The Asymptotic Theory of Transaction Costs

Lecture Notes by
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Introduction

The present lecture notes are based on several advanced courses which I gave at the University of Vienna between 2011 and 2013. In 2015 I gave a similar course ("Nachdiplom-Vorlesung") at ETH Zürich. The purpose of these lectures was to present and organize the recent progress on portfolio optimization under proportional transaction costs $\lambda > 0$. Special emphasis is given to the asymptotic behaviour when λ tends to zero.

The theme of portfolio optimization is a classical topic of Mathematical Finance, going back to the seminal work of Robert Merton in the early seventies (considering the frictionless case without transaction costs). Mathematically speaking, this question leads to a concave optimization problem under linear constraints. A technical challenge arises from the fact that – except for the case of finite probability spaces Ω – the optimization takes place over infinite–dimensional sets.

There are essentially two ways of attacking such an optimization problem.

The *primal* method consists in directly addressing the problem at hand. Following the path initiated by Robert Merton, this leads to a partial differential equation of *Hamilton–Jacobi–Bellman* type. This PDE method can also be successfully extended to the case of proportional transaction costs. Important work on this line was done by G. Constantinides [40], B. Dumas and E. Luciano [75], M. Taksar, M. J. Klass, D. Assaf [234], M. Davis and A. Norman [57], St. Shreve and M. Soner [224], just to name some of the early work on this topic.

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An alternative method consists in passing to the *dual version* of the problem. This dual method is also sometimes called the *martingale method* as one now optimizes over the constraint variables, which in the present context turn out to be martingales (or their generalizations such as super-martingales). In the course of the analysis an important role is played by the *Legendre transform* or *conjugate function* V of the utility function U appearing in the primal version of the problem.

In these notes we focus on the dual method as well as the interplay between the dual and the primal problem. This approach yields to a central concept of our approach, namely the concept of a *shadow price process*. Mathematically speaking, this is an infinite-dimensional generalization of the fundamental concept of a Lagrange multiplier. It has a clear economic interpretation as a price process which – without transaction costs – yields the same optimal portfolio as the original price process under transaction costs. This gives a direct link of the present optimization theory under transaction costs with the more classical frictionless theory.

This brings us to a major open challenge for Mathematical Finance which constitutes much of the motivation for the present notes and the underlying research. The question is how to design an economically as well as mathematically meaningful framework to deal with financial models which are based on *fractional Brownian motion*.

This variant of the basic concept of Brownian motion, was introduced in 1940 by A. Kolmogorov under the name of *Wiener spiral*. It was strongly advocated by B. Mandelbrot more than 50 years ago as a more realistic approach to financial data than models based on classical Brownian motion, such as the Black-Scholes model.

But there are fundamental problems which until today make it impossible to reconcile these models with the main stream of Mathematical Finance, which is based on the paradigmatic assumption of *no arbitrage*. In fact, fractional Brownian motion fails to be a semi-martingale. It is wellknown ([64], Theorem 7.2) that processes which fail to be semi-martingales always allow for arbitrage. Hence it does not make any economic sense to apply *no arbitrage* arguments, e.g. in the context of option pricing, if already the underlying model for the stock price process violates this paradigm.

One way to get out of this deadlock is the consideration of transaction costs. It was shown in [107] that the consideration of (arbitrarily small) proportional transaction costs $\lambda > 0$ makes the arbitrage possibilities disappear for the presently considered models based on fractional Brownian motion. This allows for a similar duality theory as in the frictionless case. While in the frictionless theory the dual objects are the martingale measures and their variants, their role now is taken by the λ-*consistent price systems*. However many of the classical concepts from the frictionless theory, such as replication and/or super-replication, do not make any economic sense when considering these models under transaction costs. Indeed, one can give rigorous mathematical *proofs* (the "face-lifting theorem" in [106], [168], [227]) that it is not possible to derive *any non-trivial result* from super-replication arguments in the present context of models under transaction costs.

Yet there is still hope to find a proper framework which allows to obtain non-trivial results for fractional Brownian motion. There is one financial application which does make perfect sense in the presence of transaction costs, from an economic as well as from a mathematical perspective, namely *portfolio optimization*. This is precisely the theme of the present lecture notes and we shall develop this theory quite extensively. But before doing so let us come back to the original motivation. What does portfolio optimization under transaction costs have to do with the original problem of pricing and hedging options in financial models involving fractional Brownian motion? The answer is that we have hope that finally a well-founded theory of portfolio optimization can shed some light on the original problem of pricing derivative securities via utility indifference pricing. The key fact is the existence of a *shadow price process* $\tilde{S}$ which can serve as a link to the traditional frictionless theory. In Theorem 8.4 we shall prove the existence of a shadow price process under general assumptions in the framework of models based on fractional Brownian motion. This theorem was proved only very recently in [52] and is the main and final result of the present lecture notes. In a sense, the lecture notes aim at providing and developing all the material for proving this theorem. At the same time they try to present a comprehensive introduction to the general theme of portfolio optimization under transaction costs. They are structured in the following way.

In the first two chapters we develop the theory of portfolio optimization in the elementary setting of a finite probability space. Under this assumption all relevant spaces are finite-dimensional and therefore the involved functional analysis reduces to linear algebra. These two chapters are analogous to the summer school course [215] as well as the two introductory chapters in [69] where a similar presentation was given for the frictionless case.

In chapter 3 we focus rather extensively on the most basic example: the Black-Scholes model under logarithmic utility $U(x) = \log(x)$. A classical result of R. Merton states that, in the frictionless case, the optimal strategy consists of holding a constant fraction (depending in an explicit way on the parameters of the model) of wealth in stock and the rest in bond (the "Merton line"). If we pass to transaction costs, it is also well known that one has to keep the proportion of wealth within a certain interval and much is known on this interval. Our purpose is to *exactly* determine all the quantities of interest, e.g. the width of this corridor, the effect on the indirect utility etc. The dual method allows us to calculate these quantities either in closed form as functions of the level $\lambda > 0$ of transaction costs, or as a power series of $\lambda^{\frac{1}{3}}$. In the latter case we are able to explicitly compute all the coefficients of the fractional Taylor series. The key concept for this analysis is the notion of a shadow price process. In the case of the Black-Scholes model this shadow price process can explicitly be determined. This demonstrates the power of the dual method. We have hope that the analysis of the Black-Scholes model can serve as a role model for a similar analysis for other models of financial markets, e.g. stock price processes based on fractional Brownian motion. Here is a wide open field for future research.

In chapter 4 we go systematically through the duality theory for financial markets under transaction costs. As regards the degree of generality we do not strive for the maximal one, i.e. the consideration of general càdlàg price process $S = (S_t)_{0 \leqslant t \leqslant T}$ as in [48] and [50]. Rather we confine ourselves to *continuous* processes S and – mainly for convenience – we assume that the underlying filtration is Brownian. We do so as we have fractional Brownian motion as our final application in the back of our mind. On the other hand, for this application it is important that we do *not* assume that the process S is a semi-martingale. The central result of chapter 4 is Theorem 4.22 which establishes a polar relation between two sets of random variables. On the primal side this is the set of random variables which are attained from initial wealth $x > 0$ by trading in the stock S under transaction costs λ in an admissible way. On the dual side the set consists of the so-called super-martingale deflators. The polar relation between these two sets is such that the conditions of the general portfolio optimization theorem in [161] are satisfied which allows to settle all central issues of portfolio optimization.

Chapter 5 is a kind of side-step and develops a local duality theory. It shows that several traditional assumptions in the theory of portfolio optimization can be replaced by their *local versions* without loss of generality with respect to the conclusions. A typical example is the assumption $(NFLVR)$ of “no free lunch with vanishing risk” from [65]. This well-known concept is traditionally assumed to hold true in portfolio optimization problems, e.g. in [161]. It was notably pointed out by I. Karatzas and K. Kardaras [149] that this assumption may be replaced by its local version which is the condition $(NUPBR)$ of “no unbounded profit with bounded risk”. We give equivalent formulations of this latter property in the frictionless setting (Theorem 5.6) and an analogous theorem in the setting of (arbitrarily small) transaction costs $\lambda > 0$ (Theorem 5.11).

After all these preparations we turn to the general theme of portfolio optimization under proportional transaction costs in chapter 6. The basic duality Theorem 6.2 is a consequence of these preparations and the results from [161]. In Theorem 6.5 we show the crucial fact that – under appropriate assumptions – the dual optimizer, which a priori only is a super-martingale, is in fact a local martingale. The crucial assumption underlying this theorem is that the process S satisfies the "two way crossing property", a notion introduced recently by C. Bender [14] which generalizes the concept of a continuous martingale.

In chapter 7 we show that the local martingale property established in Theorem 6.5 is the key to the existence of a shadow price process (Theorem 7.3). This insight goes back to the work of J. Cvitanic and I. Karatzas [43].

In chapter 8 we finally turn to the case of exponential fractional Brownian motion. It culminates in Theorem 8.4 where we prove that for this model there is a shadow price process. The key ingredient is a recent result by R. Peyre [195] showing that fractional Brownian motion has the "two way crossing" property.

Many people have participated in the research efforts underlying these lectures. My sincere thanks go to my co-authors over the past 10 years on this topic L. Campi, St. Gerhold, P. Guasoni, J. Muhle-Karbe, R. Peyre, M. Rásonyi, J. Yang. Special thanks go to Ch. Czichowsky. Without his endurance the six joint papers [47] - [52], which are the basis of the present research, would never have seen the light of the day. I also thank the participants of my lectures at the University of Vienna and ETH Zürich.

Contents

1 Models on Finite Probability Spaces

In this chapter we consider a stock price process $S = (S_t)_{t=0}^T$ in finite, discrete time, based on and adapted to a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, where $\mathcal{F} = \mathcal{F}_T$ is the sigma-algebra formed by all subsets of Ω. Similarly as in the introductory chapter 2 of [69] we want to develop the basic ideas of the duality theory in this technically easy setting. The extra ingredient will be the role of transaction costs. To avoid trivialities we assume $\mathbb{P}(\omega) > 0$, for all $\omega \in \Omega$.

To keep things simple we assume that there is only one stock. It takes strictly positive values, i.e., $S = (S_t)_{t=0}^T$ is an $\mathbb{R}_+$-valued process. In addition, there is a bond, denoted by $B = (B_t)_{t=0}^T$; by choosing B as numéraire we may assume that $B_t \equiv 1$.

Next we introduce transaction costs $\lambda \geqslant 0$: that is, the process $((1-\lambda)S_t, S_t)_{t=0}^T$ models the bid and ask price of the stock S respectively. The agent may buy stock at price S but, when selling stock, she only obtains a price $(1-\lambda)S$. Of course, we assume $\lambda < 1$ for obvious economic reasons.

We have chosen a very simple setting here. For a much more general framework we refer, e.g., to [129], [134], [141], [135] and [213]. These authors investigate the setting given by finitely many stocks $S^1, \ldots, S^n$, where the prices $(\pi_{ij})_{1 \leqslant i,j \leqslant n}$ of exchanging stock j into stock i are general adapted processes. A good economic interpretation for this situation is the case of n currencies where the bid and ask prices $\pi_{i,j}$ and $\pi_{j,i}$ depend on the pair (i,j) of currencies.

Here we do not strive for this generality. We do this on the one hand for didactic reasons to keep things as non-technical as possible. On the other hand we shall mainly be interested in the asymptotic theory for $\lambda \to 0$, for which our present simple setting is more natural.

Definition 1.1. *For given $S = (S_t)_{t=0}^T$ and $0 \leqslant \lambda < 1$, we associate the process of* solvency cones

$$K_t = \left\{ (\varphi_t^0, \varphi_t^1) \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^2) : \varphi_t^0 \geqslant \max(-\varphi_t^1 S_t, -\varphi_t^1 (1-\lambda) S_t \right\} \tag{1}$$

The interpretation is that an economic agent holding φ_t^0 units of bond, and φ_t^1 units of stock is *solvent* for a given stock price S_t if, after liquidating the position in stocks, the resulting position in bonds is non-negative. "Liquidating the stock" means selling φ_t^1 stocks (at price $(1-\lambda)S_t$) if $\varphi_t^1 > 0$ and buying $-\varphi_t^1$ stocks (at price S_t) if $\varphi_t^1 < 0$.

