Discontinuous Petrov-Galerkin Methods

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- discontinuous Petrov-Galerkin (dPG) methods invented by L. Demkowicz, J. Gopalakrishnan (2010)
- motivated by search for optimal test functions
- dPG can be viewed as mixed method with nonstandard test space (typically broken test functions) or minimal residual method
- main idea: too many test functions ensure an inf-sup condition, let computer handle redundant dofs
- applications: linear elasticity, Stokes equations, Maxwell, ...



1 General Framework of dPG Methods

- Inf-Sup Conditions
- dPG as Mixed Method and Minimal Residual Method
- Built-in A Posteriori Estimate

2 The Primal dPG Method for the Poisson Model Problem

- Problem Formulation
- Proof of (H1) and (H2)



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- The Primal dPG Method for the Poisson Model Problem
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- $b: X \times Y \to \mathbb{R}$ a bounded bilinear form with $||b|| = \sup_{x \in S(X)} \sup_{y \in S(Y)} b(x, y)$



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- $F \in Y^{\star} := \{G : Y \to \mathbb{R} \mid G \text{ linear and bounded}\}$
- for any $G \in Y^{\star}$, let $||G||_{Y^{\star}} := \sup_{y \in \mathcal{S}(Y)} G(y)$



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$$x \in X$$
: $b(x, \bullet) = F$ in Y

is well-posed with unique solution.



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is well-posed with unique solution. The standard theory on mixed finite element methods [Boffi, Brezzi, Fortin] shows that this is equivalent to the inf-sup condition

$$0 < \beta := \inf_{x \in \mathcal{S}(X)} \sup_{y \in \mathcal{S}(Y)} b(x, y)$$
(H1)

and non-degeneracy

$$\{0\} = N := \{y \in Y \mid b(\bullet, y) = 0 \text{ in } X\}.$$



Question: How to discretize this problem?



Question: How to discretize this problem? For any chosen finite-dimensional subspaces $X_h \subseteq X$, $Y_h \subseteq Y$, the well-posedness of the discrete problem

$$x_h \in X_h$$
: $b(x_h, \bullet) = F$ in Y_h

is equivalent to the discrete inf-sup condition

$$0 < \beta_h := \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(Y_h)} b(x_h, y_h)$$
(H2)

and non-degeneracy

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$$0 < \beta_h := \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(Y_h)} b(x_h, y_h)$$

$$\{0\} = N_h := \{y_h \in Y_h \mid b(\bullet, y_h) = 0 \text{ in } X_h\}$$

Spaces X_h and Y_h need to be well-balanced to satisfy these two conditions.

For fixed X_h ,

- a big Y_h makes it easier to satisfy discrete inf-sup condition
- but harder to guarantee discrete non-degeneracy.

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The idea of discontinuous Petrov-Galerkin methods (dPG methods):

- choose only the discrete trial space $X_h \subseteq X$,
- compute a discrete test space ⊆ Y with (H2) and non-degeneracy.

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Define the trial-to-test-operator $T: X \to Y$ by

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 for any $y \in Y$.

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For a fixed choice of discrete trial space $X_h \subseteq X$ the *idealized dPG* method utilizes the discrete test space $T(X_h) \subseteq Y$.



Variational formulation

continuous problem

 $x \in X$: $b(x, \bullet) = F$ in Y

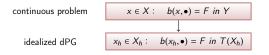


Variational formulation

$$\begin{array}{c} \text{continuous problem} \\ \hline x \in X : \quad b(x, \bullet) = F \text{ in } Y \\ \downarrow \\ \text{idealized dPG} \\ \hline x_h \in X_h : \quad b(x_h, \bullet) = F \text{ in } T(X_h) \end{array}$$



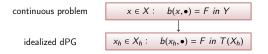
Variational formulation



 X_h and T(X_h) of idealized dPG method automatically satisfy the discrete inf-sup condition and non-degeneracy



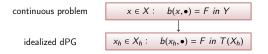




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- Remedy: discrete trial-to-test-operator $T_h : X \to Y_h$ for some discrete space $Y_h \subseteq Y$. Define $T_h : X \to Y_h$ similar to the continuous operator T by

$$(T_h x, y_h)_Y = b(x, y_h)$$
 for any $y_h \in Y_h$.



