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New Constraint Qualification and Conjugate Duality for Composed Convex Optimization Problems

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Abstract We present a new constraint qualification which guarantees strong duality between a cone-constrained convex optimization problem and its Fenchel-Lagrange dual. This result is applied to a convex optimization problem having, for a given nonempty convex cone K, as objective function a K-convex function postcomposed with a K-increasing convex function. For this so-called composed convex optimization problem, we present a strong duality assertion, too, under weaker conditions than the ones considered so far. As an application, we rediscover the formula of the conjugate of a postcomposition with a K-increasing convex function as valid under weaker conditions than usually used in the literature.

Keywords Conjugate functions · Fenchel-Lagrange duality · Composed convex optimization problems · Cone constraint qualifications

1 Introduction

A natural generalization of the optimization problems that consist in minimizing a function subject to the negativity of some other functions comes from considering the constraint functions as smaller than zero from the perspective of a partial ordering induced by a nonempty convex cone. The objective functions of the optimization problems may have different formulations, too. Many convex optimization problems arising from various directions may be formulated as minimizations of some compositions of functions subject to some constraints. We cite here [1–8] as articles dealing with composed convex optimization problems. Duality assertions

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for this kind of problems may be delivered in different ways, one of the most common consisting in considering an equivalent problem to the primal one, whose dual is easier determinable. If the desired duality results are based on conjugate functions, sometimes even a more direct way is available by obtaining a dual problem based on the conjugate function of the composed objective function of the primal, which can be written, in some situations, by using only the conjugates of the functions involved and the dual variables. Depending on the framework, the formula of the conjugate of the composed functions is taken mainly from [1, 9–11].

With this paper, we bring weaker conditions under which the known formula of the conjugate of a composed function holds when one works in $\mathbb{R}^n$. No closedness or continuity concerning the functions composed is necessary, while the interior-point regularity condition is weakened to a relation involving relative interiors. We give also an example that confirms that there are situations where our new condition is fulfilled, unlike the classical one. This result is present in our paper as an application, followed by the concrete case of calculating the conjugate of $1/F$, when F is a concave strictly positive function defined over the set of strictly positive reals. The theoretical part of the paper consists in presenting duality assertions concerning a cone-constrained convex optimization problem and its Fenchel-Lagrange dual problem. This dual problem has been introduced by Boţ and Wanka (see for example [12, 13]) as a combination of the widely-used Lagrange and Fenchel dual problems. Although recently introduced, this new duality concept has some nice applications [13–15].

The conditions under which the strong duality holds for the cone-constrained problem that we give are weaker than the interior-point Slater constraint qualifications usually considered in the literature. Thus, this strong duality assertion proves to hold also for convex problems whose constraints involve cones with empty interiors. Frenk and Kassay in [16] and Boţ, Kassay and Wanka in [12] used such constraint qualifications even under generalized convexity assumptions.

The ordinary convex program is given as a special case. Then, we consider as objective function the postcomposition of a K-increasing convex function to a K-convex function for a given nonempty closed convex cone K. Strong duality holds here under weaker conditions than so far in the literature. As a special case we consider also the unconstrained problem.

The structure of the paper follows. Section 2 is dedicated to the general cone-constrained convex optimization problem and Sect. 3 to the composed optimization problem. The application consisting in the determination of the formula of the conjugate function of a composed function follows in Sect. 4. A conclusive section closes the paper.

2 Cone-Constrained Convex Optimization Problem

2.1 Preliminaries

As usual, $\mathbb{R}^n$ denotes the n-dimensional real space for a positive integer n. Throughout this paper all the vectors are considered as column vectors. An upper index ^{T} transposes a column vector to a row one and vice versa. The *inner product* of two vectors $x=(x_1,\dots,x_n)^T$ and $y=(y_1,\dots,y_n)^T$ in the n-dimensional real space is denoted by $x^Ty=\sum_{i=1}^n x_i y_i$. For the *relative interior* of a set, we use the prefix ri, while the *effective domain* of the function $f:\mathbb{R}^n\to\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$ is

$$\mathrm{dom}(f)=\{x\in\mathbb{R}^n : f(x)<+\infty\}.$$

For a set $X\subseteq\mathbb{R}^n$, we have the *indicator function* $\delta_X:\mathbb{R}^n\to\overline{\mathbb{R}}$ defined by

$$\delta_X(x)=\begin{cases}0, & \text{if } x\in X,\\ +\infty, & \text{otherwise.}\end{cases}$$

For $D\subseteq\mathbb{R}^n$ and a function $f:D\to\mathbb{R}$, we recall the definition of the so-called *conjugate function relative to the set* D,

$$f_D^*:\mathbb{R}^n\to\overline{\mathbb{R}},\quad f_D^*(p)=\sup\{p^Tx-f(x):x\in D\}.$$

When $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ and $D=\mathrm{dom}(f)$, we obtain actually the classical *(Legendre-Fenchel) conjugate function* denoted by f^*. Concerning the conjugate functions, we have the Fenchel-Young inequality

$$f_D^*(p)+f(x)\geq p^Tx,\quad \forall x\in D,\ \forall p\in\mathbb{R}^n.$$

Let the nonempty convex cone K in $\mathbb{R}^k$. All the cones considered in this paper are assumed to contain the origin.

Definition 2.1 When $D\subseteq\mathbb{R}^k$, a function $f:D\to\mathbb{R}$ is called K*-increasing* if, for $x,y\in D$ such that $x-y\in K$, we have $f(x)\geq f(y)$.