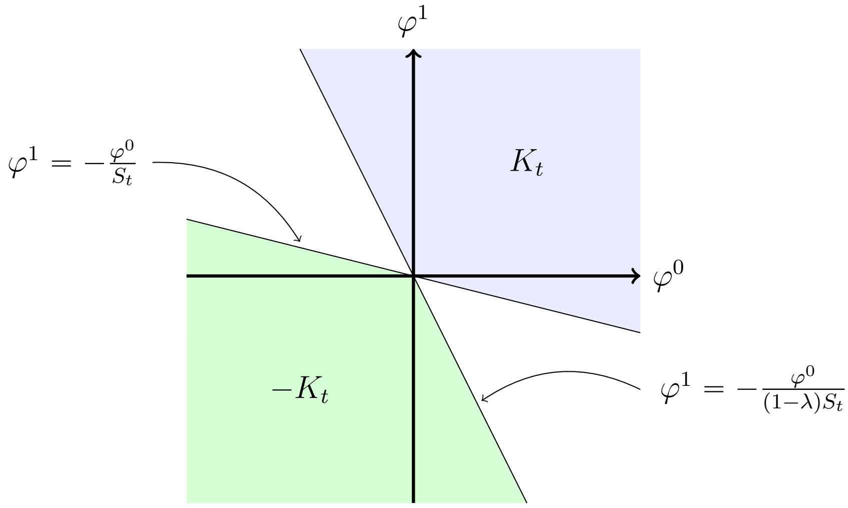


Figure 1: The solvency cone

Definition 1.2. *For given* $S = (S_t)_{t=0}^T$ *and* $0 \leqslant \lambda < 1$*, an adapted process* $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ *starting at* $(\varphi_{-1}^0, \varphi_{-1}^1) = (0,0)$ *is called* self-financing *if*

$$(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1) \in -K_t, \qquad t = 0, \ldots, T. \tag{2}$$

The relation (2) *is understood to hold* $\mathbb{P}$*-a.s., which in the present setting simply means: for each* $\omega \in \Omega$.

To motivate this definition note that the change at time t of positions in the portfolio $(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1)$ can be carried out for the bid-ask prices $\big((1-\lambda)S_t, S_t\big)$ iff $(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1) \in -K_t$.

For $(x^1, x^2) \in \mathbb{R}^2$, we call $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ self-financing and starting at (x^1, x^2) if $(\varphi_t^0 - x^1, \varphi_t^1 - x^2)_{t=-1}^T$ is self-financing and starting at $(0,0)$. We also note that it is natural in the context of transaction costs to allow for T trades (i.e. exchanges of bonds against stocks) in the T-period model $(S_t)_{t=0}^T$, which leads to the initial condition in terms of $(\varphi_{-1}^0, \varphi_{-1}^1)$.

Definition 1.3. *The process* $S = (S_t)_{t=0}^T$ *admits for* arbitrage under transaction costs $0 \leqslant \lambda < 1$ *if there is a self-financing trading strategy* $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$*, starting at* $\varphi_{-1}^0 = \varphi_{-1}^1 = 0$*, and such that*

$$(\varphi_T^0, \varphi_T^1) \geqslant (0,0), \qquad \mathbb{P}\text{-a.s.}$$

and

$$\mathbb{P}\left[(\varphi_T^0, \varphi_T^1) > (0,0)\right] > 0.$$

We say that S *satisfies the no arbitrage condition* (NA^λ) *if it does not allow for an arbitrage under transaction costs* $0 \leqslant \lambda < 1$*.*

Let us introduce the following notation. For fixed S and $\lambda > 0$, denote by $\mathcal{A}^\lambda$ the set of $\mathbb{R}^2$-valued $\mathcal{F}$-measurable random variables (φ^0, φ^1), such that there exists a self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at $(\varphi_{-1}^0, \varphi_{-1}^1) = (0, 0)$, and such that $(\varphi^0, \varphi^1) \leqslant (\varphi_T^0, \varphi_T^1)$.

Proposition 1.4. *Suppose that S satisfies (NA^λ), for fixed $0 \leqslant \lambda < 1$. Then $\mathcal{A}^\lambda$ is a closed polyhedral cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, containing $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_-)$ and such that $\mathcal{A}^\lambda \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_+) = \{0\}$.*

Proof. Fix $0 \leqslant t \leqslant T$ and an atom $F \in \mathcal{F}_t$. Recall that F is an *atom* of the finite sigma-algebra $\mathcal{F}_t$ if $E \in \mathcal{F}_t, E \subseteq F$ implies that either $E = F$ or $E = \varnothing$. Define the *ask* and *bid* elements a_F and b_F in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ as

$$a_F = \big(-S_{t|F}, 1\big)\, \mathbb{1}_F, \qquad b_F = \big((1-\lambda)S_{t|F}, -1\big)\, \mathbb{1}_F. \tag{3}$$

Note that $S_{t|F}$ is a well-defined positive number, as S_t is $\mathcal{F}_t$-measurable and F an atom of $\mathcal{F}_t$.

The elements a_F and b_F are in $\mathcal{A}^\lambda$. They correspond to the trading strategy $(\varphi_s^0, \varphi_s^1)_{s=-1}^T$ such that $(\varphi_s^0, \varphi_s^1) = (0, 0)$, for $-1 \leqslant s < t$, and $(\varphi_s^0, \varphi_s^1)$ equals a_F (resp. b_F), for $t \leqslant s \leqslant T$. The interpretation is that an agent does nothing until time t. Then, conditionally on the event $F \in \mathcal{F}_t$, she buys (resp. sells) one unit of stock and holds it until time T.

Note that an element (φ^0, φ^1) in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ is in $\mathcal{A}^\lambda$ iff there are non-negative numbers $\mu_F \geqslant 0$ and $\nu_F \geqslant 0$, where F runs through the atoms of $\mathcal{F}_t$, and $t = 0, \ldots, T$, such that

$$(\varphi^0, \varphi^1) \leqslant \sum_F (\mu_F a_F + \nu_F b_F).$$

In other words, the elements of the form (3), together with the vectors $(-1, 0)\mathbb{1}_\omega$ and $(0, -1)\mathbb{1}_\omega$, where ω runs through Ω, generate the cone $\mathcal{A}^\lambda$. It follows that $\mathcal{A}^\lambda$ is a closed polyhedral cone (see Appendix A).

The inclusion $\mathcal{A}^\lambda \supseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_-)$ is obvious, and (NA^λ) is tantamount to the assertion $\mathcal{A}^\lambda \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_+) = \{0\}$. ■

Definition 1.5. *An element $(\varphi^0, \varphi^1) \in \mathcal{A}^\lambda$ is called* maximal *if, for $((\varphi^0)', (\varphi^1)') \in \mathcal{A}^\lambda$ satisfying $((\varphi^0)', (\varphi^1)') \geqslant (\varphi^0, \varphi^1)$ a.s., we have $((\varphi^0)', (\varphi^1)') = (\varphi^0, \varphi^1)$ a.s.*

Definition 1.6. *Fix the process $S = (S_t)_{t=0}^T$ and transaction costs $0 \leqslant \lambda < 1$. A* consistent price system *is a pair $(\tilde{S}, Q)$, such that Q is a probability measure on Ω equivalent to $\mathbb{P}$, and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a martingale under Q taking its values in the bid-ask spread $[(1-\lambda)S, S]$, i.e.*

$$(1-\lambda)S_t \leqslant \tilde{S}_t \leqslant S_t, \qquad \mathbb{P}\text{-}a.s. \tag{4}$$

We denote by $\mathcal{S}^\lambda$ the set of consistent price systems.

Remark 1.7. For $\lambda = 0$ we have $\tilde{S} = S$, so that we find the classical notion of equivalent martingale measures $Q \in \mathcal{M}^e$. We shall see in Corollary 1.11 that the set of real numbers $\mathbb{E}_Q[\varphi_T^0 + \varphi_T^1 \tilde{S}_T]$, where $(\tilde{S}, Q)$ *ranges in* $\mathcal{S}^\lambda$, yields precisely the arbitrage-free prices (in terms of bonds) for the contingent claim $(\varphi_T^0, \varphi_T^1) \in L^\infty(\mathbb{R}^2)$.

Theorem 1.8. *(Fundamental Theorem of Asset Pricing): Fixing the process* $S = (S_t)_{t=0}^T$ *and transaction costs* $0 \leqslant \lambda < 1$, *the following are equivalent:*

- (i) *The no arbitrage condition* (NA^λ) *is satisfied.*
- (ii) *There is a consistent price system* $(\tilde{S}, Q) \in \mathcal{S}^\lambda$.
- (iii) *There is an* $\mathbb{R}_+^2$*-valued* $\mathbb{P}$*-martingale* $(Z_t)_{t=0}^T = (Z_t^0, Z_t^1)_{t=0}^T$ *such that* $Z_0^0 = 1$ *and*
$$\frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, S_t], \qquad t = 0, \ldots, T. \tag{5}$$

Remark 1.9. The basic idea of the above version of the Fundamental Theorem of Asset Pricing goes back to the paper [125] of Jouini and Kallal from 1995. The present version dealing with finite probability space Ω is due to Kabanov and Stricker [139].

In the case $\lambda = 0$ condition (iii) allows for the following interpretation: in this case (5) means that
$$Z_t^1 = Z_t^0 S_t. \tag{6}$$
Interpret Z_T^0 as a probability measure by letting $\frac{dQ}{d\mathbb{P}} := Z_T^0$. By Bayes' rule condition (6) and the $\mathbb{P}$-martingale property of Z^1 then is tantamount to the assertion that S is a Q-martingale.

Proof. $(ii) \Rightarrow (i)$ As usual in the context of the Fundamental Theorem of Asset Pricing, this is the easy implication, allowing for a rather obvious economic interpretation. Suppose that $(\tilde{S}, Q)$ is a consistent price system.

Let us first give the intuition: as the process $\tilde{S}$ is a martingale under Q, it is free of arbitrage (without transaction costs). Trading in S under transaction costs λ only allows for less favorable terms of trade than trading in $\tilde{S}$ without transaction costs by (4). Hence we find that S under transaction costs λ satisfies (NA^λ) *a fortiori*.

Here is the formalization of this economically obvious reasoning.

Note that, for every self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at $(\varphi_{-1}^0, \varphi_{-1}^1) = (0, 0)$ we have
$$\begin{aligned}\varphi_t^0 - \varphi_{t-1}^0 &\leqslant \min\big(-(\varphi_t^1 - \varphi_{t-1}^1)S_t, -(\varphi_t^1 - \varphi_{t-1}^1)(1-\lambda)S_t\big)\\ &\leqslant -(\varphi_t^1 - \varphi_{t-1}^1)\tilde{S}_t,\end{aligned}$$

by (4), as $(\varphi_t^0 - \varphi_{t-1}^0, \varphi_t^1 - \varphi_{t-1}^1) \in -K_t$.

Hence

$$
\begin{aligned}
(\varphi_T^0 - \varphi_{-1}^0) &= \sum_{t=0}^{T} (\varphi_t^0 - \varphi_{t-1}^0) \\
&\leqslant -\sum_{t=0}^{T} (\varphi_t^1 - \varphi_{t-1}^1) \tilde{S}_t \\
&= \sum_{t=1}^{T} \varphi_{t-1}^1 (\tilde{S}_t - \tilde{S}_{t-1}) + \varphi_{-1}^1 \tilde{S}_0 - \varphi_T^1 \tilde{S}_T.
\end{aligned}
$$

Taking expectations under Q, and using that $\varphi_{-1}^0 = \varphi_{-1}^1 = 0$, we get

$$
\mathbb{E}_Q[\varphi_T^0] + \mathbb{E}_Q[\varphi_T^1 \tilde{S}_T] \leqslant \mathbb{E}_Q\left[\sum_{t=1}^{T} \varphi_{t-1}^1 (\tilde{S}_t - \tilde{S}_{t-1})\right] = 0. \tag{7}
$$

Now suppose that $\varphi_T^0 \geqslant 0$ and $\varphi_T^1 \geqslant 0$, $\mathbb{P}$-a.s., i.e. $\varphi_T^0(\omega) \geqslant 0$ and $\varphi_T^1(\omega) \geqslant 0$, for all ω in the finite probability space Ω.

