mathbb{Z}$ such that $\tilde{\varphi}(t+2\pi) = \tilde{\varphi}(t) + 2k\pi$ for all t in $\mathbb{R}$. But $\tilde{\varphi}$ is a diffeomorphism, so there must be an $s \in \mathbb{R}$ with $\tilde{\varphi}(s) = \tilde{\varphi}(t) + 2\pi$. We get $s = t + 2m\pi$ for some integer m and $\tilde{\varphi}(s) = \tilde{\varphi}(t + 2m\pi) = \tilde{\varphi}(t) + 2mk\pi$ which implies $mk = 1$. Hence $k = \pm 1$, $+1$ for orientation preserving diffeomorphisms and -1 for orientation reversing.

$\widetilde{\mathrm{Diff}}(S^1)$ has two connected components, each one is a convex set, hence it is simply connected.

It remains to observe that P is a local homeomorphism. □

The reference for $\mathrm{Diff}(T^2)$ is [Juriev, 1993]. Let $\Gamma = \mathbb{Z}a + \mathbb{Z}b$, with a, b two independent vectors in $\mathbb{R}^2$ be a lattice and $p : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ the projection. $T^2 = \mathbb{R}^2/\Gamma$ is a two dimensional torus and p its universal covering.

Proposition. *The space*

$$\widetilde{\mathrm{Diff}}(T^2) := \{\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2 : p \circ \tilde{\varphi} = \varphi \circ p, \quad \varphi \in \mathrm{Diff}(T^2)\}$$

coincides with

$$\{\tilde{\varphi} \in \mathrm{Diff}(\mathbb{R}^2) : \tilde{\varphi}(x + ma + nb) = \tilde{\varphi}(x) + ma_1 + nb_1, \\ x \in \mathbb{R}^2, m, n \in \mathbb{Z}^2, \Gamma = \mathbb{Z}a_1 + \mathbb{Z}b_1\}$$

and the projection $P : \tilde{\varphi} \in \widetilde{\mathrm{Diff}}(T^2) \mapsto \varphi \in \mathrm{Diff}(T^2)$ is the universal covering of $\mathrm{Diff}(T^2)$.

Proof. Every $\tilde{\varphi}$ is a lifting of $\varphi \circ p$; for a given $\varphi \in \mathrm{Diff}(T^2)$, there are $\mathbb{Z}^2$ such liftings because p has $\mathbb{Z}^2$ sheets. The fiber over φ^{-1} consists of the inverses of the elements in the fiber over φ, hence $\widetilde{\mathrm{Diff}}(T^2)$ is a subgroup of $\mathrm{Diff}(\mathbb{R}^2)$ (we use the fact that all the liftings of the smooth function $\varphi \circ p$ are smooth).

For every $x \in \mathbb{R}^2$ we have $p \circ \tilde{\varphi}(x + a) = \varphi(x + a) = \varphi(p(x)) = p \circ \tilde{\varphi}(x)$ and the same for b. Hence there exist $a_1, b_1 \in \Gamma$ such that $\tilde{\varphi}(x + a) = \tilde{\varphi}(x) + a_1$ and $\tilde{\varphi}(x + b) = \tilde{\varphi}(x) + b_1$ and then for every $m, n \in \mathbb{Z}$

$$\tilde{\varphi}(x + ma + nb) = \tilde{\varphi}(x) + ma_1 + nb_1.$$

Now we show that a_1, b_1 is a pair of generators too. Suppose there exists an element $pa + qb$ in $\Gamma \setminus \mathbb{Z}a_1 + \mathbb{Z}b_1$. Because $\tilde{\varphi}$ is a diffeomorphism, we can find a $y \in \mathbb{R}^2$ such that $\tilde{\varphi}(y) = \tilde{\varphi}(x) + pa + qb$. Then $y = x + ma + nb$ for some $m, n \in \mathbb{Z}$ and so $\tilde{\varphi}(y) = \tilde{\varphi}(x) + ma_1 + nb_1$. We obtain a contradiction: $pa + qb = ma_1 + nb_1$, hence $\Gamma = \mathbb{Z}a_1 + \mathbb{Z}b_1$.

Every two pairs of generators of Γ differ by an element in $\mathrm{SL}(2,\mathbb{Z})$. $\widetilde{\mathrm{Diff}}(T^2)$ has as many connected components as Γ has independent generators. Each connected component is a convex set , hence $\widetilde{\mathrm{Diff}}(T^2)$ is simply connected.

Every fiber is isomorphic to $\mathbb{Z}^2$ and P is a local homeomorphism. $\square$

$\mathrm{Diff}(S^1)$ has two connected components, the one containing the identity is exactly $\mathrm{Diff}_+(S^1)$, the orientation preserving diffeomorphisms. The fundamental group of $\mathrm{Diff}_+(S^1)$ is isomorphic to $\mathbb{Z}$. Moreover, the inclusion $S^1 \cong \mathrm{Rot}(S^1) \subset \mathrm{Diff}_+(S^1)$ is a homotopy equivalence.

Let $\mathrm{Diff}_e(T^2)$ be the connected component of $\mathrm{Diff}(T^2)$ which contains the identity; this is a normal subgroup and the quotient is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. Its universal cover is the connected component of $\widetilde{\mathrm{Diff}}(T^2)$ which contains the identity:

$$\widetilde{\mathrm{Diff}}_e(T^2) = \{\tilde\varphi \in \mathrm{Diff}(\mathbb{R}^2) : \tilde\varphi(x+ma+nb) = \tilde\varphi(x)+ma+nb, \quad m,n\in\mathbb{Z}\}$$

There is no canonical inclusion of the rigid transformations T^2 in $\mathrm{Diff}_e(T^2)$, like in the case of the circle. Every splitting of the following exact sequence realises T^2 as a group of movements of the torus.

$$0 \longrightarrow \mathrm{Diff}_0(T^2) \longrightarrow \mathrm{Diff}_e(T^2) \xrightarrow{\gamma} T^2 \longrightarrow 0 \quad \text{where}$$

$$\mathrm{Diff}_0(T^2) := P\Big\{\tilde\varphi \in \mathrm{Diff}_e(T^2) : \int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} \tilde\varphi(x+t_1a+t_2b)dt_1dt_2 - x \in \Gamma\Big\} \quad \text{and}$$

$$\gamma(\varphi) := p\Big(\int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} \tilde\varphi(t_1a+t_2b)dt_1dt_2\Big) \in T^2.$$

We prove now the exactness of this sequence.

1.Claim: γ is surjective. For $\tilde\varphi(x) = x+c, c\in\mathbb{R}^2$ we get $\gamma(\varphi) = p(c)$.

2.Claim: $\ker\gamma = \mathrm{Diff}_0(T^2)$.

$$\ker\gamma = \big\{\varphi : \exists\tilde\varphi \text{ s.t.} \int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} \tilde\varphi(t_1a+t_2b)dt_1dt_2 \in \Gamma\big\}$$

$$= \big\{\varphi : \exists\tilde\varphi \text{ s.t.} \int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} \tilde\varphi(x+t_1a+t_2b)dt_1dt_2 - x \in \Gamma\big\} = \mathrm{Diff}_0(T^2),$$

because

$$\int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} \tilde\varphi(x+t_1a+t_2b)dt_1dt_2 = x + \int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12} \tilde\varphi(t_1a+t_2b)dt_1dt_2,$$

for all $\tilde\varphi$ in $\widetilde{\mathrm{Diff}}_e(T^2)$.

5.4. Anomaly.

The smooth manifold structure of $\mathrm{Diff}_c(M)$ cannot be described by using the exponential map, because [Pressley-Segal, 1986]:

Proposition [Koppell, 1970], [Freifeld, 1967]. *The exponential map in* $\mathrm{Diff}_+(S^1)$*, the group of orientation preserving diffeomorphisms of the circle, is neither locally injective, nor locally surjective.*

Proof. Let $\mathrm{Rot}(S^1) \subset \mathrm{Diff}_+(S^1)$ be the subgroup of rigid rotations and let R_t denote the rotation by t.
(inj) The centralizer of $R_{\frac{2\pi}{n}}$ is

$$H = \{\varphi \in \mathrm{Diff}_+(S^1) : R_{\frac{2\pi}{n}}^{-1} \circ \varphi \circ R_{\frac{2\pi}{n}} = \varphi\}$$

The 1-parameter subgroup of $\frac{d}{dt}$ is $\mathrm{Rot}(S^1)$, then the 1-parameter subgroup corresponding to $\varphi^* \frac{d}{dt}$ is $\varphi \circ \mathrm{Rot}(S^1) \circ \varphi^{-1}$. Hence $R_{\frac{2\pi}{n}} \in \varphi \circ \mathrm{Rot}(S^1) \circ \varphi^{-1}$ for every $\varphi \in H$, i.e. $R_{\frac{2\pi}{n}} = \exp(t_\varphi.\varphi^*\xi)$. Making $n \to \infty$ we see that exp is not locally injective.
(surj) A 1-parameter subgroup of $\mathrm{Diff}_+(S^1)$ without stationary points is always conjugate to $\mathrm{Rot}(S^1)$. Indeed, let $\eta = v(t)\frac{d}{dt}$ with $v(t) \neq 0, \forall t \in [0, 2\pi]$, we assume $v > 0$. Define $\psi(t) := \frac{2\pi}{\int_0^{2\pi} v(s)^{-1} ds} \int_0^t v(s)^{-1} ds$. Then $\psi \in \mathrm{Diff}_+(S^1)$ and $\psi^*\eta = T\psi \circ \eta \circ \psi^{-1} = \frac{2\pi}{\int_0^{2\pi} v(s)^{-1} ds} \frac{d}{dt}$ is a constant vector field on S^1. Hence the 1-parameter subgroup generated by η is conjugated to $\mathrm{Rot}(S^1)$.

A diffeomorphism φ with the following properties is not in the image of the exponential map:
-φ has no fixed points,
-φ has a point of order n,
-$\varphi^n \neq \mathrm{Id}$.
Indeed, suppose $\varphi = \exp \eta$. The 1-parameter subgroup generated by η is conjugate to the subgroup of rotations, because it has no stationary points. Hence $\varphi = \psi R_t \psi^{-1}$ for some diffeomorphism ψ and then $\varphi^n \neq \mathrm{Id}$ is conjugated with the rotation by nt. This contradicts the fact that φ has a point of order n.

Diffeomorphisms with the three properties exist arbitrarily close to the identity. Let $\tilde\varphi_n \in \widetilde{\mathrm{Diff}}_+(S^1)$ be defined by

$$\tilde\varphi_n(t) := t + \frac{2\pi}{n} + \varepsilon \sin nt,$$

and φ_n the diffeomorphism of S^1 induced by $\tilde\varphi_n$. In this case $\tilde\varphi_n$ has no fixed points for small ε, $(\tilde\varphi_n)^n(0) = 0$ and $((\tilde\varphi_n)^n)'(0) = (1 + n\varepsilon)^n \neq 0$ hence φ^n is not the identity. □

In [Grabowski, 1988] it is shown that $\mathrm{Diff}(M)$ contains an arcwise connected free subgroup with a continuous set of generators, which meets the image of exp only at the identity.

5.5. $\mathrm{Diff}_+(S^1)$ **does not have a complexification.**

The group $\mathrm{Diff}_+(S^1)$, like the universal cover of $\mathrm{SL}(2, \mathbb{R})$, has no complexification. Moreover, every homomorphism from $\mathrm{Diff}_+(S^1)$ into a complex Lie group is trivial.

The group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\} = \mathrm{GL}(2,\mathbb{R})/\mathbb{R}\cdot 1$ of projective transformations is contained in $\mathrm{Diff}_+(S^1)$, because S^1 can be regarded as the real projective line $\mathrm{P}\,\mathbb{R}^1$. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ defines the diffeomorphism:

$$A : [x,y] \in \mathrm{P}\,\mathbb{R}^1 = S^1 \mapsto [ax+by, cx+dy] \in \mathrm{P}\,\mathbb{R}^1 = S^1.$$

By the identification of $\mathrm{P}\,\mathbb{R}^1$ with $\mathbb{R}\cup\infty$, the projective transformations become Moebius transformations

$$\begin{array}{ccc} [x,y] & \xrightarrow{\quad A \quad} & [ax+by, cx+dy] \\ \Big\downarrow & & \Big\downarrow \\ \lambda = \dfrac{x}{y} & \xrightarrow{\quad A \quad} & \dfrac{ax+by}{cx+dy} = \dfrac{a\lambda+b}{c\lambda+d}. \end{array}$$

The universal cover of $\mathrm{PSL}(2,\mathbb{R})$ is $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and can be viewed as a subgroup of the universal covering group $\widetilde{\mathrm{Diff}}_+(S^1)$ of $\mathrm{Diff}_+(S^1)$, namely all those diffeomorphisms of $\mathbb{R}$ which cover projective transformations

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) = \{\tilde\varphi \in \mathrm{Diff}(\mathbb{R}) : p\circ\tilde\varphi = A\circ p, A \in \mathrm{PSL}(2,\mathbb{R})\}$$

$$\begin{array}{ccc} \widetilde{\mathrm{SL}}(2,\mathbb{R}) & \hookrightarrow & \widetilde{\mathrm{Diff}}_+(S^1) \\ P\Big\downarrow & & P\Big\downarrow \\ \mathrm{PSL}(2,\mathbb{R}) & \hookrightarrow & \mathrm{Diff}_+(S^1). \end{array}$$

A covering diffeomorphism $\tilde\varphi_A \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ of a projective transformation A (identified with a Moebius transformation of $\mathbb{R}\cup\infty$) can be obtained from A via the diffeomorphism

$$f : t \in (0,2\pi) \mapsto \mathrm{ctg}\,\frac{t}{2} \in \mathbb{R},$$

namely by extending $f^{-1}\circ A\circ f$ to a diffeomorphism of $\mathbb{R}$ using $\tilde\varphi_A(t+2\pi) = \tilde\varphi_A(t)+2\pi$. Each element $\tilde\varphi \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ is uniquely determined by its restriction to an open intervall.

We define $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ as the n-fold covering group of $\mathrm{PSL}(2,\mathbb{R})$. More precise, if $p_n : S^1 \to S^1, p_n(z) = z^n$ is the n-fold covering map of the circle, then

$$\mathrm{PSL}^{(n)}(2,\mathbb{R}) := \{\varphi \in \mathrm{Diff}_+(S^1) : p_n\circ\varphi = A\circ p_n, A \in \mathrm{PSL}(2,\mathbb{R})\}.$$

Lemma. *$\mathrm{PSL}^{(n)}(2,\mathbb{R}) \cong \widetilde{\mathrm{SL}}(2,\mathbb{R})/n\mathbb{Z}$, where $\mathbb{Z} \cong \{\tilde\varphi(t) = t+2k\pi, k\in\mathbb{Z}\}$, the center of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (diffeomorphisms which cover the identity on S^1).*

Proof. We define a surjective homomorphism q_n from $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ to $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ with kernel $n\mathbb{Z} = \{t \mapsto t + 2kn\pi, k \in \mathbb{Z}\}$.

$$\begin{tikzcd}
\mathbb{R} \arrow[rrr, "\tilde{\varphi}"] \arrow[dd] \arrow[dr] & & & \mathbb{R} \arrow[dl, "e^{it}"'] \arrow[dd, "e^{i\frac{t}{n}}"'] \\
& S^1 \arrow[r, "A"] & S^1 & \\
S^1 \arrow[ur] \arrow[rrr, "\varphi"] & & & S^1 \arrow[ul, "z^n"]
\end{tikzcd}$$

Every $\tilde{\varphi} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ is a cover of some $A \in \mathrm{PSL}(2,\mathbb{R})$. We assign to $\tilde{\varphi}$ an n-fold cover of A by projecting it down to $\varphi \in \mathrm{Diff}_+(S^1)$ via $t \in \mathbb{R} \mapsto e^{i\frac{t}{n}} \in S^1$ which is a universal cover of S^1. The elements which project to the identity on S^1 are exactly the translations by $2kn\pi$ with $k \in \mathbb{Z}$. $\square$

Lemma. *The kernel of any homomorphism from* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *into a complex Lie group contains* $2.\,\mathrm{Center}(\widetilde{\mathrm{SL}}(2,\mathbb{R})) = \{t \mapsto t + 4k\pi\}$.

Proof. Let $f : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to G_{\mathbb{C}}$ be a homomorphism and $f' : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_{\mathbb{C}}$. Then, because the complexification of $\mathrm{SL}(2,\mathbb{R})$ is the simply connected Lie group $\mathrm{SL}(2,\mathbb{C})$, we have the following diagrams on the Lie algebra level:

$$\begin{tikzcd}
\mathfrak{sl}(2,\mathbb{R}) \arrow[r, "f'"] \arrow[d, hook, "i"'] & \mathfrak{g}_{\mathbb{C}} \\
\mathfrak{sl}(2,\mathbb{C}) \arrow[ur, "(f')_{\mathbb{C}}"] &
\end{tikzcd}$$

and on the group level

$$\begin{tikzcd}
\widetilde{\mathrm{SL}}(2,\mathbb{R}) \arrow[r, "f"] \arrow[d, "j"'] & G_{\mathbb{C}} \\
\mathrm{SL}(2,\mathbb{C}) \arrow[ur, "h"] &
\end{tikzcd}.$$

The homomorphism j is the unique with the property $j' = i$, so j is really the projection $q_2 : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{PSL}^{(2)}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$. Hence $\ker f \supset \ker q_2 = 2.\,\mathrm{Center}(\widetilde{\mathrm{SL}}(2,\mathbb{R}))$. $\square$

As a corollary we get that $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has no complexification.

Proposition [Pressley-Segal, 1986]**.** *Every homomorphism from* $\mathrm{Diff}_+(S^1)$ *into a complex Lie group is trivial.*

Proof. Let $h : \mathrm{Diff}_+(S^1) \to G_{\mathbb{C}}$ be a homomorphism.

The kernel of a homomorphism from $\mathrm{PSL}^{(n)}$ into a complex Lie group contains $2.\,\mathrm{Center}(\mathrm{PSL}^{(n)}(2,\mathbb{R})) = \{R_{\frac{4\pi k}{n}} : k \in \mathbb{Z}\}$. Indeed, let $g : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to G_{\mathbb{C}}$ be a homomorphism and $f = g \circ q_n$. We apply the lemma and we get

$$\ker f \supset q_n(\ker f) \supset q_n(2.\,\mathrm{Center}(\widetilde{\mathrm{SL}}(2,\mathbb{R}))) = 2.\,\mathrm{Center}(\mathrm{PSL}^{(n)}(2,\mathbb{R})).$$

We apply this result to $h|\operatorname{PSL}^{(n)}(2,\mathbb{R})$ for every $n \in \mathbb{N}$ and we obtain that $\ker h \supset \{R_{\frac{4\pi k}{n}} : k \in \mathbb{Z}, n \in \mathbb{N}\}$, so $\ker h$ contains the whole subgroup of rotations. Now we apply the result of [Herman, 1971] that $\operatorname{Diff}_+(S^1)$ is simple and get the conclusion. $\square$

6. Morse moments in the diffeomorphism group

The reference for this chapter is [Kirillov, 1982] and [Kirillov, 1990].

6.1. Morse moments.
Let M be a compact orientable manifold and μ a fixed volume form on M. We denote by $G = \operatorname{Diff}(M)$ and $\mathfrak{g} = \operatorname{Vect}(M)$. Then $\mathfrak{g}^*_{reg} = \Omega^1(M) \otimes_{C^\infty(M)} \Omega^n(M)$ can be identified via μ with $\Omega^1(M)$.

Definition. Let f be a smooth section of the vector bundle $E \to M$ and S a submanifold of E. Then f intersects S *transversally* at $m \in M$ if either $f(m) \notin S$ or $f(m) \in S$ and $T_{f(m)}S + T_m f.T_m M = T_{f(m)}N$. Notation: $f \pitchfork S$ at m.

Lemma. *Let $E \to M$ be a smooth vector bundle, $S \subset E$ a submanifold and $f \in C^\infty(E)$ a smooth section with $f(m) \in S$. Suppose there is a neighbourhood U of $f(m)$ in E and a submersion $\varphi : U \to \mathbb{R}^k$ ($k = \operatorname{codim} S$) such that $S \cap U = \varphi^{-1}(0)$. Then $f \pitchfork S$ at m if and only if $\varphi \circ f$ is a submersion at m.*

We consider $J^k(\mathfrak{g}^*_{reg}) = J^k(\Omega^1(M)) = J^k(C^\infty(T^*M))$ and the coadjoint action of G on the jet space: $\operatorname{Ad}^*(\varphi) j^k_m p = j^k_{\varphi(m)}(\operatorname{Ad}^*(\varphi)p)$. Then we construct appropiate G-invariant submanifolds $S_1, \cdots, S_n$ of $J^k(\mathfrak{g}^*_{reg})$ to define Morse moments.

Definition. We say that $m \in M$ is a *singular point* of the regular moment α if $j^k_m(\alpha) \in \cup_{i=1}^n S_i$ and a *regular point* otherwise. The singularity $m \in M$ is called nondegenerate if $j^k\alpha \pitchfork S_i$ at m for $i = 1, \cdots, n$ and α is called a *Morse moment* if all its singularities are nondegenerate.

The G-invariance of the submanifolds S_i assures the G-invariance of the property to be a Morse moment. Hence we have the notion of Morse coadjoint orbits. The set of Morse moments is dense in $\mathfrak{g}^*_{reg}$ because of the following:

Thom's transversality theorem. *Let $E \to M$ be a smooth vector bundle and S_i submanifold of $J^{k_i}(E) := J^{k_i}(C^\infty(E))$ for $i = 1, \dots, n$. Then the set of smooth sections $f \in C^\infty(E)$ satisfying $j^{k_i} f \pitchfork S_i$ for $i = 1, \dots, n$ is a dense subset of $C^\infty(E)$ in the compact C^∞-topology.*

6.2. The classification of Morse moments in the case $\dim M = 1$.
Since M is compact, this is the case of the circle. The tangent bundle of S^1 is trivial, hence $\mathfrak{g}$ and $\mathfrak{g}^*_{reg}$ can be identified with $C^\infty(S^1, \mathbb{R})$. The adjoint and coadjoint action are special cases of the following action of $G = \operatorname{Diff}_+(S^1)$ on $C^\infty(S^1, \mathbb{R})$:

$$\varphi.f = (f \circ \varphi^{-1})(\varphi' \circ \varphi^{-1})^n, \quad n \in \mathbb{Z}.$$

For $n \in \mathbb{Z}_+$ this is the action of G on $C^\infty(\otimes^n(TS^1))$ and for $n \in \mathbb{Z}_-$ the action on $C^\infty(\otimes^{-n}(T^*S^1))$. Since $\mathfrak{g} = C^\infty(TS^1)$ the space of vector fields on S^1 and

$\mathfrak{g}^*_{reg} = C^\infty(T^*S^1 \otimes T^*S^1) = Q(S^1)$ the space of quadratic differentials on S^1, we have

$$\operatorname{Ad}(\varphi)f = (\varphi^{-1})^* f = (f \circ \varphi^{-1})(\varphi' \circ \varphi^{-1})$$
$$\operatorname{Ad}^*(\varphi)p = (p \circ \varphi^{-1})(\varphi' \circ \varphi^{-1})^{-2}$$

For the definition of Morse moments we take S as follows: the target map $\tau : J^1(\mathfrak{g}^*_{reg}) = J^1(S^1, \mathbb{R}) \to \mathbb{R}$ is a submersion, hence $S := \tau^{-1}(0)$ is a codimension 1 submanifold of $J^1(S^1, \mathbb{R})$. A regular moment $p \in C^\infty(S^1, \mathbb{R})$ is called Morse if $j^1 p$ is transversal to S.