Variational formulation

continuous problem	$x \in X$: $b(x, \bullet) = F$ in Y
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idealized dPG	$x_h \in X_h$: $b(x_h, \bullet) = F$ in $T(X_h)$
practical dPG	$x_h \in X_h$: $b(x_h, \bullet) = F$ in $T_h(X_h)$

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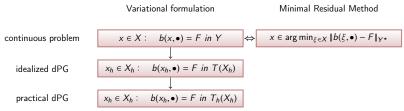
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- X_h and T_h(X_h) ⊆ Y_h satisfy non-degeneracy but not necessarily the discrete inf-sup condition
- continuous inf-sup condition holds $\rightarrow Y_h$ big enough leads to the discrete inf-sup condition

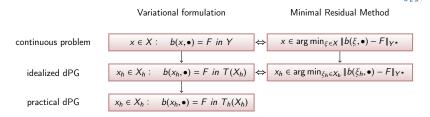
$$0 < \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(Y_h)} b(x_h, y_h) = \inf_{x_h \in \mathcal{S}(X_h)} \sup_{y_h \in \mathcal{S}(T_h(X_h))} b(x_h, y_h).$$

dPG methods...seen as minimization





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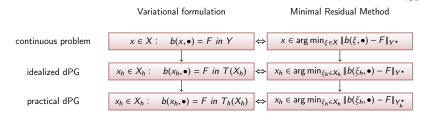
equivalence by calculation of Gâteaux derivative

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9/27

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dPG methods...seen as minimization



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Theorem

Suppose (H1) and (H2) hold. Then the solution $x_h \in X_h$ to the practical dPG method satisfies

$$\|x - x_h\|_X \le \frac{\|b\|}{\beta_h} \min_{\xi_h \in X_h} \|x - \xi_h\|_X.$$



Idealized dPG method has built-in a posteriori estimate

$$\beta \|x - x_h\|_X \le \|b(x_h, \bullet) - F\|_{Y^*} \le \|b\| \|x - x_h\|_X,$$

i.e.

$$\|x-x_h\|_X\approx \|b(x_h,\bullet)-F\|_{Y^\star}.$$

Is such an estimate possible for the practical dPG method as well?



For an a posteriori estimate of the practical dPG method, we need some Fortin operator $\Pi: Y \to Y_h$.

Existence of $\Pi: Y \to Y_h$ linear and bounded projection with $b(X_h, (1 - \Pi)Y) = 0$



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Existence of $\Pi: Y \to Y_h$ linear and bounded projection with $b(X_h, (1 - \Pi)Y) = 0$

Lemma

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Proof.

 $(H3) \implies (H2).$



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Existence of $\Pi : Y \to Y_h$ linear and bounded projection with $b(X_h, (1 - \Pi)Y) = 0$

Lemma

Suppose (H1) holds. Then (H2) and (H3) are equivalent.

Proof.

(H3) \implies (H2). Continuity of Π implies $1/\|y\|_Y \le \|\Pi\|/\|\Pi y\|_Y$ and (H1) shows

$$0 < \beta \leq \inf_{\substack{x_h \in \mathcal{S}(X_h) \ y \in Y, y \neq 0}} \sup_{\substack{b(x_h, y) \\ \|y\|_Y}} \leq \|\Pi\| \inf_{\substack{x_h \in \mathcal{S}(X_h) \ y \in Y, y \neq 0}} \frac{b(x_h, \Pi y)}{\|\Pi y\|_Y}$$
$$\leq \|\Pi\| \inf_{\substack{x_h \in \mathcal{S}(X_h) \ y_h \in Y_h, y_h \neq 0}} \frac{b(x_h, y_h)}{\|y_h\|_Y}.$$
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Lemma

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 $\begin{array}{l} \mathsf{Proof.} \\ (H2) \implies (H3). \end{array}$



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Proof.

 $(H2) \implies (H3)$. Condition (H2) and non-degeneracy imply that $B: T_h(X_h) \to X_h^{\star}, BTx_h := b(\bullet, Tx_h)$ is an isomorphism. Let $G: Y \to X_h^{\star}, Gy := b(\bullet, y)$. Define $\Pi := B^{-1} \circ G : Y \to T_h(X_h)$ linear and bounded.

$$\Pi(Tx_h) = B^{-1}b(\bullet, Tx_h) = B^{-1}BTx_h = Tx_h$$

$$y \in Y \text{ satisfies } b(\bullet, y) = G(y) = B(\Pi y) = b(\bullet, \Pi y) \text{ in } X_h$$