Using the fact that Q is equivalent to $\mathbb{P}$, i.e. $Q(\omega) > 0$ for all $\omega \in \Omega$, we conclude from (7) that $\varphi_T^0(\omega) = 0$ and $\varphi_T^1(\omega)\tilde{S}_T(\omega) = 0$, for all $\omega \in \Omega$. Observing that $\tilde{S}_T$ is strictly positive by the assumption $\lambda < 1$, for each $\omega \in \Omega$, we also have $\varphi_T^1(\omega) = 0$ so that S satisfies (NA^λ).

$(i) \Rightarrow (iii)$ Now suppose that S satisfies (NA^λ). By Proposition 1.4 we know that $\mathcal{A}^\lambda$ is a closed convex cone in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ such that

$$
\mathcal{A}^\lambda \cap L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}_+^2) = \{0\}.
$$

Claim: There is an element $Z = (Z^0, Z^1) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, verifying $Z^0(\omega) > 0$ and $Z^1(\omega) \geqslant 0$, for all $\omega \in \Omega$, and normalized by $\mathbb{E}[Z^0] = 1$, such that

$$
\langle (\varphi_T^0, \varphi_T^1), (Z^0, Z^1) \rangle = \mathbb{E}_{\mathbb{P}}[\varphi_T^0 Z^0 + \varphi_T^1 Z^1] \leqslant 0, \quad \text{for all } (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda. \tag{8}
$$

Indeed, fix $\omega \in \Omega$, and consider the element $(\mathbb{1}_\omega, 0) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ which is not an element of $\mathcal{A}^\lambda$.

By Hahn-Banach and the fact that $\mathcal{A}^\lambda$ is closed and convex (Prop. 1.4), we may find, for fixed $\omega \in \Omega$, an element $Z_\omega \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ separating $\mathcal{A}^\lambda$ from $(\mathbb{1}_\omega, 0)$. As $\mathcal{A}^\lambda$ is a cone, we may find Z_ω such that

$$
\langle (\mathbb{1}_\omega, 0), (Z_\omega^0, Z_\omega^1) \rangle > 0,
$$

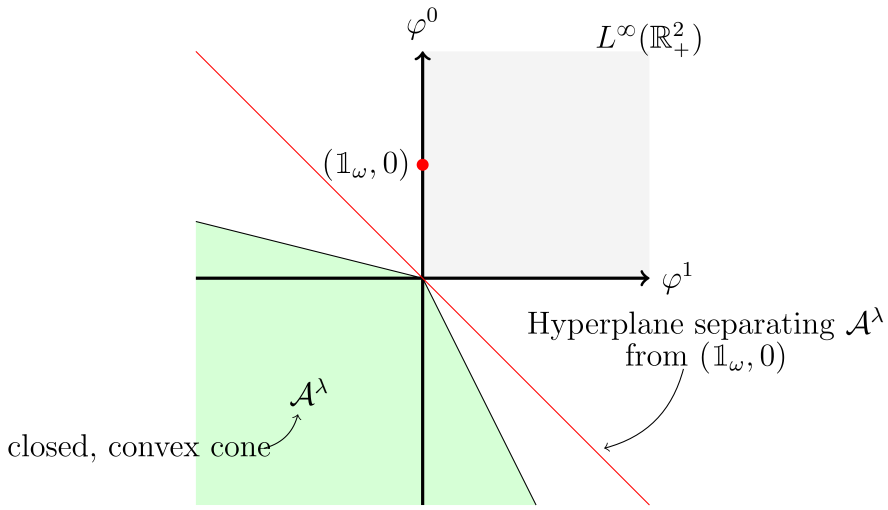


Figure 2: Regarding the proof of Thm 1.10

while

$$Z_{\omega|\mathcal{A}^\lambda} \leqslant 0.$$

The first inequality simply means that the element $Z^0_\omega \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ takes a strictly positive value on ω, i.e.

$$Z^0_\omega(\omega) > 0.$$

As $\mathcal{A}^\lambda$ contains the negative orthant $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; -\mathbb{R}^2_+)$, the second inequality implies that

$$Z^0_\omega \geqslant 0, \qquad Z^1_\omega \geqslant 0.$$

Doing this construction for each $\omega \in \Omega$ and defining

$$Z = \sum_{\omega \in \Omega} \mu_\omega Z_\omega,$$

where $(\mu_\omega)_{\omega\in\Omega}$ are strictly positive scalars such that the first coordinate of Z has $\mathbb{P}$-expectation equal to one, we obtain an element $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ such that

$$Z_{|\mathcal{A}^\lambda} \leqslant 0, \tag{9}$$

which is tantamount to (8), and

$$Z^0 > 0, \quad Z^1 \geqslant 0,$$

proving the claim.

We associate to Z the $\mathbb{R}^2$-valued martingale $(Z_t)_{t=0}^T$ by

$$Z_t = \mathbb{E}[Z|\mathcal{F}_t], \qquad t = 0, \ldots, T.$$

We have to show that $\frac{Z_t^1}{Z_t^0}$ takes its values in the bid-ask-spread $[(1-\lambda)S_t, S_t]$ of S_t. Applying (9) to the element a_F defined in (3), for an atom $F \in \mathcal{F}_t$, we obtain

$$\left\langle (Z_t^0, Z_t^1), (-S_{t|F}, 1)\mathbb{1}_F \right\rangle = \mathbb{E}\left[(-S_{t|F} Z_t^0 + Z_t^1)\mathbb{1}_F\right] \leqslant 0.$$

In the last line we have used the $\mathcal{F}_t$-measurability of $S_t \mathbb{1}_F$ to conclude that $0 \geqslant \mathbb{E}[(-S_{t|F} Z_t^0 + Z_t^1)\mathbb{1}_F] = \mathbb{E}[(-S_{t|F} Z_{t|F}^0 + Z_{t|F}^1)\mathbb{1}_F]$. As $S_{t|F}$, $Z_{t|F}^0$, and $Z_{t|F}^1$ are constants, we conclude that

$$-S_{t|F} Z_{t|F}^0 + Z_{t|F}^1 \leqslant 0,$$

i.e.

$$\frac{Z_{t|F}^1}{Z_{t|F}^0} \leqslant S_{t|F}.$$

As this inequality holds true for each $t = 0, \ldots, T$ and each atom $F \in \mathcal{F}_t$ we have shown that

$$\frac{Z_t^1}{Z_t^0} \in \,]-\infty, S_t] \qquad t = 0, \ldots, T.$$

Applying the above argument to the element b_F in (3) instead of to a_F we obtain

$$\frac{Z_t^1}{Z_t^0} \in [(1-\lambda)S_t, \infty[, \qquad t = 0, \ldots, T, \tag{10}$$

which yields (5).

Finally we obtain from (10), and the fact that $\lambda < 1$, that $(Z_t^1)_{t=0}^T$ also takes strictly positive values.

$(iii) \Rightarrow (ii)$ Defining the measure Q on $\mathcal{F}$ by

$$\frac{dQ}{d\mathbb{P}} = Z_T^0$$

we obtain a probability measure equivalent to $\mathbb{P}$.

Define the process $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ by

$$\tilde{S}_t = \frac{Z_t^1}{Z_t^0}.$$

By (5) the process $\tilde{S}$ takes its values in the bid-ask-spread of S. To verify that $\tilde{S}$ is a Q-martingale it suffices to note that this property is equivalent to $\tilde{S} Z^0$ being a $\mathbb{P}$-martingale. As $Z^1 = \tilde{S} Z^0$ this is indeed the case. ■

We denote by $\mathcal{B}^\lambda \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ the *polar* of $\mathcal{A}^\lambda$ (see Definition A.3 in the Appendix), i.e.

$$\mathcal{B}^\lambda := (\mathcal{A}^\lambda)^\circ = \{Z = (Z^0, Z^1) : \big\langle(\varphi^0, \varphi^1), (Z^0, Z^1)\big\rangle = \mathbb{E}_{\mathbb{P}}\left[\varphi^0 Z^0 + \varphi^1 Z^1\right] \leqslant 0, \\ \text{for all } (\varphi^0, \varphi^1) \in \mathcal{A}^\lambda\}.$$

As $\mathcal{A}^\lambda$ is a closed polyhedral cone in a finite-dimensional space, its polar $\mathcal{B}^\lambda$ is so too (Proposition A.3). As $\mathcal{A}^\lambda$ contains the negative orthant $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; -\mathbb{R}^2_+) = \{(\varphi^0_T, \varphi^1_T) : \varphi^0_T \leqslant 0, \varphi^1_T \leqslant 0\}$, we have that $\mathcal{B}^\lambda$ is contained in the positive orthant $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2_+)$.

Corollary 1.10. *Suppose that S satisfies (NA^λ) under transaction costs $0 \leqslant \lambda < 1$. Let $Z = (Z^0, Z^1) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ and associate to Z the martingale $Z_t = \mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t]$, where $t = 0, \ldots, T$.*

Then $Z \in \mathcal{B}^\lambda$ iff $Z \geqslant 0$ and $\tilde{S}_t := \frac{Z^1_t}{Z^0_t} \in [(1-\lambda)S_t, S_t]$ on $\{Z^0_t \neq 0\}$ and $Z^1_t = 0$ on $\{Z^0_t = 0\}$, for every $t = 0, \ldots, T$.

Dually, an element $(\varphi^0, \varphi^1) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ is in $\mathcal{A}^\lambda$ iff for every consistent price system $(\tilde{S}, Q)$ we have

$$\mathbb{E}_Q[\varphi^0 + \varphi^1 \tilde{S}_T] \leqslant 0. \tag{11}$$

Proof. If $Z = (Z^0, Z^1)$ is in $\mathcal{B}^\lambda$ we have $Z \geqslant 0$ by the paragraph preceding the corollary. Repeating the argument in the proof of the Fundamental Theorem 1.8, conditionally on $\{Z^0_t \neq 0\}$, we obtain that $\tilde{S}_t := \frac{Z^1_t}{Z^0_t}$ indeed takes values in the bid-ask interval $[(1-\lambda)S_t, S_t]$ on $\{Z^0_t \neq 0\}$ for $t = 0, \ldots, T$.

As regards the set $\{Z^0_t = 0\}$ fix an atom $F_t \in \mathcal{F}_t$, with $F_t \subseteq \{Z^0_t = 0\}$. Observe again that $(-S_{t|F_t}, 1)\mathbb{1}_{F_t} \in \mathcal{A}^\lambda$ as in the preceding proof. As $Z \in (\mathcal{A}^\lambda)^\circ$ we get

$$\langle(-S_t, 1)\mathbb{1}_{F_t}, (0, Z^1_t)\rangle \leqslant 0,$$

which readily implies that Z^1_t also vanishes on F_t.

Conversely, if $Z = (Z^0, Z^1)$ satisfies $Z \geqslant 0$ and $\frac{Z^1_t}{Z^0_t} \in [(1-\lambda)S_t, S_t]$ (with the above caveat for the case $Z^0_t = 0$), we have that

$$\langle \mathbb{1}_{F_t}(-S_t, 1), (Z^0, Z^1)\rangle \leqslant 0,$$

and

$$\langle \mathbb{1}_{F_t}\left((1-\lambda)S_t, -1\right), (Z^0, Z^1)\rangle \leqslant 0,$$

for every atom $F_t \in \mathcal{F}_t$. As the elements on the left hand side generate the cone $\mathcal{A}^\lambda$ we conclude that $Z \in \mathcal{B}^\lambda$.