Definition 2.2 Given a subset $X\subseteq\mathbb{R}^n$, a function $F:X\to\mathbb{R}^k$ is called K*-convex* if, for any x and $y\in X$ and $\lambda\in[0,1]$, one has

$$\lambda F(x)+(1-\lambda)F(y)-F(\lambda x+(1-\lambda)y)\in K.$$

2.2 Duality for the Cone-Constrained Convex Optimization Problem

Let X be a nonempty convex subset of $\mathbb{R}^n$, C a nonempty convex cone in $\mathbb{R}^m$, $f:X\to\mathbb{R}$ a convex function and $g:X\to\mathbb{R}^m$ a C-convex function, where $g=(g_1,\dots,g_m)^T$. The primal optimization problem that we consider is

$$(\mathrm{P})\qquad \inf_{x\in X,\, g(x)\in -C} f(x).$$

To (P) we attach a dual problem, which can be obtained by perturbations [12, 13, 15] or as we derive it within the proof of the strong duality theorem. It is called the *Fenchel-Lagrange dual* problem and is formulated as follows:

$$(\mathrm{D})\qquad \sup_{q\in C^*,\, p\in\mathbb{R}^n}\{-f_X^*(p)-(q^Tg)_X^*(-p)\},$$

where for $q=(q_1,\dots,q_m)^T$, $q^Tg : X\to\mathbb{R}$ is the function defined by

$$q^Tg(x)=\sum_{j=1}^m q_jg_j(x),\quad \forall x\in X.$$

For an optimization problem (P), we denote by v(P) its optimal objective value.

The so-called weak duality holds between (P) and (D), i.e. v(P) ≥ v(D). The proof arises straightforwardly from the construction of the dual. For the strong duality statement, we introduce the following constraint qualification [16]:

$$\text{(CQ)}\quad 0\in\operatorname{ri}(g(X)+C).$$

Theorem 2.1 *If* (CQ) *is fulfilled then* v(P) = v(D) *and the dual* (D) *has an optimal solution if* v(P) > $-\infty$.

Proof The Lagrange dual problem to (P) is

$$(\mathrm{D^L})\quad \sup_{q\in C^*}\inf_{x\in X}[f(x)+q^Tg(x)].$$

According to [16], (CQ) ensures the coincidence of v(P) and v($\mathrm{D^L}$), moreover guaranteeing the existence of an optimal solution $\bar q$ to ($\mathrm{D^L}$) when v(P) > $-\infty$.

Now let us write the Fenchel dual problem to the inner infimum in ($\mathrm{D^L}$). For $q\in C^*$, both f and q^Tg are real-valued convex functions defined on X, so in order to apply rigorously Fenchel's duality theorem [17] we have to consider their convex extensions to $\mathbb{R}^n$, say $\tilde f$ and $\widetilde{q^Tg}$, which take the value $+\infty$ outside X. As

$$\operatorname{dom}(\tilde f)=\operatorname{dom}(\widetilde{q^Tg})=X\quad\text{and}\quad \operatorname{ri}(X)\neq\emptyset,$$

due to the convexity of the nonempty set X we have (cf. Theorem 31.1 in [17])

$$\inf_{x\in X}[f(x)+q^Tg(x)]=\inf_{x\in\mathbb{R}^n}[\tilde f(x)+\widetilde{q^Tg}(x)]=\sup_{p\in\mathbb{R}^n}\{-\tilde f^*(p)-\widetilde{q^Tg}^*(-p)\},$$

with the existence of a $\bar p$ where the supremum in the right-hand side is attained granted. As

$$\tilde f^*(p)=f_X^*(p)\quad\text{and}\quad \widetilde{q^Tg}^*(-p)=(q^Tg)_X^*(-p),\quad\forall p\in\mathbb{R}^n,$$

it is clear that

$$\mathrm{v(P)}=\sup_{q\in C^*}\inf_{x\in X}[f(x)+q^Tg(x)]=\sup_{q\in C^*,p\in\mathbb{R}^n}\{-f_X^*(p)-(q^Tg)_X^*(-p)\}.$$

In case v(P) is finite, because of the existence of an optimal solution for the Lagrange dual and the Fenchel dual, we get

$$\begin{aligned}\mathrm{v(P)}&=\sup_{q\in C^*}\inf_{x\in X}[f(x)+q^Tg(x)]=\inf_{x\in X}[f(x)+\bar q^Tg(x)]\\&=-f_X^*(\bar p)-(\bar q^Tg)_X^*(-\bar p),\end{aligned}$$

which means exactly that (D) has an optimal solution $(\bar{p}, \bar{q})$. □

Remark 2.1 One may notice that the constraint qualification (CQ) is sufficient to ensure strong duality for both Lagrange and Fenchel-Lagrange dual problems.

Necessary and sufficient optimality conditions regarding (P) and (D) follow. The proof of the following theorem is similar to those in [13], so we omit it here.

Theorem 2.2

(a) *If* (CQ) *holds and* (P) *has an optimal solution* $\bar{x}$, *then* (D) *has an optimal solution* $(\bar{p}, \bar{q})$, $\bar{p} \in \mathbb{R}^n$, $\bar{q} \in C^*$ *and the following optimality conditions are satisfied*:
  (i) $f_X^*(\bar{p}) + f(\bar{x}) = \bar{p}^T \bar{x}$,
  (ii) $(\bar{q}^T g)_X^*(-\bar{p}) + \bar{q}^T g(\bar{x}) = -\bar{p}^T \bar{x}$,
  (iii) $\bar{q}^T g(\bar{x}) = 0$.