Proposition. *A regular moment p is Morse if and only if all its zeros are simple.*

Proof. Let m be a zero of p. Because τ is a submersion, we can apply the lemma and get $j^1 p \pitchfork S$ at m if and only if $\tau \circ j^1 p = p$ is a submersion at m, i.e. m is a simple zero. $\square$

A Morse moment always has an even number of zeros because S^1 is compact and has Euler characteristic 0.

Lemma. *Let $\mathfrak{g}^*_{reg,+} = C^\infty(S^1, \mathbb{R}_+)$. This is a G-invariant set (open cone) of $\mathfrak{g}^*_{reg}$ and $I_0 : p \in \mathfrak{g}^*_{reg,+} \to \int_0^{2\pi} \sqrt{p(t)}dt \in \mathbb{R}_+$ is exactly the orbit map. All the orbits in $\mathfrak{g}^*_{reg,+}$ are diffeomorphic to $\operatorname{Diff}_+(S^1)/\operatorname{Rot}(S^1)$.*

Proof. I_0 is a G-invariant function:

$$I_0(\operatorname{Ad}^*(\varphi)p) = \int_0^{2\pi} \sqrt{p(\varphi^{-1}(t))(\varphi'(\varphi^{-1}(t)))^{-2}}dt = \int_0^{2\pi} \sqrt{p(t)}dt = I_0(p).$$

Every moment p is G-equivalent to $I_0(p)^2$, namely $\operatorname{Ad}^*(\varphi)p = I_0(p)^2$ where the diffeomorphism φ is given by

$$\varphi(t) = \frac{1}{I_0(p)} \int_0^t \sqrt{p(s)}ds.$$

Every coadjoint orbit in $\mathfrak{g}^*_{reg,+}$ contains a constant moment, say P_0, so the isotropy group is $\operatorname{Rot}(S^1)$:

$$p_0 = p_0(\varphi' \circ \varphi^{-1})^{-2} \Leftrightarrow \varphi' = 1 \Leftrightarrow \varphi \in \operatorname{Rot}(S^1)$$

hence the coadjoint orbit is $\operatorname{Diff}_+(S^1)/\operatorname{Rot}(S^1)$. $\square$

Classification theorem. *Let $\mathfrak{g}^*_{reg,n} := \{p \in \mathfrak{g}^*_{reg} : p \text{ has } 2n \text{ zeros}\}$ for all $n \in \mathbb{N}$. This is a partition of $\mathfrak{g}^*_{reg}$ and a complete set of invariants for $\mathfrak{g}^*_{reg,n}$ is:*
(i) $I_0 = \int_0^{2\pi} \sqrt{|p(t)|}dt$ and $\varepsilon = \operatorname{sgn} p$, in the case $n = 0$;
(ii) $I_1, J_1, \cdots, I_n, J_n$ up to permutations, where we put $I_k := \int_{t_{2k-1}}^{t_{2k}} \sqrt{p(t)}dt$, $J_k := \int_{t_{2k}}^{t_{2k+1}} \sqrt{-p(t)}dt$, in the case $n \neq 0$.

Proof. The case $n = 0$ follows from the lemma.

Let $n \geq 1$, $p, q \in \mathfrak{g}^*_{reg,n}$ with the same invariants $I_1, J_1, \cdots, I_n, J_n$. We denote by $t_1, \cdots, t_{2n}$ the zeros of p and by $s_1, \cdots, s_{2n}$ the zeros of q. We can define $\varphi \in G$ by $\varphi(t_k) = s_k$ and $\int_{t_k}^t \sqrt{|p(t)|}dt = \int_{s_k}^{\varphi(t)} \sqrt{|q(t)|}dt$ for $t \in [t_k, t_{k+1}]$. This implies $\varphi'(t) = \sqrt{\frac{p(t)}{q(\varphi(t))}}$, i.e. $q = \operatorname{Ad}^*(\varphi)p$. $\square$

Proposition. *The isotropy algebra of a moment $p \in \mathfrak{g}^*_{reg,n}$, for $n \neq 0$ is $\mathfrak{g}_p = 0$. The corresponding coadjoint orbits are diffeomorphic to* $\mathrm{Diff}_+(S^1)$ *modulo some discrete groups.*

Proof. By differentiating the coadjoint action $\mathrm{Ad}^*(\varphi)p = (p(\varphi')^{-2})\varphi^{-1}$ we get

$$\mathrm{ad}^*(h)p = \frac{d}{dt}\Big|_0 (p((\mathrm{Fl}^h_t)')^{-2})) \circ \mathrm{Fl}^h_{-t} = -2ph' - p'h$$

Hence $h \in \mathfrak{g}_p$ if and only if it satisfies the differential equation $2ph' + p'h = 0$, this means h^2p is constant on $I_1, \cdots, J_n$ and $h(t_k) = 0$. Taking the limit $t \to t_k$ we obtain that all the constants are equal to 0, so $\mathfrak{g}_p = 0$. $\square$

6.3. The local description of Morse moments in the case $\dim M = 2$**.**

$J^1(\mathfrak{g}^*_{reg}) = J^1(C^\infty(T^*M))$ is a manifold of dimension 8. Let (U, u) be a coordinate chart at $m \in M$ with $u(m) = 0$. This chart induces a chart on $J^1(\Omega^1(M))$ in which $j^1_m(\alpha)$ is given by $(x, y; a, b; a_x, a_y, b_x, b_y)$ with $(x, y) = u(m)$, $a = A(x, y)$, $b = B(x, y)$, $\begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} = \frac{D(A,B)}{D(x,y)}$ the jacobian matrix at (x, y) of (A, B) if $\alpha = Adx + Bdy$ in the chart (U, u).

The target map $\tau : J^1(C^\infty(T^*M)) \to T^*M$ is transversal to the zero section $0_M \subset T^*M$, hence $S_1 := \tau^{-1}(0_M)$ is a 6-dimensional submanifold of $J^1(\mathfrak{g}^*_{reg})$. The exterior derivative of 1-forms induces a map between jet spaces $d : J^1(\Omega^1(M)) \to J^1(\Omega^2(M))$, the second space can be identified with $J^1(C^\infty(M, \mathbb{R}))$ via μ. The composition $\tau \circ d : J^1(\Omega^1(M)) \to \mathbb{R}$ is a submersion given in local coordinates by: $(x, y; a, b; a_x, a_y, b_x, b_y) \mapsto b_x - a_y$. Hence $S_2 = S_1 \cap (\tau \circ d)^{-1}(0)$ is a 5-dimensional submanifold of $J^1(\mathfrak{g}^*_{reg})$. In local coordinates $(x, y; a, b; a_x, a_y, b_x, b_y)$ these submanifolds are:

S_1: $a = b = 0$;

S_2: $a = b = 0$, $b_x - a_y = 0$.

and we define S_3 by

S_3: $a = b = 0$, $a_x b_y - (\frac{a_y + b_x}{2})^2 = 0$.

S_3 is a well defined 5-dimensional submanifold of $J^1(\mathfrak{g}^*_{reg})$. Indeed, if an element $j^1_m(\alpha) \in S_1$ is given by $X = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}$ in the chart determined by u, then in another chart, determined by u', it is given by $X' = Q^t X Q$, where $Q \in \mathrm{GL}(2, \mathbb{R})$ is the Jacobian matrix at $u(m)$ of the coordinate change $u' \circ u^{-1}$. The expression $a_x b_y - (\frac{a_y + b_x}{2})^2$ is the determinant of the symmetric part of X and by a change of coordinates it only multiplies by $(\det Q)^2$, hence the definition of S_3 makes sense.

We prove now the G-invariance of these submanifolds. S_1 is clearly G-invariant and the invariance of S_2 follows from the relation $(\varphi^{-1})^* d\alpha = d(\varphi^{-1})^*\alpha$. Now let u and u' be coordinate charts at m and $\varphi(m)$ respectively and Q the Jacobian matrix at $u(m)$ of $u' \circ \varphi \circ u^{-1}$. An element $j^1_m(\alpha) \in S^1$ given in the chart u by $(u(m); 0; X)$ goes to $j^1_{\varphi(m)}((\varphi^{-1})^*\alpha) \in S_1$, given in the chart u' by $(u'(\varphi(m)); 0; X')$ where $X' = Q^t X Q \det Q$. This is exactly the transformation law for quadratic forms, the factor $\det Q$ comes from the action of G on $\Omega^n(M)$. The same relation exists between the symmetric parts of X and X', so we get the G-invariance of S_3 too.

Proposition. *A Morse moment α is characterised by the fact that at every point $m \in M$ either $j^1_m\alpha$ is not in S_1, or $j^1_m\alpha$ is in S_1 but not in $S_2 \cup S_3$ and* $\det X \neq 0$. *In local coordinates this means either $(a,b) \neq (0,0)$ or*

$$a = b = 0, b_x - a_y \neq 0, a_x b_y - (\frac{a_y + b_x}{2})^2 \neq 0, a_x b_y - a_y b_x \neq 0$$

.

Proof. The moment α is Morse means that $j^1\alpha \pitchfork S_1, S_2, S_3$ at every $m \in M$. This means either $\alpha(m) \neq 0$ or $\alpha(m) = 0$ and
(a) transversality to $S_1 = \tau^{-1}(0_M) \Leftrightarrow \tau \circ j^1\alpha = \alpha$ is a submersion at $m \Leftrightarrow \det X = \frac{D(A,B)}{D(x,y)} \neq 0$.
(b) transversality to S_2 and S_3 just means $j^1_m\alpha \notin S_2, S_3$ because $\dim S_i + \dim M = 5 + 2 = 7 < 8 = \dim J^1(\Omega^1(M))$. □

Theorem. *The local description of Morse moments in the case* $\dim M = 2$*: around every point of M one of the following situations occurs:*
If $\alpha(m) = 0$:
(1) *$\alpha = du$ and the foliation picture is: nonintersecting lines;*
If $\alpha(M) \neq 0$:
(2) *$\alpha = r^2 d(a\vartheta + \ln r)$ and the foliation picture is a spiral;*
(3) *$\alpha = \frac{1}{f(x,y)} d(|x|^{1+a}|y|^{1-a})$ and the foliation picture is a knot or a saddle.*

Proof. (1) If $\alpha(m) \neq 0$, there exists a neighbourhood of m where $\alpha \neq 0$ and therefore $\ker\alpha$ defines there a 1-dimensional distribution which is always integrable. The foliation must be locally of the form $u = \text{const}$, that means $du = 0$ on the leaves. Then $du = f\alpha$ for some function f without zeros.

If $\alpha(m) = 0$, the matrix X satisfies all the properties of the preceding proposition. Because the coadjoint action on S_1 is $Q.X = Q^t X Q \det Q$, we can find a moment α' equivalent to α, such that $j^1_m\alpha'$ has the matrix $X' = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ with $a \neq 0$ (there must be an antisymmetric part) and $\varepsilon = \pm 1$ (the symmetric part has nonzero determinant).

(2) The case $\varepsilon = 1$ (index $+1$). Here $\alpha = a(xdy - ydx) + (xdx + ydy)$ plus terms of order higher than 1. In polar coordinates (r, ϑ) we get $\alpha = ar^2 d\vartheta + rdr = r^2 d(a\vartheta + \ln r)$. The foliation picture $u = \text{const}$ is formed by the spirals

$$\ln r = -a\vartheta + \text{const}$$

.

(2) The case $\varepsilon = -1$. We can find an Ad^*-equivalent moment with $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Up to higher order terms $\alpha = (1+a)ydx + (1-a)xdy$. An integrating factor for α is $f(x,y) = |\frac{x}{y}|^a \operatorname{sgn}(xy)$. So $\alpha = \frac{1}{f}((1+a)|x|^a|y|^{1-a}(\operatorname{sgn} x)dx + (1-a)|x|^{1+a}|y|^{-a}(\operatorname{sgn} y)dy) = \frac{1}{f} d(|x|^{1+a}|y|^{1-a})$. The foliation picture $u = \text{const}$ is

$$|y| = c|x|^{\frac{a+1}{a-1}}.$$

If $|a| < 1$ we have a saddle ($\det X < 0$) (index -1) and if $|a| > 1$ we have a knot ($\det X > 0$) (index +1). $\square$

M is a compact 2-dimensional manifold, hence an orientable surface of genus g. The Euler characteristic is $2-2g$, hence the singularities of Morse moments occur pairwise. The torus is the only such surface that has a moment without singularities. In general we have much more saddles than spirals and knots, because for $g > 1$ $2-2g < 0$ is the sum of the indices over all the singularities.

7. Coadjoint orbits for the Virasoro-Bott group

7.1. Central extensions of $\mathrm{Vect}(S^1)$**.**

In this paragraph we show following [Pressley-Segal, 1986] that

$$\dim H^2(\mathrm{Vect}(S^1)) = 1,$$

i.e. there exists, up to isomorphisms, a unique nontrivial central extension of $\mathrm{Vect}(S^1)$ by $\mathbb{R}$. If the manifold M has $\dim M \geq 2$, then there exist no non-trivial central extensions of $\mathrm{Vect}(M)$ (Fuchs).

Let $\mathrm{Vect}_{\mathbb{C}}(S^1) = \mathrm{Vect}_{\mathbb{C}}(S^1) \otimes \mathbb{C}$ be the complexification of $\mathrm{Vect}(S^1)$ and $\alpha : \mathrm{Vect}_{\mathbb{C}}(S^1) \times \mathrm{Vect}_{\mathbb{C}}(S^1) \to \mathbb{C}$ a complex bilinear, skewsymmetric form, the complexification of a cocycle on $\mathrm{Vect}(S^1)$. An unconditional basis for the nuclear Fréchet space $\mathrm{Vect}_{\mathbb{C}}(S^1)$ is

$$\{L_n = e^{int}\frac{d}{dt} : n \in \mathbb{N}\},$$

so α is uniquely determined by the complex numbers $\alpha_{p,q} = \alpha(L_p, L_q)$.

In 3.6 we had $[\mathrm{Ad}^*(\varphi)\alpha] = [\alpha]$ for every diffeomorphism φ of S^1, in particular for all rotations. Hence

$$\beta = \int_{S^1} \mathrm{Ad}^*(R_t)\alpha dt$$

is a $\mathrm{Rot}(S^1)$-invariant cocycle cohomologous to α. Without loss of generality we can then assume that α is invariant under the group of rotations. This assures that $\alpha_{p,0} = 0$ for $p = 0$. Indeed

$$\begin{aligned}\alpha_{p,0} = \alpha(L_p, L_0) &= \mathrm{Ad}^*(R_\vartheta)\alpha(L_p, L_0) \\ &= \alpha(R_\vartheta^* e^{ipt}\frac{d}{dt}, R_\vartheta^*\frac{d}{dt}) \\ &= \alpha(e^{ip(t-\vartheta)}\frac{d}{dt}, \frac{d}{dt}) = e^{-ip\vartheta}\alpha_{p,0}.\end{aligned}$$

Writing the cocycle condition for L_0, L_p, L_q and using $[L_p, L_q] = i(q-p)L_{p+q}$ we get

$$(q-p)\alpha_{p+q,0} - q\alpha_{q,p} + p\alpha_{p,q} = 0.$$

The antisymmetry of α implies $\alpha_{p,q} = -\alpha_{q,p}$ and the relation becomes

$$(p-q)\alpha_{p+q,0} = (p+q)\alpha_{p,q}.$$

We obtain $\alpha_{p,q} = 0$ if $p + q \neq 0$. The cocycle condition for $L_p, L_q, L_{-(p+q)}$ gives

$$(q-p)\alpha_{p+q,-(p+q)} - (p+2q)\alpha_{-p,p} + (2p+q)\alpha_{-q,q} = 0.$$

Denoting $\alpha_p = \alpha_{p,-p} = -\alpha_{-p,p}$, we obtain a recurrence formula for α_p:

$$(p-q)\alpha_{p+q} = (p+2q)\alpha_p - (2p+q)\alpha_q,$$

which determines all the α_p in terms of α_1 and α_2. The general solution is $\alpha_p = \lambda p^3 + \mu p$ for some real λ, μ. Hence the general form of a $\mathrm{Rot}(S^1)$-invariant cocycle α is:

(*) $$\alpha(L_p, L_q) = \begin{cases} 0, & \text{for } p+q \neq 0 \\ \lambda p^3 + \mu p, & \text{for } p+q = 0. \end{cases}$$

A trivial $\mathrm{Rot}(S^1)$-invariant cocycle is

$$\alpha_0(\xi, \eta) = \int_0^{2\pi} [\xi, \eta] dt = 2 \int_0^{2\pi} \xi d\eta.$$

It has the form

$$\alpha_0(L_p, L_q) = \begin{cases} 0, & \text{for } p+q \neq 0 \\ 4\pi i p, & \text{for } p+q = 0. \end{cases}$$

So the value of μ in (*) is unimportant.

The *Virasoro cocycle* is defined by

$$\omega(\xi, \eta) = \int_0^{2\pi} \xi' d\eta'.$$

Evaluated on the basis it gives

$$\omega_{p,q} = -ipq^2 \int_0^{2\pi} e^{i(p+q)t} dt = \begin{cases} 0, & \text{for } p+q \neq 0 \\ -2\pi i p^3, & \text{for } p+q = 0. \end{cases}$$

The Virasoro cocycle is a $\mathrm{Rot}(S^1)$-invariant cocycle whose cohomology class generates $H^2(\mathrm{Vect}(S^1))$.

7.2. The Virasoro-Bott extension of $\mathrm{Diff}_+(S^1)$.

For every $\varphi \in \mathrm{Diff}_+(S^1)$ we denote by $\varphi' : S^1 \to \mathbb{R}_+$ the map given by $T_t\varphi . \frac{d}{dt}|_t = \varphi'(t) \frac{d}{dt}|_{\varphi(t)}$.

Lemma. *The map*

$$c : \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \to S^1$$
$$c(\varphi, \psi) := \int_{S^1} \log(\varphi \circ \psi)' d \log \psi',$$

is a group cocycle. It is called the Bott cocycle and the extended group is called the Virasoro-Bott extension of $\mathrm{Diff}_+(S^1)$.

Proof. Because $\int_{S^1} \log\psi' d\log\psi' = \frac{1}{2}\int_{S^1} d(\log\psi')^2 = 0$, we have

$$c(\varphi,\psi) = \int_{S^1} \log(\varphi'\circ\psi)d\log\psi'.$$

We verify the cocycle equation

$$\begin{aligned}
&-c(\varphi_1\circ\varphi_2,\varphi_3) - c(\varphi_1,\varphi_2) + c(\varphi_1,\varphi_2\circ\varphi_3) + c(\varphi_2,\varphi_3)\\
&= -\int_{S^1} \log(\varphi_1\circ\varphi_2\circ\varphi_3)' d\log\varphi_3' - \int_{S^1}\log(\varphi_1\circ\varphi_2)'d\log\varphi_2'\\
&+\int_{S^1}\log(\varphi_1\circ\varphi_2\circ\varphi_3)'d\log(\varphi_2\circ\varphi_3)' + \int_{S^1}\log(\varphi_2\circ\varphi_3)'d\log\varphi_3'\\
&= -\int_{S^1}\log(\varphi_1'\circ\varphi_2)d\log\varphi_2' + \int_{S^1}(\log(\varphi_1'\circ\varphi_2\circ\varphi_3)+\log(\varphi_2\circ\varphi_3)')d\log(\varphi_2'\circ\varphi_3)\\
&+\int_{S^1}\log(\varphi_2\circ\varphi_3)'d\log\varphi_3' = \int_{S^1}\log(\varphi_2\circ\varphi_3)'d\log((\varphi_2'\circ\varphi_3)\varphi_3') = 0. \quad \square
\end{aligned}$$

From this calculation we see that also $c : \mathrm{Diff}_+(S^1)\times\mathrm{Diff}_+(S^1)\to\mathbb{R}$ is a group cocycle and we could extend $\mathrm{Diff}_+(S^1)$ also by $\mathbb{R}$.

Proposition. *The Lie algebra cocycle corresponding to the Bott cocycle*

$$c(\varphi,\psi) := \frac{1}{2}\int_{S^1}\log(\varphi\circ\psi)'d\log\psi'$$

is the Virasoro cocycle

$$\omega(\xi,\eta) = \int_{S^1}\xi' d\eta',\ \textit{for}\ \xi,\eta\in\mathrm{Vect}(S^1).$$

Proof. In 3.3 we proved that a Lie algebra cocycle can be obtained from the group cocycle by differentiation:

$$\omega(\xi,\eta) = \partial_1\partial_2 c(\mathrm{Id},\mathrm{Id})(\xi,\eta) - \partial_1\partial_2 c(\mathrm{Id},\mathrm{Id})(\eta,\xi) = \ddot{c}(\xi,\eta) - \ddot{c}(\eta,\xi).$$

Then

$$\partial_1 c(\mathrm{Id},\psi)\xi = \frac{1}{2}\int_{S^1}\log'(\mathrm{Id}'\circ\psi)(\xi'\circ\psi)d\log\psi' = \frac{1}{2}\int_{S^1}(\xi'\circ\psi)d\log\psi'$$

and

$$\partial_1\partial_2 c(\mathrm{Id},\mathrm{Id})(\xi,\eta) = \frac{1}{2}\int_{S^1}(\xi'\circ\mathrm{Id})d\log'(\mathrm{Id}')\eta' + \frac{1}{2}\int_{S^1}\xi''\eta d\log\mathrm{Id}' = \frac{1}{2}\int_{S^1}\xi'd\eta'.$$

Finally we get

$$\omega(\xi,\eta) = \frac{1}{2}\int_{S^1}\xi'd\eta' - \frac{1}{2}\int_{S^1}\eta'd\xi' = \int_{S^1}\xi'd\eta'. \quad \square$$

7.3. The coadjoint action of the Virasoro-Bott group.

There are two possibilities to compute the coadjoint action in the extended group:

1.Method. A direct computation of h like in 3.5;

2.Method. To notice that $H^1(\mathrm{Vect}(S^1)) = 0$ and to use the result of 3.6.