Let us now pass to the dual point of view. By the bipolar theorem (see Proposition A.1 in the appendix) and the fact that $\mathcal{A}^\lambda$ is closed and convex in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, we have $(\mathcal{A}^\lambda)^{\circ\circ} = (\mathcal{B}^\lambda)^\circ = \mathcal{A}^\lambda$. Hence $(\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda = (\mathcal{A}^\lambda)^{\circ\circ}$ iff for every $Z = (Z^0, Z^1) \in \mathcal{B}^\lambda$ we have

$$\mathbb{E}_{\mathbb{P}}[Z^0\varphi_T^0 + Z^1\varphi_T^1] \leqslant 0. \tag{12}$$

This is equivalent to (11) if we have that Z^0 is strictly positive as in this case $\frac{dQ}{d\mathbb{P}} := Z^0/\mathbb{E}_{\mathbb{P}}[Z^0]$ and $\tilde{S}_t = E_{\mathbb{P}}[Z^1|\mathcal{F}_t]/\mathbb{E}_{\mathbb{P}}[Z^0|\mathcal{F}_t]$ well-defines a consistent price system.

In the case when Z^0 may also assume the value zero, a little extra care is needed to deduce (11) from (12). By assumption (NA^λ) and the Fundamental Theorem 1.8 we know that there is $\bar{Z} = (\bar{Z}^0, \bar{Z}^1) \in \mathcal{B}^\lambda$ such that $\bar{Z}^0$ and $\bar{Z}^1$ are strictly positive. Given an arbitrary $Z = (Z^0, Z^1) \in \mathcal{B}^\lambda$ and $\mu \in]0, 1]$ we have that the convex combination $\mu\bar{Z} + (1-\mu)Z$ is in $\mathcal{B}^\lambda$ and strictly positive. Hence we may deduce the validity of (12) from (11) for $\mu\bar{Z} + (1-\mu)Z$. Sending μ to zero we conclude that $(\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda$ iff (11) is satisfied, for all consistent price systems $(\tilde{S}, Q)$. ∎

Corollary 1.11. *(Superreplication Theorem): Fix the process* $S = (S_t)_{t=0}^T$, *transaction costs* $0 \leqslant \lambda < 1$, *and suppose that* (NA^λ) *is satisfied. Let* $(\varphi^0, \varphi^1) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ *and* $(x^0, x^1) \in \mathbb{R}^2$ *be given. The following are equivalent.*

(*i*) $(\varphi^0, \varphi^1) = (\varphi_T^0, \varphi_T^1)$ *for some self-financing trading strategy* $(\varphi_t^0, \varphi_t^1)_{t=0}^T$ *starting at* $(\varphi_{-1}^0, \varphi_{-1}^1) = (x^0, x^1)$.

(*ii*) $\mathbb{E}_Q[\varphi^0 + \varphi^1\tilde{S}_T] \leqslant x^0 + x^1\tilde{S}_0$, *for every consistent price system* $(\tilde{S}, Q) \in \mathcal{S}^\lambda$.

Proof. Condition (*i*) is equivalent to $(\varphi^0 - x^0, \varphi^1 - x^1)$ being in $\mathcal{A}^\lambda$. By Corollary 1.10 this is equivalent to the inequality

$$\mathbb{E}_Q\left[(\varphi^0 - x^0) + (\varphi^1 - x^1)\tilde{S}_T\right] \leqslant 0,$$

for every $(\tilde{S}, Q) \in \mathcal{S}^\lambda$ which in turn is tantamount to (*ii*). ∎

We now specialize the above result, considering only the case of trading strategies $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at some $(\varphi_{-1}^0, \varphi_{-1}^1) = (x, 0)$, i.e. without initial holdings in stock. Similarly we require that at terminal time T the position in stock is liquidated, i.e., $(\varphi_T^0, \varphi_T^1)$ satisfies $\varphi_T^1 = 0$.

We call $\mathcal{C}^\lambda$ the cone of claims (in units of bonds), attainable from initial endowment $(0, 0)$:

$$\begin{aligned} \mathcal{C}^\lambda &= \{\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \exists (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda \text{ s.t. } \varphi_T^0 \geqslant \varphi^0, \varphi_T^1 \geqslant 0\} \\ &= \mathcal{A}^\lambda \cap L_0^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2). \end{aligned} \tag{13}$$

In the last line we denote by $L^\infty_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ the subspace of $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ formed by the elements (φ^0, φ^1) with $\varphi^1 = 0$. We may and shall identify $L^\infty_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ with $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

The present notation $\mathcal{C}^\lambda$ corresponds, for $\lambda = 0$, to the notation of [69], where the cone of contingent claims attainable at prize 0 (without transaction costs) is denoted by C.

By (13) and Proposition 1.4 we conclude that $\mathcal{C}^\lambda$ is a closed polyhedral cone. Using analogous notation as in [161], we denote by $\mathcal{D}^\lambda$ the polar of $\mathcal{C}^\lambda$. By elementary linear algebra we obtain from (13) the representation

$$\mathcal{D}^\lambda = \{Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : \text{ there is } Z = (Z^0, Z^1) \in \mathcal{B}^\lambda \text{ with } Y = Z^0\}, \quad (14)$$

which is a polyhedral cone in $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{M}^\lambda$ the probability measures in $\mathcal{D}^\lambda$, i.e.

$$\mathcal{M}^\lambda = \mathcal{D}^\lambda \cap \{Y : \|Y\|_1 = \mathbb{E}_\mathbb{P}[Y] = 1\}.$$

The Superreplication Theorem 1.11 now specializes into a very familiar form, if we start with initial endowment $(x, 0)$ consisting only of bonds, and liquidate all positions in stock at terminal date T.

Corollary 1.12. *(one-dimensional Superreplication Theorem): Fix the process $S = (S_t)_{t=0}^T$, transaction costs $0 \leqslant \lambda < 1$, and suppose that (NA^λ) is satisfied. Let $\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $x \in \mathbb{R}$ be given. The following are equivalent.*

(i) $\varphi^0 - x$ *is in* $\mathcal{C}^\lambda$.

(ii) $\mathbb{E}_Q[\varphi^0] \leqslant x$, *for every* $Q \in \mathcal{M}^\lambda$.

Proof. Condition (i) is equivalent to $(\varphi^0 - x, 0)$ being in $\mathcal{A}^\lambda$. This in turn is equivalent to $\mathbb{E}_Q[\varphi^0 - x] \leqslant 0$, for every $Q \in \mathcal{M}^\lambda$, which is the same as (ii). ■

Formally, the above corollary is in perfect analogy to the superreplication theorem in the frictionless case (see, e.g., ([69], Th. 2.4.2)). The reader may wonder whether – in the context of this corollary – there is any difference at all between the frictionless and the transaction cost case.

In fact, there is a subtle difference: in the frictionless case the set $\mathcal{M} = \mathcal{M}^0$ of martingale probability measures Q for the process S has the following remarkable *concatenation property*: let $Q', Q'' \in \mathcal{M}$ and associate the density process $Y'_t = \mathbb{E}[\frac{dQ'}{d\mathbb{P}}|\mathcal{F}_t]$, and $Y''_t = \mathbb{E}[\frac{dQ''}{d\mathbb{P}}|\mathcal{F}_t]$. For a stopping time τ we define the concatenated process

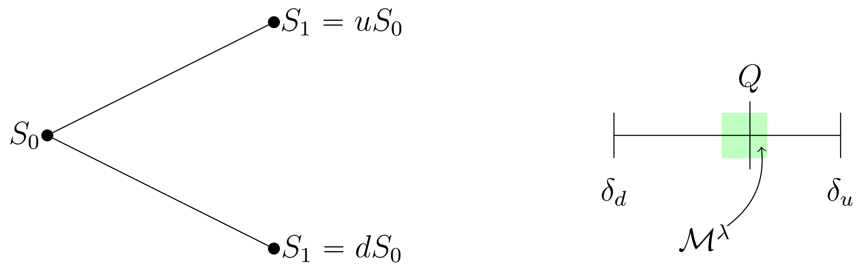
$$Y_t = \begin{cases} Y'_t, & \text{for } 0 \leqslant t \leqslant \tau, \\ Y'_\tau \frac{Y''_t}{Y''_\tau} & \text{for } \tau \leqslant t \leqslant T. \end{cases} \quad (15)$$

We then have that $\frac{dQ}{d\mathbb{P}} = Y_T$ again defines a probability measure under which S is a martingale, as one easily checks. This concatenation property turns out to be crucial for several aspects of the frictionless theory.

For $\lambda > 0$ the sets $\mathcal{M}^\lambda$ do not have this property any more. But apart from this drawback the sets $\mathcal{M}^\lambda$ share the properties of $\mathcal{M}$ of being a closed polyhedral subset of the simplex of probability measures on $(\Omega, \mathcal{F})$. Hence all the results pertaining only to the latter aspect, e.g. much of the duality theory, carry over from the frictionless to the transaction cost case, at least in the present setting of finite Ω. This applies in particular to the theory of utility maximization treated in the next chapter.

We end this chapter by illustrating the set $\mathcal{M}^\lambda$ for two very elementary examples.

Example 1.13. (One Period Binomial Model; for notation see, e.g., [69][Ex.3.3.1]): The process S can only move from S_0 to uS_0 or dS_0 where $0 < d < 1 < u$. In the traditional case, without transaction costs, there is a unique equivalent martingale measure $Q = (q_u, q_d)$ determined by the two equations



$$\mathbb{E}_{\mathbb{Q}}[S_1] = S_0 = uS_0q_u + dS_0q_d = uS_0q_u + dS_0(1 - q_u) \tag{16}$$
$$1 = uq_u + d(1 - q_u), \tag{17}$$

which gives the well-known formulas for the riskless probability $Q = (q_u, q_d)$.

$$q_u = \tfrac{1-d}{u-d} \quad \text{and} \quad q_d = 1 - q_u = \tfrac{u-1}{u-d}. \tag{18}$$

Introducing proportional transaction costs, we are looking for a consistent price system $(\tilde{S}, Q)$, where $\tilde{S}$ is a Q-martingale and

$$(1 - \lambda)S_t \leqslant \tilde{S}_t \leqslant S_t, \qquad t \in \{0, 1\}. \tag{19}$$

We therefore have:

$$\mathbb{E}_Q[\tilde{S}_1] = \underbrace{\tilde{S}_0}_{\geqslant(1-\lambda)S_0} = q_u\tilde{S}_1(u) + q_d\tilde{S}_1(d) \leqslant q_u uS_0 + q_d dS_0, \tag{20}$$

and therefore $q_u u + q_d d \geqslant (1-\lambda)$. Using analogue inequalities in the other direction and the fact that $q_u = 1 - q_d$ we obtain by elementary calculations lower and upper bounds for q_u:

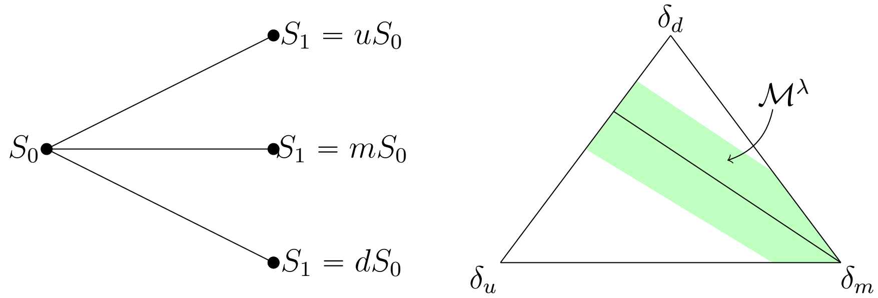
$$\max\left(\frac{\frac{1-\lambda}{1} - d}{u-d}, 0\right) \leqslant q_u \leqslant \min\left(\frac{\frac{1}{1-\lambda} - d}{u-d}, 1\right). \tag{21}$$

For $\lambda \mapsto 0$, this interval shrinks to the point $q_u = \frac{1-d}{u-d}$ which is the unique frictionless probability (18).

