FARKAS-TYPE RESULTS WITH CONJUGATE FUNCTIONS

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Abstract. We present some new Farkas-type results for inequality systems involving a finite as well as an infinite number of convex constraints. For this, we use two kinds of conjugate dual problems, namely an extended Fenchel-type dual problem and the recently introduced Fenchel-Lagrange dual problem. For the latter, which is a "combination" of the classical Fenchel and Lagrange duals, the strong duality is established.

Key words. Farkas-type results, conjugate duality, finitely and infinitely many convex constraints

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1. Introduction. The Farkas lemma [6] states that a linear inequality $c^T x \leq 0$ is a consequence of a linear system $Ax \leqq 0$ if and only if c is a nonpositive linear combination of elements of A. This result has played an important role in the development of linear programming and optimization theory. During the last two decades a number of Farkas-type results have been given in the literature with applications to more general nonlinear programming problems and nonsmooth optimization problems (see [22], [11], [9], [10], [15], [8], [13], [18], [7], [14], [16], [12]).

In this paper we present some new Farkas-type results for inequality systems involving finitely as well as infinitely many convex constraints. The approach is based on the theory of conjugate duality for convex optimization problems. Two dual problems play an important role in our investigations, namely an extended Fenchel-type dual problem (see, for example, [17]) and the so-called Fenchel-Lagrange dual problem. The last one is a "combination" of the classical Fenchel and Lagrange dual problems and has been introduced and extensively studied recently by the authors of this paper (cf. [19], [2], [20], [21], [1], etc.). The construction of the Fenchel-Lagrange dual is described here in detail since we use a conjugacy approach which is based on perturbation theory (cf. [4]). Then we introduce a constraint qualification whose fulfillment is sufficient in order to guarantee strong duality. The strong duality assertion is also proved.

The results we are going to present here underline the connections that exist between Farkas-type results and theorems of alternative and, on the other hand, the theory of duality. Furthermore, we bring some generalizations to some recently published results due to Jeyakumar (cf. [12]).

The paper is organized as follows. In Section 2 we present definitions and preliminary results that will be used later in the paper. In Section 3 we introduce a convex optimization problem and construct its Fenchel-Lagrange dual. Then we prove the existence of strong duality between these two problems. Section 4 provides some Farkas-type results, obtained by using the Fenchel-Lagrange dual problem, for inequality systems involving a finite number of convex constraints. Finally, in Section 5 we give some Farkas-type results for inequality systems involving an infinite number of convex constraints. There we use an extended version of the classical Fenchel dual problem.

2. Notation and preliminaries. In this section we describe the notations we use throughout this paper and present preliminary results. All vectors will be column vectors. A column vector will be transposed to a row vector by an upper

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index ^{T}. The inner product of two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in the n-dimensional real space $\mathbb{R}^n$ will be denoted by $x^T y = \sum_{i=1}^n x_i y_i$. For a set $X \subseteq \mathbb{R}^n$ we shall denote the *closure*, the *convex hull* and the *relative interior* of X by $cl(X)$, $co(X)$ and $ri(X)$, respectively. Similarly, we shall denote the *cone* and the *convex cone* generated by the set X by $cone(X) = \bigcup\limits_{\lambda \geq 0} \lambda X$, respectively, $coneco(X) = \bigcup\limits_{\lambda \geq 0} \lambda co(X)$.

Furthermore, for the set $X \subseteq \mathbb{R}^n$, the *indicator function* $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

while the *support function* $\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$\sigma_X(p) = \sup_{x \in X} p^T x.$$

Considering now a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by

$$dom(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$$

its *effective domain*, by

$$epi(f) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$$

its *epigraph* and by $cl(f)$ its *closure*, namely the function whose epigraph is the closure of $epi(f)$ in $\mathbb{R}^{n+1}$. We say that f is proper if $dom(f) \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for f the so-called *conjugate relative to the set* X

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}},\ f_X^*(p) = \sup_{x \in X} \{p^T x - f(x)\}.$$

When X equals the whole space $\mathbb{R}^n$, the conjugate relative to the set X becomes the classical *conjugate function of* f (the Fenchel-Moreau conjugate)

$$f^* : \mathbb{R}^n \to \overline{\mathbb{R}},\ f^*(p) = \sup_{x \in \mathbb{R}^n} \{p^T x - f(x)\}.$$

Assuming that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper function, we have the following elementary result for its conjugate

$$(\alpha f)^*(p) = \alpha f^* \left(\frac{1}{\alpha} p \right) \ \forall p \in \mathbb{R}^n \ \forall \alpha > 0. \tag{2.1}$$

Two results, which play an important role in this paper, follow. But first the following definition is necessary.

Definition 2.1. *Let the functions $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be given. The function $f_1 \Box \cdots \Box f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$, defined by*

$$f_1 \Box \cdots \Box f_m(x) := \inf \left\{ \sum_{i=1}^m f_i(x_i) : \sum_{i=1}^m x_i = x \right\},$$

is called the infimal convolution function of $f_1, ..., f_m$.

THEOREM 2.2 (cf. Theorem 16.4 in [17]). *Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. Then*

$$(cl(f_1) + \cdots cl(f_m))^* = cl(f_1^* \square \cdots \square f_m^*).$$

If the sets $ri(dom(f_i)), i = 1, ..., m$, have a point in common, then

$$\left(\sum_{i=1}^m f_i \right)^* (p) = (f_1^* \square \cdots \square f_m^*)(p) = \inf \left\{ \sum_{i=1}^m f_i^*(p_i) : \sum_{i=1}^m p_i = p \right\},$$

where for each $p \in \mathbb{R}^n$ the infimum is attained.