The final result is $h(\varphi) = S(\varphi^{-1})dt^2$ and then

$$\mathrm{Ad}^*(\varphi)(p,c) = (\mathrm{Ad}^*(\varphi)p + cS(\varphi^{-1})dt^2, c)$$

where $p \in \mathfrak{g}^*_{reg}, c \in \mathbb{R}, \varphi \in G$ and $S(\varphi) : S^1 \to \mathbb{R}$ is the Schwartzian derivative of φ, or we take some $\tilde{\varphi}$ who covers φ (see 5.3) and build the 2π-periodic function $S(\tilde{\varphi}) : \mathbb{R} \to \mathbb{R}$.

Some facts about the Schwartzian derivative. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then $S(f) := \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$ measures the deviation of f from being a Moebius transformation, this means

$$S(f) = 0 \Leftrightarrow f(x) = \frac{ax+b}{cx+d} \text{ for some real } a, b, c, d.$$

Indeed

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = 0$$

$$\Leftrightarrow y = \frac{f''}{f'} \text{ satisfies the differential equation } y' = \frac{y^2}{2}$$

$$\Leftrightarrow \frac{f''}{f'}(x) = -\frac{2}{x+c} \Leftrightarrow f'(x) = \frac{1}{(cx+d)^2} \Leftrightarrow f(x) = \frac{ax+b}{cx+d}.$$

The Schwartz derivative of a composition is:

$$S(f \circ g) = (S(f) \circ g)(g')^2 + S(g),$$

then we can deduce the Schwartz derivative of an inverse:

$$S(f^{-1}) = -(S(f) \circ f^{-1})(f^{-1})'^2 = -\frac{S(f)}{f'^2} \circ f^{-1}.$$

1.Method. The long computation

$$h(\varphi^{-1}) = d_2c(\varphi, \mathrm{Id}) + c(\varphi, \varphi^{-1})^{-1} d_1c(\varphi, \varphi^{-1}).T_e\lambda_\varphi.$$

Recall that $c(\varphi, \psi) = \frac{1}{2}\int_{S^1} \log(\varphi' \circ \psi) d\log\psi'$. We have $c(\varphi, \varphi^{-1}) = 0$ and $T_e\lambda_\varphi.\xi = \frac{d}{dt}\Big|_0 \varphi \circ \mathrm{Fl}^\xi_t = T\varphi.\xi = \varphi'\xi$. Then

$$\begin{aligned}
d_2c(\varphi, \mathrm{Id})\xi &= \frac{1}{2}\int_{S^1} \log'(\varphi').\xi d\log(\mathrm{Id}') + \frac{1}{2}\int_{S^1} \log\varphi' d(\log'(\mathrm{Id}').\xi') \\
&= \frac{1}{2}\int_{S^1} \log\varphi' d\xi' = -\frac{1}{2}\int_{S^1} \xi' d\log\varphi' \\
d_1c(\varphi, \varphi^{-1})\varphi'\xi &= \frac{1}{2}\int_{S^1} \log'(\varphi' \circ \varphi^{-1})((\varphi'\xi)' \circ \varphi^{-1}) d\log(\varphi^{-1})' \\
&= \frac{1}{2}\int_{S^1} \frac{(\varphi'\xi)'}{\varphi'} \circ \varphi^{-1} d\log(\frac{1}{\varphi'} \circ \varphi^{-1}) = \frac{1}{2}\int_{S^1} \frac{\varphi''\xi + \frac{1}{2}\varphi'\xi'}{\varphi'} d\log(\frac{1}{\varphi'}) \\
&= -\frac{1}{2}\int_{S^1} (\frac{\varphi''}{\varphi'})^2 \xi dt - \frac{1}{2}\int_{S^1} \xi' d\log\varphi'
\end{aligned}$$

Finally

$$h(\varphi^{-1})\xi = -\frac{1}{2}\int_{S^1} (\frac{\varphi''}{\varphi'})^2 \xi dt - \int_{S^1} \frac{\varphi''}{\varphi'}\xi' dt = \int_{S^1} S(\varphi)\xi dt,$$

hence $h(\varphi^{-1}) = S(\varphi)dt^2$.

2.Method. We have $H^1(\mathrm{Vect}(S^1)) = 0$ (this follows from 5.1), so we can determine uniquely $h(\varphi)$ from the relation $dh(\varphi) = \omega - \mathrm{Ad}^*(\varphi)\omega$.

The second long computation

$$\begin{aligned}
\mathrm{Ad}^*(\varphi)\omega(\xi,\eta) &= \omega(\varphi^*\xi, \varphi^*\eta) = \int_0^{2\pi} (\frac{\xi\circ\varphi}{\varphi'})' d(\frac{\eta\circ\varphi}{\varphi'})' \\
&= \int_0^{2\pi} (\xi' - F\xi)\circ\varphi \cdot d(\eta' - F\eta)\circ\varphi, \text{ where } F := \frac{\varphi''}{(\varphi')^2} \\
&= \int_0^{2\pi} (\xi' - F\xi) d(\eta' - F\eta) \\
&= \int_0^{2\pi} \xi' d\eta' + \int_0^{2\pi} -\xi'(F'\eta + F\eta')dt \\
&\quad + \int_0^{2\pi} \eta'(F'\xi + F\xi')dt + \int_0^{2\pi} F\xi(F'\eta + F\eta')dt \\
&= \int_0^{2\pi} \xi' d\eta' + \int_0^{2\pi} F'(\xi\eta' - \xi'\eta)dt + \int_0^{2\pi} \frac{F^2}{2}(\xi\eta' - \xi'\eta)dt \\
&= \omega(\xi,\eta) + \int_0^{2\pi} (F' + \frac{F^2}{2})(\xi\eta' - \xi'\eta)dt.
\end{aligned}$$

But $F' + \frac{F^2}{2} = \frac{\varphi'''\varphi' - \frac{3}{2}\varphi''^2}{(\varphi')^4}\circ\varphi^{-1} = \frac{S(\varphi)}{\varphi'^2}\circ\varphi^{-1} = -S(\varphi^{-1})$. Hence

$$\begin{aligned}
dh(\varphi)(\xi,\eta) &= \omega(\xi,\eta) - \mathrm{Ad}^*(\varphi)\omega(\xi,\eta) = \int_0^{2\pi} S(\varphi^{-1})(\xi\eta' - \xi'\eta)dt \\
&= \langle S(\varphi^{-1})dt^2, [\xi,\eta]\rangle = -\langle S(\varphi^{-1})dt^2, \mathrm{ad}(\xi)\eta\rangle = d(S(\varphi^{-1})dt^2)(\xi,\eta).
\end{aligned}$$

We get $h(\varphi) = S(\varphi^{-1})dt^2$ and

$$\mathrm{Ad}^*(\varphi^{-1})(pdt^2, c) = (((p\circ\varphi)\varphi'^2 + cS(\varphi))dt^2, c).$$

7.4. The isotropy group of a moment.

Let $G = \mathrm{Diff}_+(S^1)$ and $\mathfrak{g} = \mathrm{Vect}(S^1)$ its Lie algebra. Let $G_{(p,c)}$ be the isotropy group of the regular moment $(p,c) \in \tilde{\mathfrak{g}}^*_{reg} = \mathfrak{g}^*_{reg} = Q(S^1)\oplus\mathbb{R}$ under the coadjoint action in the extended group. Its Lie algebra is

$$\mathfrak{g}_{(p,c)} = \{\xi \in \mathfrak{g} : c\xi''' + 2p\xi' + p'\xi = 0\}$$

because

$$\langle \mathrm{ad}^*(\xi)(p,c), \begin{pmatrix}\eta\\ \mu\end{pmatrix}\rangle = \langle (p,c), -\mathrm{ad}(\xi)\begin{pmatrix}\eta\\ \mu\end{pmatrix}\rangle = -\langle (p,c), \begin{pmatrix}-[\xi,\eta]\\ \omega(\xi,\eta)\end{pmatrix}\rangle$$

$$= \langle p, [\xi,\eta]\rangle - c\int_0^{2\pi} \xi' d\eta' = \int_0^{2\pi} p(\xi\eta' - \xi'\eta) + c\xi''\eta' = -\int_0^{2\pi} (c\xi''' + 2p\xi' + p'\xi)\eta.$$

So

$$\mathrm{ad}^*(\xi)(p,c) = -(c\xi''' + 2p\xi' + p'\xi, 0).$$

Proposition [Kirillov, 1982]**.** *Every moment (p,c) has nontrivial stabilizer, that means the equation $c\xi''' + 2p\xi' + p'\xi = 0$ has at least one nontrivial 2π-periodic solution.*

Proof. Let L be the space of all solutions. This is a 3-dimensional Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$, because L is the stabilizer of $\mathfrak{g}_{(p,c)}$ viewed as a Lie algebra action of $\mathrm{Vect}(\mathbb{R})$. We denote by ξ_{-1}, ξ_0, ξ_1 the basis of solutions with initial conditions $\xi_k(t) = t^{k+1} + o(t^2)$. Then the commutation relations between them are:

$$\begin{aligned}
[\xi_0, \xi_{-1}] &= -\xi_{-1} + \frac{p(0)}{c}\xi_1 \\
[\xi_0, \xi_1] &= \xi_1 \\
[\xi_1, \xi_{-1}] &= -2\xi_0.
\end{aligned}$$

For example the first equality follows from:

$$\begin{aligned}
[\xi_0, \xi_{-1}](0) &= \xi_0(0)\xi'_{-1}(0) - \xi'_0(0)\xi_{-1}(0) = -1 \\
[\xi_0, \xi_{-1}]'(0) &= \xi_0(0)\xi''_{-1}(0) - \xi''_0(0)\xi_{-1}(0) = 0 \\
[\xi_0, \xi_{-1}]''(0) &= \xi_0(0)\xi'''_{-1}(0) - \xi'_0(0)\xi''_{-1}(0) - \xi'''_0(0)\xi_{-1}(0) - \xi''_0(0)\xi'_{-1}(0) \\
&= -\xi''_0(0) = \frac{2p(0)}{c}.
\end{aligned}$$

So we see that $L \cong \mathfrak{sl}(2,\mathbb{R})$.

Let T be the automorphism of the Lie algebra L defined by the shift $\xi(t) \mapsto \xi(t+2\pi)$. The fixed points of T are exactly the periodic solutions of the differential equation $c\xi''' + 2p\xi' + p'\xi = 0$. But every automorphism of $\mathfrak{sl}(2,\mathbb{R})$ has fixed points, because for every $T : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})$ there exists a $g \in \mathrm{GL}(2,\mathbb{R})$ such that $T(X) = gXg^{-1}$ and $X_0 = g - \frac{1}{2}\,\mathrm{tr}(g)I \in \mathfrak{sl}(2,\mathbb{R})$ is a fixed point for T. □

7.5. Coadjoint orbits containing constant moments.

The results of this paragraph can be found in [Kirillov, 1982]. In the special case $p = pdt^2$ constant, the stabilizing algebra $\mathfrak{g}_{(p,c)}$ consists of those vector fields which satisfy the equation

$$c\xi''' + 2p\xi' = 0$$

Proposition. *The isotropy Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ if $p = \frac{n^2}{2}c$ for some $n \in \mathbb{N}$, and isomorphic to $\mathbb{R}$ otherwise.*

Proof. The differential equation $y'' + ay = 0$ has nontrivial 2π-periodic solutions if and only if $a = n^2$, namely linear combinations of $\cos nt$ and $\sin nt$. Hence the 2π-periodic solutions of the equation $c\xi''' + 2p\xi' = 0$ are constants if $\frac{2p}{c} \neq n^2$ and are generated by 1, $\frac{1}{n}\cos nt$, $\frac{1}{n}\sin nt$ if $\frac{2p}{c} = n^2$.

$$\mathfrak{g}_{(p,c)} = \begin{cases} \langle \frac{d}{dt} \rangle, & \text{for } p \neq \frac{n^2}{2}c \\ \langle \frac{d}{dt}, \frac{1}{n}\cos nt\frac{d}{dt}, \frac{1}{n}\sin nt\frac{d}{dt} \rangle, & \text{for } p = \frac{n^2}{2}c. \end{cases}$$

From the commutation relations

$$\begin{aligned}
[\frac{d}{dt},\frac{1}{n}\sin nt\frac{d}{dt}] &= -\frac{1}{n}\cos nt\frac{d}{dt}\\
[\frac{d}{dt},\frac{1}{n}\cos nt\frac{d}{dt}] &= \frac{1}{n}\sin nt\frac{d}{dt}\\
[\frac{1}{n}\cos nt\frac{d}{dt},\frac{1}{n}\sin nt\frac{d}{dt}] &= \frac{d}{dt}
\end{aligned}$$

we see that the last Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. $\square$

Proposition. *The isotropy group of a constant moment* $(p,c)\in\mathfrak{g}^*_{reg}$ *is the subgroup* $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ *of* $\mathrm{Diff}_+(S^1)$*, of* n*-fold coverings of elements in* $\mathrm{PSL}(2,\mathbb{R})$ *if* $p=\frac{n^2}{2}cdt^2$, $n\in\mathbb{N}$, *and the subgroup of rotations for all other choices of* p.

Proof. 1.Method. To detect the Lie subgroups of $\mathrm{Diff}_+(S^1)$ corresponding to the isotropy Lie algebras determined in the preceding proposition. (Every finite dimensional Lie algebra of smooth vector fields on M arises from a Lie subgroup of $\mathrm{Diff}(M)$).

The 1-parameter subgroup in $\mathrm{Diff}_+(S^1)$ of $\frac{d}{dt}$ is $\mathrm{Rot}(S^1)$ the subgroup of rotations. The asociated 1-parameter subgroup to $\sin t\frac{d}{dt}$ is $\{x\mapsto 2\operatorname{arctg}(e^a\operatorname{tg}\frac{x}{2}) : a\in\mathbb{R}\}$. The 1-parameter subgroup of $\cos t\frac{d}{dt}$ is $\{x\mapsto 2\operatorname{arctg}\frac{(e^a+1)\operatorname{tg}\frac{x}{2}+(e^a-1)}{(e^a-1)\operatorname{tg}\frac{x}{2}+(e^a+1)} : a\in\mathbb{R}\}$. In fact we extend these maps defined around 0 uniquely to $\mathbb{R}$ by $\tilde\varphi(x+2\pi)=\tilde\varphi(x)+2\pi$. As mappings from $\mathbb{R}\cup\infty=S^1$ to itself they are Moebius transformations:

$$\begin{aligned}
x\mapsto& x+a\\
x\mapsto& \frac{x}{e^a}\\
x\mapsto& \frac{(e^a+1)x+e^a-1}{(e^a-1)x+e^a+1}
\end{aligned}$$

and they generate the 3-dimensional Moebius group. Hence the Lie subalgebra corresponding to $\mathrm{PSL}(2,\mathbb{R})$ is $\langle\frac{d}{dt},\sin t\frac{d}{dt},\cos t\frac{d}{dt}\rangle$.

Lifted to $\mathbb{R}$, the n-fold covering map of the circle p_n becomes the multiplication by n, hence the vector fields $\frac{d}{dt}$, $\sin t\frac{d}{dt}$, $\cos t\frac{d}{dt}$ are p_n-related to $\frac{d}{dt}$, $\frac{1}{n}\sin nt\frac{d}{dt}$, $\frac{1}{n}\cos nt\frac{d}{dt}$. Then

$$p_n\circ\mathrm{Fl}^{p_n^*\xi}_t=\mathrm{Fl}^{\xi}_t\circ p_n$$

and we conclude from the first part of the proof that the Lie subalgebra

$$\langle\frac{d}{dt},\frac{1}{n}\sin nt\frac{d}{dt},\frac{1}{n}\cos nt\frac{d}{dt}\rangle$$

arises from

$$\mathrm{PSL}^{(n)}(2,\mathbb{R})=\{\varphi\in\mathrm{Diff}_+(S^1) : p_n\circ\varphi=A\circ p_n, A\in\mathrm{PSL}(2,\mathbb{R})\}.$$

2.Method. For $p = \frac{n^2}{2}c$, this result can be obtained directly on the group level. Recall that the coadjoint action on $\mathfrak{g}^*_{reg}$ is

$$\mathrm{Ad}^*(\varphi^{-1})(pdt^2, c) = (((p \circ \varphi){\varphi'}^2 + cS(\varphi))dt^2, c),$$

(the left side of the equality we can actually also compute using an arbitrary covering $\tilde{\varphi}$ of φ). Hence in the case where $p = \frac{n^2}{2}c$ is constant, the isotropy group is

$$G_{(p,c)} = \{\varphi \in \mathrm{Diff}_+(S^1) : \frac{n^2}{2}({\varphi'}^2 - 1) + S(\varphi) = 0\}.$$

Next we will show that these diffeomorphisms are n-fold coverings of Moebius transformations. We have the following equivalences:

$$\begin{aligned}\tilde{\varphi} \in \widetilde{\mathrm{SL}}(2,\mathbb{R}) &\Leftrightarrow f \circ \tilde{\varphi} \circ f^{-1} \text{ is a Moebius transformation}\\ &\Leftrightarrow S(f \circ \tilde{\varphi} \circ f^{-1}) = 0\\ &\Leftrightarrow \frac{1}{2}({\tilde{\varphi}'}^2 - 1) + S(\tilde{\varphi}) = 0.\end{aligned}$$

The computation uses $S(f) = \frac{1}{2}$:

$$\begin{aligned}S(f \circ \tilde{\varphi} \circ f^{-1}) &= (S(f \circ \tilde{\varphi}) \circ f^{-1})((f^{-1})')^2 + S(f^{-1})\\ &= ((S(f) \circ \tilde{\varphi})(\tilde{\varphi}')^2 + S(\tilde{\varphi}) - S(f)) \circ f^{-1}((f^{-1})')^2\\ &= (\frac{1}{2}((\tilde{\varphi}')^2 - 1) + S(\tilde{\varphi})) \circ f^{-1}((f^{-1})')^2\end{aligned}$$

So we get in the case $n = 1$ that $G_{(p,c)} = \mathrm{PSL}(2,\mathbb{R})$. For the general case we show that

$$\mathrm{PSL}^{(n)}(2,\mathbb{R}) = \{\varphi \in \mathrm{Diff}_+(S^1) : \frac{n^2}{2}((\varphi')^2 - 1) + S(\varphi) = 0\}.$$

Indeed, the condition $p_n \circ \varphi = A \circ p_n$ which is the definition of $\mathrm{PSL}^{(n)}(2,\mathbb{R})$, transcribes to $n\tilde{\varphi}(t) = \tilde{\varphi}_A(nt)$ in $\widetilde{\mathrm{Diff}}_+(S^1)$. The result follows then from the equivalences:

$$\begin{aligned}S(\tilde{\varphi})(t) &= S(\frac{1}{n}\,\mathrm{Id} \circ \tilde{\varphi}_A \circ n\,\mathrm{Id})(t) = S(\tilde{\varphi}_A \circ n\,\mathrm{Id})(t)\\ = n^2 S(\tilde{\varphi}_A)(nt) &= -\frac{n^2}{2}((\tilde{\varphi}'_A)^2(nt) - 1) = -\frac{n^2}{2}((\tilde{\varphi}')^2(t) - 1).\end{aligned}$$

We find that the isotropy group of $(\frac{n^2}{2}c, c)$ is $\mathrm{PSL}^{(n)}(2,\mathbb{R})$. $\square$

Corollary. *All constant moments are pairwise nonequivalent.*

Proof. Let $p_1 = p_1 dt^2, p_2 = p_2 dt^2$ be constant moments and suppose there exists a $\varphi \in \mathrm{Diff}_+(S^1)$ such that $\mathrm{Ad}^*(\varphi)(p_1, c) = (p_2, c)$. Then the isotropy groups $G_{(p_1,c)}$ and $G_{(p_2,c)}$ are conjugate by φ. This can occur only if both are equal to $\mathrm{Rot}(S^1)$ and φ normalizes $\mathrm{Rot}(S^1)$. Then for every $a \in \mathbb{R}$, there exists a $b(a) \in \mathbb{R}$ with

$\varphi\circ R_a\circ\varphi^{-1}=R_{b(a)}$, i.e. $\tilde\varphi(t+a)=\tilde\varphi(t)+b(a)$ for every t. We get $\tilde\varphi'$ is constant, hence $\varphi\in\operatorname{Rot}(S^1)$. The relation $\operatorname{Ad}^*(\varphi)(p_1,c)=(p_2,c)$ gives

$$p_1=p_2(\tilde\varphi')^2+cS(\tilde\varphi)=p_2 1+c0=p_2. \quad\square$$

Conclusion: To every pair of real numbers (p,c) corresponds a different coadjoint orbit $\mathcal{O}_{p,c}$:

$$\mathcal{O}_{p,c}\cong\operatorname{Diff}_+(S^1)/\operatorname{Rot}(S^1),\ \text{if } p\neq\frac{n^2}{2}c$$
$$\mathcal{O}_{p,c}\cong\operatorname{Diff}_+(S^1)/\operatorname{PSL}^{(n)}(2,\mathbb{R}),\ \text{if } p=\frac{n^2}{2}.$$

7.6. Locally projective structures on S^1.

Definition. A projective transformation in $\mathbb{R}$ is a restriction of an element in $\widetilde{\operatorname{SL}}(2,\mathbb{R})$ to an open set. A local projective structure on S^1 is a complete atlas of S^1 with projective transformations as transition mappings.

Remark. Every projective transformation in $\mathbb{R}$ is the restriction of a unique element in $\widetilde{\operatorname{SL}}(2,\mathbb{R})$.