Theorem Suppose (H1) and (H3) hold. Then any $\xi_h \in X_h$ satisfies

$$\begin{split} \beta \|x - x_h\|_X &\leq \|\Pi\| \|b(x_h, \bullet) - F\|_{Y_h^{\star}} + \|F \circ (1 - \Pi)\|_{Y^{\star}} \\ &\leq 2\|b\| \|\Pi\| \|x - x_h\|_X. \end{split}$$

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Proof. (H1) and (H3) imply $\beta ||x - x_h||_X \le ||F - b(\xi_h, \bullet)||_{Y^*}$ and

$$\|F - b(\xi_h, \bullet)\|_{Y^{\star}} \leq \sup_{y \in \mathcal{S}(Y)} F((1 - \Pi)y) + \sup_{y \in \mathcal{S}(Y)} F(\Pi y) - b(\xi_h, \Pi y).$$

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Second estimate follows from continuity of $b, \ \|\Pi\| = \|1 - \Pi\|$ and

$$F((1-\Pi)y) = b(x,(1-\Pi)y) = b(x-\xi_h,(1-\Pi)y).$$

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14/27

Computation of dPG Solution



- $\{\Phi_1, \ldots, \Phi_J\}$ basis of X_h
- $\{\Psi_1, \ldots, \Psi_K\}$ basis of Y_h
- $A \in \mathbb{R}^{K \times J}$, $A_{kj} := b(\Phi_j, \Psi_k)$, $k = 1, \dots, K$, $j = 1, \dots, J$, matrix for bilinear form
- $M \in \mathbb{R}^{K \times K}$, $M_{k\ell} := (\Psi_k, \Psi_\ell)_Y$, $k, \ell = 1, ..., K$, matrix for scalar product on Y
- $b \in \mathbb{R}^{K}$, $b_k := F(\Psi_k)$, $k = 1, \dots, K$, vector for right-hand side

For $x \in \mathbb{R}^J$ coefficient vector for solution $x_h = \sum_{j=1}^J x_j \Phi_j$, the linear system of equation reads

$$A^{\top}M^{-1}Ax = A^{\top}M^{-1}b.$$

Bigger Y_h guarantees inf-sup condition, but computation will be more expensive! Concept of broken test functions leads to block-diagonal M.

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Theorem (Traces of H^1 -functions)

For $U \subseteq \mathbb{R}^n$ open and bounded Lipschitz domain, there exists continuous, linear $\gamma_0 : H^1(U) \to L^2(\partial U)$ with

$$\gamma_0 w = w|_{\partial U}$$
 for all $w \in H^1(U) \cap C^0(\overline{U})$.





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Set

$$H^{1/2}(\partial U) := \gamma_0(H^1(U)) \text{ and } H^{-1/2}(\partial U) := (H^{1/2}(\partial U))^*.$$





Theorem (Normal traces of H(div)-functions) For $U \subseteq \mathbb{R}^n$ open and bounded Lipschitz domain, there exists continuous, linear $\gamma_v : H(\operatorname{div}, U) \to H^{-1/2}(\partial U)$, with

$$\gamma_{\nu}q = q|_{\partial U} \cdot \nu \quad \text{ for all } q \in C^{\infty}(\overline{U}; \mathbb{R}^n).$$

Any $q \in H(\operatorname{div}, U)$ and $w \in H^1(U)$ satisfies

$$\langle \gamma_{\nu} q, \gamma_0 w \rangle_{\partial U} = (q, \nabla w)_U + (\operatorname{div} q, w)_U.$$





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 $\gamma_{\nu}^{\mathcal{T}}q \coloneqq (t_{\mathcal{T}})_{\mathcal{T}\in\mathcal{T}}, \quad \text{with } t_{\mathcal{T}} \coloneqq \gamma_{\nu}(q|_{\mathcal{T}}) \quad \text{ for all } \mathcal{T}\in\mathcal{T}.$





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$$H^{-1/2}(\partial \mathcal{T}) \mathrel{\mathop:}= \gamma_v^{\mathcal{T}} H(\mathsf{div}, \Omega)$$

is a Hilbert space with minimal extension norm

 $\|t\|_{H^{-1/2}(\partial\mathcal{T})} = \min\{\|q\|_{H(\operatorname{div})} \mid q \in H(\operatorname{div},\Omega), \gamma_{\nu}^{\mathcal{T}}q = t\}.$