On the other hand, for λ sufficiently close to 1, this interval equals $[0,1]$, i.e. $\mathcal{M}^\lambda$ consists of all convex, combinations of the Dirac measures δ_d and δ_u. In an intermediate range of λ, the set $\mathcal{M}^\lambda$ is an interval containing the measure $Q = q_u\delta_u + q_d\delta_d$ in its interior (see the above sketch).

Example 1.14. (One period trinomial model):

In this example (compare [69, Ex.3.3.4]) we consider *three* possible values for S_1: apart from the possibilities $S_1 = uS_0$ and $S_1 = dS_0$, where again $0 < d < 1 < u$, we also allow for an intermediate case $S_1 = mS_0$. For notational simplicity we let $m = 1$.



In the frictionless case we have, similar to the binomial model, for any martingale measure Q, that

$$\mathbb{E}_Q[S_1] = S_0 = uS_0q_u + S_0q_m + dS_0q_d \tag{22}$$
$$1 = uq_u + dq_d + (1 - q_u - q_d), \tag{23}$$

which yields one degree of freedom among all probabilities (q_u, q_m, q_d), for the cases of an *up, medium* or *down* movement of S_0. The corresponding set $\mathcal{M}$ of martingale measures for S in the set of convex combinations of the Dirac measures $\{\delta_u, \delta_m, \delta_q\}$ therefore is determined by the triples (q_u, q_m, q_d)

of non-negative numbers, where $0 \leqslant q_m \leqslant 1$ is arbitrary and where q_u and q_d are determined via

$$q_u + q_d = 1 - q_m, \qquad (u-1)q_u + (d-1)q_d = 0. \tag{24}$$

This corresponds to the line through δ_m in the above sketch.

We next introduce transaction costs $0 \leqslant \lambda \leqslant 1$, and look for the set of consistent probability measures. In analogy to (20) we obtain the inequalities:

$$\mathbb{E}_{\mathbb{Q}}[\tilde{S}_1] = \overbrace{\tilde{S}_0}^{\geqslant (1-\lambda)S_0} = q_u \tilde{S}_1(u) + q_m \tilde{S}_1(m) + q_d \tilde{S}_1(d) \tag{25}$$

$$\leqslant [q_u u S_0 + q_m S_0 + q_d d S_0]\,. \tag{26}$$

Together with the inequality in the direction this gives us again a lower and upper bound:

$$-\lambda \leqslant q_u(u-1) + q_d(d-1) \leqslant \frac{\lambda}{1-\lambda}. \tag{27}$$

Hence $\mathcal{M}^\lambda$ is given by the shaded area in the above sketch which is confined by two lines, parallel to the line given by (24).

2 Utility Maximization under Transaction Costs: the Case of Finite Ω

In this chapter we again adopt the simple setting of a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ as in chapter 1. In addition to the ingredients of the previous chapter, i.e. the stock price process $S = (S_t)_{t=0}^T$ and the level of transaction costs $0 \leqslant \lambda < 1$, we also fix a utility function

$$U : D \to \mathbb{R}. \tag{28}$$

The domain D of U will be either $D =]0, \infty[$ or $D =]-\infty, \infty[$, and U is supposed to be a concave, $\mathbb{R}$-valued (hence continuous), increasing function on D. We also assume that U is strictly concave and differentiable on the interior of D. This assumption is not very essential but avoids to speak about subgradients instead of derivatives and allows for the uniqueness of solutions. More importantly, we assume that U satisfies the Inada conditions

$$\lim_{x \searrow x_0} U'(x) = \infty, \qquad \lim_{x \nearrow \infty} U'(x) = 0, \tag{29}$$

where $x_0 \in \{-\infty, 0\}$ denotes the left boundary of D.

Remark 2.1. Some widely studied examples for utility functions include:

- $U(x) = \log(x)$,
- $U(x) = \frac{x^{1/2}}{1/2}$ or, more generally, $U(x) = \frac{x^\gamma}{\gamma}$, for $\gamma \in]0,1[$,
- $U(x) = \frac{x^{-1}}{-1}$ or, more generally, $U(x) = \frac{x^\gamma}{\gamma}$, for $\gamma \in]-\infty, 0[$,
- $U(x) = -\exp(-x)$, or, more generally, $U(x) = -\exp(-\mu x)$, for $\mu > 0$.

The first three examples pertain to the domain $D =]0, \infty[$, while the fourth pertains to $D =]-\infty, \infty[$.

We also fix an initial endowment $x \in D$, denoted in units of bond. The aim is to find a trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ maximizing expected utility of terminal wealth (measured in units of bond). More formally, we consider the optimization problem

$$(P_x) \qquad \begin{aligned} &\mathbb{E}[U(x + \varphi_T^0)] \to \max! \\ &\varphi_T^0 \in \mathcal{C}^\lambda \end{aligned} \tag{30}$$

In (P_x) the random variables φ_T^0 run through the elements of C^λ, i.e. such that there is a self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$, starting at $\varphi_{-1}^0, \varphi_{-1}^1 = (0, 0)$.

The interpretation is that an agent, whose preferences are modeled by the utility function U, starts with x units of bond (and no holdings in stock). She then trades at times $t = 0, \ldots, T-1$, and at terminal date T she liquidates her position in stock so that $\varphi_T^1 = 0$ (this equality constraint clearly is equivalent to the inequality constraint $\varphi_T^1 \geqslant 0$ when solving the problem (P_x)). She then evaluates the performance of her trading strategy in terms of the expected utility of her final holdings φ_T^0 in bond.

Of course, we could formulate the utility maximization problem in greater generality. For example, we could consider initial endowments (x, y) in bonds as well as in stocks, instead of restricting to the case $y = 0$. We also could replace the requirement $\varphi_T^1 \geqslant 0$ by introducing a utility function $\mathcal{U}(x, y)$ defined on an appropriate domain $D \subseteq \mathbb{R}^2$ and consider

$$(P_{x,y}) \qquad \mathbb{E}[\mathcal{U}(\varphi_T^0, \varphi_T^1)] \to \max!$$

where $(\varphi_T^0, \varphi_T^1)$ runs through all terminal values of trading strategies $(\varphi_t^0, \varphi_t^1)_{t=-1}^T$ starting at $(\varphi_{-1}^0, \varphi_{-1}^1) = (x, y)$.

Note that (28) corresponds to the two-dimensional utility function

$$\mathcal{U}(x, y) = \begin{cases} U(x), & \text{if } y \geqslant 0, \\ -\infty, & \text{if } y < 0. \end{cases} \tag{31}$$

We refer to [60] and [135] for a thorough treatment of such a more general framework. For the present purposes we prefer, however, to remain in the realm of problem (30) as this allows for easier and crisper formulations of the results.

Using (28) and Corollary 1.12, we can reformulate (P_x) as a concave maximization problem under linear constraints:

$$(P_x) \qquad \mathbb{E}[U(x+\varphi_T^0)] \to \max! \qquad \varphi_T^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \tag{32}$$

$$\mathbb{E}_Q[\varphi_T^0] \leqslant 0, \qquad Q \in \mathcal{M}^\lambda. \tag{33}$$

As $\mathcal{M}^\lambda$ is a compact polyhedron we can replace the infinitely many constraints (33) by finitely many: it is sufficient that (33) holds true for the extreme points $(Q^1, \dots, Q^M)$ of $\mathcal{M}^\lambda$.

We now are precisely in the well-known situation of utility optimization as in the frictionless case, which in the present setting reduces to a concave optimization problem on the finite-dimensional vector space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ under linear constraints. Proceeding as in ([69, section 3.2]) we obtain the following basic duality result, where V denotes the conjugate function (the Legendre transform up to the choice of signs) of U

$$V(y) = \sup_{x \in D}\{U(x) - xy\}, \qquad y > 0. \tag{34}$$

Theorem 2.2. *(compare [69, Th. 3.2.1]): Fix* $0 \leqslant \lambda < 1$ *and suppose that in the above setting the* (NA^λ) *condition is satisfied for some fixed* $0 \leqslant \lambda < 1$.

Denote by u *and* v *the value functions*

$$\begin{aligned} u(x) &= \sup\left\{\mathbb{E}[U(x+\varphi_T^0)] : (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda, \varphi_T^1 \geqslant 0\right\} \qquad (35)\\ &= \sup\{\mathbb{E}[U(x+\varphi_T^0)] : \varphi_T^0 \in \mathcal{C}^\lambda\}, \qquad x \in D. \end{aligned}$$

$$\begin{aligned} v(y) &= \inf\{\mathbb{E}[V(y\tfrac{dQ}{d\mathbb{P}})] : Q \in \mathcal{M}^\lambda\} \qquad (36)\\ &= \inf\{\mathbb{E}[V(Z_T^0)] : Z_T = (Z_T^0, Z_T^1) \in \mathcal{B}^\lambda, \mathbb{E}[Z_T^0] = y\}, \quad y > 0. \end{aligned}$$

Then the following statements hold true:

(i) The value functions $u(x)$ *and* $v(y)$ *are mutually conjugate, and the indirect utility function* $u : D \to \mathbb{R}$ *is smooth, concave, increasing, and satisfies the Inada conditions* (29).

(ii) For $x \in D$ and $y > 0$ such that $u'(x) = y$, the optimizers $\hat{\varphi}^0_T = \hat{\varphi}^0_T(x) \in \mathcal{C}^\lambda$ and $\hat{Q} = \hat{Q}(y) \in \mathcal{M}^\lambda$ in (35) and (36) exist, are unique, and satisfy

$$x + \hat{\varphi}^0_T = I\left(y \tfrac{d\hat{Q}}{d\mathbb{P}}\right), \quad y\tfrac{d\hat{Q}}{d\mathbb{P}} = U'\left(x + \hat{\varphi}^0_T\right), \tag{37}$$

where $I = (U')^{-1} = -V'$ denotes the "inverse" function. The measure $\hat{Q}$ is equivalent to $\mathbb{P}$, i.e. $\hat{Q}$ assigns a strictly positive mass to each $\omega \in \Omega$.

(iii) The following formulae for u' and v' hold true

$$u'(x) = \mathbb{E}_{\mathbb{P}}\left[U'\left(x + \hat{\varphi}^0_T(x)\right)\right], \qquad v'(y) = \mathbb{E}_{\hat{Q}(y)}\left[V'\left(y\tfrac{d\hat{Q}(y)}{d\mathbb{P}}\right)\right], \tag{38}$$

$$x\ u'(x) = \mathbb{E}_{\mathbb{P}}\left[\left(x + \hat{\varphi}^0_T(x)\right)U'\left(x + \hat{\varphi}^0_T(x)\right)\right], \quad y\ v'(y) = \mathbb{E}_{\mathbb{P}}\left[y\tfrac{d\hat{Q}(y)}{d\mathbb{P}}V'\left(y\tfrac{d\hat{Q}(y)}{d\mathbb{P}}\right)\right]. \tag{39}$$

Proof. We follow the reasoning of [69, section 3.2]. Denote by $\{\omega_1, \ldots, \omega_N\}$ the elements of Ω. We may identify a function $\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with the vector $(\xi_n)_{n=1}^N = (\varphi^0(\omega_n))_{n=1}^N \in \mathbb{R}^N$.