Proposition. *The problem of classifying coadjoint orbits with $c\neq 0$ in the Virasoro-Bott group is the same as that of determining all local projective structures on S^1.*

Proof. First recall that $\widetilde{\operatorname{SL}}(2,\mathbb{R})=\{\tilde\varphi\in\widetilde{\operatorname{Diff}}(S^1):S(\tilde\varphi)=\frac12-\frac12(\tilde\varphi')^2\}$. Let $\{f_\alpha:U_\alpha\subset S^1\to\mathbb{R}\}$ be a complete projective atlas on S^1. We define $p_\alpha=cS(f_\alpha)+\frac{c}{2}(f_\alpha')^2:U_\alpha\subset S^1\to\mathbb{R}$. The transition mappings $f_\alpha\circ f_\beta^{-1}$ are projective transformations, i.e.

$$S(f_\alpha\circ f_\beta^{-1})=\frac12-\frac12((f_\alpha\circ f_\beta^{-1})')^2.$$

Hence

$$\begin{aligned}\frac1c p_\alpha&=S(f_\alpha\circ f_\beta^{-1}\circ f_\beta)+\frac12((f_\alpha\circ f_\beta^{-1}\circ f_\beta)')^2\\&=(S(f_\alpha\circ f_\beta^{-1})\circ f_\beta)(f_\beta')^2+S(f_\beta)+\frac12((((f_\alpha\circ f_\beta^{-1})')^2\circ f_\beta)(f_\beta')^2\\&=S(f_\beta)+\frac12(f_\beta')^2=\frac1c cp_\beta\end{aligned}$$

on their common domain of definition and so they can be pieced together to give a smooth map $p:S^1\to\mathbb{R}$.

A projectively equivalent complete atlas is of the form $\{f_\alpha\circ\varphi:\varphi^{-1}(U_\alpha)\subset S^1\to\mathbb{R}\}$ for a fixed diffeomorphism φ of S^1. The smooth map $q:S^1\to\mathbb{R}$ defined by this atlas is:

$$\begin{aligned}\frac1c q&=S(f_\alpha\circ\varphi)+\frac12((f_\alpha\circ\varphi)')^2\\&=(S(f_\alpha)\circ\varphi)(\varphi')^2+S(\varphi)+\frac12((f_\alpha')^2\circ\varphi)(\varphi')^2=\frac1c(p\circ\varphi)(\varphi')^2+S(\varphi),\end{aligned}$$

hence (p, c) and (q, c) lie in the same coadjoint orbit.

For the converse let's consider $(p, c) \in \tilde{\mathfrak{g}}^*_{reg}$. The ordinary differential equation of 3-rd order

$$p = cS(f) + \frac{1}{2}c(f')^2$$

has local solutions $\{f_\alpha : U_\alpha \subset S^1 \to \mathbb{R}\}$. We show that they form a complete projective atlas on S^1:

$$\begin{aligned} S(f_\alpha \circ f_\beta^{-1}) &= (S(f_\alpha) \circ f_\beta^{-1})((f_\beta^{-1})')^2 + S(f_\beta^{-1}) \\ &= ((S(f_\alpha) - S(f_\beta)) \circ f_\beta^{-1})((f_\beta^{-1})')^2 \\ &= \frac{1}{2}(((f'_\beta)^2 - (f'_\alpha)^2) \circ f_\beta^{-1})((f_\beta^{-1})')^2 = \frac{1}{2} - \frac{1}{2}((f_\alpha \circ f_\beta^{-1})')^2 \end{aligned}$$

hence $f_\alpha \circ f_\beta^{-1}$ is a projective transformation in $\mathbb{R}$. The completeness of this atlas: let $\tilde{\varphi} \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$, i.e. $S(\tilde{\varphi}) = \frac{1}{2} - \frac{1}{2}(\tilde{\varphi}')^2$, and f_α a chart of this atlas. Then $\tilde{\varphi} \circ f_\alpha$ is again a chart of this atlas because

$$\begin{aligned} S(\tilde{\varphi} \circ f_\alpha) + \frac{1}{2}((\tilde{\varphi} \circ f_\alpha)')^2 &= (S(\tilde{\varphi}) \circ f_\alpha)(f'_\alpha)^2 + S(f_\alpha) + \frac{1}{2}(\tilde{\varphi} \circ f_\alpha)(f'_\alpha)^2 \\ &= S(f_\alpha) + \frac{1}{2}(f'_\alpha)^2 = \frac{p}{c}. \end{aligned}$$

It is clear that the two constructions are dual, hence the conclusion. $\square$

7.7. The conjugacy classes in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

Proposition [Kuiper, 1954]**.** *The equivalence classes of locally projective structures on S^1 are in one-to-one correspondence with conjugacy classes in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.*

Proof. Let Z be S^1 equiped with a locally projective structure $\{f_\alpha\}$ and $\tilde{Z}$ a connected component of the topological space of all germs of all coordinate charts in the complete projective atlas with the topology determined by the base

$$N_f = \{\mathrm{germ}_z(f) : z \in \mathrm{Dom}(f) \subset S^1\}.$$

The mapping $\mathrm{germ}_z(f) \in \tilde{Z} \mapsto f(z) \in \mathbb{R}$ is a homeomorphism, because in one connected component we can find only germs of coordinate maps obtained by extending a fixed coordinate map (extending also around the circle). Let g be the coordinate map obtained by extending f exactly one time around the circle ($\mathrm{Dom}(f) = \mathrm{Dom}(g)$) and let $h = gf^{-1}$ be the projective transformation on $\mathbb{R}$ representing the coordinate change.

The mapping

$$\mathrm{germ}_z(f) \in \tilde{Z} \mapsto z \in Z$$

is a universal covering projection. The fundamental group of Z operates on $\tilde{Z}$ as a group of projective transformations generated by the element $h \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$:

$$h^n : \mathrm{germ}_z(f) \mapsto \mathrm{germ}_z(h^n \circ f), n \in \mathbb{Z}.$$

Another connected component $\tilde{Z}_1$, determined by the coordinate map f_1, defines in this way an element $h_1 \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ conjugate to h, because $f_1 = \tilde{\varphi} f$ for some $\tilde{\varphi} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ implies $g_1 = \tilde{\varphi} g$ and then $h_1 = g_1 f_1^{-1} = \tilde{\varphi} h \tilde{\varphi}^{-1}$.

So the locally projective structure $\{f_\alpha\}$ defines a conjugacy class $[h]$ in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. An equivalent structure $f_\alpha \circ \varphi$ with $\varphi \in \mathrm{Diff}_+(S^1)$, defines the same conjugacy class because $(g \circ \varphi) \circ (f \circ \varphi)^{-1} = g \circ f^{-1}$.

Let $\hat{h}$ be a conjugacy class in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We define

$$Z := \mathbb{R}/\{h^n : n \in \mathbb{Z}\}$$

with the locally projective structure induced by the (unique) obvious locally projective structure on $\mathbb{R}$ defined by $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The two constructions are dual. □

In [Segal, 1981] there is a direct proof to the fact that the coadjoint orbits in the Virasoro Bott group are in bijection with conjugacy classes in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

The conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$ are of three types:
–elliptic, if $|\operatorname{tr} A| < 2$;
–hyperbolic, if $|\operatorname{tr} A| > 2$;
–parabolic, if $|\operatorname{tr} A| = 2$.

Correspondingly, the conjugacy classes in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ are of three types. They are determined by the trace and the component in which they lie. The elliptic classes are all in the image of the exponential map and have dimension 2. The parabolic classes are 1-dimensional, the hyperbolic classes 2-dimensional. Only those in the 0-component lie on 1-parameter subgroups.

A coadjoint orbit has a constant representative if and only if the corresponding conjugacy class in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ lies in the image of exp [Segal, 1981].

The picture of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and its conjugacy classes is obtained after rotating the following picture around the horizontal axis. The image of the exponential map is the complement of the dashed part:

7.8. Hill's equation and superalgebras.

This section follows [Kirillov, 1982]. With every regular moment (p, c), $c \neq 0$ we can associate an auxiliary equation, known as *Hill's equation*:

$$2cu''(t) + p(t)u(t) = 0$$

We consider $u = u(t)\sqrt{\frac{d}{dt}}$ as the square root of a vector field (or a density of weight $-\frac{1}{2}$) and the natural action of $\mathrm{Diff}_+(S^1)$:$\mathrm{Ad}^*(\varphi^{-1}) = \varphi^* u = (u \circ \varphi)(\varphi')^{-\frac{1}{2}}$.

Lemma. *If u is a solution of the Hill equation associated to (p,c), then $\mathrm{Ad}^*(\varphi^{-1})u$ is a solution of the Hill equation associated to $(q,c) = \mathrm{Ad}^*(\varphi^{-1})(p,c)$, for every $\varphi \in \mathrm{Diff}_+(S^1)$.*

Proof.

$$(\varphi^* u)' = (u' \circ \varphi)(\varphi')^{\frac{1}{2}} - \frac{1}{2}(u \circ \varphi)(\varphi')^{-\frac{3}{2}}\varphi''$$
$$(\varphi^* u)'' = (u'' \circ \varphi)(\varphi')^{\frac{3}{2}} + \frac{3}{4}(u \circ \varphi)(\varphi')^{-\frac{5}{2}}(\varphi'')^2 - \frac{1}{2}(u \circ \varphi)(\varphi')^{-\frac{3}{2}}\varphi'''.$$

Using $2cu'' + pu = 0$, we get

$$2c(\varphi^* u)'' = ((-pu) \circ \varphi)(\varphi')^{\frac{3}{2}} + \frac{3}{2}c(u \circ \varphi)(\varphi')^{-\frac{5}{2}}(\varphi'')^2$$
$$- c(u \circ \varphi)(\varphi')^{-\frac{3}{2}}\varphi''' = -(\varphi^* u)[(p \circ \varphi)(\varphi')^2 + cS(\varphi)].$$

Hence $\varphi^* u$ is a solution of the equation $2cv'' + qv = 0$. $\square$

The known fact that the product of any two solutions of Hill's equation is a solution of the equation $c\xi''' + 2p\xi' + p'\xi = 0$ which characterizes the isotropy Lie algebra, has an interpretation in the language of superalgebras.

The *Ramon superalgebra* γ is the set of pairs (ξ, u) with ξ a vector field on S^1 and u the square root of a vector field on S^1 with the operations

$$\left[\xi\frac{d}{dt}, \eta\frac{d}{dt}\right] = -(\xi\eta' - \xi'\eta)\frac{d}{dt}$$
$$\left[\xi\frac{d}{dt}, u\sqrt{\frac{d}{dt}}\right] = -(\xi u' - \frac{1}{2}\xi' u)\sqrt{\frac{d}{dt}}$$
$$\left[u\sqrt{\frac{d}{dt}}, v\sqrt{\frac{d}{dt}}\right] = -2uv\frac{d}{dt}$$

The even part of the Ramon superalgebra is $\mathfrak{g} = \mathrm{Vect}(S^1)$ and the Virasoro cocycle

$$\omega(\xi, \eta) = \int_0^{2\pi} \xi' d\eta'$$

on $\mathfrak{g}$ can be extended to the whole superalgebra by

$$\omega(u\sqrt{\frac{d}{dt}}, v\sqrt{\frac{d}{dt}}) = 4\int_0^{2\pi} uv dt$$
$$\omega(\xi\frac{d}{dt}, u\sqrt{\frac{d}{dt}}) = 0$$

The extension $\tilde{\gamma}$ is again a superalgebra with even part the Virasoro algebra $\tilde{\mathfrak{g}}$ and odd part the densities of degree $-\frac{1}{2}$. The bracket operation in $\tilde{\gamma}$ is

$$[(\xi, \lambda, u), (\eta, \mu, v)]$$
$$= (-(\xi\eta' - \xi'\eta + 2uv)\frac{d}{dt}, \int_0^{2\pi}(\xi'\eta'' + 4u'v'), -(\xi v' - \frac{1}{2}\xi' v - \eta u' + \frac{1}{2}\eta' u)\sqrt{\frac{d}{dt}})$$

for $(\xi, \lambda), (\eta, \mu) \in \tilde{\mathfrak{g}}$ and u, v $-\frac{1}{2}$-densities. This induces a coadjoint action ad^* of γ on the super-moment-space $\tilde{\gamma}^*$, because the extension is central.

Proposition. *The isotropy superalgebra of the regular moment* $(p,c,0)$ *is*

$$\gamma_{(p,c)} = \{(\xi\frac{d}{dt}, u\sqrt{\frac{d}{dt}}) : c\xi''' + 2p\xi' + p'\xi = 0, 2cu'' + pu = 0\}$$

Proof.

$$\begin{aligned}\langle \mathrm{ad}^*(\xi,u)(p,c,0),(\eta,\mu,v)\rangle &= \langle (p,c,0), -[(\xi,0,u),(\eta,\mu,v)]\rangle \\ &= \langle p, \xi\eta' - \xi'\eta + 2uv\rangle - c\int_0^{2\pi}(\xi'\eta'' + 4u'v') \\ &= \int_0^{2\pi}(p\xi\eta' - p\xi'\eta - c\xi'\eta'') + 2\int_0^{2\pi}(puv - 2cu'v') \\ &= \langle (-(c\xi''' + 2p\xi' + p'\xi), 0, 2(2cu'' + pu)), (\eta,\mu,v)\rangle\end{aligned}$$

for every (η,μ,v) in the Ramon superalgebra. We get the coadjoint action

$$\mathrm{ad}^*(\xi,u)(p,c,0) = (-(c\xi''' + 2p\xi' + p'\xi), 0, 2(2cu'' + pu))$$

and the desired form of the stabilizing superalgebra. $\square$

Now two solutions u_1, u_2 of the equation

$$2cu'' + pu = 0$$

define two elements $(0,0,u_1\sqrt{\frac{d}{dt}}), (0,0,\sqrt{\frac{d}{dt}})$ of the even part of the superalgebra $\gamma_{(p,c)}$, hence

$$\left[u_1\sqrt{\frac{d}{dt}}, u_2\sqrt{\frac{d}{dt}}\right] = -2u_1u_2\frac{d}{dt}$$

belongs again to $\gamma_{(p,c)}$. This means that u_1u_2 is a solution of

$$c\xi''' + 2p\xi' + p'\xi = 0.$$

8. The coadjoint orbit $\mathbf{Diff}_+(S^1)/\mathbf{Rot}(S^1)$

8.1. The symplectic structure.

Let G be a (possibly infinite dimensional) Lie group with Lie algebra $\mathfrak{g}$. There exists a canonical symplectic structure on each coadjoint orbit

$$\mathcal{O}_\alpha = \mathrm{Ad}^*(G) \cong G/G_\alpha, \quad \alpha \in \mathfrak{g}^*.$$

Construction. We define an alternating bilinear form on $\mathfrak{g}$ by

$$\omega_\alpha(X,Y) = \langle \alpha, [X,Y]\rangle.$$

The kernel of ω_α is the isotropy Lie algebra $\mathfrak{g}_\alpha$, because $\omega_\alpha(X,Y) = -\langle \operatorname{ad}^*(X)\alpha, Y\rangle$. Then ω_α projects to a weakly nondegenerate alternating bilinear form on $\mathfrak{g}/\mathfrak{g}_\alpha$, denoted also ω_α. This is G_α-invariant, so it defines a G-invariant 2-form on G/G_α. By identifying $T_\beta\mathcal{O}_\alpha = \{\zeta_X(\beta) : X \in \mathfrak{g}\}$ this means

$$\omega_\alpha(\zeta_X(\beta), \zeta_Y(\beta)) = \langle \beta, [X,Y]\rangle.$$

Indeed, if $\beta = \operatorname{Ad}^*(g)\alpha$, then $\omega(\zeta_X(\beta), \zeta_Y(\beta))= \omega(\zeta_{\operatorname{Ad}(g^{-1})X}(\alpha), \zeta_{\operatorname{Ad}(g^{-1})Y}(\alpha)) = \langle \alpha, [\operatorname{Ad}(g^{-1})X, \operatorname{Ad}(g^{-1})Y]\rangle = \langle \operatorname{Ad}^*(g)\alpha, [X,Y]\rangle= \langle \beta, [X,Y]\rangle$.

This is a closed non-degenerate form, hence a symplectic form on the orbit. ω is closed because

$$\begin{aligned} d\omega(\zeta_X,\zeta_Y,\zeta_Z) &= \zeta_X\omega(\zeta_Y,\zeta_Z) + \zeta_Y\omega(\zeta_Z,\zeta_X) + \zeta_Z\omega(\zeta_X,\zeta_Y) \\ &\quad - \omega([\zeta_X,\zeta_Y],\zeta_Z) - \omega([\zeta_Y,\zeta_Z],\zeta_X) - \omega([\zeta_Z,\zeta_X],\zeta_Y) \end{aligned}$$

and the identities

$$(\zeta_X\omega(\zeta_Y,\zeta_Z))(\beta) = \frac{d}{dt}\Big|_0 \omega(\zeta_Y,\zeta_Z)(\operatorname{Ad}^*(\exp tX).\beta)$$
$$= \frac{d}{dt}|_0\langle \operatorname{Ad}^*(\exp tX).\beta, [Y,Z]\rangle = \langle \operatorname{ad}^*(X).\beta, [Y,Z]\rangle = -\langle \beta, [X,[Y,Z]]\rangle$$
$$\omega([\zeta_X,\zeta_Y],\zeta_Z)(\beta) = \omega(\zeta_{[X,Y]},\zeta_Z)(\beta) = \langle \beta, [[X,Y],Z]\rangle$$

reduce the closedness condition to the Jacoby identity. ω is non degenerate because $\omega(\zeta_X(\beta),\zeta_Y(\beta)) = \beta([X,Y]) = 0$ for all $Y \in \mathfrak{g}$ is equivalent to $Y \in \mathfrak{g}_\beta$, i.e. $\zeta_Y(\beta) = 0$.

Moreover, the action of G on $\mathcal{O}_\alpha$ is Hamiltonian with symplectic moment the inclusion $\mathcal{O}_\alpha \hookrightarrow \mathfrak{g}^*$. This means $\zeta_X = H_{\operatorname{ev}_X}$:

$$\begin{aligned} d\operatorname{ev}_X .\zeta_Y(\beta) &= \frac{d}{dt}|_0 \operatorname{ev}_X(\operatorname{Ad}^*(\exp tY).\beta) = \langle \operatorname{ad}^*(Y).\beta, X\rangle \\ &= \langle \beta, [X,Y]\rangle = \omega(\zeta_X(\beta),\zeta_Y(\beta)) = i_{\zeta_X}\omega.\zeta_Y(\beta), \forall \zeta_Y(\beta) \in T_\beta\mathcal{O}, \end{aligned}$$

and the map $X \in \mathfrak{g} \mapsto \operatorname{ev}_X \in C^\infty(\mathcal{O}_\alpha, \mathbb{R})$ is a Lie algebra homomorphism:

$$\{\operatorname{ev}_X, \operatorname{ev}_Y\} = \omega(H_{\operatorname{ev}_X}, H_{\operatorname{ev}_Y}) = \omega(\zeta_X,\zeta_Y) = \operatorname{ev}_{[X,Y]}.$$

Now we consider the case of the Virasoro-Bott group and the coadjoint orbit of the constant moment $(p,c) = (p(dt)^2, c) \in \tilde{\mathfrak{g}}^*_{reg}$ with $\frac{2p}{c} \neq n^2$, $n \in \mathbb{N}$, which is diffeomorphic to $\operatorname{Diff}_+(S^1)/\operatorname{Rot}(S^1)$.

The references for the rest of this chapter are [Kirillov, 1990] and [Kirillov, 1990]. We compute the bilinear form $\omega_{(p,c)}$:

$$\begin{aligned} \omega_{(p,c)}(\xi,\eta) &= \langle (p.c), \begin{pmatrix} [\xi,\eta] \\ \omega(\xi,\eta) \end{pmatrix} \rangle \\ &= p\int_0^{2\pi} (\xi\eta' - \xi'\eta)dt + c\int_0^{2\pi} \xi'\eta'' dt \\ &= \int_0^{2\pi} \xi(2p\eta' - c\eta'')dt, \quad \forall \xi\eta \in \mathfrak{g}. \end{aligned}$$

If we expand ξ, η in Fourier series

$$\xi(t) = \sum_{k\in\mathbb{Z}} \xi_k e^{ikt}, \eta(t) = \sum_{k\in\mathbb{Z}} \eta_k e^{ikt},$$

(the coefficients $\xi_k, \eta_k \in \mathbb{C}$ verify $\bar{\xi}_k = \xi_{-k}, \bar{\eta}_k = \eta_{-k}$) we get

$$\begin{aligned}
\omega_{(p,c)}(\xi,\eta) &= \int_0^{2\pi} (\sum_{k\in\mathbb{Z}} \xi_k e^{ikt})(2pi \sum_{l\in\mathbb{Z}} l\eta_l e^{ilt} - ci^3 \sum_{l\in\mathbb{Z}} l^3 \eta_l e^{ilt}) dt \\
&= i \sum_{k\geq 1} (2p(-k) + c(-k)^3) \xi_k \eta_{-k} \\
&= -i \sum_{k\geq 1} (2pk + ck^3) \xi_k \bar{\eta}_k \\
&= Im(\sum_{k\geq 1} (2pk + ck^3) \xi_k \bar{\eta}_k).
\end{aligned}$$

The tangent space at (p,c) to the orbit can be identified with $\operatorname{Vect}(S^1)/\mathbb{R}$, i.e. the space $C_0^\infty(S^1)$ of smooth functions on the circle with zero integral. This means the 0-coefficient ξ_0 in the Fourier expansion is zero.

This two-parameter family of non-degenerate alternating bilinear forms $\omega_{(p,c)}$ on $\operatorname{Vect}(S^1)/\mathbb{R}$ form by $\operatorname{Diff}_+(S^1)$-invariance a two-parameter family of different homogeneous symplectic structures on $\operatorname{Diff}_+(S^1)/\operatorname{Rot}(S^1)$.

8.2. The almost complex structure.

The Hilbert transformation operator J on $C_0^\infty(S^1) \cong \operatorname{Vect}(S^1)/\mathbb{R}$ is defined by

$$J(\xi)(s) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\xi(s) - \xi(t)}{\operatorname{tg} \frac{s-t}{2}} dt.$$

For a Fourier series this means $J(\sum_{k\neq 0} \xi_k e^{ikt}) = \sum_{k\neq 0} i\xi_k (\operatorname{sgn} k) e^{ikt}$ and we see that $J^2 = -I$. Because J is also $\operatorname{Rot}(S^1)$-invariant, it defines a $\operatorname{Diff}_+(S^1)$-invariant almost complex structure on the homogeneous space $\operatorname{Diff}_+(S^1)/\operatorname{Rot}(S^1)$. The eigenspace corresponding to the eigenvalue i is

$$V_+ := \{\sum_{k\neq 0} \xi_k e^{ikt} : i\xi_k = i\xi_k \operatorname{sgn} k\} = \{\sum_{k>0} \xi_k e^{ikt}\},$$

the space of boundary values of holomorphic functions in $D := \{z \in \mathbb{C} : |z| < 1\}$ and the eigenspace corresponding to the eigenvalue $-i$ is $V_- = (V_+)^\perp$, where the orthogonality is relative to the scalar product

$$\langle \sum \xi_k e^{ikt}, \sum \eta_k e^{ikt} \rangle = \sum \xi_k \bar{\eta}_k.$$

J satisfies the integrability condition $[J,J] = 0$, but in the infinite dimensional case this doesn't assure the existence of a complex structure.