A direct consequence of this theorem comes next.

COROLLARY 2.3. *Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. Then*

$$epi \left(\left(\sum_{i=1}^m cl(f_i) \right)^* \right) = cl \left(\sum_{i=1}^m epi(f_i^*) \right). \tag{2.2}$$

If the sets $ri(dom(f_i)), i = 1, ..., m$, have a point in common, then

$$epi \left(\left(\sum_{i=1}^m f_i \right)^* \right) = \sum_{i=1}^m epi(f_i^*). \tag{2.3}$$

3. The Fenchel-Lagrange dual problem. In this section we assume that X is a nonempty convex subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper convex function and $g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function, with g_i also convex for $i = 1, ..., m$. Using them we introduce the following convex optimization problem, further called primal problem,

$$(P) \quad \inf_{x \in C} f(x),$$

$$C = \{x \in X : g(x) \leqq 0\}.$$

As usual, $g(x) \leqq 0$ means $g_i(x) \leq 0$ for all $i = 1, ..., m$.

In order to determine the so-called Fenchel-Lagrange dual problem of (P) we need to introduce the perturbation function (cf. [19], [2], [20], [21], [1])

$$\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}, \Phi_{FL}(x, y, z) = \begin{cases} f(x + y), & \text{if } x \in X, g(x) \leqq z, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables y and z. Following the path of the perturbation method described in [4] and [17] the next step is to calculate the conjugate function to Φ. Let us proceed with the definition for $\Phi_{FL}^* : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\begin{aligned} \Phi_{FL}^*(x^*, p, q) &= \sup_{\substack{x, y \in \mathbb{R}^n, \\ z \in \mathbb{R}^k}} \left\{ x^{*T} x + p^T y + q^T z - \Phi(x, y, z) \right\} \\ &= \sup_{\substack{x \in X, y \in \mathbb{R}^n, \\ g(x) \leqq z}} \left\{ x^{*T} x + p^T y + q^T z - f(x + y) \right\}. \end{aligned}$$

In order to ease the calculations let us introduce the new variables r instead of y and s instead of z by

$$r := x + y, \ s := z - g(x)$$

and therefore the supremum above can be separated into a sum of three suprema

$$\Phi_{FL}^*(x^*,p,q) = \sup_{s\in\mathbb{R}_+^m} q^T s + \sup_{r\in\mathbb{R}^n}\Big\{p^T r - f(r)\Big\} + \sup_{x\in X}\Big\{(x^*-p)^T x + q^T g(x)\Big\}$$
$$= \begin{cases} f^*(p) - \inf\limits_{x\in X}\Big\{(p-x^*)^T x - q^T g(x)\Big\}, & \text{if } q \in -\mathbb{R}_+^m, \\ +\infty, & \text{otherwise,}\end{cases}$$

where $\mathbb{R}_+^m = \{z : z\in\mathbb{R}^m, 0 \leqq z\}$.

According to [4] the dual problem to (P) is

$$(D_{FL}) \quad \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}^m}} \Big\{ -\Phi_{FL}^*(0,p,q)\Big\},$$

that becomes in our case after changing the sign of the variable q

$$(D_{FL}) \quad \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}_+^m}} \Big\{ -f^*(p) + \inf_{x\in X}[p^T x + q^T g(x)]\Big\}$$

or, equivalently,

$$(D_{FL}) \quad \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}_+^m}} \Big\{ -f^*(p) - (q^T g)_X^*(-p)\Big\}. \tag{3.1}$$

The dual problem (D_{FL}) has been introduced by Wanka and Boţ in [19] and has been extensively studied in [2], [20], [21], [1], etc. We call it the Fenchel-Lagrange dual problem because, as one may observe, it is a "combination" of the well-known Lagrange and Fenchel dual problems.

It is obvious from the construction of the dual that the weak duality assertion between (P) and (D_{FL}), i. e. the value of the primal objective function at any feasible point is greater than or equal to the value of the dual objective function at any dual feasible point, always holds. By strong duality we understand the situation in which the optimal objective values of the primal and dual are equal and (D_{FL}) has an optimal solution. Unlike weak duality, strong duality can fail in the general case. To avoid this undesired situation, we introduce a constraint qualification that guarantees the validity of strong duality in case this constraint qualification is fulfilled. First let us divide the index set $\{1,...,m\}$ into two subsets,

$$L := \Bigg\{ i\in\{1,...,m\} : g_i : \mathbb{R}^n \to \mathbb{R} \text{ is an affine function}\Bigg\}$$

and $N := \{1,...,k\}\backslash L$. The constraint qualification follows

$$(CQ) \;\; \exists x' \in ri(X)\cap ri(dom(f)) : \begin{cases} g_i(x') \leq 0, & i\in L,\\ g_i(x') < 0, & i\in N.\end{cases}$$

We are ready now to formulate the strong duality assertion. Before that let us denote by $v(P)$ and $v(D_{FL})$ the optimal objective values of the primal and dual problem, respectively.