(b) *If* $\bar{x}$ *is a feasible point to* (P) *and* $(\bar{p}, \bar{q})$ *is feasible to* (D) *fulfilling the optimality conditions* (i)–(iii), *then* v(P) = v(D) *and the mentioned feasible points are optimal solutions of the corresponding problems*.

Remark 2.2 Let us notice that (b) applies without any convexity assumption as well as constraint qualification.

The constraint qualification (CQ) seems quite hard to be verified sometimes, that is why we provide the following equivalent formulation to it.

Theorem 2.3 *The constraint qualification* (CQ) *is equivalent to*

$$(\mathrm{CQ}') \quad 0 \in g(\mathrm{ri}(X)) + \mathrm{ri}(C).$$

Proof Consider the set $M := \{(x, y) : x \in X, y - g(x) \in C\}$, which is easily provable to be convex. For each $x \in X$ consider another set, $M_x := \{y \in \mathbb{R}^m : (x, y) \in M\}$. When $x \notin X$ it is obvious that $M_x = \emptyset$, while in the opposite case we have $y \in M_x \Leftrightarrow y - g(x) \in C \Leftrightarrow y \in g(x) + C$, so we conclude that $M_x = g(x) + C$, if $x \in X$, otherwise being the empty set. Therefore M_x is also convex for any $x \in X$. Let us characterize the relative interior of the set M. According to Theorem 6.8 in [17] we have $(x, y) \in \mathrm{ri}(M)$ if and only if $x \in \mathrm{ri}(X)$ and $y \in \mathrm{ri}(M_x)$. On the other hand, for any $x \in \mathrm{ri}(X) \subseteq X$, $y \in \mathrm{ri}(M_x)$ means actually

$$y \in \mathrm{ri}(g(x) + C) = g(x) + \mathrm{ri}(C),$$

so we can write further

$$\mathrm{ri}(M) = \{(x, y) : x \in \mathrm{ri}(X), y - g(x) \in \mathrm{ri}(C)\}.$$

Consider the linear transformation $A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $A(x, y) = y$. We prove that $A(M) = g(X) + C$. Take first an element $y \in A(M)$. It follows that there is an $x \in X$ such that $y - g(x) \in C$, which yields $y \in g(X) + C$. Reversely, for any $y \in g(X) + C$

there is an $x \in X$ such that $y \in g(x) + C$, so $y - g(x) \in C$. This means $(x, y) \in M$, followed by $y \in A(M)$.

For the relative interior in (CQ) we have (by Theorem 6.6 in [17])

$$\mathrm{ri}(g(X) + C) = \mathrm{ri}(A(M)) = A(\mathrm{ri}(M)) = g(\mathrm{ri}(X)) + \mathrm{ri}(C),$$

so (CQ) and (CQ′) are equivalent. □

Remark 2.3 Let us notice that (CQ′) has been mentioned to guarantee strong duality between (P) and its Lagrange dual in [18]. From Theorem 2.3 one can see that the constraint qualification of Frenk and Kassay [16] is equivalent to the one due to Wolkowicz [18]. As proved in Theorem 2.1. this condition closes moreover the duality gap between (P) and its Fenchel-Lagrange dual (which is generally bigger than the one between (P) and its Lagrange dual).

We give an example that shows that a relaxation of (CQ′) by considering the whole set X instead of its relative interior does not guarantee strong duality.

Example 2.1 (see also [19]) Consider the convex functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_2$ and $g(x_1, x_2) = x_1$ and the convex set

$$X = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 2; 3 \le x_2 \le 4 \text{ if } x_1 = 0; 1 < x_2 \le 4 \text{ if } x_1 > 0\}.$$

Formulate the problem

$$(\mathrm{P}_1) \qquad \inf_{x \in X, g(x) = 0} f(x).$$

This problem fits into our scheme for $C = \{0\}$. (CQ) becomes here $0 \in \mathrm{ri}([0, 2] + 0) = (0, 2)$, that is false, while the condition $0 \in g(X) + \mathrm{ri}(C)$ holds, being in this case $0 \in [0, 2]$. The optimal objective value of (P_1) is 3, while the one of its dual is 1.

Thus we see that a relaxation of (CQ′) by considering $g(X)$ instead of $g(\mathrm{ri}(X))$ does not imply strong duality. Frenk and Kassay have shown in [16] that if there is an $y_0 \in \mathrm{aff}(g(X))$ such that $g(X) \subseteq y_0 + \mathrm{aff}(C)$ then $0 \in g(X) + \mathrm{ri}(C)$ becomes equivalent to (CQ). For any $M \subseteq \mathbb{R}^n$, $\mathrm{aff}(M)$ means the affine hull of the set M.