Proposition. *On* $\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$ *there is only one integrable almost complex structure which is* $\mathrm{Diff}_+(S^1)$*-invariant.*

Proof. To give a G-invariant almost complex structure J on G/H is the same as to give an H-invariant decomposition

$$(\mathfrak{g}/\mathfrak{h})^{\mathbb{C}} = V_+ \oplus V_-, \text{where } V_- = \bar{V}_+.$$

Here V_+ and V_- are the eigenspaces of $J^{\mathbb{C}}$ at 0 corresponding to the eigenvalues i and $-i$.

Let $\pi^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to (\mathfrak{g}/\mathfrak{h})^{\mathbb{C}}$ be the complexification of the canonical projection π. J is integrable if and only if $\mathfrak{p}_+ := (\pi^{\mathbb{C}})^{-1}(V_+)$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Indeed: let $l : G \times (G/H) \to G/H$ and ζ_X the fundamental vector field induced by this action. The vector fields $R_X \times 0$ on $G \times (G/H)$ and ζ_X on G/H are l-related for every $X \in \mathfrak{g}$. The complex distribution on G/H obtained by translating V_+ with the G-action is generated by ζ_X, where $X \in \mathfrak{p}_+$. J is integrable if and only if this distribution is integrable. This is equivalent to the integrability of the distribution in G generated by R_X with $X \in \mathfrak{p}_+$, i.e. $[\mathfrak{p}_+, \mathfrak{p}_+] \subset \mathfrak{p}_+$.

Hence it suffices to find all $\mathrm{Rot}(S^1)$-invariant decompositions of

$$C_0^\infty(S^1)^{\mathbb{C}} = \{\sum_{k\neq 0} \xi_k e^{ikt} : \xi_k \in \mathbb{C} \text{ fast falling}\}.$$

Such a decomposition has the form $V_A \oplus V_{-A}$, where $V_A = \{\sum_{k\in A} \xi_k e^{ikt}\}$ with $A \subset \mathbb{Z} - \{0\}$, because

$$\mathrm{Ad}(R_\vartheta)(\sum \xi_k e^{ikt}) = \sum \xi_k e^{ik\vartheta} e^{ikt}.$$

The condition that $\mathfrak{p}_+ = \{\sum_{k\in A\cup\{0\}} \xi_k e^{ikt}\}$ is a Lie algebra gives

$$k, l \in A \cup \{0\}, k \neq l \Rightarrow k + l \in A \cup \{0\}.$$

We use here the fact that

$$[e^{ikt}\frac{d}{dt}, e^{ilt}\frac{d}{dt}] = i(l-k)e^{i(k+l)t}\frac{d}{dt}$$

Hence $A = \mathbb{N}$ or $A = -\mathbb{N}$ and the required decomposition is unique. $\square$

8.3. The complex structure.

The set of univalent functions

$$\mathcal{F} := \{f : \bar{D} \to \mathbb{C} \text{ holomorphic } : f(0) = 0, f'(0) = 1, f \text{ injective}\}$$

is an infinite dimensional complex manifold; the coordinate map is

$$z + c_1 z + c_2 z^2 + \cdots \mapsto (c_1, c_2, \cdots), c_i \in \mathbb{C}.$$

Proposition. *The spaces* $\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$ *and* $\mathcal{F}$ *are homeomorphic.*

Proof. Let $f \in \mathcal{F}$. By the Riemann mapping theorem, there exists a uniquely defined, modulo $\mathrm{Rot}(S^1)$, biholomorphic mapping $g : \hat{\mathbb{C}} - D \to \hat{\mathbb{C}} - f(D)$ such that $g(\infty) = \infty$. Then $\gamma = f^{-1} \circ g : \partial\bar{D} \to \partial\bar{D}$ is an element of $\mathrm{Diff}_+(S^1)$ defined modulo $\mathrm{Rot}(S^1)$.

Conversely, let $\gamma \in \mathrm{Diff}_+(S^1)$. The complement of $\bar{D}$ in the Riemann sphere $\hat{\mathbb{C}}$, denoted A, is homeomorphic to D, so we can glue $\bar{D}$ and $\bar{A}$ along their boundaries using γ and we get $\hat{\mathbb{C}}_\gamma := \bar{D} \sqcup_\gamma \bar{A}$. The analytic structure on $\hat{\mathbb{C}}_\gamma$ is given by the condition

$$F : \hat{\mathbb{C}}_\gamma \to \hat{\mathbb{C}} \text{ analytic} \Leftrightarrow \begin{cases} F|D, F|A & \text{analytic} \\ F & \text{continuous.} \end{cases}$$

The Riemann surface $\hat{\mathbb{C}}_\gamma$ is topologically a sphere and depends only on the class of γ in $\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$. By Riemann's uniformization theorem, $\hat{\mathbb{C}}_\gamma$ is biholomorphically equivalent to the Riemann sphere $\hat{\mathbb{C}}$. The automorphisms of $\hat{\mathbb{C}}$ form the group of Moebius transformations which is 3-dimensional, hence we have 3 free complex parameters in the choice of the biholomorphic map $F : \hat{\mathbb{C}}_\gamma \to \hat{\mathbb{C}}$ and the conditions $F(0) = 0$, $F(\infty) = \infty$, $F'(0) = 1$ uniquely define F. We assign to the class $\gamma.\mathrm{Rot}(S^1)$ the univalent function $f = F|\bar{D} \in \mathcal{F}$. This is the inverse of the map defined earlier because if $g := F|\bar{A}$, then $f \circ \gamma = g$ on S^1. $\square$

Moreover, the homeomorphism constructed above and its inverse are smooth (they map smooth curves into smooth curves), so the spaces $\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$ and $\mathcal{F}$ are diffeomorphic. Hence on $\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$ there is a complex structure: the one induced from $\mathcal{F}$. Is this diffeomorphism $\mathrm{Diff}_+(S^1)$-invariant? If yes, then it integrates exactly the almost complex structure given by the Hilbert transformation.

8.4. The Kähler structure.

For every pair (p, c) with $\frac{2p}{c} \neq n^2$, the complex structure J is compatible with the symplectic structure $\omega_{(p,c)}$:

$$\omega_{(p,c)}(J\xi, J\eta) = \omega_{(p,c)}(\sum_{k\neq 0} i\xi_k(\operatorname{sgn} k)e^{ikt}, \sum_{k\neq 0} i\eta_k(\operatorname{sgn} k)e^{ikt})$$
$$= -i\sum_{k\geq 1}(2pk + ck^3)(i\xi_k \operatorname{sgn} k)\overline{(i\eta_k \operatorname{sgn} k)} = -i\sum_{k\geq 1}(2pk + ck^3)\xi_k\bar{\eta}_k = \omega_{(p,c)}(\xi, \eta),$$

hence they define a family of Kähler structures on $\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$:

$$\sum_{k\geq 1}(2pk + ck^3)dc_k d\bar{c}_k.$$

9. Subgroups of the diffeomorphism group

9.1. Diffeomorphisms preserving a given structure.

Definition. Let ω be a differential p-form on M.

A diffeomorphism φ of M is said to be an *automorphism of* ω if $\varphi^*\omega = \omega$ and a *conformal transformation of* ω if $\varphi^*\omega = \rho\omega$ for some positive valued function ρ on M.

An *infinitesimal automorphism of* ω is a vector field ξ on M such that for every element (t, x) of the domain of the flow of ξ,

$$(\mathrm{Fl}^\xi_t)^*\omega(x) = \omega(x);$$

and a *conformal infinitesimal transformation of* ω, if

$$(\mathrm{Fl}^\xi_t)^*\omega(x) = \rho(t, x)\omega(x)$$

for a positive valued smooth function ρ.

Proposition. *A vector field ξ is an infinitesimal automorphism of ω if and only if*

$$\mathcal{L}_\xi\omega = 0$$

and a conformal infinitesimal transformation if and only if

$$\mathcal{L}_\xi\omega = \lambda\omega$$

where λ is a smooth function on M. The function λ is related to ρ by

$$\rho(t, x) = \exp(\int_0^t \lambda \circ \mathrm{Fl}^\xi_\tau(x)d\tau).$$

Proof. The assertion that ξ is a (conformal) infinitesimal automorphism implies $\mathcal{L}_\xi\omega = 0$ (resp. $\mathcal{L}_\xi\omega = \lambda\omega$) follows from

$$\frac{d}{dt}\Big|_0 (\mathrm{Fl}^\xi_t)^*\omega = \mathcal{L}_\xi\omega.$$

The converse in the first case: let $\mathcal{L}_\xi\omega = 0$, then $(\mathrm{Fl}^\xi_t)^*\omega$ is constant in t:

$$\frac{d}{dt}((\mathrm{Fl}^\xi_t)^*\omega) = \frac{d}{ds}\Big|_0 (\mathrm{Fl}^\xi_{s+t})^*\omega = (\mathrm{Fl}^\xi_t)^*\frac{d}{ds}\Big|_0 (\mathrm{Fl}^\xi_s)^*\omega = (\mathrm{Fl}^\xi_t)^*(\mathcal{L}_\xi\omega) = 0$$

hence $(\mathrm{Fl}^\xi_t)^*\omega = (\mathrm{Fl}^\xi_0)^*\omega = \omega$.

The converse in the conformal case: let ξ satisfy $\mathcal{L}_\xi\omega = \lambda\omega$. Then

$$\frac{d}{dt}((\mathrm{Fl}^\xi_t)^*\omega) = (\mathrm{Fl}^\xi_t)^*(\mathcal{L}_\xi\omega) = (\mathrm{Fl}^\xi_t)^*(\lambda\omega) = (\lambda \circ \mathrm{Fl}^\xi_t)(\mathrm{Fl}^\xi_t)^*\omega$$

and so the curve $t \mapsto C(t) = (\mathrm{Fl}^\xi_t)^*\omega(x)$ in $\wedge^p T^*_x M$ satisfies the ordinary differential equation:

$$\begin{cases} \frac{d}{dt}C(t) = (\lambda \circ \mathrm{Fl}^\xi_t(x))C(t) \\ C(0) = \omega(x) \end{cases}$$

whose unique solution is $t \mapsto \rho(t,x)\omega(x)$, where

$$\rho(t,x) = \exp(\int_0^t \lambda \circ \mathrm{Fl}^\xi_\tau(x) d\tau).$$

Hence $(\mathrm{Fl}^\xi_t)^* \omega(x) = \rho(t,x)\omega(x)$. $\square$

The set of infinitesimal automorphisms of ω and the set of conformal infinitesimal automorphisms of ω are Lie subalgebras of $\mathrm{Vect}(M)$. We give the proof in the conformal case: let $\xi, \eta \in \mathrm{Vect}(M)$ such that $\mathcal{L}_\xi \omega = \lambda_\xi \omega$, $\mathcal{L}_\eta \omega = \lambda_\eta \omega$, for $\lambda_\xi, \lambda_\eta$ functions on M. Then $\mathcal{L}_{[\xi,\eta]}\omega = [\mathcal{L}_\xi, \mathcal{L}_\eta]\omega = [\xi(\lambda_\eta) - \eta(\lambda_\xi)]\omega$, hence $[\xi,\eta]$ is again a conformal infinitesimal automorphism of ω. However the group of automorphisms, resp. conformal automorphisms of ω in general do not form a Lie subgroup of $\mathrm{Diff}(M)$, because it is not always a manifold. This can be done by putting some non-degeneracy conditions on ω.

(a) The group of symplectomorphisms.

Let (M,ω) be a connected smooth symplectic manifold, i.e. ω is a closed 2-form on M such that ω^n is a volume form. The space of infinitesimal automorphisms of ω:

$$\mathrm{Vect}_c(M,\omega) = \{\xi \in \mathrm{Vect}_c(M) : \mathcal{L}_\xi \omega = 0\}$$

is a Lie subalgebra of $\mathrm{Vect}_c(M)$. In [Michor, 1980] it is shown that the group of symplectomorphisms

$$\mathrm{Diff}_c(M,\omega) = \{\varphi \in \mathrm{Diff}_c(M) : \varphi^*\omega = \omega\}$$

is a Lie subgroup of $\mathrm{Diff}_c(M)$ and, if M is compact, its Lie algebra is the space of infinitesimal automorphisms of ω.

The vector space $\mathrm{Vect}_c(M,\omega)$ can be identfied with $Z^1_c(M)$, the space of closed differential 1-forms on M with compact support, because the correspondence $\xi \mapsto i_\xi \omega$ is an isomorphism between vector fields and 1-forms by which the infinitesimal automorphisms of ω correspond to the closed 1-forms

$$d(i_\xi \omega) = \mathcal{L}_\xi \omega = 0.$$

(b) The group of volume preserving diffeomorphisms.

Let μ be a volume form on the compact manifold M. Let $\mathrm{Diff}_c(M,\mu) = \{\varphi \in \mathrm{Diff}_c(M) : \varphi^*\mu = \mu\}$ be the volume preserving diffeomorphisms (the automorphisms of μ) and $\mathrm{Vect}_c(M,\mu) = \{\xi \in \mathrm{Vect}_c(M) : \mathcal{L}_\xi \mu = 0\}$ the Lie algebra of infinitesimal automorphisms of μ.

The divergence of a vector field ξ is the unique function div such that $\mathcal{L}_\xi \mu = (\mathrm{div}\,\xi)\mu$, so the infinitesimal automorphisms are the zero divergence vector fields.

It follows from the next theorem that $\mathrm{Diff}_c(M,\mu)$ is a submanifold of $\mathrm{Diff}_c(M)$, hence a Lie subgroup with Lie algebra $\mathrm{Vect}_c(M,\mu)$.

Theorem. [Ebin-Marsden, 1970]. *Let M be a compact orientable manifold and μ_0 a volume form on M with total mass 1. Then* $\mathrm{Diff}(M)$ *splits smoothly into*

$\mathrm{Diff}(M) = \mathrm{Diff}(M, \mu_0) \times \mathrm{Vol}(M)$, *where* $\mathrm{Vol}(M)$ *is the space of all volume forms with total mass 1.*

Proof. We construct first a smooth mapping $\tau : \mathrm{Vol}(M) \to \mathrm{Diff}(M)$ such that $\tau(\mu)^*\mu_0 = \mu$. Let $\mu_1 \in \mathrm{Vol}(M)$ and $\mu_t = \mu_0 + t(\mu_1 - \mu_0)$. We search for a curve $t \mapsto \varphi_t$ in $\mathrm{Diff}(M)$ with $\varphi_t^*\mu_t = \mu_0$ and we will find one as the evolution operator of a well chosen time dependent vector field X_t, i.e. $\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$, $\varphi_0 = \mathrm{Id}$. Then

$$0 = \frac{d}{dt}(\varphi_t^*\mu_t) = \varphi_t^*\mathcal{L}_{X_t}\mu_t + \varphi_t^*(\mu_1 - \mu_0)$$

implies $\mathcal{L}_{X_t}\mu_t = \mu_0 - \mu_1$. Because $\int_M(\mu_1 - \mu_0) = 0$, there exists an $(n-1)$-form ω such that $di_{X_t}\mu_t = \mu_0 - \mu_1 = d\omega$ and we can choose by using Hodge theory (so the condition M compact is necessary) ω depending smoothly on μ_1. Then it is sufficient to choose X_t as the unique time dependent vector field satisfying $i_{X_t}\mu_t = \omega$ (this is possible because μ_t is a volume form for every t). Now we denote by φ_t the evolution operator of this X_t and by going back we find $\frac{d}{dt}(\varphi_t^*\mu_t) = 0$, hence $\varphi_1^*\mu_1 = \mu_0$. The mapping τ can be defined as $\tau(\mu_1) = \varphi_1^{-1}$ and it is smooth because it maps smooth curves into smooth curves.

Now let the mapping $\Psi : \mathrm{Diff}(M) \to \mathrm{Diff}(M, \mu) \times \mathrm{Vol}(M)$ be given by $\Psi(\varphi) := (\varphi \circ \tau(\varphi^*\mu_0)^{-1}, \varphi^*\mu_0)$. Its inverse is $\Psi^{-1}(\psi, \mu) = \psi \circ \tau(\mu)$ and both are smooth because τ is smooth (see also chapter 2.), hence the conclusion. □

Corollary. *For a compact orientable manifold all the groups* $\mathrm{Diff}(M, \mu)$ *are diffeomorphic.*

Proof. From the theorem it follows that $\mathrm{Diff}(M)$ acts transitvely on the space of volume forms of total mass c. Let μ_1 and μ_2 be volume forms with mass c, then the groups $\mathrm{Diff}(M, \mu_1)$ and $\mathrm{Diff}(M, \mu_2)$ are the isotropy groups of μ_1, respectively μ_2, hence they are conjugated subgroups in $\mathrm{Diff}(M)$. If c is a constant, then $\mathrm{Diff}(M, \mu) = \mathrm{Diff}(M, c\mu)$. These two facts solve the problem. □

(c) The group of contact diffeomorphisms.

Let (M, α) be a strict contact manifold, i.e. α is a 1-form on M such that $\alpha \wedge (d\alpha)^n$ is a volume form. It has been shown for a compact manifold in [Ratiu-Schmid, 1981] that the group of contact diffeomorphisms

$$\mathrm{Diff}_c(M, \alpha) = \{\varphi \in \mathrm{Diff}_c(M) : \varphi^*\alpha = \rho\alpha, \rho \in C^\infty(M)\}$$

is a Lie group with Lie algebra the space of conformal infinitesimal transformations

$$\mathrm{Vect}_c(M, \alpha) = \{\xi \in \mathrm{Vect}_c(M) : \mathcal{L}_\xi\alpha = \lambda\alpha, \lambda \in C^\infty(M)\}.$$

Proposition. *The map*

$$\xi \in \mathrm{Vect}(M, \alpha) \mapsto f_\xi = \alpha(\xi) \in C^\infty(M)$$

is an isomorphism.

First we need a lemma.

Lemma. *There exists a unique vector field ϵ on M, called the Reeb vector field on M, which satisfies the following conditions:*

$$i_\epsilon \alpha = 1 \quad , \quad i_\epsilon d\alpha = 0.$$

Proof of the lemma. We show that these conditions are equivalent to

$$i_\epsilon(\alpha \wedge (d\alpha)^n) = (d\alpha)^n$$

and from the fact that $\alpha \wedge (d\alpha)^n$ is a volume form we get the existence and uniqueness of ϵ.

The direct implication is evident. For the reverse let $i_\epsilon(\alpha \wedge (d\alpha)^n) = (d\alpha)^n$. Because $i_\epsilon \circ i_\epsilon = 0$, we have $i_\epsilon (d\alpha)^n = 0$. Consequently $(d\alpha)^n = i_\epsilon(\alpha \wedge (d\alpha)^n) = (i_\epsilon\alpha)(d\alpha)^n$, therefore $i_\epsilon\alpha = 1$. On the other hand $0 = i_\epsilon(d\alpha)^n = n(i_\epsilon d\alpha) \wedge (d\alpha)^{n-1}$, and $(d\alpha)^n \neq 0$, therefore $i_\epsilon d\alpha = 0$. $\square$

The tangent bundle may be decomposed into

$$TM = \ker d\alpha \oplus \ker \alpha.$$

Indeed, $\ker d\alpha$ is of rank 1 (generated by ε), $\ker \alpha$ is of rank $2n$ (called the horizontal bundle) and the intersection is 0 because if $i_\xi\alpha = 0$ and $i_\xi d\alpha = 0$, then $i_\xi(\alpha \wedge (d\alpha)^n) = 0$ and this implies $\xi = 0$. It follows that every vector field is uniquely determined by $i_\xi\alpha$ and $i_\xi d\alpha$. The unique decomposition of a vector field ξ is: $\xi = (i_\xi\alpha)\epsilon + (\xi - (i_\xi\alpha)\epsilon)$.

Proof of the proposition. Every infinitesimal contactomorphism ξ is completely determined by the function $f_\xi = \alpha(\xi)$, because the relation $\mathcal{L}_\xi\alpha = \lambda\alpha$ may be written

$$df_\xi + i_\xi d\alpha = \lambda\alpha.$$

Since $i_\epsilon d\alpha = 0$ and $i_\epsilon\alpha = 1$, after applying i_ϵ we get $\lambda = i_\epsilon(df_\xi)$. Finally $i_\xi d\alpha = i_\epsilon(df_\xi)\alpha - df_\xi$ depends only on f_ξ, also $i_\xi\alpha = f_\xi$, and together they uniquely determine ξ.

It remains to verify that every ξ defined in this way starting with an arbitrary smooth function f, is an infinitesimal contactomorphism:

$$\mathcal{L}_\xi\alpha = i_\xi d\alpha + d(i_\xi\alpha) = i_\epsilon(df)\alpha - df + df = i_\epsilon(df)\alpha. \quad \square$$

This isomorphism restricts to an isomorphism between $\mathrm{Vect}_c(M,\alpha)$ and the space of smooth functions on M with compact support $C_c^\infty(M)$.

9.2. Splitting subgroups.

Let (M,ω) be a symplectic manifold. The symplectic form induces an isomorphism $\xi \mapsto i_\xi\omega$

$$\omega : \mathrm{Vect}(M) \to \Omega^1(M).$$

The space $\mathrm{Vect}(M,\omega)$ is the preimage of $Z^1(M)$ and the space of *Hamiltonian vector fields* $\mathrm{Ham}(M,\omega)$ is defined as the preimage of $B^1(M)$ under this isomorphism. The Hamiltonian vector field with generating function $u \in C^\infty(M)$ is denoted by H_u.

Remark. $\mathrm{Ham}(M,\omega)$ is a Lie subalgebra of $\mathrm{Vect}(M,\omega)$ because

$$[\mathrm{Vect}(M,\omega),\mathrm{Vect}(M,\omega)] \subset \mathrm{Ham}(M,\omega).$$

This follows from $i_{[\xi,\eta]}\omega = [\mathcal{L}_\xi, i_\eta]\omega = \mathcal{L}_\xi i_\eta\omega = di_\xi i_\eta\omega + i_\xi di_\eta\omega = d(\omega(\eta,\xi))$ for every ξ,η corresponding to closed 1-forms.

Let (M,μ) be a smooth manifold with a volume form. This induces an isomorphism $\xi \mapsto i_\xi\mu$

$$\mu : \mathrm{Vect}(M) \to \Omega^{m-1}(M).$$

The preimage of $Z^{m-1}(M)$ is the Lie subalgebra $\mathrm{Vect}(M,\mu)$. The preimage of $B^{m-1}(M)$ is denoted $B(M,\mu)$.