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For $t \in \prod_{T \in \mathcal{T}} H^{-1/2}(\partial T)$ and $v \in H^1(\mathcal{T})$ define

$$\langle t, v \rangle_{\partial \mathcal{T}} := \sum_{T \in \mathcal{T}} \langle t_T, \gamma_0 v \rangle_{\partial T}.$$



$\Omega\subseteq\mathbb{R}^2$ open Lipschitz domain with polygonal boundary Seek $u:\Omega\to\mathbb{R}$ with

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial \Omega.$$



- multiply equation with test function v,
- \blacksquare integrate by parts on Ω
- test function $v \in H_0^1(\Omega)$

$$\int_{\Omega} f v \, \mathrm{d}x = \int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial \Omega} v \nabla u \cdot v \, \mathrm{d}s$$



- multiply equation with test function v,
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- boundary integral vanishes

$$\int_{\Omega} f v \, \mathrm{d}x = \int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}x - \underbrace{\int_{\partial \Omega} v \nabla u \cdot v \, \mathrm{d}s}_{=0}$$



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- boundary integral introduces new variable t

$$\int_{T} f v \, \mathrm{d}x = \int_{T} \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial T} v \underbrace{\nabla u \cdot v}_{t_{T}} \mathrm{d}s$$



- multiply equation with test function v,
- integrate by parts on $T \in \mathcal{T}$
- test function $v \in H^1(\mathcal{T})$
- boundary integral introduces new variable t
- sum over all elements

$$\int_{T} f v \, \mathrm{d}x = \int_{T} \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial T} v \underbrace{\nabla u \cdot v}_{t_{T}} \mathrm{d}s$$

Primal dPG formulation seeks $u \in H_0^1(\Omega), t \in H^{-1/2}(\partial \mathcal{T})$ with

$$(f,v)_{\Omega} = (\nabla u, \nabla_{NC}v)_{\Omega} - \langle t, v \rangle_{\partial \mathcal{T}} \text{ for all } v \in H^1(\mathcal{T}).$$



Theorem Any $t \in H^{-1/2}(\partial \mathcal{T})$ satisfies

$$\|t\|_{H^{-1/2}(\partial \mathcal{T})} \leq \sup_{v \in H^1(\mathcal{T}), v \neq 0} \frac{\langle t, v \rangle_{\partial \mathcal{T}}}{\|v\|_{H^1(\mathcal{T})}}.$$



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Proof.

Let $v \in H^1(\mathcal{T})$ on each $\mathcal{T} \in \mathcal{T}$ weak solution to

$$-\Delta v + v = 0$$
 in T and $\nabla v \cdot v = t_T$ on ∂T .

With $q := \nabla_{NC} v \in H(\operatorname{div}, \Omega)$, it holds $||t||_{H^{-1/2}(\partial \mathcal{T})} \leq ||q||_{H(\operatorname{div})}$, div q = v, and $||q||_{H(\operatorname{div})} = ||v||_{H^1(\mathcal{T})}$. Integration by parts shows

$$\langle t, v \rangle_{\partial \mathcal{T}} = (q, \nabla_{NC} v)_{\Omega} + (\operatorname{div} q, v)_{\Omega} = ||q||_{H(\operatorname{div})}^{2}.$$



Theorem

The spaces $X := H_0^1(\Omega) \times H^{-1/2}(\partial \mathcal{T})$, $Y := H^1(\mathcal{T})$ and the bilinear form $b : X \times Y$, $b(u, t; v) := (\nabla u, \nabla_{NC} v)_{\Omega} - \langle t, v \rangle_{\partial \mathcal{T}}$ satisfy (H1).



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Proof.

The Friedrichs inequality implies

$$\|\nabla u\|_{L^{2}(\Omega)} \lesssim \sup_{v \in H_{0}^{1}(\Omega), v \neq 0} \frac{(\nabla u, \nabla v)_{\Omega}}{\|v\|_{H^{1}(\mathcal{T})}} = \sup_{v \in H_{0}^{1}(\Omega), v \neq 0} \frac{b(u, t; v)}{\|v\|_{H^{1}(\mathcal{T})}}$$

The duality lemma and the triangle inequality show

$$\|t\|_{H^{-1/2}(\partial\mathcal{T})} \leq \sup_{v \in Y, v \neq 0} \frac{b(u,t;v)}{\|v\|_{H^1(\mathcal{T})}} + \sup_{v \in Y, v \neq 0} \frac{(\nabla u, \nabla_{NC}v)_{\Omega}}{\|v\|_{H^1(\mathcal{T})}}. \quad \Box$$



Recall

$$\begin{split} &X=H^1_0(\Omega)\times H^{-1/2}(\partial\mathcal{T}),\\ &Y=H^1(\mathcal{T}). \end{split}$$

The discrete spaces read

$$\begin{split} X_h &:= S_0^1(\mathcal{T}) \times P_0(\mathcal{E}) \subseteq X, \\ Y_h &:= P_1(\mathcal{T}) \subseteq Y. \end{split}$$



Theorem The discrete spaces X_h and Y_h satisfy (H2).