2.3 Ordinary Convex Programs as Special Cases

The ordinary convex programs may be included among the problems to which the duality assertions formulated earlier are applicable. Consider such an ordinary convex program

$$\begin{array}{lll} (\mathrm{P_o}) & \inf & f(x), \\ & \text{s.t.} & x \in X, \quad g_i(x) \le 0, \quad i = 1, \dots, r, \\ & & \qquad\qquad g_j(x) = 0, \quad j = r+1, \dots, m, \end{array}$$

where $X \subseteq \mathbb{R}^n$ is a nonempty convex set, $0 \le r \le m$, f and g_i, $i = 1, \dots, r$, are convex real-valued functions defined on X and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = r+1, \dots, m$, are affine functions. Denote $g = (g_1, \dots, g_m)^T$. This problem is a special case of (P) when we consider the cone $C=\mathbb{R}^r_+\times\{0\}^{m-r}$. The Fenchel-Lagrange dual problem to $(\mathrm{P_o})$ is

$$(\mathrm{D_o})\qquad \sup_{q\in\mathbb{R}^r_+\times\mathbb{R}^{m-r},\,p\in\mathbb{R}^n}\{-f_X^*(p)-(q^Tg)_X^*(-p)\}.$$

The constraint qualification that assures strong duality is in this case

$$(\mathrm{CQ_o})\quad 0\in\mathrm{ri}(g(X)+\mathbb{R}^r_+\times\{0\}^{m-r}),$$

equivalent to $0\in g(\mathrm{ri}(X))+\mathrm{ri}(\mathbb{R}^r_+\times\{0\}^{m-r})$, i.e.

$$(\mathrm{CQ_o})\quad \exists x'\in\mathrm{ri}(X): g_i(x')<0 \text{ if } i=1,\dots,r, \text{ and } g_j(x')=0 \text{ if } j=r+1,\dots,m,$$

which is exactly the sufficient condition given in [17] to state strong duality between $(\mathrm{P_o})$ and its Lagrange dual problem

$$(\mathrm{D_o^L})\qquad \sup_{q\in\mathbb{R}^r_+\times\mathbb{R}^{m-r}}\inf_{x\in X}[f(x)+q^Tg(x)].$$

As the following theorem shows, $(\mathrm{CQ_o})$ is a sufficient condition for strong duality also for the Fenchel-Lagrange dual.

Theorem 2.4 *If* $(\mathrm{CQ_o})$ *holds, then* $\mathrm{v}(\mathrm{P_o})=\mathrm{v}(\mathrm{D_o})$ *and the dual has an optimal solution if* $\mathrm{v}(\mathrm{P_o})>-\infty$.

Remark 2.4 For the following ordinary convex problem, where "$\leqq$" is the partial ordering introduced by the m-dimensional positive orthant,

$$(\mathrm{P_o'})\qquad \inf_{x\in X,\,g(x)\leqq 0} f(x),$$

strong Lagrange duality has been given under the constraint qualification (cf. [17, 19])

$$(\mathrm{CQ_o'})\quad \exists x'\in\mathrm{ri}(X):\quad \begin{aligned} &g_i(x')<0, && i=1,\dots,r,\\ &g_j(x')\le 0, && j=r+1,\dots,m.\end{aligned}$$

Considering $(\mathrm{P_o'})$ as a special case of (P), by taking $C=\mathbb{R}^m_+$, (CQ) must not always be fulfilled even if $(\mathrm{CQ_o'})$ holds.

Let us prove that for an appropriate choice of the cone C there exists an equivalent formulation of $(\mathrm{P_o'})$

$$(\mathrm{P_o'})\qquad \inf_{x\in X,\,g(x)\in -C} f(x),$$

for which $(\mathrm{CQ_o'})$ implies the fulfilment of (CQ), too.

Consider $(\mathrm{CQ_o'})$ fulfilled and take the set $I:=\{i\in\{r+1,\dots,m\}: x\in X \text{ such that } g(x)\leqq 0\Rightarrow g_i(x)=0\}$. When $I=\emptyset$ then for each $i\in\{r+1,\dots,m\}$ there is an

$x^i \in X$ feasible to (P'_o) such that $g_i(x^i) < 0$. Take the cone $C = \mathbb{R}^m_+$. By Theorem 6.1 in [17],

$$x^0 = \sum_{i=r+1}^{m} \frac{1}{m-r+1} x^i + \frac{1}{m-r+1} x' \in \mathrm{ri}(X)$$

and, for any $j \in \{1, \dots, m\}$, we have

$$g_j(x^0) \leq \sum_{i=r+1}^{m} \frac{1}{m-r+1} g_j(x^i) + \frac{1}{m-r+1} g_j(x') < 0.$$

Thus x^0 satisfies (CQ).

When $I \neq \emptyset$, without loss of generality as we perform at most a reindexing of the functions g_j, $r+1 \leq j \leq m$, let $I = \{r+l, \dots, m\}$, where l is a positive integer smaller than $m-r$. This means that for $j \in \{r+l, \dots, m\}$ follows $g_j(x) = 0$ if $x \in X$ and $g(x) \leqq 0$. Then (P) becomes (P'_o) choosing $C = \mathbb{R}^{r+l-1}_+ \times \{0\}^{m-r-l+1}$. For each $j \in \{r+1, \dots, r+l-1\}$ there is an x^j feasible to (P'_o) such that $g_j(x^j) < 0$. Taking

$$x^0 = \sum_{i=r+1}^{r+l-1} \frac{1}{l} x^i + \frac{1}{l} x',$$

we have as above that $x^0 \in \mathrm{ri}(X)$ and $g_j(x^0) < 0$ for any $j \in \{1, \dots, r+l-1\}$ and $g_j(x^0) = 0$ for $j \in I$ (because of the affinity of the functions g_j, $r+1 \leq j \leq m$), which is exactly what (CQ) asserts.

Consequently there is always a choice of the cone C which guarantees that for the reformulated problem (CQ) stands when $(\mathrm{CQ}'_\mathrm{o})$ is valid.