Theorem 3.1. *Assume that $v(P) > -\infty$. Provided that the constraint qualification (CQ) is fulfilled, there is strong duality between problems (P) and (D_{FL}), i. e. their optimal objective values are equal and the dual problem has an optimal solution.*

Proof. We can write the problem (P) equivalently

$$(P) \quad \inf_{\substack{x \in X \cap dom(f),\\ g(x) \leqq 0}} f(x).$$

By Theorem 6.5 in [17], (CQ) yields

$$x' \in ri\big(X \cap dom(f)\big) = ri(X) \cap ri\big(dom(f)\big).$$

Theorem 5.7 in [5] states under the present hypotheses the existence of a $\bar{q} \geqq 0$ such that the Lagrange duality holds, namely

$$v(P) = \max_{q \geqq 0} \inf_{x \in X \cap dom(f)} \big[f(x) + q^T g(x)\big] = \inf_{x \in X \cap dom(f)} \big[f(x) + \bar{q}^T g(x)\big].$$

Defining

$$h : \mathbb{R}^n \to \overline{\mathbb{R}},\ h(x) = \begin{cases} \bar{q}^T g(x), & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X, \end{cases}$$

we can rewrite the right-hand side term in the following way

$$v(P) = \inf_{x \in \mathbb{R}^n} \big[f(x) + h(x)\big].$$

Because $ri(dom(f)) \cap ri(dom(h)) = ri(dom(f)) \cap ri(X) \neq \emptyset$, by Theorem 31.1 (Fenchel's Duality Theorem) in [17], there exists a $\bar{p} \in \mathbb{R}^n$ such that this infimum is equal to

$$\begin{aligned} v(P) &= \max_{p \in \mathbb{R}^n} \big[-f^*(p) - h^*(-p)\big] = -f^*(\bar{p}) - h^*(-\bar{p}) \\ &= -f^*(\bar{p}) - \sup_{x \in \mathbb{R}^n} \big\{-\bar{p}^T x - h(x)\big\} \\ &= -f^*(\bar{p}) - \sup_{x \in X} \big\{-\bar{p}^T x - \bar{q}^T g(x)\big\} \\ &= -f^*(\bar{p}) - (\bar{q}^T g)^*_X(-\bar{p}). \end{aligned} \tag{3.2}$$

In the right-hand term of (3.2) one may recognize the objective function of (D_{FL}) at $(\bar{p}, \bar{q})$. From weak duality it follows that the supremum of (D_{FL}) is attained at $(\bar{p}, \bar{q})$, which is therefore an optimal solution of the dual problem. □

Remark 3.2. Let us notice that in the proof above we have first proved that under the fulfillment of (CQ) there holds strong duality between the primal problem and its Lagrange dual problem. Then we proved the existence of strong duality between the inner infimum of the Lagrange dual at its outer maximum and its Fenchel dual problem, the last one proving to be exactly the Fenchel-Lagrange dual problem we introduced earlier.

4. Some new Farkas-type results for finitely many constraints. In the following we give some new Farkas-type results for inequality systems involving finitely many convex constraints. The main theorem yields a new dual characterization for this kind of results by using the duality concept introduced above. Moreover, it generalizes some recently published results due to Jeyakumar in [12].

Let X be again a nonempty convex subset of $\mathbb{R}^n$ and $I = \{1, ..., m\}$ be an index set. Like in the previous section, let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper and convex function and $g_i : \mathbb{R}^n \to \mathbb{R}, i \in I$, be real-valued convex functions. By $g : \mathbb{R}^n \to \mathbb{R}^m$ we denote the vector-valued function defined by $g(x) := (g_1(x), ..., g_m(x))^T\ \forall x \in \mathbb{R}^n$. We formulate now the main result of this section.

THEOREM 4.1. *Let the constraint qualification (CQ) be fulfilled. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0\ \forall i \in I \Rightarrow f(x) \geq 0$.
(ii) *There exist* $p \in \mathbb{R}^n$ *and* $q \in \mathbb{R}^m, q \geqq 0$, *such that*

$$f^*(p) + (q^T g)^*_X(-p) \leq 0.$$

Proof. $(ii) \Rightarrow (i)$. Choose $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m, q \geqq 0$, such that $f^*(p) + (q^T g)^*_X(-p) \leq 0$ or, equivalently, $-f^*(p) - (q^T g)^*_X(-p) \geq 0$. The optimal objective value $v(D_{FL})$ of the optimization problem

$$(D_{FL}) \quad \sup_{\substack{p \in \mathbb{R}^n \\ q \geqq 0}} \left\{-f^*(p) - (q^T g)^*_X(-p)\right\}$$

is greater than or equal to zero, which implies that the optimal objective value $v(P)$ of the problem (P) must be greater than or equal to zero, too. We recall that by weak duality the following inequality is true $v(P) \geq v(D_{FL})$. Therefore for all $x \in X$ such that $g_i(x) \leq 0\ \forall i \in I$, we have $f(x) \geq 0$ and so (i) is fulfilled.

$(i) \Rightarrow (ii)$. Assuming now that (i) is true, it follows that the optimal objective value of the problem (P) is greater than or equal to zero. On the other hand, the constraint qualification (CQ) being fulfilled, we obtain by Theorem 3.1 that there exists (p, q), $p \in \mathbb{R}^n, q \in \mathbb{R}^m, q \geqq 0$, an optimal solution to (D_{FL}), such that

$$v(P) = v(D_{FL}) = -f^*(p) - (q^T g)^*_X(-p) \geq 0.$$

This proves the validity of (ii). □

Remark 4.2. For the implication $(ii) \Rightarrow (i)$ the constraint qualification (CQ) is not necessary.

As an immediate consequence of Theorem 4.1 we get the following theorem of the alternative.

COROLLARY 4.3. *Let the constraint qualification* (CQ) *be fulfilled. Then either the inequality system*

(I) $x \in X, g_i(x) \leq 0\ \forall i \in I, f(x) < 0$

has a solution or the system

(II) $f^*(p) + (q^T g)^*_X(-p) \leq 0, p \in \mathbb{R}^n, q \geqq 0$

has a solution, but never both.