Theorem

The discrete spaces X_h and Y_h satisfy (H2).

Proof.

- given $x_h = (u_C, t_0) \in X_h$, let $q_{RT} \in RT_0(\mathcal{T}), \gamma_v^{\mathcal{T}}(q_{RT}) = t_0$
- choose $v_1 = -\operatorname{div} q_{RT} + (\nabla u_C \Pi_0 q_{RT}) \cdot (\bullet \operatorname{mid}(\mathcal{T})) \in Y_h$

integration by parts shows

$$b(u_{C}, t_{0}; v_{1}) = (\nabla u_{C} - q_{RT}, \nabla_{NC} v_{1})_{\Omega} - (\operatorname{div} q_{RT}, v_{1})_{\Omega}$$
$$= \|\nabla u_{C} - \Pi_{0} q_{RT}\|_{L^{2}(\Omega)}^{2} + \|\operatorname{div} q_{RT}\|_{L^{2}(\Omega)}^{2}$$

Since
$$P_0(\mathcal{T})$$
 is orthogonal to $(\bullet - \operatorname{mid}(\mathcal{T}))$ in $L^2(\Omega)$,
 $\|v_1\|_{H^1(\mathcal{T})}^2 \leq (1 + h_{\max}^2)b(u_C, t_0; v_1).$



Proof.

- recall $b(u_C, t_0; v_1) = \|\nabla u_C \Pi_0 q_{RT}\|_{L^2(\Omega)}^2 + \|\operatorname{div} q_{RT}\|_{L^2(\Omega)}^2$
- Helmholtz decomposition leads to $\alpha_C \in S_0^1(\mathcal{T})$, $\beta_{CR} \in CR^1(\mathcal{T})$ with $\nabla u_C - \Pi_0 q_{RT} = \nabla \alpha_C + \operatorname{Curl}_{NC} \beta_{CR}$. Orthogonality in Helmholtz and integration by parts shows

$$\begin{aligned} \|\nabla(u_C - \alpha_C)\|_{L^2(\Omega)}^2 &= (\nabla(u_C - \alpha_C), q_{RT})_{\Omega} = -(u_C - \alpha_C, \operatorname{div} q_{RT})_{\Omega} \\ &\lesssim \|\nabla(u_C - \alpha_C)\|_{L^2(\Omega)} \|\operatorname{div} q_{RT}\|_{L^2(\Omega)}. \end{aligned}$$

■ triangle inequality implies $\begin{aligned} \|\nabla u_{C}\|_{L^{2}(\Omega)} &\leq \|\nabla (u_{C} - \alpha_{C})\|_{L^{2}(\Omega)} + \|\nabla \alpha_{C}\|_{L^{2}(\Omega)} \leq \\ \|\operatorname{div} q_{RT}\|_{L^{2}(\Omega)} + \|\nabla u_{C} - \Pi_{0}q_{RT}\|_{L^{2}(\Omega)} \leq b(u_{C}, t_{0}; v_{1})^{1/2} \text{ and} \\ \|t_{0}\|_{H^{-1/2}(\partial \mathcal{T})} &\leq \|q_{RT}\|_{H(\operatorname{div})} \leq \|\Pi_{0}q_{RT}\|_{L^{2}(\Omega)} + \|\operatorname{div} q_{RT}\|_{L^{2}(\Omega)} \leq \\ b(u_{C}, t_{0}; v_{1})^{1/2}. \end{aligned}$



- idea of dPG: choose discrete trial space, compute discrete test space
- idealized dPG: inf-sup stable, but not practical
- practical dPG inf-sup stable for Y_h big enough
- practical dPG has built-in a priori and a posteriori error control
- application to Poisson as primal dPG with broken test functions
- continuous inf-sup follows from stability of non-broken functions
- discrete inf-sup utilizes discrete Helmholtz decomposition