3 Convex Optimization Problem with Composed Objective Function

3.1 Cone-Constrained Case

Let K and C be nonempty convex cones in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, and X a nonempty convex subset of $\mathbb{R}^n$. Take $f : D \to \mathbb{R}$ to be a K-increasing convex function, with D a convex subset of $\mathbb{R}^k$, $F : X \to \mathbb{R}^k$ a K-convex function and $g : X \to \mathbb{R}^m$ a C-convex function. Moreover, we impose the feasibility condition $F(X) \subseteq D$. The problem we consider within this section is

$$(\mathrm{P_c}) \qquad \inf_{x \in X, g(x) \in -C} f(F(x)).$$

One could formulate a dual problem to it directly from the general case, since $f \circ F$ is a convex function. Unfortunately, the existing formulae which allow to write the conjugate of $f \circ F$ as a combination of the conjugates of f and F ask the functions to be closed even in some particular cases [9, 10]. To avoid this too strong requirement we formulate the following problem equivalent to $(\mathrm{P_c})$, in the sense that their optimal objective values coincide,

$$\begin{array}{llll} (\mathrm{P}'_{\mathrm{c}}) & \inf & f(y), & \\ & \text{s.t.} & x \in X, & g(x) \in -C, \\ & & y \in D, & F(x) - y \in -K. \end{array}$$

Proposition 3.1 $\mathrm{v}(\mathrm{P_c}) = \mathrm{v}(\mathrm{P}'_{\mathrm{c}})$.

Proof Let x be feasible to $(\mathrm{P_c})$. For $y = F(x)$, one has $F(x) - y = 0 \in -K$ and $y \in F(X) \subseteq D$. Thus (x, y) is feasible to $(\mathrm{P}'_{\mathrm{c}})$ and $f(F(x)) = f(y) \geq \mathrm{v}(\mathrm{P}'_{\mathrm{c}})$. Since this is valid for any x feasible to $(\mathrm{P_c})$ the relation $\mathrm{v}(\mathrm{P_c}) \geq \mathrm{v}(\mathrm{P}'_{\mathrm{c}})$ follows.

On the other hand, for (x, y) feasible to $(\mathrm{P}'_{\mathrm{c}})$ we have $x \in X$ and $g(x) \in -C$, so x is feasible to $(\mathrm{P_c})$. As f is K-increasing we get $\mathrm{v}(\mathrm{P_c}) \leq f(F(x)) \leq f(y)$. Taking the infimum on the right-hand side over (x, y) feasible to $(\mathrm{P}'_{\mathrm{c}})$ we get $\mathrm{v}(\mathrm{P_c}) \leq \mathrm{v}(\mathrm{P}'_{\mathrm{c}})$. □

The problem $(\mathrm{P}'_{\mathrm{c}})$ can be written as a special case of (P) with the objective function $A : X \times D \to \mathbb{R}$, $A(x, y) = f(y)$, the constraint function $B : X \times D \to \mathbb{R}^m \times \mathbb{R}^k$, $B(x, y) = (g(x), F(x) - y)$ and the cone $C \times K$, nonempty and convex in $\mathbb{R}^m \times \mathbb{R}^k$. We also use $(C \times K)^* = C^* \times K^*$. The Fenchel-Lagrange dual problem to $(\mathrm{P}'_{\mathrm{c}})$ is

$$\begin{array}{llll} (\mathrm{D}'_{\mathrm{c}}) & \sup & \{-A^*_{X \times D}(p, s) - ((\alpha, \beta)^T B)^*_{X \times D}(-p, -s)\}, & \\ & \text{s.t.} & \alpha \in C^*, \qquad \beta \in K^*, & (p, s) \in \mathbb{R}^n \times \mathbb{R}^k. \end{array}$$

Easy calculations yield

$$A^*_{X \times D}(p, s) = f^*_D(s) + \delta^*_X(p)$$

and

$$((\alpha, \beta)^T B)^*_{X \times D}(-p, -s) = (\alpha^T g + \beta^T F)^*_X(-p) + \delta^*_D(\beta - s), \quad \forall (p, s) \in \mathbb{R}^n \times \mathbb{R}^k.$$

Thus, the dual becomes

$$(\mathrm{D}'_{\mathrm{c}}) \qquad \sup_{\alpha \in C^*, \beta \in K^*} \Big\{ \sup_{p \in \mathbb{R}^n} \{-\delta^*_X(p) - (\alpha^T g + \beta F)^*_X(-p)\} + \sup_{s \in \mathbb{R}^k} \{-f^*_D(s) - \delta^*_D(\beta - s)\}\Big\},$$

which turns into

$$(\mathrm{D}'_{\mathrm{c}}) \qquad \sup_{\alpha \in C^*, \beta \in K^*} \{-f^*_D(\beta) - (\alpha^T g + \beta^T F)^*_X(0)\}.$$

Applying Theorem 16.4 in [17] for $\alpha \in C^*$, $\beta \in K^*$, we get

$$(\alpha^T g + \beta^T F)^*_X(0) = \inf\{(\beta^T F)^*_X(u) + (\alpha^T g)^*_X(-u) : u \in \mathbb{R}^n\},$$

so the final form of the dual problem is

$$(\mathrm{D_c}) \qquad \sup_{\alpha \in C^*, \beta \in K^*, u \in \mathbb{R}^n} \{-f^*_D(\beta) - (\beta^T F)^*_X(u) - (\alpha^T g)^*_X(-u)\}.$$

The constraint qualification (CQ′) becomes in this case

$$(\mathrm{CQ_c}) \quad 0 \in B(\mathrm{ri}(X \times D)) + \mathrm{ri}(C \times K),$$

which is equivalent to

$$(\mathrm{CQ_c}) \quad \exists x' \in \mathrm{ri}(X)\colon \quad g(x') \in -\mathrm{ri}(C) \quad \text{and} \quad F(x') \in \mathrm{ri}(D) - \mathrm{ri}(K).$$

Using Theorems 2.1, 2.2 and Proposition 3.1 the strong duality statement and the optimality conditions follow.