Denote by $Q^1, \ldots, Q^M$ the extremal points of the compact polyhedron $\mathcal{M}^\lambda$ and, for $1 \leqslant m \leqslant M$, by $(q^m_n)_{n=1}^N = (Q^m[\omega_n])_{n=1}^N$ the weights of Q^m. We may write the Lagrangian for the problem (32) as

$$\begin{aligned} L(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_M) &= \sum_{n=1}^N p_n U(\xi_n) - \sum_{m=1}^M \eta_m \left(\sum_{n=1}^N q^m_n \xi_n - x\right) \\ &= \sum_{n=1}^N p_n \left(U(\xi_n) - \sum_{m=1}^M \frac{\eta_m q^m_n}{p_n}\xi_n\right) + x\sum_{m=1}^M \eta_m. \end{aligned}$$

Here x is the initial endowment in bonds, which will be fixed in the sequel. The variables ξ_n vary in $\mathbb{R}$, the variables η_m in $\mathbb{R}_+$. Our aim is to find the (hopefully uniquely existing) saddle point $(\hat{\xi}_1, \ldots, \hat{\xi}_N, \hat{\eta}_1, \ldots, \hat{\eta}_M)$ of L which will give the primal optimizer via $x + \hat{\varphi}^0_T(\omega_n) := \hat{\xi}_n$, as well as the dual optimizer via $y\hat{Q} = \sum_{m=1}^M \hat{\eta}_m Q^m$, where $y = \sum_{m=1}^M \hat{\eta}_m$ so that $\hat{Q} \in \mathcal{M}^\lambda$.

In order to do so we shall consider $\max_\xi \min_\eta L(\xi, \eta)$ as well as $\min_\eta \max_\xi L(\xi, \eta)$. Define

$$\begin{aligned} \Phi(\xi_1, \ldots, \xi_N) &= \inf_{\eta_1, \ldots, \eta_M} L(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_M) \\ &= \inf_{y>0, Q\in\mathcal{M}^\lambda} \left\{\sum_{n=1}^N p_n \left(U(\xi_n) - \frac{yq_n}{p_n}\xi_n\right) + yx\right\} \end{aligned}$$

Again the relation between $(\eta_1, \dots, \eta_M)$ and $y > 0$ and $Q \in \mathcal{M}^\lambda$ is given via $y = \sum_{m=1}^M \eta_m$ and $Q = \sum_{m=1}^M \frac{\eta_m}{y} Q^m$, where we denote by q_n the weights $q_n = Q[\omega_n]$.

Note that $\Phi(\xi_1, \dots, \xi_N)$ equals the target functional (30) if $(\xi_1, \dots, \xi_N)$ is admissible, i.e. satisfies (33), and -∞ otherwise. Identifying the elements $\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $(\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, this may be written as

$$\Phi(\varphi^0) = \begin{cases} \mathbb{E}[U(\varphi^0)], \text{ if } \mathbb{E}_Q[\varphi^0] \leqslant x \text{ for all } Q \in \mathcal{M}^\lambda \\ -\infty, \text{ otherwise.} \end{cases} \tag{40}$$

Let us now pass from the max min to the min max: identifying $(\eta_1, \dots, \eta_M)$ with (y, Q) as above, define

$$\begin{aligned} \Psi(y, Q) &= \sup_{\xi_1, \dots, \xi_N} L(\xi_1, \dots, \xi_N, y, Q) \\ &= \sup_{\xi_1, \dots, \xi_N} \sum_{n=1}^N p_n \left(U(\xi_n) - y \tfrac{q_n}{p_n} \xi_n \right) + xy \\ &= \sum_{n=1}^N p_n \sup_{\xi_n} \left(U(\xi_n) - y \tfrac{q_n}{p_n} \xi_n \right) + xy \\ &= \sum_{n=1}^N p_n V \left(y \tfrac{q_n}{p_n} \right) + xy \\ &= \mathbb{E}_{\mathbb{P}} \left[V \left(y \tfrac{dQ}{d\mathbb{P}} \right) \right] + xy. \end{aligned}$$

We have used above the definition (30) of the conjugate function V of U.

Defining

$$\Psi(y) = \inf_{Q \in \mathcal{M}^\lambda} \Psi(y, Q) \tag{41}$$

we infer from the compactness of $\mathcal{M}^\lambda$ that, for $y > 0$, there is a minimizer $\hat{Q}(y)$ in (41). From the strict convexity of V (which corresponds to the differentiability of U as we recall in the appendix) we infer, as in [69], section 3.2, that $\hat{Q}(y)$ is unique and $\hat{Q}(y)[\omega] > 0$, for each $\omega \in \Omega$.

Finally, we minimize $y \mapsto \Psi(y)$ to obtain the optimizer $\hat{y} = \hat{y}(x)$ by solving

$$\Psi'(\hat{y}) = 0. \tag{42}$$

Denoting by $v(y)$ the dual value function which is obtained from $\Psi(y)$ by dropping the term xy, i.e.

$$v(y) = \inf_{Q \in \mathcal{M}^\lambda} \mathbb{E} \left[V(y \tfrac{dQ}{d\mathbb{P}}) \right],$$

we obtain from (42) the relation

$$v'(\hat{y}(x)) = -x.$$

The uniqueness of $\hat{y}(x)$ follows from the strict convexity of v which, in turn, is a consequence of the strict convexity of V (see Proposition B.4 of the appendix).

Turning back to the Lagrangian $L(\xi_1, \ldots, \xi_N, y, Q)$, the first order conditions

$$\tfrac{\partial}{\partial \xi_n} L(\xi_1, \ldots, \xi_N, y, Q)|_{\hat{\xi}_1, \ldots, \hat{\xi}_N, \hat{y}, \hat{Q}} = 0 \tag{43}$$

for a saddle point yield the following equations for the primal optimizers $\hat{\xi}_1, \ldots, \hat{\xi}_N$

$$U'(\hat{\xi}_n) = \hat{y} \tfrac{\hat{q}_n}{p_n}, \tag{44}$$

where $\hat{y} = \hat{y}(x)$ and $\hat{Q} = \hat{Q}(\hat{y}(x))$. By the Inada conditions (29), as well as the smoothness and strict concavity of U, equation (44) admits unique solutions $(\hat{\xi}_1, \ldots, \hat{\xi}_N) = (\hat{\xi}_1(x), \ldots, \hat{\xi}_N(x))$.

Summing up, we have found a unique saddle point $(\hat{\xi}_1, \ldots, \hat{\xi}_N, \hat{y}, \hat{Q})$ of the Lagrangian L. Denoting by $\hat{L} = \hat{L}(x)$ the value

$$\hat{L} = L(\hat{\xi}_1, \ldots, \hat{\xi}_N, \hat{y}, \hat{Q})$$

we infer from the concavity of L in $\xi_1, \ldots, \xi_N$ and convexity in y and Q that

$$\max_{\xi} \min_{y,Q} L = \min_{y,Q} \max_{\xi} L = \hat{L}. \tag{45}$$

It follows from (40) that $\hat{L}$ is the optimal value of the primal problem (P_x) in (30), i.e.

$$u(x) = \sum_{n=1}^{N} p_n U(\hat{\xi}_n) = \hat{L}. \tag{46}$$

The second equality in (45) yields

$$\hat{L} = \Psi(\hat{y}) = v(\hat{y}) + x\hat{y}. \tag{47}$$

Equations (46) and (47), together with the concavity (resp. convexity) of u (resp. (v)) and $v'(\hat{y}) = -x$ are tantamount to the fact that the functions u and v are conjugate.

We thus have shown (i) of Theorem 2.2. The listed qualitative properties of u are straightforward to verify (compare [69], section 3.2). Item (ii) now follows from the above obtained existence and uniqueness of the saddle point $(\hat{\xi}_1, \ldots, \hat{\xi}_N, \hat{y}, \hat{Q})$ and (iii) again is straightforward to check as in [69]. ■

We remark that in the above proof we did not apply an abstract minimax theorem guaranteeing the existence of a saddle point of the Lagrangian. Rather we directly found the saddle point by using the first order conditions, very much as we did in high school: differentiate and set the derivative to zero! The assumptions of the theorem are designed in such a way to make sure that this method yields a unique solution.

We now adapt the idea of *market completion* as developed in [151] to the present setting. Fix the initial endowment $x \in D$, and $y = u'(x)$. Define a frictionless financial market, denoted by AS, in the following way. For each fixed $\omega \in \Omega$, the *Arrow security* AS^ω, paying $AS^\omega_T = \mathbb{1}_\omega$ units of bond at time $t = T$, is traded (without transaction costs) at time $t = 0$ at price $AS^\omega_0 := \hat{Q}(y)[\omega]$. In other words, AS^ω pays one unit of bond at time T if ω turns out at time T to be the true state of the world, and zero otherwise. We define, for each $\omega \in \Omega$, the price *process* of AS^ω as the $\hat{Q}(y)$-martingale

$$AS^\omega_t = \mathbb{E}_{\hat{Q}(y)}[\mathbb{1}_\omega|\mathcal{F}_t], \qquad t = 0, \ldots, T.$$

The set $\mathcal{C}^A$, where A stands for *K. Arrow*, of claims attainable at price zero in this complete, frictionless market equals the half-space of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathcal{C}^A = H_{\hat{Q}(y)} = \{\varphi^0_T \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \mathbb{E}_{\hat{Q}(y)}[\varphi^0_T] \leqslant 0\}. \tag{48}$$

Indeed, every $\varphi^0_T \in H_{\hat{Q}(y)}$ may trivially be written as a linear combination of Arrow securities

$$\begin{aligned}\varphi^0_T &= \sum_{\omega\in\Omega} \varphi^0_T(\omega)\mathbb{1}_{\{\omega\}} \\ &= \sum_{\omega\in\Omega} \varphi^0_T(\omega) AS^\omega_T(\omega)\end{aligned}$$

which may be purchased at time $t = 0$ at price

$$\sum_{\omega\in\Omega} \varphi^0_T(\omega) AS^\omega_0(\omega) = \mathbb{E}_{\hat{Q}(y)}[\varphi^0_T] \leqslant 0.$$

The Arrow securities AS^ω are quite different from the original process $S = (S_t)^T_{t=0}$ or, more precisely, the process of bid-ask intervals $([(1-\lambda)S_t, S_t])^T_{t=0}$. But we know from the fact that $\hat{Q}(y) \in \mathcal{M}^\lambda$ that

$$\mathcal{C}^\lambda \subseteq \mathcal{C}^A = H_{\hat{Q}(y)}. \tag{49}$$

In prose: the contingent claims φ^0_T attainable at price 0 in the market S under transaction costs λ are a subset of the contingent claims φ^0_T attainable at price zero in the frictionless Arrow market AS.

The message of the next theorem is the following: although the complete, frictionless market AS offers better terms of trade than S (under transaction costs λ), the economic agent applying the utility function U (28) will choose as her terminal wealth the same optimizer $\hat{\varphi}^0_T \in \mathcal{C}^\lambda$, although she can choose in the bigger set $\mathcal{C}^A$.

Theorem 2.3. *Fix $S = (S_t)_{t=0}^T$, transaction costs $0 \leqslant \lambda < 1$ such that (NA^λ) is satisfied, as well as $U : D \to \mathbb{R}$ verifying* (29) *and $x \in D$. Using the notation of Theorem 2.2, let $y = u'(x)$ and denote by $\hat{Q}(y) \in \mathcal{M}^\lambda$ the dual optimizer in* (36).

Define the optimization problem

$$(P_x^A) \qquad \begin{aligned} &\mathbb{E}[U(x + \varphi^0_T)] \to \max! \\ &\mathbb{E}_{\hat{Q}(y)}[\varphi^0_T] \leqslant 0, \end{aligned} \tag{50}$$

where φ^0_T ranges through all D-valued, $\mathcal{F}_T$-measurable functions.

The optimizer $\hat{\varphi}^0_T(x)$ of the above problem exists, is unique, and coincides with the optimizer of problem (P_x) defined in (30) *and given by* (37).

Proof. As $\hat{Q}(y) \in \mathcal{M}^\lambda$ we have that $\hat{Q}(y)|_{\mathcal{C}^\lambda} \leqslant 0$ so that $\hat{Q}(y)|_{x+\mathcal{C}^\lambda} \leqslant x$. It follows from (50) that in (P_x^A) we optimize over a larger set than in (P_x).