The next assertion offers an alternative formulation for the statement (ii) in Theorem 4.1.

PROPOSITION 4.4. *The statement* (ii) *in Theorem 4.1 is equivalent to*

$$0 \in epi(f^*) + coneco\left(\bigcup_{i \in I} epi(g_i^*)\right) + epi(\sigma_X). \tag{4.1}$$

Proof. $(ii) \Rightarrow (4.1)$. Let $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m, q \geqq 0$, be such that $f^*(p) + (q^T g)^*_X(-p) \leq 0$.

At first we assume that $q = 0$. This implies that $f^*(p) + \sup_{x \in X}(-p^T x) \leq 0$. Therefore $\sigma_X(-p) = \sup_{x \in X}(-p^T x) \leq -f^*(p)$ and so $(-p, -f^*(p)) \in epi(\sigma_X)$. This leads to the following relation

$$\begin{aligned} 0 = (p, f^*(p)) + (-p, -f^*(p)) &\in epi(f^*) + epi(\sigma_X) \\ &\subseteq epi(f^*) + coneco\left(\bigcup_{i \in I} epi(g_i^*)\right) + epi(\sigma_X). \end{aligned}$$

When $q \neq 0$ the set of indices $I_q := \{i \in I : q_i \neq 0\}$ is not empty. On the other hand, because $f^*(p) + (q^T g)^*_X(-p) \leq 0$, there exists an $r \in \mathbb{R}$ such that

$$f^*(p) \leq r \leq -(q^T g)^*_X(-p),$$

which yields

$$(p, r) \in epi(f^*) \text{ and } (-p, -r) \in epi\Big((q^T g)^*_X\Big). \tag{4.2}$$

From the definition of the conjugate function relative to the set X we have

$$\begin{aligned}(q^T g)^*_X(p) &= \sup_{x \in X} \Big\{p^T x - q^T g(x)\Big\} = \sup_{x \in \mathbb{R}^n} \Big\{p^T x - \big(q^T g(x) + \delta_X(x)\big)\Big\} \\ &= (q^T g + \delta_X)^*(p)\ \forall p \in \mathbb{R}^n,\end{aligned}$$

where δ_X is the indicator function of X. Obviously, $(q^T g)^*_X = (q^T g + \delta_X)^*$. Moreover, the fact that X is a convex set implies that the relative interior of $X = dom(\delta_X)$ is not empty. By Corollary 2.3 we have further

$$(-p, -r) \in epi\Big((q^T g + \delta_X)^*\Big) = epi\Big((q^T g)^*\Big) + epi(\delta^*_X). \tag{4.3}$$

By (2.1) and (2.3) we get

$$epi\Big((q^T g)^*\Big) = epi\left(\left(\sum_{i \in I_q} q_i g_i\right)^*\right) = \sum_{i \in I_q} epi\Big((q_i g_i)^*\Big) = \sum_{i \in I_q} q_i epi(g^*_i).$$

Therefore

$$epi\Big((q^T g)^*\Big) = \sum_{i \in I_q} q_i epi(g^*_i) = \left(\sum_{i \in I_q} q_i\right) \sum_{i \in I_q} \frac{q_i}{\sum\limits_{i \in I_q} q_i} epi(g^*_i)$$

$$\subseteq \left(\sum_{i \in I_q} q_i\right) co\left(\bigcup_{i \in I} epi(g^*_i)\right) \subseteq coneco\left(\bigcup_{i \in I} epi(g^*_i)\right).$$

So, by (4.2), (4.3) and, because $\delta^*_X = \sigma_X$, we obtain

$$0 \in epi(f^*) + coneco\left(\bigcup_{i \in I} epi(g^*_i)\right) + epi(\sigma_X).$$

(4.1) $\Rightarrow$ (ii). Now let be $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that $(p, r) \in epi(f^*)$ and $(-p, -r) \in coneco\left(\bigcup_{i \in I} epi(g^*_i)\right) + epi(\sigma_X)$. Further, there exists $\lambda \geq 0$ such that $(-p, -r) \in \lambda co\left(\bigcup_{i \in I} epi(g^*_i)\right) + epi(\sigma_X)$.

If $\lambda = 0$, then $(-p, -r) \in epi(\sigma_X) = epi(\delta^*_X)$ and, by taking $q \in \mathbb{R}^m, q := 0$, we get $(q^T g)^*_X(-p) = (0)^*_X(-p) = \sigma_X(-p) \leq -r$. It follows that $f^*(p) + (q^T g)^*_X(-p) \leq r - r = 0$.

Assuming now that $\lambda > 0$, there exist some $\mu_i \geq 0, i = 1, ..., m$, $\sum_{i=1}^m \mu_i = 1$, such that $(-p, -r) \in \lambda \sum_{i=1}^m \mu_i epi(g^*_i) + epi(\sigma_X)$. Let us consider now the following set of indices $I_\mu := \{i \in I : \mu_i \neq 0\}$. The relation above can be written as

$$\begin{aligned}(-p,-r) \in \sum_{i\in I_\mu} \lambda\mu_i epi(g_i^*) + epi(\delta_X^*) &= \sum_{i\in I_\mu} epi\Big((\lambda\mu_i g_i)^*\Big) + epi(\delta_X^*)\\ &= epi\left(\Big(\sum_{i\in I_\mu}\lambda\mu_i g_i\Big)^*\right) + epi(\delta_X^*) = epi\left(\Big(\sum_{i=1}^m \lambda\mu_i g_i\Big)^*\right) + epi(\delta_X^*)\\ &= epi\left(\Big(\sum_{i=1}^m \lambda\mu_i g_i + \delta_X\Big)^*\right) = epi\Big((q^T g)_X^*\Big),\end{aligned}$$

for $q := (\lambda\mu_1, ..., \lambda\mu_m)^T$. Therefore we found a pair $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m_+$ such that $f^*(p) + (q^T g)_X^*(-p) \leq r - r = 0$. □

Corollary 4.5. *Let be $u \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and let us assume that there exists $x' \in ri(X)$ such that $g_i(x') \leq 0$ $\forall i \in L$ and $g_i(x') < 0$ $\forall i \in N$. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0$ $\forall i \in I \Rightarrow u^T x \leq \alpha$.