Theorem 3.1 *If* $(\mathrm{CQ_c})$ *holds then* $\mathrm{v}(\mathrm{P_c}) = \mathrm{v}(\mathrm{D_c})$ *and the dual has an optimal solution if* $\mathrm{v}(\mathrm{P_c}) > -\infty$.

Theorem 3.2

(a) *If* $(\mathrm{CQ_c})$ *is fulfilled and* $(\mathrm{P_c})$ *has an optimal solution* $\bar{x}$, *then the dual* $(\mathrm{D_c})$ *has an optimal solution* $(\bar{u}, \bar{\alpha}, \bar{\beta})$, $\bar{u} \in \mathbb{R}^n$, $\bar{\alpha} \in C^*$, $\bar{\beta} \in K^*$ *and the following optimality conditions are satisfied*:
 (i) $f_D^*(\bar{\beta}) + f(F(\bar{x})) = \bar{\beta}^T F(\bar{x})$,
 (ii) $(\bar{\beta}^T F)_X^*(\bar{u}) + \bar{\beta}^T F(\bar{x}) = \bar{u}^T \bar{x}$,
 (iii) $(\bar{\alpha}^T g)_X^*(-\bar{u}) + \bar{\alpha}^T g(\bar{x}) = -\bar{u}^T \bar{x}$,
 (iv) $\bar{\alpha}^T g(\bar{x}) = 0$.

(b) *If* $\bar{x}$ *is feasible to* $(\mathrm{P_c})$ *and* $(\bar{u}, \bar{\alpha}, \bar{\beta})$ *is feasible to* $(\mathrm{D_c})$ *fulfilling the optimality conditions* (i)–(iv), *then* $\mathrm{v}(\mathrm{P_c}) = \mathrm{v}(\mathrm{D_c})$ *and the mentioned feasible points are optimal solutions of the corresponding problems*.

3.2 Unconstrained Case

We give now duality assertions for the unconstrained problem having as objective function the postcomposition of a K-increasing convex function to a K-convex function, problem treated under different conditions in [7]. Taking in $(\mathrm{P_c})$ g the null function and $C = \{0\}^m$, we get the unconstrained primal problem

$$(\mathrm{P_u}) \quad \inf_{x \in X} f(F(x)).$$

Its Fenchel-Lagrange dual problem is

$$(\mathrm{D_u}) \quad \sup_{\beta \in K^*} \{-f_D^*(\beta) - (\beta^T F)_X^*(0)\},$$

while the constraint qualification $(\mathrm{CQ_c})$ turns into (it can be obtained also from Theorem 2.8 in [20])

$$(\mathrm{CQ_u}) \quad \exists x' \in \mathrm{ri}(X)\colon \quad F(x') \in \mathrm{ri}(D) - \mathrm{ri}(K).$$

Theorem 3.3 *If* $(\mathrm{CQ_u})$ *holds*, *then* $\mathrm{v}(\mathrm{P_u}) = \mathrm{v}(\mathrm{D_u})$ *and the dual has an optimal solution if* $\mathrm{v}(\mathrm{P_u}) > -\infty$.

4 Conjugate Function of a Postcomposition with a K-Increasing Convex Function

An interesting application of the duality assertions presented so far is the calculation of the conjugate function of a postcomposition of a K-convex function with a K-increasing convex function, for K a nonempty convex cone. In this section we obtain for this conjugate the classical formula [1, 9–11], but under weaker conditions than known so far.

Let K be a nonempty convex cone in $\mathbb{R}^k$, E a nonempty convex subset of $\mathbb{R}^n$, $f:\mathbb{R}^k \to \overline{\mathbb{R}}$ a K-increasing convex function and $F:E\to\mathbb{R}^k$ a K-convex function such that $F(E)\cap \operatorname{dom}(f)\neq\emptyset$. We determine the formula of the conjugate function $(f\circ F)^*_E$ as a function of f^* and F^*_E. For $p\in\mathbb{R}^n$, we have

$$(f\circ F)^*_E(p) = \sup_{x\in E}\{p^T x - f(F(x))\} = -\inf_{x\in E}\{f(F(x)) - p^T x\}$$
$$= -\inf_{x\in X}\{f(F(x)) - p^T x\},$$

where $X=\{x\in E: F(x)\in\operatorname{dom}(f)\}$, which is a convex set. Consider the functions $A:\operatorname{dom}(f)\times\mathbb{R}^n\to\mathbb{R}$, $A(z,y)=f(z)-p^Ty$ and $B:X\to\mathbb{R}^k\times\mathbb{R}^n$, $B(x)=(F(x),x)$. Obviously, A is convex and $(K\times\{0\}^n)$-increasing, B is $(K\times\{0\}^n)$-convex and $B(X)\subseteq F(X)\times X\subseteq \operatorname{dom}(f)\times\mathbb{R}^n$. The infimum problem above becomes

$$(\mathrm{P_a})\quad \inf_{x\in X}\{f(F(x))-p^Tx\} = \inf_{x\in X} A(B(x)).$$

Its Fenchel-Lagrange dual problem is

$$(\mathrm{D_a})\quad \sup_{\beta\in K^*,\gamma\in\mathbb{R}^n}\{-A^*_{\operatorname{dom}(f)\times\mathbb{R}^n}(\beta,\gamma) - ((\beta,\gamma)^T B)^*_X(0)\},$$

while the constraint qualification necessary for strong duality is

$$(\mathrm{CQ_a})\quad \exists x'\in \operatorname{ri}(X):\quad B(x')\in\operatorname{ri}(\operatorname{dom}(f)\times\mathbb{R}^n)-\operatorname{ri}(K\times\{0\}^n),$$

simplifiable to

$$(\mathrm{CQ_a})\quad 0\in F(\operatorname{ri}(X))-\operatorname{ri}(\operatorname{dom}(f))+\operatorname{ri}(K).$$