Denote by $\hat{\varphi}^0_T = \hat{\varphi}^0_T(x)$ the optimizer of (P_x) which uniquely exists by Theorem 2.2. Denote by $\hat{y} = \hat{y}(x)$ the corresponding Lagrange multiplier $\hat{y} = u'(x)$. We shall now show that $\hat{Q}(\hat{y})$ induces the *marginal utility pricing functional.*

Fix $1 \leqslant k \leqslant N$ and consider the variation functional corresponding to ω_k

$$v_k(h) = \mathbb{E}\left[U(\hat{\varphi}^0_T + h\mathbb{1}_{\omega_k})\right] = \sum_{\substack{n=1 \\ n\neq k}}^{N} p_n U(\hat{\xi}_n) + p_k U(\hat{\xi}_k + h), \qquad h \in \mathbb{R}.$$

The function v_k is strictly concave and its derivative at $h = 0$ satisfies by (44)

$$v_k'(0) = p_k \hat{y} \frac{\hat{q}_k}{p_k} = \hat{y}\hat{q}_k.$$

Let $\zeta \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \zeta \neq 0$ be such that $\mathbb{E}_{\hat{Q}}[\zeta] = 0$. The variation functional v_ζ

$$v_\zeta(h) = \mathbb{E}\left[U(\hat{\varphi}^0_T + h\zeta)\right] = \sum_{k=1}^{N} p_k U(\hat{\xi}_k + h\zeta_k), \qquad h \in \mathbb{R},$$

has as derivative

$$v'_\zeta(h) = \sum_{k=1}^{N} p_k U'(\hat{\xi}_k + h\zeta_k)\zeta_k.$$

Hence

$$\begin{aligned} v'_\zeta(0) &= \sum_{k=1}^{N} p_k \underbrace{U'(\hat{\xi}_k)}_{=\hat{y}\frac{\hat{q}_k}{p_k}} \zeta_k = \sum_{k=1}^{N} \hat{y}\hat{q}_k\zeta_k \\ &= \hat{y}\mathbb{E}_{\hat{Q}}[\zeta] \\ &= 0 \end{aligned}$$

The function $h \mapsto v_\zeta(h)$ is strictly concave and therefore attains its unique maximum at $h = 0$.

Hence, for every $\varphi^0_T \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \varphi^0_T \neq \hat{\varphi}^0_T$ such that $\mathbb{E}_{\hat{Q}}[\varphi^0_T] = x$ we have

$$\mathbb{E}[U(\varphi^0_T)] < \mathbb{E}[U(\hat{\varphi}^0_T)].$$

Indeed, it suffices to apply the previous argument to $\zeta = \varphi^0_T - \hat{\varphi}^0_T$. Finally, by the monotonicity of U, the same inequality holds true for all $\varphi^0_T \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}_{\hat{Q}}[\varphi^0_T] < x$.

The proof of Theorem 2.3 now is complete. ■

In the above formulation of Theorems 2.2 and 2.3 we have obtained the unique primal optimizer $\hat{\varphi}^0_T$ only in terms of the final holdings in bonds; similarly the unique dual optimizer $\hat{Q}$ is given in terms of a probability measure which corresponds to a one dimensional density $Z^0 = \frac{d\hat{Q}}{d\mathbb{P}}$. What are the "full" versions of these optimizers in terms of $(\varphi^0_T, \varphi^1_T) \in \mathcal{A}^\lambda$, i.e., in terms of bond and stock, resp. $(Z^0, Z^1) \in \mathcal{B}^\lambda$ which is an $\mathbb{R}^2_+$-valued martingale? As regards the former, we mentioned already that it is economically obvious (and easily checked mathematically) that the unique optimizer $(\varphi^0_T, \varphi^1_T) \in \mathcal{A}^\lambda$ corresponding to $\hat{\varphi}^0_T \in \mathcal{C}^\lambda$ in (35) simply is $(\varphi^0_T, \varphi^1_T) = (\hat{\varphi}^0_T, 0)$, i.e. the optimal holding in stock at terminal date T is zero. As regards the optimizer $(Z^0, Z^1) \in \mathcal{B}^\lambda$ in (36) corresponding to the optimizer $\hat{Q} \in \mathcal{M}^\lambda$ the situation is slightly more tricky. By the definition (14) of $\mathcal{D}^\lambda$, for given $\hat{Z}^0 \in \mathcal{D}^\lambda$ there is $\hat{Z}^1 \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ such that $(\hat{Z}^0, \hat{Z}^1) \in \mathcal{B}^\lambda$. But this $\hat{Z}^1$ need not be unique, even in very regular situations as shown by the subsequent easy example. Hence the "shadow price process" $(\tilde{S}_t)_{t=0}^T = \left(\frac{\hat{Z}^1_t}{\hat{Z}^0_t}\right)_{t=0}^T$ need not be unique. The terminology "shadow price" will be explained below, and will be formally defined in 2.7.

Example 2.4. In the above setting suppose that $(S_t)_{t=0}^T$ is a martingale under the measure $\mathbb{P}$. Then it is economically obvious (and easily checked) that it is optimal not to trade at all (even under transaction costs $\lambda = 0$). More formally, we obtain $u(x) = U(x), v(y) = V(y)$ and, for $x \in D$, the unique optimizers in Theorem 2.2 are given by $\hat{\varphi}_t \equiv (X, 0)$, and $\hat{Q} = \mathbb{P}$, as well as $\hat{y} = U'(x)$. For the optimal shadow price process $\tilde{S}$ we may take $\tilde{S} = S$. But this choice is not unique. In fact, we may take any $\mathbb{P}$-martingale $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ taking values in the bid-ask spread $([(1-\lambda)S_t, S_t])_{t=0}^T$.

In the setting of Theorem 2.2 let $(\hat{Z}_T^0, \hat{Z}_T^1)$ be an optimizer of (36) and denote by $\hat{\tilde{S}}$ the process

$$\hat{\tilde{S}}_t = \frac{\mathbb{E}[\hat{Z}_T^1|\mathcal{F}_t]}{\mathbb{E}[\hat{Z}_T^0|\mathcal{F}_t]}, \qquad t = 0, \ldots, T,$$

which is a martingale under $\hat{Q}(y)$. We shall now justify why we have called this process a *shadow price process* for S under transaction costs λ.

Fix $x \in D$ and $y = u'(x)$. To alleviate notation we write $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ for $\hat{\tilde{S}}(y)$ and Q for $\hat{Q}(y)$. Denote by $\mathcal{C}^{\tilde{S}}$ the cone of random variables φ_T^0 dominated by a contingent claim of the form $(H \cdot \tilde{S})_T$, i.e.

$$\mathcal{C}^{\tilde{S}} = \{\varphi_T^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) : \varphi_T^0 \leqslant (H \cdot \tilde{S})_T, \text{ for some } H \in \mathcal{P}\}.$$

Here we use standard notation from the frictionless theory. The letter $\mathcal{P}$ denotes the space of *predictable* $\mathbb{R}$-valued trading strategies $(H_t)_{t=1}^T$, i.e. H_t is $\mathcal{F}_{t-1}$-measurable, and $(H \cdot \tilde{S})_T$ denotes the stochastic integral

$$(H \cdot \tilde{S})_T = \sum_{t=1}^T H_t(\tilde{S}_t - \tilde{S}_{t-1}). \tag{51}$$

In prose: $\mathcal{C}^{\tilde{S}}$ denotes the cone of random variables φ_T^0 which can be super-replicated in the financial market $\tilde{S}$ without transaction costs and with zero initial endowment.

Lemma 2.5. *Using the above notation and assuming that S satisfies (NA^λ) we have*

$$\mathcal{C}^\lambda \subseteq \mathcal{C}^{\tilde{S}} \subseteq \mathcal{C}^A \tag{52}$$

Proof. The first inclusion was already shown in the proof of the Fundamental Theorem 1.8; it corresponds to the fact that trading without transaction costs on $\tilde{S}$ yields better terms of trade than trading on S under transaction costs λ.

As regards the second inclusion note that, for $(H \cdot \tilde{S})_T$ as in (51), we have

$$\mathbb{E}_Q[(H \cdot \tilde{S})_T] = 0,$$

whence $(H \cdot \tilde{S})_T$ belongs to $\mathcal{C}^A$ by (48). As $\mathcal{C}^A$ also contains the negative orthant $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; -\mathbb{R}^2_+)$ we obtain $\mathcal{C}^{\tilde{S}} \subseteq \mathcal{C}^A$. ■

Corollary 2.6. *Using the above notation and assuming that S satisfies (NA^λ), the optimization problem*

$$(P_x^{\tilde{S}}) \qquad \mathbb{E}[U(x + \varphi_T^0)] \to \max! \tag{53}$$

$$\varphi_T^0 \leqslant (H \cdot \tilde{S})_T, \quad \textit{for some } H \in \mathcal{P}. \tag{54}$$

has the same unique optimizer $\hat{\varphi}_T^0$ as the problem (P_x) defined in (30) as well as the problem (P_x^A) defined in (50).

If the λ-self-financing process $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t=0}^T$, starting at zero is a maximizer for problem (P_x) then

$$\hat{H}_t = \hat{\varphi}_{t-1}^1, \qquad t = 1, \dots, T$$

defines a maximizer for problem $(P_x^{\tilde{S}})$ and we have

$$(\hat{H} \cdot \tilde{S})_T = \sum_{t=1}^T \hat{H}_t(\tilde{S}_t - \tilde{S}_{t-1}) = \hat{\varphi}_T^0 \tag{55}$$

and more generally,

$$\begin{aligned}(\hat{H} \cdot \tilde{S})_t &= \hat{\varphi}_t^0 + \hat{\varphi}_t^1 \tilde{S}_t \\ &= \hat{\varphi}_{t-1}^0 + \hat{\varphi}_{t-1}^1 \tilde{S}_t, \qquad t = 1, \dots, T.\end{aligned} \tag{56}$$

Proof. The first part follows form (52) and Theorem 2.3.

As regards the second part, let us verify (56) by induction. Rewrite these equations as

$$(\hat{H} \cdot \tilde{S})_t = \hat{\varphi}_{t-1}^0 + \hat{\varphi}_{t-1}^1 \tilde{S}_t + a_t \tag{57}$$

$$(\hat{H} \cdot \tilde{S})_t = \hat{\varphi}_t^0 + \hat{\varphi}_t^1 \tilde{S}_t + b_t \tag{58}$$

We have to show that the elements $a_t, b_t \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ are all zero. Obviously $a_0 = 0$. As inductive hypothesis assume that $0 = a_0 \leqslant b_0 = a_1 \leqslant \ldots \leqslant b_{t-1} = a_t$. We claim that $a_t \leqslant b_t$. Indeed, $(\hat{\varphi}_t^0, \hat{\varphi_t}^1)$ is obtained from $(\hat{\varphi}_{t-1}^0, \hat{\varphi}_{t-1}^1)$ by trading at price S_t or $(1-\lambda)S_t$, depending on whether

$\hat{\varphi}_t{}^1 - \hat{\varphi}^1_{t-1} \geqslant 0$ or $\hat{\varphi}_t{}^1 - \hat{\varphi}^1_{t-1} \leqslant 0$. As $\tilde{S}_t$ takes values in $[(1-\lambda)S_t, S_t]$ we get in either case

$$(\hat{\varphi}_t{}^1 - \hat{\varphi}^1_{t-1})\tilde{S}_t + (\hat{\varphi}^0_t - \hat{\varphi}^0_{t-1}) \leqslant 0,$$

which gives $a_t \leqslant b_t$.

To complete the inductive step we have to show that $b_t = a_{t+1}$, i.e.

$$(\hat{H} \cdot \tilde{S})_{t+1} - (\hat{H} \cdot \tilde{S})_t = \hat{\varphi}_t{}^1(\tilde{S}_{t+1} - \tilde{S}_t).$$

As the left hand side equals $\hat{H}_{t+1}(\tilde{S}_{t+1} - \tilde{S}_t)$ this follows from the definition $\hat{H}_{t+1} = \hat{\varphi}^1_t$.