(ii) There exists $q \in \mathbb{R}^m, q \geqq 0$, such that $(q^T g)_X^*(u) \leq \alpha$.

Proof. Considering $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \alpha - u^T x$, we get

$$f^*(p) = \begin{cases} -\alpha, & \text{if } p = -u,\\ +\infty, & \text{otherwise.}\end{cases}$$

The equivalence follows from Theorem 4.1. □

Remark 4.6. For $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \alpha - u^T x$, the epigraph of f^* is nothing but the set $\{(-u, -\alpha)\} + \{0\} \times \mathbb{R}_+$. By Proposition 4.4, the statement (ii) in Corollary 4.5 can be equivalently written as

$$(u,\alpha) \in coneco\left(\bigcup_{i\in I} epi(g_i^*)\right) + epi(\sigma_X) + \{0\} \times \mathbb{R}_+$$

and, because $epi(\sigma_X) + \{0\} \times \mathbb{R}_+ = epi(\sigma_X)$, we get

$$(u,\alpha) \in coneco\left(\bigcup_{i\in I} epi(g_i^*)\right) + epi(\sigma_X).$$

Remark 4.7. If we reconsider Corollary 4.5 for $X = \mathbb{R}^n$, the following statements turn out to be equivalent:

(i) $x \in \mathbb{R}^n, g_i(x) \leq 0$ $\forall i \in I \Rightarrow u^T x \leq \alpha$.

(ii) There exists $q \in \mathbb{R}^m, q \geqq 0$, such that $(q^T g)^*(u) \leq \alpha$.

Because $epi(\sigma_{\mathbb{R}^n}) = \{0\} \times \mathbb{R}_+$, both (i) and (ii) are equivalent to

$$(u,\alpha) \in coneco\left(\bigcup_{i\in I} epi(g_i^*)\right) + \{0\} \times \mathbb{R}_+. \tag{4.4}$$

Assuming now that $u \neq 0$, relation (4.4) can be further equivalently written as

$$(u,\alpha) \in coneco\left(\bigcup_{i\in I} epi(g_i^*)\right),$$

which is exactly the result obtained in [12]. Thus our results extend those of Jeyakumar, because the equivalences hold here under much weaker conditions, namely even if the set $\{x \in \mathbb{R}^n : g_i(x) < 0\ \forall i \in I\}$ is empty.

The readers interested in the applicability of the Farkas-type results presented in this section are referred to the forthcoming paper [3]. There we characterize the containment of a nonempty polyhedral set in an arbitrary polyhedral set, in a reverse-polyhedral set and in a reverse-convex set determined by convex quadratic constraints. Moreover some famous theorems of the alternative are rediscovered as special cases of our results.

5. Some new Farkas-type results for infinitely many constraints. In this section we consider Farkas-type results for inequality systems involving infinitely many convex constraints. To achieve them we use an extended Fenchel-type dual problem for which strong duality holds, provided that a regularity condition is fulfilled. Under the same hypotheses we give then equivalent formulations by using the epigraphs of the conjugates of the functions involved. Similar results obtained by using epigraphs are stated next, this time under closedness assumptions.

Let I be an arbitrary index set, $g_i : \mathbb{R}^n \to \mathbb{R}$ be convex functions, for $i \in I$, and let G denote the set $\{x \in \mathbb{R}^n : g_i(x) \leq 0 \ \forall i \in I\}$. For $X \subseteq \mathbb{R}^n$ a nonempty convex set and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ a proper convex function let us consider the primal optimization problem

$$(P^\infty) \quad \inf_{x \in C} f(x),$$

$$C := X \cap G = \{x \in X : g_i(x) \leq 0 \ \forall i \in I\}.$$

Moreover, let us assume that the set C is not empty.

By using the indicator functions δ_G and δ_X of the sets G and X, respectively, we may formulate (P^∞) as follows

$$(P^\infty) \quad \inf_{x \in \mathbb{R}^n} \Big(f(x) + \delta_G(x) + \delta_X(x) \Big) = -\Big(f + \delta_G + \delta_X \Big)^*(0).$$

If $ri(dom(f)) \cap ri(G) \cap ri(X) \neq \emptyset$, then, by Theorem 2.2, it follows that the extended Fenchel-type dual problem to (P^∞)

$$(D_F^\infty) \quad \sup_{p_1, p_2 \in \mathbb{R}^n} \Big\{ -f^*(p_1 + p_2) - \delta_G^*(-p_1) - \delta_X^*(-p_2) \Big\}$$

has an optimal solution and the optimal objective values of (P^∞) and (D_F^∞) coincide, i.e. $v(P^\infty) = v(D_F^\infty)$.