Because

$$F(X)=F(E)\cap\operatorname{dom}(f),$$

the last formula is rewritable as (cf. Theorem 2.3)

$$(\mathrm{CQ_a})\quad \operatorname{ri}(F(E)\cap\operatorname{dom}(f)+K)\cap\operatorname{ri}(\operatorname{dom}(f))\neq\emptyset.$$

To determine a formulation of the dual problem that contains only the conjugates of f and F relative to E, we have to determine the conjugate functions involved in the dual problem. For all $(\beta,\gamma)\in K^*\times\mathbb{R}^n$, we have $A^*_{\operatorname{dom}(f)\times\mathbb{R}^n}(\beta,\gamma)=\sup\{\beta^Tz - f(z): z\in\operatorname{dom}(f)\}+\sup\{\gamma^Ty+p^Ty: y\in\mathbb{R}^n\}$, thus $A^*_{\operatorname{dom}(f)\times\mathbb{R}^n}(\beta,\gamma)=f^*(\beta)$, if

$\gamma = -p$, and $A^*_{\mathrm{dom}(f)\times\mathbb{R}^n}(\beta,\gamma) = +\infty$, otherwise, and $((\beta,\gamma)^T B)^*_X(0) = \sup\{0 - \beta^T F(x) - \gamma^T x : x \in X\} = (\beta^T F)^*_X(-\gamma)$. As the plus infinite value is not relevant for $A^*_{\mathrm{dom}(f)\times\mathbb{R}^n}$ in ($\mathrm{D_a}$), we take further $\gamma = -p$ and the dual problem becomes

$$(\mathrm{D_a}) \qquad \sup_{\beta\in K^*}\{-f^*(\beta) - (\beta^T F)^*_X(p)\}.$$

When ($\mathrm{CQ_a}$) is satisfied, there is strong duality between ($\mathrm{P_a}$) and ($\mathrm{D_a}$), so we have

$$\begin{aligned}(f\circ F)^*_E(p) &= -\inf_{x\in X}\{f(F(x)) - p^T x\}\\ &= -\max_{\beta\in K^*}\{-f^*(\beta) - (\beta^T F)^*_X(p)\}.\end{aligned}$$

Therefore,

$$(f\circ F)^*_E(p) = \min_{\beta\in K^*}\{f^*(\beta) + (\beta^T F)^*_X(p)\}. \tag{1}$$

Unlike [9, 10] no closedness assumption regarding f or F is necessary for the validity of formula (1). Let us prove now that the condition ($\mathrm{CQ_a}$) is weaker than the one required in the literature ([9] for instance), which is in this case

$$F(E)\cap \mathrm{int}(\mathrm{dom}(f)) \neq \emptyset. \tag{2}$$

Assuming (2) true let z' be one of the common elements of these sets. It follows that $\mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$, so $\mathrm{ri}(\mathrm{dom}(f)) = \mathrm{int}(\mathrm{dom}(f))$. We have also $z' \in F(E)\cap\mathrm{int}(\mathrm{dom}(f)) \subseteq F(E)\cap\mathrm{dom}(f) \subseteq F(E)\cap\mathrm{dom}(f) + K$. On the other hand $F(E)\cap\mathrm{dom}(f) + K = F(X) + K$, which is convex, so it has nonempty relative interior. Take $z'' \in \mathrm{ri}(F(X)+K)$.

According to Theorem 6.1 in [17], for any $\lambda \in (0,1]$ one has $(1-\lambda)z' + \lambda z'' \in \mathrm{ri}(F(X)+K)$. As $z' \in \mathrm{int}(\mathrm{dom}(f))$, which is an open set, there is a $\bar\lambda \in (0,1]$ such that $\bar z = (1-\bar\lambda)z' + \bar\lambda z'' \in \mathrm{int}(\mathrm{dom}(f)) = \mathrm{ri}(\mathrm{dom}(f))$. Therefore $\bar z \in \mathrm{ri}(F(E)\cap\mathrm{dom}(f)+K)\cap\mathrm{ri}(\mathrm{dom}(f))$, i.e. ($\mathrm{CQ_a}$) is fulfilled.

An example where our condition ($\mathrm{CQ_a}$) is applicable, while (2) fails follows.

Example 4.1 Take $k=2$, $E=\mathbb{R}$, $K=\{0\}\times\mathbb{R}_+$, $F:\mathbb{R}\to\mathbb{R}^2$, defined by $F(x) = (0,x)$, $\forall x\in\mathbb{R}$ and $f:\mathbb{R}^2\to\overline{\mathbb{R}}$ given for any pair $(x,y)\in\mathbb{R}^2$ by $f(x,y)=y$, if $x=0$, and $f(x,y)=+\infty$, otherwise. F is K - convex, f is proper convex and K-increasing and one has $K^*=\mathbb{R}\times\mathbb{R}_+$, $F(E)=\mathrm{dom}(f)=\{0\}\times\mathbb{R}$, $\mathrm{int}(\mathrm{dom}(f))=\emptyset$ and $\mathrm{ri}(\mathrm{dom}(f))=\{0\}\times\mathbb{R}$. We have $X=E=\mathbb{R}$ and the feasibility condition $F(X)\cap\mathrm{dom}(f)\neq\emptyset$ is obviously satisfied. Thus the conjugates regarding X and E are actually classical conjugate functions.