Remark. $B(M,\mu)$ is a Lie subalgebra of $\mathrm{Vect}(M,\mu)$. Indeed, for $\xi,\eta \in \mathrm{Vect}(M,\mu)$ we have $i_{[\xi,\eta]}\mu = d(i_\xi i_\eta\mu)$, hence

$$[\mathrm{Vect}(M,\mu),\mathrm{Vect}(M,\mu)] \subset B(M,\mu).$$

The exact sequence

$$0 \to B^p(M) \hookrightarrow Z^p(M) \to H^p(M) \to 0$$

gives exact sequences of Lie algebras:

$$\begin{gathered}0 \to \mathrm{Ham}(M,\omega) \hookrightarrow \mathrm{Vect}(M,\omega) \to H^1(M) \to 0\\ 0 \to B(M,\mu) \hookrightarrow \mathrm{Vect}(M,\mu) \to H^{m-1}(M) \to 0,\end{gathered}$$

where on $H^1(M)$ and on $H^{m-1}(M)$ we put the trivial Lie algebra structure. The last two remarks assure that the morphisms

$$\begin{gathered}\xi \in \mathrm{Vect}(M,\omega) \mapsto [i_\xi\omega] \in H^1(M)\\ \xi \in \mathrm{Vect}(M,\mu) \mapsto [i_\xi\mu] \in H^{m-1}(M)\end{gathered}$$

are Lie algebra morphisms.

Proposition. *For a compact manifold, the spaces of closed p-forms and exact p-forms are splitting subspaces in the space of p-forms.*

Proof. Let g be a Riemannian metric on the compact manifold M, δ the codifferential and $\mathrm{Harm}^p(M) = \ker(\Delta = d\delta + \delta d)$ the space of harmonic p-forms. Then we have the following Hodge decomposition

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M) \oplus \mathrm{Harm}^p(M).$$

More precisely, $\alpha = d\delta G\alpha + \delta dG\alpha + H\alpha$, where H is the projection on $\mathrm{Harm}^p(M)$ and G the Green operator: the projection on $\mathrm{Harm}^p(M)^\perp$. Because $Z^p(M) = B^p(M) \oplus \mathrm{Harm}^p(M)$, we get that both $B^p(M)$ and $Z^p(M)$ are splitting subspaces of $\Omega^p(M)$ $\square$

Let M be compact. Then the two commutator groups: $[\mathrm{Diff}(M,\omega),\mathrm{Diff}(M,\omega)]$ and $[\mathrm{Diff}(M,\mu),\mathrm{Diff}(M,\mu)]$ are Lie subgroups of $\mathrm{Diff}(M)$ by [Ratiu-Schmid, 1981] with Lie algebras (see also 9.3):

$$[\mathrm{Vect}(M,\omega),\mathrm{Vect}(M,\omega)] = \mathrm{Ham}(M,\omega)$$
$$[\mathrm{Vect}(M,\mu),\mathrm{Vect}(M,\mu)] = B(M,\mu)$$

respectively. By the isomorphisms ω, resp. μ, the Lie subalgebras $\mathrm{Vect}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$, resp. $\mathrm{Vect}(M,\mu)$ and $B(M,\mu)$, correspond to the spaces of closed and exact 1-forms, resp. $(m-1)$-forms, hence by the proposition they are splitting subspaces of $\mathrm{Vect}(M)$. We just proved the following

Corollary. *If M is compact, then* $\mathrm{Diff}(M,\omega)$, $[\mathrm{Diff}(M,\omega),\mathrm{Diff}(M,\omega)]$, $\mathrm{Diff}(M,\mu)$ *and* $\mathrm{Diff}(M,\mu),\mathrm{Diff}(M,\mu)$ *are splitting subgroups of* $\mathrm{Diff}(M)$.

9.3. The commutator algebra of some subalgebras of $\mathrm{Vect}(M)$**.**

Lemma. *The Lie algebras* $\mathrm{Vect}_c(\mathbb{R}^n,\mu) = B_c(\mathbb{R}^n,\mu)$ *and* $\mathrm{Vect}_c(\mathbb{R}^{2n+1},\alpha)$ *are perfect, where* $\mu = dx^1\wedge\cdots\wedge dx^n$ *is the standard volume form on* $\mathbb{R}^n$ *and* $\alpha = dx^{2n+1} + \sum_{i=1}^n x^1 dx^{n+i}$ *is the standard contact form on* $\mathbb{R}^{2n+1}$.

Proof. See [Arnold, 1969] and [Rozenfeld, 1970]. □

Proposition. *1. Let (M,α) be a contact manifold. Then* $\mathrm{Vect}_c(M,\alpha)$ *is a perfect Lie algebra.*
2. Let (M,μ) be a manifold with a volume form. Then the commutator algebra of $\mathrm{Vect}_c(M,\mu)$ *is* $B_c(M,\mu)$*. Moreover,* $B_c(M,\mu)$ *is perfect.*

Proof. 1. Let $\mathcal{O} = \{U_\nu : \nu\in I\}$ be an open cover of M with canonical coordinate domains and $(\varphi_\nu)_{\nu\in I}$ a partition of unity subordinated to $\mathcal{O}$. Let $\xi\in\mathrm{Vect}_c(M,\alpha)$ and $f=\alpha(\xi)$. The support of ξ equals the support of f and is covered by finitely many sets of $\mathcal{O}$, say $\operatorname{supp}\xi\subset U_1\cup\cdots\cup U_k$. Let $f_j=\varphi_j f$ and $\xi_j\in\mathrm{Vect}_c(M,\alpha)$ corresponding to f_j by the isomorphism in 9.1(c). Then $\xi=\sum_{j=1}^k\xi_j$ and because the support of ξ_j lies in a canonical coordinate domain we can apply the lemma and get the conclusion.

2. Let $\mathcal{O} = \{U_\nu : \nu\in I\}$ be an open cover of M with canonical coordinate domains and $(\varphi_\nu)_{\nu\in I}$ a partition of unity subordinated to $\mathcal{O}$. Let $\xi\in B_c(M,\mu)$ and choose $\sigma\in\Omega_c^{n-2}(M)$ such that $i_\xi\mu = d\sigma$. The support of σ contains the support of ξ and is covered by finitely many sets of $\mathcal{O}$, say $\operatorname{supp}\sigma\subset U_1\cup\cdots\cup U_k$. Let $\sigma_j=\varphi_j\sigma$ and $\xi_j\in B_c(M,\mu)$ corresponding to $d\sigma_j$ by the isomorphism μ in 9.2. Then $\xi=\sum_{j=1}^k\xi_j$ and because the support of ξ_j lies in a canonical coordinate domain we can apply the lemma and get the conclusion. □

Let (M,ω) be a symplectic manifold.

Definition. The *symplectic pairing* is the alternating bilinear form of $H_c^1(M)$

$$\langle[\alpha],[\beta]\rangle := \int_M \alpha\wedge\beta\wedge\omega^{n-1} \text{ for } \alpha,\beta\in Z_c^1(M).$$

The symplectic pairing is trivial for the cotangent bundle T^*N with the canonical symplectic structure, because in this case the symplectic form is exact.

Proposition. *The symplectic pairing is nonsingular for compact Kähler manifolds.*

Proof. Let M be a compact Kähler manifold. Then the Lefschetz Theorem assures that the map

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

which maps $[\beta]$ to $[\beta \wedge \omega^k]$ is an isomorphism. In particular $[\beta] \in H^1(M) \mapsto [\beta \wedge \omega^{n-1}] \in H^{2n-1}(M)$ is an isomorphism. Using also the Poincare duality

$$([\alpha], [\mu]) \in H_c^1(M) \times H^{2n-1}(M) \mapsto \int_M \alpha \wedge \mu \in \mathbb{R}$$

we get the non-degeneracy of the symplectic pairing. □

Remark. The generating function of the Lie bracket of $\xi, \eta \in \mathrm{Vect}_c(M, \omega)$ is $\omega(\eta, \xi)$ and satisfies the relation

$$\int_M \omega(\eta, \xi)\omega^n = n\langle [i_\eta \omega], [i_\xi \omega] \rangle$$

Indeed, using the properties of the inner product we get

$$0 = i_\xi(i_\eta \omega \wedge \omega^n) = i_\xi i_\eta \omega \wedge \omega^n - i_\eta \omega \wedge i_\xi(\omega^n) = \omega(\eta, \xi)\omega^n - n i_\eta \omega \wedge i_\xi \omega \wedge \omega^{n-1}.$$

Then $\omega(\eta, \xi)\omega^n = n i_\eta \omega \wedge i_\xi \omega \wedge \omega^{n-1}$.

The space of Hamiltonian vector fields having generating function with integral zero, is denoted by $\mathrm{Ham}_c^0(M, \omega)$ and is a Lie subalgebra of $\mathrm{Ham}_c(M, \omega)$. This follows from the preceding remark. If M is compact, then $\mathrm{Ham}(M, \omega) = \mathrm{Ham}^0(M, \omega)$ because we can replace any generating function u by the generating function $u_0 = u - \int\limits_M u\omega^n$ with integral zero. If M is not compact, then $\mathrm{Ham}_c(M, \omega)/\,\mathrm{Ham}_c^0(M, \omega) \cong \mathbb{R}$ because in this case the Hamiltonian vector field with compact support determines uniquely the generating function with compact support and $\int\limits_M : \mathrm{Ham}_c(M, \omega) \to \mathbb{R}$ which maps $\xi = H_u$ to $\int\limits_M u\omega^n$ is a linear, surjective map with kernel $\mathrm{Ham}_c^0(M, \omega)$.

Lemma [Arnold, 1969]**.** *Let $\omega = \sum_{i=1}^n dx^i \wedge dx^{n+i}$ be the standard symplectic form on $\mathbb{R}^{2n}$. Then the commutator algebra of $\mathrm{Vect}_c(\mathbb{R}^{2n}, \omega) = \mathrm{Ham}_c(\mathbb{R}^{2n}, \omega)$ is the space $\mathrm{Ham}_c^0(\mathbb{R}^{2n}, \omega)$. Moreover, $\mathrm{Ham}_c^0(\mathbb{R}^{2n}, \omega)$ is a perfect Lie algebra.*

Proof. Let $H_u \in \mathrm{Ham}_c^0(\mathbb{R}^{2n}, \omega)$, i.e. $\int_{\mathbb{R}^{2n}} u\omega^n = 0$. It follows $u\omega^n = d\psi$ for an $(n-1)$-form ξ with compact support, which written in coordinates gives $u = \sum\limits_{i=1}^{2n} \frac{\partial z_i}{\partial x^i}$, if $\psi = n! \sum\limits_{i=1}^{2n} (-1)^i z_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{2n}$. Taking

$$v_i = z_i$$
$$w_i = \begin{cases} -x^{i+1}, & \text{if } i \text{ odd} \\ x^{i-1}, & \text{if } i \text{ even} \end{cases}$$

we obtain the Poisson bracket (see 10.1)

$$\{v_i, w_i\} = \sum_{l=1}^{n} \left(\frac{\partial v_i}{\partial x^{2l}} \frac{\partial w_i}{\partial x^{2l-1}} - \frac{\partial v_i}{\partial x^{2l-1}} \frac{\partial w_i}{\partial x^{2l}}\right) = \frac{\partial z_i}{\partial x^i}.$$

Hence $u = \sum_{i=1}^{2n} \{v_i, w_i\}$.

To obtain elements H_{v_i}, H_{w_i} in $\mathrm{Ham}^0_c(\mathbb{R}^{2n}, \omega)$ we must:
–multiply w_i by a function h identically 1 on the support of ψ and with compact support.
–add to v_i and w_i bump functions with disjoint compact supports in the complement of $\operatorname{supp} u$; so we make the integral of v_i and w_i to be zero. The Poisson brackets remain unchanged, hence $H_u = \sum_{i=1}^{2n} [H_{v_i}, H_{w_i}] \in [\mathrm{Ham}^0_c(\mathbb{R}^{2n}, \omega), \mathrm{Ham}^0_c(\mathbb{R}^{2n}, \omega)]$. □

Lemma. $\mathrm{Ham}^0_c(M, \omega)$ *is a perfect Lie algebra.*

Proof. Let $\xi = H_u \in \mathrm{Ham}^0_c(M, \omega)$, i.e. $\operatorname{supp} u$ is compact and $\int_M u\omega^n = 0$. There exists a $(2n-1)$-form ψ with compact support such that $u\omega^n = d\psi$.

Let $\mathcal{O} = \{U_\nu : \nu \in I\}$ be an open cover of M with canonical coordinate domains and $(\varphi_\nu)_{\nu\in I}$ a partition of unity subordinated to $\mathcal{O}$. The support of ψ is covered by finitely many sets of $\mathcal{O}$, say $\operatorname{supp} u \subset \operatorname{supp} \psi \subset U_1 \cup \cdots \cup U_k$. Let $\psi_j = \varphi_j \psi$. Because ω^n is a volume form, there exists a function u_j such that $d\psi_j = u_j \omega^n$. Then $\operatorname{supp} u_j \subset \operatorname{supp} \psi_j \subset U_j$ and $u = \sum_{j=1}^{k} u_j$, because

$$u\omega^n = d\psi = d\left(\sum_{j=1}^{k} \psi_j\right) = \sum_j d\psi_j = \left(\sum_j u_j\right)\omega^n.$$

We also have $\int_M u_j \omega^n = \int_M d\psi_j = 0$, hence H_u is the finite sum of $H_{u_j} \in \mathrm{Ham}^0_c(M, \omega)$ with support in a canonical coordinate domain. Now we can apply the preceding lemma and we are done. □

For a compact manifold M it follows that $\mathrm{Ham}(M, \omega)$ is a perfect Lie algebra.

Theorem [Calabi, 1970]**.** *If M is a non-compact symplectic manifold, then:*
1. The commutator algebra of $\mathrm{Vect}_c(M, \omega)$ *is either* $\mathrm{Ham}^0_c(M, \omega)$ *or* $\mathrm{Ham}_c(M, \omega)$*, depending on whether the symplectic pairing is trivial or not.*
2. $\mathrm{Ham}(M, \omega)$ *is a perfect algebra. In particular,* $\mathrm{Ham}(M, \omega)$ *is the commutator algebra of* $\mathrm{Vect}(M, \omega)$.

Proof. 1. The last lemma assures that

$$\begin{aligned} \mathrm{Ham}^0_c(M, \omega) &= [\mathrm{Ham}^0_c(M, \omega), \mathrm{Ham}^0_c(M, \omega)] \\ &\subset [\mathrm{Vect}_c(M, \omega), \mathrm{Vect}_c(M, \omega)] \subset \mathrm{Ham}_c(M, \omega). \end{aligned}$$

Because $\mathrm{Ham}_c(M, \omega) / \mathrm{Ham}^0_c(M, \omega) \cong \mathbb{R}$, we have two possibilities: the commutator group $[\mathrm{Ham}_c(M, \omega), \mathrm{Ham}_c(M, \omega)]$ is either $\mathrm{Ham}_c(M, \omega)$ or $\mathrm{Ham}^0_c(M, \omega)$. The second case arises if and only if the symplectic pairing is trivial; see the remark about the symplectic pairing.

2. Let $\xi = H_u \in \mathrm{Ham}(M, \omega)$. There exists a $(2n-1)$-form ψ on M such that $u\omega^n = d\psi$ because $H^{2n}(M) = 0$. For every open cover $\mathcal{O}$ of M, there exists a number $p \leq \dim M + 1 = 2n + 1$ and a refinement $\mathcal{V}$ of $\mathcal{O}$:

$$\mathcal{V} = \{U_\nu : \nu \in J = J_1 \cup \cdots \cup J_p \text{ partition}\}$$

such that the open sets $(U_\nu)_{\nu \in J_k}$ are pairwise disjoint for a fixed $k = 1, \ldots, p$. We choose an open cover $\mathcal{O}$ consisting of relatively compact canonical coordinate domains. Then $\mathcal{V}$ has the same properties. Let $(\varphi_\nu)_{\nu \in J}$ be a partition of unity subordinated to $\mathcal{V}$.

Like in the proof of the lemma we obtain a decomposition $u = \sum_{\nu \in J} u_\nu$ with the properties: $u_\nu \omega^n = d\psi_\nu$, $\operatorname{supp} u_\nu \subset \operatorname{supp} \psi_\nu \subset U_\nu$ and we get functions $v_\nu^i, w_\nu^i, i = 1, \ldots, 2n$ with support in U_ν such that

$$u_\nu = \sum_{i=1}^{2n} \{v_\nu^i, w_\nu^i\}.$$

Let $u_k = \sum_{\nu \in J_k} u_\nu$, $v_k^i = \sum_{\nu \in J_k} v_\nu^i$, $w_k^i = \sum_{\nu \in J_k} w_\nu^i$ (these sums have in every point only one term). The supports of v_ν^i and w_ν^j are disjoint for $i \neq j$, therefore $\{v_\nu^i, w_\nu^j\} \equiv 0$ and

$$u_k = \sum_{i=1}^{2n} \{v_k^i, w_k^i\}.$$

Thus $H_u = \sum_{k=1}^{p} H_{u_k} = \sum_{k=1}^{p} \sum_{i=1}^{2n} [H_{v_k^i}, H_{w_k^i}] \in [\mathrm{Ham}(M, \omega), \mathrm{Ham}(M, \omega)]$. $\square$

9.4. n-transitivity.

Lemma. *Let $c : (-\varepsilon, 1 + \varepsilon) \to M^m$ be a smooth embedding. Then every 1-form (respectively $(m-1)$-form) along $c([0, 1])$ can be extended to an exact 1-form (respectively $(m-1)$-form) on M with compact support in a tubular neighborhood of the image of c.*

Proof. There exists a tubular neighborhood of $c(-\varepsilon, 1 + \varepsilon)$, i.e. a diffeomorphism from $(-\varepsilon, 1 + \varepsilon) \times \mathbb{R}^{m-1}$ to an open neighborhood U of the image of c in M which on $(-\varepsilon, 1+\varepsilon) \times \{0\}$ coincides with c, and whose inverse $u : U \to (-\varepsilon, 1+\varepsilon) \times \mathbb{R}^{m-1}$ we may use as a chart with $u(c(t)) = (t, 0)$.

(i) The case of a 1-form.
A 1-form along c is given by $\sigma(t) = \sum_{i=1}^{m} a_i(t) du^i|_{c(t)}$ for $t \in [0, 1]$, where $a_i : [0, 1] \to \mathbb{R}$ are smooth and we may extend them smoothly to $a_i : (-\varepsilon, 1 + \varepsilon) \to \mathbb{R}$. Consider the function $f : U \to \mathbb{R}$, given by

$$f = A_1(u^1) + u^2 a_2(u^1) + \cdots + u^m a_m(u^1),$$

where $A_1(t) = \int_0^t a_1(s)ds$. Then $df(c(t)) = \sigma(t)$. Let $h, k : \mathbb{R} \to \mathbb{R}$ be smooth bump functions such that $\operatorname{supp} h \subset (-\delta, \delta)$, $\operatorname{supp} k \subset (-\varepsilon, 1+\varepsilon)$, $h = 1$ in a neighborhood of 0, and $k = 1$ in a neighborhood of $[0, 1]$. Then

$$\tilde{f} := k(u^1)h(u^2)\dots h(u^m)f$$

has compact support in U, so we extend it by 0 to the whole of M, and $d\tilde{f} = df$ near $c([0, 1])$, so $d\tilde{f}$ is also an extension of σ.

(ii) The case of an $(m-1)$-form.
An $(m-1)$-form along c is given by

$$\sigma(t) = \sum_{i=1}^{m} b_i(t) du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n|_{c(t)}$$

where $b_i : [0, 1] \to \mathbb{R}$ are smooth functions which we may extend smoothly to $(-\varepsilon, 1+\varepsilon)$. Let us write $m = 2k$ or $m = 2k+1$. Then the following $(m-2)$-form $\beta \in \Omega^{m-2}(U)$ satisfies $d\beta|_{c(t)} = \omega(t)$.

$$\begin{aligned}
\beta &= \sum_{i=1}^{k} \beta_i du^1 \wedge \dots \wedge du^{2(i-1)} \wedge du^{2i+1} \wedge \dots \wedge du^m + \bar{\beta} du^1 \wedge \dots \wedge du^{m-2}, \\
\beta_1 &= u^2 b_1(u^1) + \int_0^{u^1} b_2(t)dt, \\
\beta_i &= u^{2i} b_{2i-1}(u^1) + u^{2i-1} b_{2i}(u^1) \quad \text{ for } 2 \le i \le k, \\
\bar{\beta} &= \begin{cases} -u^{m-1} b_m(u^1) & \text{for } m = 2k+1. \\ 0 & \text{for } m = 2k \end{cases}
\end{aligned}$$

Then $\tilde{\beta} := k(u^1)h(u^2)\dots h(u^m)\beta$, where h, k are bump functions as above, has compact support in U, so it may be extended by 0 to the whole of M, and since $\tilde{\beta} = \beta$ near $c([0, 1])$ we still have $d\tilde{\beta}|_{c(t)} = \sigma(t)$. $\square$

Theorem. *Let (M^m, μ) be a connected smooth manifold of dimension $m \ge 2$ with a positive volume density. Then the group $\operatorname{Diff}_c(M, \mu)$ of all smooth volume preserving diffeomorphisms of M with compact support acts n-transitively on M, for each finite n.*

Proof. By the argument used at the end of the proof of the proposition in 5.2 it suffices to show, that there exists $\varphi \in \operatorname{Diff}_c(M, \mu)$ with $\varphi(x_i) = y_i$, for any $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in M.

Having fixed the points, we may find an orientable connected open subset of M containing all points and replace M by this set. So without loss we assume that M is orientable.

For some $\varepsilon > 0$ let $c_i : (-\varepsilon, 1+\varepsilon) \to M$, $i = 1, \dots, n$ be smooth embeddings with $c_i(0) = x_i$, $c_i(1) = y_i$ which do not intersect. We choose pairwise disjoint tubular neighborhoods U_i of $c_i(-\varepsilon, 1+\varepsilon)$, $i = 1, \dots, n$.