Remark 5.1. The extended Fenchel-type dual problem (D_F^∞) can also be obtained using the approach described in Section 3 by considering as perturbation function

$$\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}, \Phi_F(x, y) = \begin{cases} f(x + y), & \text{if } x + z \in G, x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables y and z. The relations between the classical Fenchel, Lagrange and the Fenchel-Lagrange dual problems have been investigated in [19] and [2].

By using the extended Fenchel-type dual problem we can prove the following Farkas-type result.

THEOREM 5.2. *Assume that $ri(dom(f)) \cap ri(G) \cap ri(X) \neq \emptyset$. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0 \ \forall i \in I \Rightarrow f(x) \geq 0$.

(ii) There exist $p_1, p_2 \in \mathbb{R}^n$ such that $f^*(p_1 + p_2) + \delta_G^*(-p_1) + \delta_X^*(-p_2) \leq 0$.

Proof. $(ii) \Rightarrow (i)$. Let $p_1, p_2 \in \mathbb{R}^n$ such that $f^*(p_1+p_2)+\delta_G^*(-p_1)+\delta_X^*(-p_2) \leq 0$. The optimal objective value $v(D_F^\infty)$ of the extended Fenchel-type dual problem

$$(D_F^\infty) \quad \sup_{p_1,p_2 \in \mathbb{R}^n} \Big\{ - f^*(p_1+p_2) - \delta_G^*(-p_1) - \delta_X^*(-p_2) \Big\}$$

is greater than or equal to zero. By weak duality we have that the optimal objective value $v(P^\infty)$ of the primal problem (P^∞) must also be greater than or equal to zero. This implies that for all $x \in C$, $f(x) \geq 0$ and so (i) is fulfilled.

$(i) \Rightarrow (ii)$. Assuming now that (i) is true it follows that $v(P^\infty) \geq 0$. The regularity condition being fulfilled, Theorem 2.2 guarantees the existence of (p_1, p_2), an optimal solution to (D_F^∞), such that

$$v(P^\infty) = -\Big(f + \delta_G + \delta_X\Big)^*(0) = -f^*(p_1+p_2) - \delta_G^*(-p_1) - \delta_X^*(-p_2) = v(D_F^\infty) \geq 0.$$

Therefore (i) is also fulfilled. □

Remark 5.3. For the implication $(ii) \Rightarrow (i)$ the condition $ri(dom(f)) \cap ri(G) \cap ri(X) \neq \emptyset$ is not necessary.

Let us reformulate now Theorem 5.2 as a theorem of the alternative.

Corollary 5.4. *Assume that the regularity condition in Theorem 5.2 is fulfilled. Then either the inequality system*

(I) $x \in X, g_i(x) \leq 0 \ \forall i \in I, f(x) < 0$

has a solution or the system

(II) $f^*(p_1+p_2) + \delta_G^*(-p_1) + \delta_X^*(-p_2) \leq 0, p_1 \in \mathbb{R}^n, p_2 \in \mathbb{R}^n$

has a solution, but never both.

The next result provides an equivalent characterization of the statement (ii) in Theorem 5.2.

Proposition 5.5. *The statement (ii) in Theorem 5.2 is equivalent to*

$$0 \in epi(f^*) + cl\left(coneco\left(\bigcup_{i \in I} epi(g_i^*)\right)\right) + epi(\sigma_X). \tag{5.1}$$

Proof. The statement (ii) in Theorem 5.2 can equivalently be written as follows:

There exist $p_1, p_2 \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $s \in \mathbb{R}$ such that $f^*(p_1+p_2) \leq -r - s, \delta_G^*(-p_1) \leq r$ and $\delta_X^*(-p_2) \leq s \Leftrightarrow$ there exist $p_1, p_2 \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $s \in \mathbb{R}$ such that $(p_1+p_2, -r-s) \in epi(f^*)$, $(-p_1, r) \in epi(\delta_G^*)$ and $(-p_2, s) \in epi(\delta_X^*) \Leftrightarrow$

$$0 \in epi(f^*) + epi(\delta_G^*) + epi(\delta_X^*) = epi(f^*) + epi(\sigma_G) + epi(\sigma_X).$$

Using that $epi(\sigma_G) = cl\left(coneco\left(\bigcup_{i \in I} epi(g_i^*)\right)\right)$ (see, for example, [14],[16], [12]), we get (5.1). □

The next theorem provides a similar characterization to (5.1), this time under some closedness assumptions.

Theorem 5.6. *Alongside the initial hypotheses we assume that X is a closed set and f is a lower semi-continuous function. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0 \ \forall i \in I \Rightarrow f(x) \geq 0$.

(ii) $0 \in cl\left(epi(f^*) + coneco\left(\bigcup_{i \in I} epi(g_i^*)\right) + epi(\sigma_X)\right)$.