As $(f\circ F)(x) = f(0,x) = x$ $\forall x\in\mathbb{R}$, it follows $(f\circ F)^*(p)=0$, if $p=1$, and $(f\circ F)^*(p)=+\infty$, otherwise. We also have, for all $(a,b)\in\mathbb{R}\times\mathbb{R}_+$ and all $p\in\mathbb{R}$, $f^*(a,b)=0$, if $b=1$, and $f^*(a,b)=+\infty$, otherwise, and $((a,b)F)^*(p)=0$, if $b=p$, and $((a,b)F)^*(p)=+\infty$, otherwise. This yields $\min_{(a,b)\in\mathbb{R}\times\mathbb{R}_+}\{f^*(a,b) + ((a,b)F)^*(p)\}=0$, if $p=1$, otherwise being equal to $+\infty$.

Therefore the formula (1) is valid. Taking into consideration the things above, ($\mathrm{CQ_a}$) means $\{0\}\times\mathbb{R}\neq\emptyset$, while (2) is $\{0\}\times\mathbb{R}\cap\emptyset\neq\emptyset$. It is clear that the latter is false, while our new condition is satisfied. Therefore ($\mathrm{CQ_a}$) is indeed weaker than (2).

The formula of the conjugate of the postcomposition with an increasing convex function becomes for an appropriate choice of the functions and for $K=[0,+\infty)$ similar to the result given in Theorem 2.5.1 in [9]. As shown above, there is no need to impose closedness for the functions and a so strong constraint qualification like (2).

We conclude this section with a concrete problem where the results given in this paper find a good application.

Example 4.2 Let $F:E\to\mathbb{R}$ be a concave function with strictly positive values, where E is a nonempty convex subset of $\mathbb{R}^n$. We determine the value of the conjugate function of $1/F$ at some fixed $p\in\mathbb{R}^n$. According to the preceding results, we write $(1/F)_E^*(p)$ as an unconstrained composed convex problem by taking $K=(-\infty,0]$, which is a nonempty convex cone and $f:\mathbb{R}\to\overline{\mathbb{R}}$ with $f(y)=1/y$ for $y\in(0,+\infty)$ and $+\infty$ otherwise. One can notice that the concave function F is actually K-convex for this K while f is K-increasing. The constraint qualification (CQ_a) specialized for this problem is

$$(CQ_e)\quad \operatorname{ri}(F(E)\cap(0,+\infty)+(-\infty,0])\cap\operatorname{ri}((0,+\infty))\neq\emptyset,$$

which is equivalent to

$$\operatorname{ri}(F(E)+(-\infty,0])\cap(0,+\infty)\neq\emptyset.$$

By Theorem 2.3, this is nothing but

$$(CQ_e)\quad (F(\operatorname{ri}(E))+(-\infty,0))\cap(0,+\infty)\neq\emptyset,$$

which is always fulfilled since F has only strictly positive values.

Thus the formula (1) obtained before can be applied without any additional assumption. We have $(1/F)_E^*(p)=\inf\{f^*(\beta)+(\beta F)_X^*(p):\beta\le 0\}$. One has $f^*(\beta)=\sup\{\beta y-1/y: y>0\}=-2\sqrt{-\beta}$, if $\beta<0$, and $f^*(\beta)=0$, if $\beta=0$. Moreover, $X=\{x\in E: F(x)\in(0,+\infty)\}=E$ and we have $(\beta F)_X^*(p)=-\beta(-F)_E^*(p/(-\beta))$, if $\beta<0$, and $(\beta F)_X^*(p)=\delta_E^*(p)$, if $\beta=0$.

This leads to the following formula of the conjugate of $1/F$ (see also Theorem 10 in [10])

$$(1/F)_E^*(p)=\min\{\inf_{\beta>0}\{\beta(-F)_E^*(p/\beta)-2\sqrt{\beta}\},\delta_E^*(p)\}.$$

When the value of the conjugate is finite either it is equal to $\delta_E^*(p)$ or there is a $\bar\beta>0$ for which the infimum in the right-hand side is attained. The value of the infimum gives in this latter case actually the formula of the conjugate.

5 Conclusions

To a convex optimization problem with cone-convex constraints we have attached the so-called Fenchel-Lagrange dual problem. To achieve strong duality between these two problems we have used a constraint qualification due to Frenk and Kassay [16], which we proved to be equivalent to the one introduced by Wolkowicz in [18]. The ordinary convex programming problem is a special case of this problem and the weakest constraint qualification for Lagrange duality known to us is rediscovered as a particular instance. The convex optimization problem that consists in the minimization of the postcomposition of a K-increasing convex function to a K-convex function, where K is a nonempty convex cone, subject to cone-convex constraints follows. Strong duality and optimality conditions are derived also for this problem, as well as for its special case when the cone constraints are omitted. On this last problem is based the application we deliver. We rediscover the formula of the conjugate of the composition of two functions, giving weaker conditions for it than known so far in the literature. Finally, a concrete example where our theoretical results are applicable is the calculation of the conjugate function of $1/F$ for any strictly positive concave function defined on a convex set $F : E \to \mathbb{R}$.

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