We can find a Riemannian metric g on M whose volume form is μ. Then the divergence of a vector field $\xi \in \operatorname{Vect}(M)$ is $\operatorname{div} \xi = *d*\xi^\flat$, where $\xi^\flat = g(\xi) \in \Omega^1(M)$

(here we view $g : TM \to T^*M$) and $*$ is the Hodge star operator. The velocity field of the curve c_i defines an $(m-1)$-form $*(c_i' \circ c_i^{-1})^\flat$ along $c_i([0,1])$. Using the lemma we extend it to an exact $(m-1)$-form $d\alpha_i$ on M with support in U_i and we put $\alpha = \sum_{i=1}^n \alpha_i \in \Omega^{m-2}(M)$. We consider the vector field ξ uniquely given by the relation $d\alpha = *\xi^\flat$, i.e.

$$\xi = (-1)^{m+1}(*d\alpha)^\sharp = (-1)^{m+1} g^{-1} * d\alpha.$$

Then ξ is divergence free $\operatorname{div}\xi = *d*\xi^\flat = *dd\alpha = 0$ and has compact support in the union of all U_i. It also coincides on $c_i([0,1])$ with the velocity field of the curve c_i. Hence $\mathrm{Fl}_1^\xi \in \mathrm{Diff}_c(M,\mu)$ with $\mathrm{Fl}_1^\xi(x_i) = y_i$. $\square$

Theorem. *Let (M,ω) be a connected symplectic smooth manifold of dimension $m \geq 2$. Then the group $\mathrm{Diff}_c(M,\omega)$ of all smooth diffeomorphisms with compact support which preserve the symplectic form ω acts n-transitively on M, for each finite n.*

This proof will also show that the Lie subgroup of $\mathrm{Diff}_c(M,\omega)$ whose Lie algebra is the Lie algebra of compactly supported Hamiltonian vector fields acts n-transitively on M. This group has been identified as a Lie group in [Ratiu-Schmid, 1981], for compact M.

Proof. By the argument used at the end of the proof of the proposition in 5.2 it suffices to show, that there exists $\varphi \in \mathrm{Diff}_c(M,\omega)$ with $\varphi(x_i) = y_i$, for any $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in M. We take again smooth curves $c_i : (-\varepsilon, 1+\varepsilon) \to M$ with $c_i(0) = x_i$ and $c_i(1) = y_i$ which are embeddings and do not intersect. Let U_i be pairwise disjoint tubular neighborhoods of $c_i(-\varepsilon, 1+\varepsilon)$.

The velocity field of the curve c_i defines the 1-form $\alpha_i = i_{c_i'}\omega$ along the curve c_i. Using the lemma we extend this form to an exact 1-form df_i on M with $\operatorname{supp} f_i \subset U_i$. Let $f := f_1 + \dots + f_n$ and $\xi := H_f$ the Hamiltonian vector field corresponding to f. Then $\xi \in \mathrm{Vect}_c(M,\omega)$, the Lie algebra of locally Hamiltonian vector fields on M with compact support, and coincides with the velocity field $c_i' \circ c_i^{-1}$ on $c_i([0,1])$. Hence $\mathrm{Fl}_1^\xi \in \mathrm{Diff}_c(M,\omega)$ and $Fl_1^\xi(x_i) = y_i$. $\square$

Theorem. *Let M be a connected smooth manifold of dimension $m \geq 2$, and let α be a contact form on M. Then the group $\mathrm{Diff}_c(M,\alpha)$ of contact diffeomorphisms with compact support acts n-transitively on M for all finite n.*

Proof. By the argument used at the end of the proof of the proposition in 5.2 it suffices to show, that there exists $\varphi \in \mathrm{Diff}_c(M,\mu)$ with $\varphi(x_i) = y_i$, for any $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in M. For $\varepsilon > 0$ let again $c_i : (-\varepsilon, 1+\varepsilon) \to M$ be smooth embeddings with $c_i(0) = x_i$, $c_i(1) = y_i$ which do not intersect. We choose pairwise disjoint tubular neighborhoods U_i of $c_i(-\varepsilon, 1+\varepsilon)$.

Let $f_i : M \to \mathbb{R}$ be a smooth extension of $\alpha(c_i' \circ c_i^{-1}) : c_i([0,1]) \to \mathbb{R}$ with support in U_i and $f := \sum_{i=1}^n f_i \in C_c^\infty(M,\mathbb{R})$. Then the unique vector field $\xi \in \mathrm{Vect}_c(M,\alpha)$ such that $\alpha(\xi) = f$ coincides with the velocity field of the curve c_i on $c_i([0,1])$. Hence $\mathrm{Fl}_1^\xi \in \mathrm{Diff}_c(M,\alpha)$ and $\mathrm{Fl}_1^\xi(x_i) = y_i$ for $i = 1,\dots,n$. $\square$

9.5. Finite dimensional coadjoint orbits.

Lemma. *Let $E = C_c^\infty(\otimes^p TM \otimes \otimes^q T^*M)$ for M a smooth m-dimensional manifold and G a subgroup of $\operatorname{Diff}_c(M)$ with the natural action $(\varphi^{-1})^* : E \to E$, $\varphi \in G$. Then the orbit of an element of $E^* = \mathcal{D}'(M) \otimes_{C^\infty(M)} C^\infty(\otimes^p T^*M \otimes \otimes^q TM)$ with finite support under the dual action (which is again the natural action) is finite dimensional.*

Proof. Let $\delta \in E^*$ and $\operatorname{supp}\delta = \{x_1, \dots, x_N\}$. Then $(\varphi^{-1})^*\delta$ has as support a subset of $\varphi\{x_1, \dots, x_N\}$ and at every point the order is less or equal to the order of δ. Let s be the maximal order of δ over all the x_i. Then

$$\dim \mathcal{O}_\delta \le N(n + \dim \oplus_{j=0}^s L^j_{sym}(\mathbb{R}^m, \mathbb{R}^{(p+q)m})) = N\big(n + (p+q)m\binom{m+s}{m}\big),$$

which is the dimension of the space of all elements in E^* with support consisting of maximum N points and having at each point the order $\le s$. Hence $\mathcal{O}_\delta$ is finite dimensional. $\square$

Let $G = \operatorname{Diff}_c(M)$. Then the adjoint action on $\mathfrak{g} = \operatorname{Vect}_c(M) = C_c^\infty(TM)$ is the natural action, and the lemma can be applied.

Let $G = \operatorname{Diff}_c(M, \omega)$. Then $\mathfrak{g}$ can be identified with $Z_c^1(M) = B_c^1(M) \oplus H_c^1(M)$ and the adjoint action is again the natural one on 1-forms with compact support, restricted to the closed forms. Moreover G acts trivially on the cohomology group $H_c^1(M)$, because if $\sigma \in Z_c^1(M)$, then $\mathcal{L}_\xi[\sigma] = [\mathcal{L}_\xi\sigma] = [di_\xi\sigma] = 0$ for every vector field ξ, hence $[\varphi^*\sigma] = [\sigma]$ for every diffeomorphism φ. The inclusion of the dual space $B_c^1(M)^* = (\Omega_c^0(M)/Z_c^0(M))^* = \{\delta \in \Omega_c^0(M)^* : \delta|_{Z_c^0(M)} = 0\}$ in $\Omega_c^0(M)^*$ is equivariant. So we can apply the lemma in the case $E = \Omega_c^0(M)$ to get the finite dimensionality of the coadjoint orbit in $\operatorname{Diff}_c(M, \omega)$ of an element with finite support.

Exactly in the same way, applying the lemma for $E = \Omega_c^{m-2}(M)$, we get the result for $\operatorname{Diff}_c(M, \mu)$.

Theorem [Kirillov, 1974]**.** *Let G be one of the groups:* $\operatorname{Diff}_c(M)$, $\operatorname{Diff}_c(M,\omega)$ *or* $\operatorname{Diff}_c(M,\mu)$*. Then a coadjoint orbit $\mathcal{O}_\delta$ is finite dimensional if and only if the distribution $\delta \in \mathfrak{g}^*$ has finite support.*

Proof. One direction has just been proved. For the converse let $\delta \in \mathfrak{g}^*$ be an infinitely supported distribution. We will show that for any natural number N, there exist N linearly independent vectors in the tangent space to the orbit $\mathcal{O}_\delta$ at δ, which is $\operatorname{ad}^*(\mathfrak{g})\delta \subset \mathfrak{g}^*$. Let $x_1, \dots, x_N \in \operatorname{supp}\delta$ and let $U_1, \dots, U_N$ be disjoint canonical coordinate domains, centered at $x_1, \dots, x_N$ respectively. Then there exist $\xi_1, \dots, \xi_N \in \mathfrak{g}$ vector fields on M with $\operatorname{supp}\xi_i \subset U_i$ and $\langle\delta, \xi_i\rangle \ne 0$. The Lie algebras $\operatorname{Vect}_c(\mathbb{R}^n)$ and $\operatorname{Vect}_c(\mathbb{R}^n, \mu)$ are perfect (see 5.1 and 9.3), hence in the case $\mathfrak{g}$ is $\operatorname{Vect}_c(M)$ or $\operatorname{Vect}_c(M, \mu)$, we can find $\eta_i^{(l)}, \zeta_i^{(l)} \in \mathfrak{g}$ vector fields with support in U_i such that $\xi_i = \sum_{l=1}^{k_i}[\eta_i^{(l)}, \zeta_i^{(l)}]$. The same is true in the case $\mathfrak{g} = \operatorname{Vect}_c(M, \omega)$, because we can add to ξ_i a Hamiltonian vector field with support in $U_i - \{x_i\}$ to obtain an element $\xi_i' \in \operatorname{Ham}_c^0(M, \omega)$ and still $\langle\delta, \xi_i'\rangle = \langle\delta, \xi_i\rangle \ne 0$. Now we can use the lemma in 9.3 which says that $\operatorname{Ham}_c^0(\mathbb{R}^{2n}, \omega)$ is perfect.

Because $\langle \delta, \xi_i \rangle \neq 0$, there exist $l_i \in \{1, \ldots, k_i\}$ with

$$\langle \mathrm{ad}^*(\zeta_i^{(l_i)}).\delta, \eta_i^{(l_i)} \rangle = \langle \delta, [\eta_i^{(l_i)}, \zeta_i^{(l_i)}] \rangle \neq 0.$$

So we have found N linearly independent tangent vectors at δ, namely $\mathrm{ad}^*(\zeta_i^{(l_i)}).\delta$. Indeed, they are not zero and have disjoint supports. Hence $\mathcal{O}_\delta$ is infinite dimensional. $\square$

10. The group of symplectomorphisms

10.1. Multivalued functions.

Theorem. *Let (M, ω) be a symplectic manifold. Then the space of smooth functions on M with the Poisson bracket $\{f, g\} := \omega(H_g, H_f)$ is a Lie algebra. Considering also the trivial Lie algebra structure on $H^0(M)$ and $H^1(M)$, there is an exact sequence of Lie algebras and Lie algebra homomorphisms:*

$$0 \longrightarrow H^0(M) \xrightarrow{\alpha} C^\infty(M) \xrightarrow{H} \mathrm{Vect}(M, \omega) \xrightarrow{\gamma} H^1(M) \longrightarrow 0$$

where α is the embedding of the locally constant functions, H_f is the unique vector field with $i_{H_f}\omega = df$ and $\gamma(\xi) = [i_\xi \omega]$. In particular the following sequence is also exact

$$0 \longrightarrow H^0(M) \longrightarrow C^\infty(M) \longrightarrow \mathrm{Ham}(M, \omega) \longrightarrow 0.$$

Proof. From the first remark in 9.2 we get $[\xi, \eta] = H_{\omega(\eta,\xi)}$. In particular $[H_f, H_g] = H_{\omega(H_g, H_f)} = H_{\{f,g\}}$, so H is a Lie algebra homomorphism. The other two mappings in the sequence are also Lie algebra homomorphisms because $\{c_1, c_2\} = 0$ for locally constant functions and $\gamma([\xi, \eta]) = \gamma(H_{\omega(\eta,\xi)}) = 0$.

$(C^\infty(M), \{,\})$ is a Lie algebra. Indeed

$$\begin{aligned}
\{f, g\} &= \omega(H_g, H_f) = -\omega(H_f, H_g) = -\{g, f\} \\
\{\{f, g\}, h\} &= H_{\{f,g\}}h = [H_f, H_g]h = H_f H_g h - H_g H_f h \\
&= H_f\{g, h\} - H_g\{f, h\} = \{f, \{g, h\}\} - \{g, \{f, h\}\}.
\end{aligned}$$

The exactness of the sequence at $\mathrm{Vect}(M, \omega)$: $\gamma(\xi) = 0$ if and only if $i_\xi \omega = df$ for some smooth function f, i.e. $\xi = H_f$. The exactness at the other stages is obvious. $\square$

Every locally Hamiltonian vector field $\xi \in \mathrm{Vect}(M, \omega)$ posesses locally a generating function because $i_\xi \omega$ is closed, hence locally exact. So ξ defines a multivalued function on M.

Definition. A *multivalued function* on M is a smooth function g on $\tilde{M}$, the universal covering space of M, such that $g \circ \alpha - g$ is constant for every covering transformation $\alpha \in \mathrm{Aut}(\tilde{M}) = \{\alpha : \tilde{M} \to \tilde{M} : p \circ \alpha = p\}$. We denote by $C^\infty_m(M)$ the set of multivalued functions on M.

For every multivalued function g, dg is a 1-form on $\tilde{M}$, invariant under all covering transformations:

$$\alpha^*(dg) = d(g\circ\alpha) = d(g+\text{const.}) = dg,$$

hence it projects onto a closed 1-form ϑ on M. Indeed, the fact that $dg = p^*\vartheta$ implies $p^*d\vartheta = dp^*\vartheta = ddg = 0$ and because p^* is injective we obtain $d\vartheta = 0$. Identifying the closed forms with the locally hamiltonian vector fields, we get the map $K : C^\infty_m(M) \to \mathrm{Vect}(M,\omega)$, an analogue to the Hamilton map $H : C^\infty(M) \to \mathrm{Ham}(M,\omega)$.

Proposition. *There is an exact sequence*

$$0 \longrightarrow H^0(\tilde{M}) \xrightarrow{\alpha} C^\infty_m(M) \xrightarrow{K} \mathrm{Vect}(M,\omega) \longrightarrow 0,$$

where α is the embedding of the locally constant functions on $\tilde{M}$ into $C^\infty_m(M) \subset C^\infty(\tilde{M})$.

Proof. The exactness at $C^\infty_m(M)$: $K(g) = 0$ if and only if ϑ, the projection of dg, is zero. This means $dg = 0$, i.e. G is locally constant. Now we prove the surjectivity of K. Let $\xi \in \mathfrak{g}$ and $\vartheta = i_\xi\omega \in Z^1(M)$. The lifted 1-form $p^*\vartheta$ is a closed form on $\tilde{M}$, hence exact. Every smooth function on $\tilde{M}$ such that $p^*\vartheta = dg$ is a multivalued function on M: $d(g\circ\alpha) = \alpha^*dg = \alpha^*p^*\vartheta = \alpha^*\vartheta = dg$ and so $\xi = K(g)$. $\square$

Consequence. *The following diagram is commutative and its lines are exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\tilde{M}) & \xrightarrow{\alpha} & C^\infty(\tilde{M}) & \xrightarrow{H} & \mathrm{Vect}(\tilde{M},\tilde{\omega}) & \longrightarrow & 0\\
 & & \mathrm{Id}\uparrow & & \uparrow & & p^*\uparrow & & \\
0 & \longrightarrow & H^0(\tilde{M}) & \xrightarrow{\alpha} & C^\infty_m(M) & \xrightarrow{K} & \mathrm{Vect}(M,\omega) & \longrightarrow & 0\\
 & & p^*\uparrow & & p^*\uparrow & & \uparrow & & \\
0 & \longrightarrow & H^0(M) & \xrightarrow{\alpha} & C^\infty(M) & \xrightarrow{H} & \mathrm{Ham}(M,\omega) & \longrightarrow & 0
\end{array}$$

where $\tilde{\omega} = p^*\omega$ is the lifted symplectic structure on $\tilde{M}$ and $p^* : \mathrm{Vect}(M,\omega) \to \mathrm{Vect}(\tilde{M},\tilde{\omega})$ is the lifting of vector fields.

Proof. The exactness of the second line follows from the proposition and the exactness of the other two lines follows from the theorem, after noticing that $H^1(\tilde{M}) = 0$. Let g be a multivalued function on M. Then $\vartheta = i_{K(g)}\omega$ and $dg = i_{H(g)}\tilde{\omega}$, hence the relation $p^*\circ K(g) = H(g)$ translated in terms of 1-forms is $p^*\vartheta = dg$ the definition of $K(g)$. The relation $K\circ p^*(f) = H(f)$ for a smooth function f on M is true because the projection of $d(p^*f)$ is df. $\square$

In the case M is connected we have

$\mathrm{Ham}(M,\omega) \cong C^\infty(M)/\mathbb{R} \cong C^\infty_0(M)$ the space of functions with integral zero,

$\mathrm{Vect}(M,\omega) \cong C^\infty_m(M)/\mathbb{R}$,

$\mathrm{Vect}(M,\omega)/\,\mathrm{Ham}(M,\omega)' \cong H^1(M)$

and the following sequence is exact:

$$0 \to \mathrm{Ham}(M, \omega) \to \mathrm{Vect}(M, \omega) \to H^1(M) \to 0.$$

For the universal covering $p\colon \tilde{M} \to M$ there is a canonical identification of the group of covering transformations with the fundamental group of M.

$$\alpha \in \mathrm{Aut}(\tilde{M}) \leftrightarrow [c] \in \pi_1(M).$$

Here c is a loop in M, the projection of a path $\tilde{c}$ in $\tilde{M}$ starting at x and ending at $\alpha(x)$.

Proposition. *For every multivalued function g on M, the assignment*

$$\alpha \in \mathrm{Aut}(\tilde{M}) \cong \pi_1(M) \mapsto g \circ \alpha - g \in \mathbb{R}$$

is a group homomorphism. In fact

$$g \circ \alpha - g = \langle [\vartheta], h(\alpha) \rangle$$

where $\vartheta = i_{K(g)}\omega$ and h is the Hurewitz homomorphism $h : \pi_1(M) \to H_1(M)$, $h([c]) = c_*[S^1]$; here $[S^1]$ is the generator of $H_1(S^1)$.

Proof. Let $\alpha \in \mathrm{Aut}(\tilde{M}), \alpha \equiv [c]$, and $\tilde{c}$ a lift of c starting at x. Then:

$$\begin{aligned} g \circ \alpha - g &= g(\alpha(x)) - g(x) \\ &= \int_0^1 \tilde{c}^* dg = \int_0^1 \tilde{c}^* p^* \vartheta = \int_0^1 (p \circ \tilde{c})^* \vartheta = \int_0^1 c^* \vartheta \\ &= \frac{1}{2\pi} \int_{S^1} c^* \vartheta = \langle c_*[\vartheta], [S^1] \rangle \\ &= \langle [\vartheta], c^*[S^1] \rangle = \langle [\vartheta], h([c]) \rangle = \langle [\vartheta], h(\alpha) \rangle. \quad \Box \end{aligned}$$

Example. The multivalued functions on the torus T^2.

Let $p : \mathbb{R}^2 \to T^2, p(x, y) = (e^{ix}, e^{iy})$ be the universal covering of the torus. The covering transformations $\mathrm{Aut}(\mathbb{R}^2) \cong \mathbb{Z}^2$ are

$$\alpha(x, y) = (x + 2m\pi, y + 2n\pi), \quad (m, n) \in \mathbb{Z}^2$$

and the group homomorphisms from $\mathbb{Z}^2$ to $\mathbb{R}$ are

$$(m, n) \mapsto a'm + b'n, \quad a', b' \in \mathbb{R}.$$

By the preceding proposition, every multivalued function on the torus satisfies:

$$g(x + 2\pi m, y + 2\pi n) - g(x, y) = 2\pi(am + bn), \quad a, b \in \mathbb{R}.$$

Then $f(x,y) = g(x,y) - (ax+by)$ is a smooth function on $\mathbb{R}$ invariant under all covering transformations, hence f is $(2\pi, 2\pi)$-periodic and can therefore be written as a Fourier series

$$f(x,y) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} c_{k_1k_2} e^{i(k_1x+k_2y)}, \quad c_{k_1k_2} \in \mathbb{C} \text{ fast falling}, \quad \bar{c}_{k_1,k_2} = c_{-k_1,-k_2}.$$

Conclusion

$$C^\infty_m(T^2) = \{f : \mathbb{R}^2 \to \mathbb{R}^2 : f(x,y) = ax+by + \sum_{(k_1,k_2)\in\mathbb{Z}^2} c_{k_1k_2} e^{i(k_1x+k_2y)}\}.$$

We consider on T^2 the projection ω of the standard form $dx \wedge dy$ on $\mathbb{R}^2$. This is a symplectic form. An unconditional basis for the nuclear Fréchet space $\mathrm{Vect}(T^2,\omega) \cong C^\infty_m(T^2)/\mathbb{R}$ is

$$E_1 = K(x) = -\frac{\partial}{\partial y}$$
$$E_2 = K(y) = \frac{\partial}{\partial x}$$
$$E_{k_1k_2} = K(e^{i(k_1x+k_2y)}) = ie^{i(k_1x+k_2y)}(k_2\frac{\partial}{\partial x} - k_1\frac{\partial}{\partial y}), \text{ for } (k_1,k_2) \in \mathbb{Z}^2 - \{(0,0)\}.$$

We denote the function $e^{i(k_1x+k_2y)}$ by $e_{k_1k_2}$. Then the Hamiltonian vector fields $E_{k_1k_2} = H_{e_{k_1k_2}}$ form an unconditional basis for $\mathrm{Ham}(T^2,\omega)$.

10.2. Central extensions of $\mathrm{Vect}(M,\omega)$ **which leave a certain scalar product invariant.**

Let (M,ω) a symplectic compact manifold of dimension $2n$, $\mathfrak{g}$ the Lie algebra of locally Hamiltonian vector fields and $\mathfrak{g}'$ the Lie subalgebra of Hamiltonian vector fields. By choosing a linear splitting of the exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{g}' \xrightarrow{i} \mathfrak{g} \xrightarrow{\gamma} H^1(M) \longrightarrow 0$$

we can identify $\mathfrak{g}$ and $H^1(M) \oplus \mathfrak{g}'$ as vector spaces. On the vector space $\tilde{\mathfrak{g}} = H^1(M) \oplus \mathfrak{g}' \oplus H_1(M)$ there is a natural scalar product determined by the pairing between $H^1(M)$ and $H_1(M)$, and by the scalar product on $\mathfrak{g}' : \langle f, g\rangle = \int_M fg\omega^n$. Explicitly:

$$((a^*,f,a),(b^*,g,b)) = \langle a^*,b\rangle + \langle b^*,a\rangle + \langle f,g\rangle \tag{1}$$

Any Lie algebra structure extension on $\tilde{\mathfrak{g}}$ can be defined with the help of a cocycle on $\mathfrak{g} = H^1(M) \oplus \mathfrak{g}'$ with values in $H_1(M)$: $c \in \mathrm{Hom}(\wedge^2\mathfrak{g}, H_1(M))$.

$$0 \longrightarrow H_1(M) \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$
$$\left[\begin{pmatrix}\xi\\a\end{pmatrix}, \begin{pmatrix}\eta\\b\end{pmatrix}\right] = \begin{pmatrix}[\xi,\eta]\\c(\xi,\eta)\end{pmatrix}, \xi,\eta \in \mathfrak{g}, a,b \in H_1(M).$$

Problem: For which Lie algebra structure extensions on $\tilde{\mathfrak{g}}$ is the scalar product $(,)$ ad-invariant? This means $([l_1, l_2], l_3) = (l_1, [l_2, l_3])$.