Proof. As we have seen, statement (i) is rewritable as

$$\inf_{x\in\mathbb{R}^n}\Big(f(x)+\delta_G(x)+\delta_X(x)\Big)=-\Big(f+\delta_G+\delta_X\Big)^*(0)\geq 0$$

or, equivalently,

$$0\in epi\left((f+\delta_G+\delta_X)^*\right).$$

The sets X and $G=\bigcap\limits_{i\in I}\{x\in\mathbb{R}^n : g_i(x)\leq 0\}$ are closed and therefore the functions δ_G and δ_X are lower semi-continuous. Thus by (2.2) in Corollary 2.3 we have

$$epi\left((f+\delta_G+\delta_X)^*\right)=epi\left((cl(f)+cl(\delta_G)+cl(\delta_X))^*\right)=$$

$$cl\left(epi(f^*)+epi(\delta_G^*)+epi(\delta_X^*)\right)=cl\left(epi(f^*)+epi(\sigma_G)+epi(\sigma_X)\right)=$$

$$cl\left(epi(f^*)+cl\left(coneco\left(\bigcup_{i\in I}epi(g_i^*)\right)\right)+epi(\sigma_X)\right). \tag{5.2}$$

Since $cl(S_1+cl(S_2))=cl(S_1+S_2)$ for arbitrary sets S_1 and S_2 in $\mathbb{R}^n$, we get

$$0\in cl\left(epi(f^*)+cl\left(coneco\left(\bigcup_{i\in I}epi(g_i^*)\right)\right)+epi(\sigma_X)\right)=$$

$$cl\left(epi(f^*)+coneco\left(\bigcup_{i\in I}epi(g_i^*)\right)+epi(\sigma_X)\right).$$

□

Remark 5.7. If $X=\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}$, the assumptions in both Theorem 5.2 and Theorem 5.6 are valid, namely

(i) $ri(dom(f))\cap ri(G)\cap ri(X)\neq\emptyset$,
(ii) X is closed and f is lower semi-continuous.

Having that

$$epi(f^*)+epi(\sigma_{\mathbb{R}^n})=epi(f^*)+\{0\}\times\mathbb{R}_+=epi(f^*),$$

the statement

$(i) x\in\mathbb{R}^n, g_i(x)\leq 0\ \forall i\in I\Rightarrow f(x)\geq 0$

turns out to be equivalent to

$$0\in cl\left(epi(f^*)+coneco\left(\bigcup_{i\in I}epi(g_i^*)\right)\right)=epi(f^*)+cl\left(coneco\left(\bigcup_{i\in I}epi(g_i^*)\right)\right),$$

which is nothing else than Theorem 4.1 in [12].

The next theorem provides a Farkas-type result involving a difference of convex functions.

THEOREM 5.8. *Let $h:\mathbb{R}^n\to\overline{\mathbb{R}}$ be a proper convex function with the property that C is a subset of $ri(dom(h))$. Assume that $ri(dom(f))\cap ri(G)\cap ri(X)\neq\emptyset$. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0 \ \forall i \in I \Rightarrow h(x) - f(x) \leq 0.$

(ii) $\forall u \in \mathbb{R}^n \ \exists p_1^u, p_2^u \in \mathbb{R}^n$ *such that* $f^*(u + p_1^u + p_2^u) + \delta_G^*(-p_1^u) + \delta_X^*(-p_2^u) \leq h^*(u)$.

Proof. By Theorem 7.4 and Theorem 12.2 in [17] we have that $\forall x \in C \ h(x) = cl(h)(x) = h^{**}(x) = \sup\limits_{u \in \mathbb{R}^n} \{u^T x - h^*(u)\}$. This implies that (i) is rewritable as

$$\forall u \in \mathbb{R}^n \ \forall x \in C \ f(x) - u^T x + h^*(u) \geq 0. \tag{5.3}$$

Applying Theorem 5.2 we obtain, equivalently, that for all $u \in \mathbb{R}^n$ there exist $p_1^u, p_2^u \in \mathbb{R}^n$ (even if $h^*(u) = +\infty$) such that

$$\sup_{x \in \mathbb{R}^n} \Big\{ (p_1^u + p_2^u)^T x - \big(f(x) - u^T x + h^*(u) \big) \Big\} + \delta_G^*(-p_1^u) + \delta_X^*(-p_2^u) \leq 0 \Leftrightarrow$$

$$f^*(u + p_1^u + p_2^u) + \delta_G^*(-p_1^u) + \delta_X^*(-p_2^u) \leq h^*(u).$$

□

Proposition 5.9. *The statement (ii) in Theorem 5.8 is equivalent to*

$$epi(h^*) \subseteq epi(f^*) + cl\left(coneco \left(\bigcup_{i \in I} epi(g_i^*) \right) \right) + epi(\sigma_X). \tag{5.4}$$

Proof. $(ii) \Rightarrow (5.4)$. Let $(u, r) \in epi(h^*) \Leftrightarrow h^*(u) \leq r$. By (ii) in Theorem 5.8 there exist $p_1^u, p_2^u \in \mathbb{R}^n$ such that

$$r \geq h^*(u) \geq f^*(u + p_1^u + p_2^u) + \delta_G^*(-p_1^u) + \delta_X^*(-p_2^u).$$

Therefore

$$(u, r) = (u + p_1^u + p_u^2, r - \delta_G^*(-p_1^u) - \delta_X^*(-p_2^u)) + (-p_1^u, \delta_G^*(-p_1^u)) + (-p_2^u, \delta_X^*(-p_2^u)$$

$$\in epi(f^*) + epi(\delta_G^*) + epi(\delta_X^*) = epi(f^*) + cl\left(coneco \left(\bigcup_{i \in I} epi(g_i^*) \right) \right) + epi(\sigma_X)$$

and so (5.4) is verified.