Let $(e_i)_{i=1,\ldots,b_1}, b_1 = b_1(M)$ be a basis of $H^1(M)$ and $\xi_i = s(e_i) \in \mathfrak{g}$. Given a cocycle which fulfills this requirement, we can define an antisymmetric tensor A by

$$a_{ijk} = \langle e_i, c(\xi_j, \xi_k)\rangle, i, j, k = 1, \ldots, b_1.$$

Proof of the antisymmetry:

$$\begin{aligned}\left(\begin{pmatrix}\xi_i\\0\end{pmatrix}, \left[\begin{pmatrix}\xi_j\\0\end{pmatrix}, \begin{pmatrix}\xi_k\\0\end{pmatrix}\right]\right) &= \left(\begin{pmatrix}\xi_i\\0\end{pmatrix}, \begin{pmatrix}[\xi_j, \xi)k]\\c(\xi_j, \xi_k)\end{pmatrix}\right)\\ &= \langle e_i, c(\xi_j, \xi_k)\rangle = a_{ijk}\end{aligned}$$

$$\begin{aligned}\text{and } (l_1, [l_2, l_3]) &= -(l_1, [l_3, l_2]) \text{ by antisymmetry of } [\ ,\ ]\\ &= -(l_2, [l_1, l_3]) \text{ by ad-invariance of } (\ ,\ ).\end{aligned}$$

Hence $A \in \mathrm{Hom}(\wedge^3 H^1(M), \mathbb{R}) \cong \wedge^3 H_1(M)$.

Proposition [Kirillov, 1990]**.** *The cocycle is completely determined by this tensor.*

Proof. A calculation gives us:

$$\langle e_i, c(\xi_j + H_f, \xi_k + H_g)\rangle = \langle e_k, c(\xi_i, \xi_j + H_f)\rangle + \int_M \omega(\xi_i, \xi_j + H_f) g\omega^n.$$

Indeed:

$$\begin{aligned}&\langle e_i, c(\xi_j + H_f, \xi_k + Hg)\rangle && \text{def. of } (\ ,\ )\\ &= \left(\begin{pmatrix}\xi_i\\0\end{pmatrix}, \begin{pmatrix}[\xi_j + H_f, \xi_k + H_g]\\c(\xi_j + H_f, \xi_k + H_g)\end{pmatrix}\right) && \text{def. of } [\ ,\ ] \text{ on } \tilde{g}\\ &= \left(\begin{pmatrix}\xi_i\\0\end{pmatrix}, [\begin{pmatrix}\xi_j + H_f\\0\end{pmatrix}, \begin{pmatrix}\xi_k + H_g\\0\end{pmatrix}]\right) && \text{ad-invar. of } (\ ,\ )\\ &= \left(\left[\begin{pmatrix}\xi_i\\0\end{pmatrix}, \begin{pmatrix}\xi_j + H_f\\0\end{pmatrix}\right], \begin{pmatrix}\xi_k + H_g\\0\end{pmatrix}\right) && \text{def. of } [\ ,\ ] \text{ on } \tilde{g}\\ &= \left(\begin{pmatrix}H_{\omega(\xi_i, \xi_j + H_f)}\\c(\xi_i, \xi_j + H_f)\end{pmatrix}, \begin{pmatrix}\xi_k + H_g\\0\end{pmatrix}\right) && \text{def. of } (\ ,\ )\\ &= \langle e_k, c(\xi_i, \xi_j + H_f)\rangle + \int_M \omega(\xi_i, \xi_j + H_f) g\omega^n.\end{aligned}$$

From this assertion it follows also that

$$\begin{aligned}\langle e_k, c(\xi_i, \xi_j + H_f)\rangle &= \langle e_j, c(\xi_k, \xi_i)\rangle + \int_M \omega(\xi_k, \xi_i) f\omega^n\\ &= a_{ijk} + \int_M \omega(\xi_k, \xi_i) f\omega^n\end{aligned}$$

Using also $\omega(\xi_i, H_f) = -\xi_i f$, we get

$$\langle e_i, c(\xi_j + H_f, \xi_k + H_g)\rangle = a_{ijk} + \int_M (f\xi_i g + g\omega(\xi_i, \xi_j) - f\omega(\xi_i, \xi_k))\omega^n \tag{2}$$

Every $\xi \in \mathfrak{g}$ can be written as $\xi = [\xi - s(\gamma(\xi))] + s(\gamma(\xi)) = H_\varphi + \sum_{i=1}^{b_1} \lambda_i \xi_i$, and (e_i) is a basis in $H^1(M) = (H_1(M))^*$, hence the relation (2) defines the cocycle c. $\square$

Question. Does the formula (2) always define a Lie algebra cocycle? We should verify the cocycle identity:

$$c([\xi, \xi'], \xi'') + c([\xi', \xi''], \xi) + c([\xi'', \xi], \xi') = 0. \tag{3}$$

Lemma. *Let c be given by (2). Then the restriction of c to the Lie subalgebra $\mathfrak{g}'$ of Hamiltonian vector fields is always a cocycle.*

Proof. In this case the defining relation of c becomes:

$$\langle e_i, c(H_f, H_g)\rangle = \int_M f\xi_i g\omega^n.$$

Then $\langle e_i, c([H_f, H_{f'}], H_{f''})\rangle = \int_M \{f, f'\}\xi_i f''\omega^n$ and the cocycle identity is equivalent to

$$\int_M (\{f, f'\}\xi_i f'' + \{f', f''\}\xi_i f + \{f'', f\}\xi_i f')\omega^n = 0.$$

To show this we need a few identities:

$$\int_M \{f, f'\}f''\omega^n = \int_M f\{f', f''\}\omega^n \tag{i}$$

Indeed:

$$-\int_M \{f, f'\}f''\omega^n + \int_M f\{f', f''\}\omega^n = \int_M \{f', ff''\}\omega^n = \int_M df' \wedge d(ff'') \wedge \omega^{n-1} = 0.$$

$$\xi\{f, f'\} = \{\xi f, f'\} + \{f, \xi f'\}, \quad \forall \xi \in \mathfrak{g} \tag{ii}$$

(a generalization of the Jacobi identity)

$$\begin{aligned}
\xi\{f, f'\}\omega &= \mathcal{L}_\xi(\{f, f'\}\omega)\{f, f'\}\mathcal{L}_\xi\omega \\
&= \mathcal{L}_\xi(d\varphi \wedge df') \\
&= \mathcal{L}_\xi(df) \wedge df' + df \wedge \mathcal{L}_\xi(df') \\
&= d(\mathcal{L}_\xi f) \wedge df' + df \wedge d(\mathcal{L}_\xi f') \\
&= \{\xi f, f'\}\omega + \{f, \xi f'\}\omega
\end{aligned}$$

(iii) $$\int\limits_M \xi f\omega^n = \int\limits_M \mathcal{L}_\xi(f\omega^n) = 0 \quad \forall \xi \in \mathfrak{g}$$

Thus

$$\begin{aligned}\int_M \{f,f'\}\xi_i f''\omega^n &= \int_M \xi_i(\{f,f'\}f'')\omega^n - \int_M f''\xi_i\{f,f'\}\omega^n \\ &= -\int_M f''\{\xi_i f, f'\}\omega^n - \int_M f''\{f,\xi_i f'\}\omega^n \\ &= -\int_M \{f',f''\}\xi_i f\omega^n - \int_M \{f'',f\}\xi_i f'\omega^n. \quad \square\end{aligned}$$

Lemma. *If $b_1(M) = 2$, the cocycle identity (3) is satisfied if and only if*

(5) $$\int_M (\{f,g\}\omega + df \wedge dg)(\xi_1,\xi_2)\omega^n = 0$$

for all f, g in $C^\infty(M)$ and ξ_1, ξ_2 in $\mathfrak{g}$.

Proof. Let (e_1, e_2) be a basis of $H^1(M)$ and $\xi_1 = s(e_1), \xi_2 = s(e_2)$. Because the cocycle relation is linear in every argument and because every locally Hamiltonian vector field is a linear combination of ξ_1, ξ_2 and some H_f, it is sufficient to verify it for them.

We have

$$\begin{aligned}\langle e_i, c([\xi,\xi'],\xi'')\rangle &= \langle e_i, c(H_{\omega(\xi,\xi')-\int_M \omega(\xi,\xi')\omega^n}, \xi'')\rangle \\ &= \int_M \omega(\xi,\xi')\omega(\xi'',\xi''')\omega^n - \int_M \omega(\xi,\xi')\omega^n \int_M \omega(\xi'',\xi''')\omega^n\end{aligned}$$

Then the cocycle identity (3) is equivalent to:

(4) $$\sum_{cyclic(\xi,\xi',\xi'')} \left(\int_M \omega(\xi,\xi')\omega(\xi'',\xi''')\omega^n - \int_M \omega(\xi,\xi')\omega^n \int_M \omega(\xi'',\xi''')\omega^n\right)$$

where every ξ, ξ', ξ'', ξ''' is ξ_1, ξ_2 or a Hamiltonian vector field.

Case 1: None of them are Hamiltonian.

$$\omega(\xi_1,\xi_2)\omega(\xi_1,\xi_2) + \omega(\xi_2,\xi_1)\omega(\xi_1,\xi_2) + \omega(\xi_1,\xi_1)\omega(\xi_2,\xi_2) = 0$$

and (4) is satisfied.

Case 2: One of them is Hamiltonian. From (iii) we get $\int_M \omega(H_f,\xi)\omega^n = 0 \quad, \forall \xi \in \mathfrak{g}$. Then (4) becomes

$$\int_M (\omega(\xi_1,\xi_2)\omega(H_f,\xi_1) + \omega(H_f,\xi_1)\omega(\xi_2,\xi_1) + \omega(\xi_2,H_f)\omega(\xi_1,\xi_1))\omega^n = 0$$

and is always satisfied.

Case 3: Two of them are Hamiltonian. Then (4) becomes

$$\int_M (\omega(\xi_1, H_f)\omega(H_g, \xi_2) + \omega(H_g, \xi_1)\omega(H_f, \xi_2) + \omega(H_f, H_g)\omega(\xi_1, \xi_2))\omega^n$$
$$= -\int_M (\xi_1 f \xi_2 g - \xi_1 g \xi_2 f + \{f, g\}\omega(\xi_1, \xi_2))\omega^n$$
$$= -\int_M (\{f, g\}\omega + df \wedge dg)(\xi_1, \xi_2)\omega^n = 0.$$

This is the relation (5), that should be satisfied. In the case all three of them are Hamiltonian, the cocycle relation is always satisfied by the lemma. □

10.3. Examples and counterexamples.

The torus T^2. By the remark in 9.3 which says that $\omega(\xi, \eta)\omega^n = n i_\xi \omega \wedge i_\eta \omega \wedge \omega^{n-1}$ for $\xi, \eta \in \mathfrak{g}$, we get $\{f, g\}\omega = dg \wedge df$ and the relation (5) in the lemma is satisfied. Hence c is a cocycle.

Symplectic forms on the torus are the same as volume forms, hence symplectomorphisms are just area preserving transformations. Then, by a result in paragraph 9.1, all the symplectomorphism groups are isomorphic.

Therefore we take $T^2 = \mathbb{R}^2/\Gamma$ with the lattice Γ generated by $(0, 2\pi)$ and $(2\pi, 0)$, the universal covering $p(x, y) = (e^{ix}, e^{iy})$ and the sympectic form the projection of $dx \wedge dy$. A basis for the Lie algebra $\mathfrak{g} = \mathrm{Vect}(T^2, \omega)$ is (see 10.1)

$$E_1 = -\frac{\partial}{\partial y}, E_2 = \frac{\partial}{\partial x}, E_{k_1 k_2} = H_{e_{k_1 k_2}} = ie^{i(k_1 x + k_2 y)}\left(k_2 \frac{\partial}{\partial x} - k_1 \frac{\partial}{\partial y}\right)$$

with $(k_1, k_2) \in \mathbb{Z}^2 - \{(0, 0)\}$. The commutation relations in $\mathfrak{g}$ are:

$$\begin{aligned}
[E_1, E_2] &= 0 \\
[E_1, E_{k_1 k_2}] &= -ik_2 E_{k_1 k_2} \\
[E_2, E_{k_1 k_2}] &= ik_1 E_{k_1 k_2} \\
[E_{k_1 k_2}, E_{l_1 l_2}] &= \begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} E_{k_1 + k_2, l_1 + l_2}.
\end{aligned}$$

Next we choose a splitting s of the exact sequence

$$0 \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow H^1(T^2) \longrightarrow 0$$

namely $s([dx]) = E_1, s([dy]) = E_2$.

The relation (2) defines a unique cocycle c on $\mathfrak{g}$ with values in $H_1(M)$, because $\wedge^3 H_1(T^2) = 0$:

$$\langle e_i, c(E_j + H_f, E_k + H_g)\rangle = \int_{T^2} (f E_i g + g\omega(E_i, E_j) - f\omega(E_i, E_k))\omega$$

Let $[c_1],[c_2] \in H_1(T^2)$ be the dual basis to $[dx],[dy] \in H^1(T^2)$ relative to the pairing $\langle [c],[\alpha]\rangle = \int_c \alpha$. Then

$$\begin{aligned}
c(E_{k_1k_2}, E_{l_1l_2}) &= (\int_{T^2} e_{k_1k_2}(E_1 e_{l_1l_2})\omega)[c_1] + (\int_{T^2} e_{k_1k_2}(E_2 e_{l_1l_2})\omega)[c_2] \\
&= (\int_{T^2} -il_2 e_{k_1+l_1,k_2+l_2}\omega)[c_1] + (\int_{T^2} il_1 e_{k_1+l_1,k_2+l_2}\omega)[c_2] \\
&= i\delta(k_1+l_1)\delta(k_2+l_2)(k_2[c_1]-k_1[c_2]) \\
c(E_j, E_{k_1k_2}) &= (\int_{T^2} e_{k_1k_2}\omega(E_1,E_j)\omega)[c_1] + (\int_{T^2} e_{k_1k_2}\omega(E_2,E_j)\omega)[c_2] \\
&= 0 \text{ because } \int_{T^2} e^{i(k_1x+k_2y)}dx\wedge dy = 0 \text{ for } (k_1,k_2)\neq(0,0) \\
c(E_i,E_j) &= 0.
\end{aligned}$$

The restriction of the cocycle c to the Lie algebra $\mathfrak{g}'$ of Hamiltonian vector fields, which can be identified with the zero integral functions on T^2, i.e. $(2\pi,2\pi)$-periodic functions on $\mathbb{R}^2$ with $\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy = 0$, is

$$c(f,g) = -\left(\int_0^{2\pi}\int_0^{2\pi} f\frac{\partial g}{\partial y}\right)[c_1] + \left(\int_0^{2\pi}\int_0^{2\pi} f\frac{\partial g}{\partial x}\right)[c_2] \in H_1(T^2).$$

We can obtain a 2-parameter family of $\mathbb{R}$-valued cocycles $c_{\alpha\beta}$ on $\mathfrak{g}' \cong C_0^\infty(T^2)$ by pairing the cocycle c with elements in $H^1(T^2)$.

$$c_{\alpha\beta} = \langle \beta[dx]-\alpha[dy], c\rangle : C_0^\infty(T^2)\times C_0^\infty(T^2)\to\mathbb{R}.$$

Hence

$$c_{\alpha\beta}(f,g) = \int_0^{2\pi}\int_0^{2\pi} f(\alpha\frac{\partial g}{\partial x}+\beta\frac{\partial g}{\partial y})dxdy \in \mathbb{R}.$$

The torus T^4.

For the symplectic form $\omega = \frac{1}{2}(dx_1\wedge dx_2 + dx_3\wedge dx_4)$, the cocycle relation (3) is satisfied. We choose as generators of $H^1(T^4)$ the 1-forms dx_1, dx_2, dx_3, dx_4 and correspondingly the locally Hamiltonian vector fields $\xi_1 = -\frac{\partial}{\partial x_2}$, $\xi_2 = \frac{\partial}{\partial x_1}$, $\xi_3 = -\frac{\partial}{\partial x_4}$, $\xi_4 = \frac{\partial}{\partial x_3}$. For all possible choices of ξ,ξ',ξ'',ξ''', the cocycle relation (2) is satisfied. The non-trivial cases are:

(i) H_f, H_g, ξ_1, ξ_2

$$\begin{aligned}
&\int_{T^4}((\xi_1 f)(\xi_2 g) - (\xi_1 g)(\xi_2 f) + \{f,g\})\omega^2 = 0 \quad \Leftrightarrow \\
&\qquad \int_{T^2} dx_1 dx_2 \int_{T^2}\{f,g\}_{dx_3\wedge dx_4} dx_3\wedge dx_4 = 0.
\end{aligned}$$

The last relation is true because in general $\int_M \{f,g\}\omega^n = 0$.

(ii) ξ_1,ξ_2,ξ_3,ξ_4

$$\begin{aligned}
\int_{T^4}\omega(\xi_1,\xi_2)\omega(\xi_3,\xi_4)\omega^2 - \int_{T^4}\omega(\xi_1,\xi_2)\omega^2\int_{T^4}\omega(\xi_3,\xi_4)\omega^2 &= 0 \\
\Leftrightarrow 1 - 1\cdot 1 &= 0
\end{aligned}$$

The relation (4) representing the cocycle identity is not homogeneous in ω, hence, in general, if for (M, ω) the relation (2) defines a cocycle c, then for $(M, a\omega)$, $a \in \mathbb{R} - \{-1, 0, 1\}$ this will not be true anymore. This is the case here.

Riemann surface of genus $g \geq 2$.

A Riemann surface M has an essentially unique realization as a two sheeted cover of the sphere $\hat{\mathbb{C}} \cong \mathrm{P}\,\mathbb{C}^1$, branched over $2g+2$ points. It is the Riemann surface of the algebraic curve

$$w^2 = \prod_{k=1}^{2g+2} (z - a_k), \quad a_i \neq a_j \text{ for } i \neq j$$

We think of z as a variable point in $\hat{\mathbb{C}}$ and then view M as the Riemann surface on which w is a well defined (single valued) meromorphic function. As a function of z, w is two valued.

Let $\pi : M \to \hat{\mathbb{C}}$ be the covering and $P_k \in M$ the $2g+2$ branched points. Then $\pi(P_k) = z(P_k) = a_k$. We suppose that all a_k are real numbers. We get a volume form on M, which is also a symplectic form, by lifting the volume form on $\mathrm{P}\,\mathbb{C}^1$

$$\mu = \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$$

to the form $\omega = \pi^* \mu$ on M.

A basis for the holomorphic 1-forms on M is given by

$$\alpha_k = \frac{z^{k-1} dz}{w}, \quad k = 1, \ldots, g$$

hence a basis for $H^1(M)$ is determined by the holomorphic and antiholomorphic forms α_k and $\bar{\alpha}_k$ with $k = 1, \ldots, g$. The locally Hamiltonian vector fields corresponding to them are ξ_k and $\bar{\xi}_k$, defined by $i_{\xi_k} \omega = \alpha_k$

$$\begin{aligned} \xi_k &= -\frac{2\pi}{i} \frac{z^{k-1}(1 + |z|^2)^2}{\bar{w}} \frac{\partial}{\partial \bar{z}} \\ \bar{\xi}_k &= \frac{2\pi}{i} \frac{\bar{z}^{k-1}(1 + |z|^2)^2}{\bar{w}} \frac{\partial}{\partial z} \end{aligned}$$

The cocycle relation (4) is written in terms of $\omega(\xi, \xi')$. In this case

$$\begin{aligned} \omega(\xi_j, \bar{\xi}_k) &= \frac{2\pi}{i} \frac{z^{j-1} \bar{z}^{k-1} (1 + |z|^2)^2}{|w|^2} \\ &= \frac{2\pi}{i} \frac{z^{j-1} \bar{z}^{k-1} (1 + |z|^2)^2}{\prod_{h=1}^{2g+2} |z - a_h|} \end{aligned}$$

is projectable to $\hat{\mathbb{C}}$, hence its integral over M can be written as an integral over $\hat{\mathbb{C}}$.

$$\int_M (\pi^* f) \omega = 2 \int_{\hat{\mathbb{C}}} f \mu.$$

Furthermore $\omega(\xi_j,\xi_k)=\omega(\bar\xi_j,\bar\xi_k)=0$.

If we insert the vector fields $\xi_1,\xi_2,\bar\xi_1,\bar\xi_2$, the relation (8.4) becomes

$$\int_M \omega(\xi_1,\bar\xi_1)\omega\int_M \omega(\xi_2,\bar\xi_2)\omega+\int_M \omega(\bar\xi_1,\xi_2)\omega\int_M \omega(\xi_1,\bar\xi_2)\omega=0.$$

This is equivalent to

$$\int_{\mathbb{C}}\frac{1}{\prod|z-a_k|}dzd\bar z\int_{\mathbb{C}}\frac{|z|^2}{\prod|z-a_k|}dzd\bar z+\int_{\mathbb{C}}\frac{z}{\prod|z-a_k|}dzd\bar z\int_{\mathbb{C}}\frac{\bar z}{\prod|z-a_k|}dzd\bar z=0,$$

and writing it as an integral over $\mathbb{R}^2$, we see that this relation can never be accomplished:

$$\int_{\mathbb{R}^2}\frac{dxdy}{\prod\sqrt{(x-a_k)^2+y^2}}\int_{\mathbb{R}^2}\frac{x^2+y^2}{\prod\sqrt{(x-a_k)^2+y^2}}dxdy$$
$$+\int_{\mathbb{R}^2}\frac{x+iy}{\prod\sqrt{(x-a_k)^2+y^2}}dxdy\int_{\mathbb{R}^2}\frac{x-iy}{\prod\sqrt{(x-a_k)^2+y^2}}dxdy\gneqq 0$$

because the first term is strictly greater than zero and the second term is

$$\left(\int_{\mathbb{R}^2}\frac{x}{\prod\sqrt{(x-a_k)^2+y^2}}dxdy\right)^2+\left(\int_{\mathbb{R}^2}\frac{y}{\prod\sqrt{(x-a_k)^2+y^2}}dxdy\right)^2\geq 0$$

We have found that for a surface with genus $g\geq 2$ there is no central extension of $\mathfrak{g}$ with $H_1(M)$ which leaves the scalar product (1) invariant.

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