$(5.4) \Rightarrow (ii)$. Let $u \in \mathbb{R}^n$ be fixed. If $h^*(u) = +\infty$, then (ii) is true. Assuming that $h^*(u) < +\infty$, we have $(u, h^*(u)) \in epi(h^*)$ and so there exist $(a, s) \in epi(f^*)$, $(b, t) \in cl\left(coneco \left(\bigcup\limits_{i \in I} epi(g_i^*) \right) \right) = epi(\delta_G^*)$ and $(c, z) \in epi(\sigma_X)$ such that $a + b + c = u$ and $s + t + z = h^*(u)$. Taking $p_1^u := -b$ and $p_2^u := -c$, we get

$$f^*(u + p_1^u + p_2^u) + \delta_G^*(-p_1^u) + \delta_X^*(-p_2^u) = f^*(a) + \delta_G^*(b) + \delta_X^*(c) \leq s + t + z = h^*(u).$$

□

We consider now in the hypotheses of Theorem 5.8 instead of the regularity condition $ri(dom(f)) \cap ri(G) \cap ri(X) \neq \emptyset$, the assumptions that X is a closed set and f is a lower semi-continuous function. This leads us to the following result.

Theorem 5.10. *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function with the property that C is a subset of $ri(dom(h))$. Assume that X is a closed set and f is a lower semi-continuous function. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0 \ \forall i \in I \Rightarrow h(x) - f(x) \leq 0.$

(ii) $epi(h^*) \subseteq cl\left(epi(f^*) + coneco\left(\bigcup\limits_{i \in I} epi(g_i^*) \right) + epi(\sigma_X) \right).$

Proof. As we have seen in the proof of Theorem 5.8, the statement (i) can be equivalently written as (cf. (5.3))

$$\forall u \in \mathbb{R}^n \ \forall x \in \mathbb{R}^n \ f(x) + \delta_G(x) + \delta_X(x) - u^T x + h^*(u) \geq 0 \Leftrightarrow$$

$$\forall u \in \mathbb{R}^n \ \forall x \in \mathbb{R}^n \ h^*(u) \geq u^T x - (f + \delta_G + \delta_X)(x) \Leftrightarrow$$

$$\forall u \in \mathbb{R}^n \ h^*(u) \geq (f + \delta_G + \delta_X)^*(u) \Leftrightarrow epi(h^*) \subseteq epi\left((f + \delta_G + \delta_X)^*\right).$$

The desired conclusion follows using that (cf. (5.2))

$$epi\left((f + \delta_G + \delta_X)^*\right) = cl\left(epi(f^*) + coneco\left(\bigcup_{i \in I} epi(g_i^*)\right) + epi(\sigma_X)\right).$$

□

Remark 5.11. For $X = \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ the assumptions in Theorem 5.8 and Theorem 5.10 are again valid. If $G \subseteq ri(dom(h))$, the statement

$(i) x \in \mathbb{R}^n, g_i(x) \leq 0 \ \forall i \in I \Rightarrow h(x) - f(x) \leq 0$

becomes equivalent to

$$\begin{aligned} epi(h^*) &\subseteq cl\left(epi(f^*) + coneco\left(\bigcup_{i \in I} epi(g_i^*)\right)\right) \\ &= epi(f^*) + cl\left(coneco\left(\bigcup_{i \in I} epi(g_i^*)\right)\right). \end{aligned}$$

In case $h : \mathbb{R}^n \to \mathbb{R}$, this result is nothing else than Theorem 4.2 in [12] (see also Theorem 3.1 and Theorem 3.2 in [16]).

For the last part of this section we consider under the same hypotheses as for Theorem 5.8 that $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = 0 \ \forall x \in \mathbb{R}^n$. The mentioned theorem leads us to the following result.

THEOREM 5.12. *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function with the property that $C \subseteq ri(dom(h))$. Assume that $ri(G) \cap ri(X) \neq \emptyset$. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0 \ \forall i \in I \Rightarrow h(x) \leq 0$.

(ii) $\forall u \in \mathbb{R}^n \ \exists p^u \in \mathbb{R}^n$ *such that* $h^*(u) \geq \delta_G^*(u + p^u) + \delta_X^*(-p^u)$.

Because $epi(f^*) = \{0\} \times \mathbb{R}_+$, the statement (i) in Theorem 5.12 can be equivalently written as (cf. (5.4))

$$epi(h^*) \subseteq cl\left(coneco\left(\bigcup_{i \in I} epi(g_i^*)\right)\right) + epi(\sigma_X). \tag{5.5}$$

By the same argument, Theorem 5.10 can be formulated as follows.

THEOREM 5.13. *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function with the property that C is a subset of $ri(dom(h))$. Assume that X is a closed set. Then the following statements are equivalent:*

(i) $x \in X, g_i(x) \leq 0 \ \forall i \in I \Rightarrow h(x) \leq 0$.

(ii) $epi(h^*) \subseteq cl\left(coneco\left(\bigcup_{i \in I} epi(g_i^*)\right) + epi(\sigma_X)\right)$.

For $X = \mathbb{R}^n$, we have $epi(\sigma_X) = \{0\} \times \mathbb{R}_+$ and therefore (5.5) and (ii) in Theorem 5.13 become

$$epi(h^*) \subseteq cl\left(coneco\left(\bigcup_{i \in I} epi(g_i^*)\right)\right). \tag{5.6}$$

In case $h : \mathbb{R}^n \to \mathbb{R}$, (5.6) is nothing else than Corollary 4.3 in [12].

6. Conclusion. In this paper we present some new Farkas-type results for inequality systems involving finitely as well as infinitely many convex constraints. Our approach, which employs the conjugates of the functions involved, is based on the theory of duality for convex optimization problems. An important role is played by an extended Fenchel-type dual problem as well as by the Fenchel-Lagrange dual problem. The last one has been introduced and extensively studied in the last years by the authors of this paper. The results we formulate and prove here generalize some recently published results due to Jeyakumar in [12]. Moreover, they underline the connections that exist between Farkas-type results and theorems of alternative and, on the other hand, the theory of duality.

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