Having completed the inductive step we conclude that $b_T \geqslant 0$. We have to show that $b_T = 0$. If this were not the case we would have

$$\begin{aligned}\mathbb{E}\Big[U(x + (\hat{H} \cdot \tilde{S})_T)\Big] &= \mathbb{E}\left[U(x + \hat{\varphi}^0_T + b_T)\right] \\ &> \mathbb{E}\left[U(x + \hat{\varphi}^0_T)\right],\end{aligned}$$

which contradicts the first part of the corollary, showing (55) and (56). ■

Here is the economic interpretation of the above argument: whenever $\hat{\varphi}^1_t - \hat{\varphi}^1_{t-1} \neq 0$ we must have that $\tilde{S}_t$ equals either the bid or the ask price $(1-\lambda)S_t$, resp. S_t, depending on the sign of $\hat{\varphi}^1_t - \hat{\varphi}^1_{t-1}$. More formally

$$\left\{\hat{\varphi}^1_t - \hat{\varphi}^1_{t-1} > 0\right\} \subseteq \left\{\tilde{S}_t = S_t\right\}, \tag{59}$$

$$\left\{\hat{\varphi}^1_t - \hat{\varphi}^1_{t-1} < 0\right\} \subseteq \left\{\tilde{S}_t = (1-\lambda)S_t\right\}, \qquad t = 0, \dots, T. \tag{60}$$

The predictable process $(\hat{H}_t)^T_{t=1}$ denotes the holdings of stock during the intervals $(]t-1, t])^T_{t=1}$. Inclusion (59) indicates that the utility maximizing agent, trading optimally in the frictionless market $\tilde{S}$, only increases her investment in stock when $\tilde{S}$ equals the ask price S. Inclusion (60) indicates the analogous result for the case of decreasing the investment in stock. The inclusions pertain to $\mathcal{F}_{t-1}$-measurable sets, i.e. to investment decisions done at time $t-1$, where $t-1$ ranges from 0 to T. One may check that, defining $\hat{H}_0 = \hat{H}_{T+1} = 0$, this reasoning also extends to the trading decisions at time $t = 0$ and $t = T+1$.

The reader may wonder why we index the process H by $(H_t)^{T+1}_{t=0}$, while φ is indexed by $(\varphi_t)^T_{t=-1}$. As regards H, this is the usual definition of a *predictable* process from the frictionless theory (where H_{T+1} plays no role). The reason why we shift the indexation for t by 1 will be discussed in the more general continuous time setting in chapter 4 again.

One may also turn the point of view around and start from a process $\tilde{S}$ (obtained, e.g., from an educated guess) such that the associated (frictionless) optimizer $\hat{\varphi}^1_t = \hat{H}_{t+1}$ satisfies (59) and (60), and deduce from the solution of $(P_x^{\tilde{S}})$ the solution of (P_x). In fact, this idea will turn out to work very nicely in the applications (see chapter 3 below).

Here is a formal definition [145].

Definition 2.7. *Fix a process $(S_t)_{t=0}^T$ and $0 \leqslant \lambda < 1$ such that (NA^λ) is satisfied, as well as a utility function U and an initial endowment $x \in D$ as above. In addition, suppose that $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is an adapted process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, taking its values in the bid-ask spread $([(1-\lambda)S_t, S_t])_{t=0}^T$. We call $\tilde{S}$ a* shadow price process *for S if there is an optimizer $(\hat{H}_t)_{t=1}^T$ for the frictionless market $\tilde{S}$, i.e.*

$$\mathbb{E}_{\mathbb{P}}\left[U\left(x + (\hat{H}\cdot\tilde{S})_T\right)\right] = \sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(x + (H\cdot\tilde{S})_T)\right] : H \in \mathcal{P}\right\},$$

such that

$$\left\{\Delta\hat{H}_t > 0\right\} \subseteq \{\tilde{S}_{t-1} = S_{t-1}\}, \qquad t = 1, \ldots, T, \tag{61}$$

$$\left\{\Delta\hat{H}_t < 0\right\} \subseteq \{\tilde{S}_{t-1} = (1-\lambda)S_{t-1}\}, \qquad t = 1, \ldots, T. \tag{62}$$

Theorem 2.8. *Suppose that $\tilde{S}$ is a shadow price for S, and let $\hat{H}, U, x$, and $0 \leqslant \lambda < 1$ be as in Definition 2.7.*

Then we obtain an optimal (in the sense of (30)) trading strategy $(\hat{\varphi}^0_t, \hat{\varphi}^1_t)_{t=-1}^T$ in the market S under transaction costs λ via the identification $\hat{\varphi}^0_{-1} = \hat{\varphi}^1_{-1} = 0$ and

$$\hat{\varphi}^1_{t-1} = \hat{H}_t, \qquad t = 1, \ldots, T, \tag{63}$$

$$\hat{\varphi}^0_{t-1} = -\hat{\varphi}^1_{t-1}\tilde{S}_{t-1} + (\hat{H}\cdot\tilde{S})_{t-1}, \qquad t = 1, \ldots, T, \tag{64}$$

as well as $\hat{\varphi}^1_T = 0, \hat{\varphi}^0_T = (\hat{H}\cdot\tilde{S})_T$.

Proof. Again the proof reduces to the economically obvious fact that trading in the frictionless market $\tilde{S}$ yields better terms of trade than in the market S under transaction costs λ. This is formalized by the first inclusion in Lemma 2.5. Hence (61) and (62) imply that the frictionless trading strategy $(\hat{H}_t)_{t=1}^T$ can be transformed into a trading strategy $(\hat{\varphi}^0_t, \hat{\varphi}^1_t)_{t=-1}^T$ under transaction costs via (63) and (64). ∎

Remark 2.9. In the above analysis the notion of the *Legendre transform* played a central role.

As a side step – which may be safely skipped without missing any mathematical content – let us try to give an economic "interpretation", or rather "visualization" of the conjugate function V

$$V(y) = \sup_x (U(x) - xy), \tag{65}$$

in the present financial application. Instead of interpreting U as a function which maps money to happiness, it is more feasible for the present purpose to interpret U as a production function.

We shall only give a hypothetical mind experiment which is silly form a realistic point of view: suppose that you own a gold mine. You have the choice to invest x Euros into the (infrastructure of the) gold mine which will result in a production of $U(x)$ kilos of gold. You only can make this investment decision once, then take the resulting kilos of gold, and then the story is finished. In other words, the gold mine is a machine turning money x into gold $U(x)$. The monotonicity and concavity of U correspond to the "law of diminishing returns".

Now suppose that gold is traded at a price of y^{-1} Euros for one kilo of gold or, equivalently, y is the price of one Euro in terms of kilos of gold. What is your optimal investment into the gold mine? Clearly you should invest the amount of $\hat{x}$ Euros for which the marginal production $U'(\hat{x})$ of kilos of gold per invested Euro equals the market price y of one Euro in terms of gold, i.e. $\hat{x}$ is determined by $U'(\hat{x}) = y$.

Given the price y, we thus may interpret the conjugate function (65) as the net value $V(y)$ of your gold mine in terms of kilos of gold: it equals $V(y) = \sup_x(U(x) - xy) = U(\hat{x}) - \hat{x}y$. Indeed, starting from an initial capital of 0 Euros it is optimal for you to borrow $\hat{x}$ Euros and invest them into the mine so that it produces $U(\hat{x})$ many kilos of gold. Subsequently you sell $\hat{x}y$ many of those kilos of gold to obtain $\hat{x}$ Euros which you use to pay back the loan. In this way you end up with a net result of $U(\hat{x}) - \hat{x}y$ kilos of gold.

Summing up, $V(y)$ equals the net value of your gold mine in terms of kilos of gold, provided that the price of a kilo of gold equals y^{-1} Euros and that you invest optimally.

Let us next try to interpret the inversion formula

$$U(x) = \inf_y (V(y) + xy).$$

Suppose that you have given the gold mine to a friend, whom we might call the "devil", and he promises to give you in exchange for the mine its net value in gold, i.e. $V(y)$ many kilos of gold, if the market price of one kilo of gold turns out to be y^{-1}. Fix $y > 0$. If you own – contrary to the situation considered in the interpretation of V above – an initial capital of x Euros and want to transform all your wealth, i.e. the claims to the devil plus the x Euros, into gold, the total amount of kilos of gold then equals

$$V(y) + xy.$$

Fix your initial capital of x Euros. If the devil is able to manipulate the market, then he might be evil and choose the price y in such a way that your resulting position in gold is minimized, i.e.

$$V(y) + xy \mapsto min!, \qquad y > 0.$$

Again, the optimal $\hat{y}$ (i.e. the meanest choice of the devil) is determined by the first order condition $V'(\hat{y}) = -x$. The duality relation

$$U(x) = \inf_y (V(y) + xy) = V(\hat{y}) + x\hat{y}$$

thus may interpreted in the following way: if the devil does the choice of y which is least favourable for you, then you will earn the same amount of gold as if you would have done by keeping the mine and investing your x Euros directly into the mine. In both cases the result equals $U(x)$ kilos of gold.

Next we try to visualize the theme of Theorem 2.2: we not only consider the utility function U, but also the financial market S under transaction costs λ. In this variant of the above story you invest into the goldmine at time T to transform an investment of ξ units of Euros into $U(\xi)$ many kilos of gold. At time $t = 0$ you start with an initial capital of x Euros and you are allowed to trade in the financial market S under transaction costs λ by choosing a trading strategy φ. This will result in a random variable of $\xi(\omega) = x + \varphi_T^0(\omega)$ Euros which you can transform into $U(x + \varphi_T^0(\omega))$ kilos of gold. Passing to the optimal strategy $\hat{\varphi}_T^0$ you therefore obtain $U(x + \hat{\varphi}_T^0(\omega))$ many kilos of gold if ω turns out to be the true state of the world. In average this will yield $u(x) = \mathbb{E}_{\mathbb{P}}[U(x + \hat{\varphi}_T^0)]$ many kilos of gold. We thus may consider the indirect utility function $u(x)$ as a machine which turns the original wealth of x Euros at time $t = 0$ into $u(x)$ many expected kilos of gold at time $t = T$, provided you invest optimally into the financial market S and subsequently into the gold mine also in an optimal way.

We now pass again to the dual problem, i.e., to the devil to whom you have given your gold mine. Fix your initial wealth x and first regard $u(x)$ simply as a utility function as in the first part of this remark.

We may define the conjugate function

$$v(y) = \sup_{\xi}(u(\xi) - \xi y) \tag{66}$$

and interpret it as the net value of the gold mine, denoted in expected kilos of gold, if the price y of Euro versus gold equals y at time $t = 0$. Indeed the argument works exactly as in the first part of this remark where again we interpret u as a machine turning money into gold (measured in expectation and assuming that you trade optimally). In particular we get for the "devilish" price $\hat{y}$ at time $t = 0$, given by $\hat{y} = u'(x)$, that the devil gives you at time $t = 0$ precisely the amount of $v(\hat{y})$ kilos of gold such that $v(\hat{y}) + x\hat{y}$ equals $u(x)$, i.e. the expected kilos of gold which you could obtain by trading optimally and investing into the gold mine at time T.

But this time there is an additional feature: the devil will also do something more subtle. He offers you, alternatively, to pay $V(y(\omega))$ many kilos of gold as recompensation for leaving him the goldmine. The payment now depends on the prize $y(\omega)$ of one Euro in terms of gold at time T which may depend on the random element ω and which is only revealed at time T. The function V now is the conjugate function of the original utility function U as defined in (65).

The main message of Theorem 2.2 can be resumed in prose as follows

(a) there is a choice of "devilish" prices $\hat{y}(\omega)$ given by the marginal utility of the optimal terminal wealth

$$\hat{y}(\omega) = U'(x + \hat{\varphi}^0_T(\omega)), \quad \omega \in \Omega.$$

(b) There is a probability measure $\hat{Q}$ on Ω such that

$$\hat{y}(\omega) = \hat{y}\frac{d\hat{Q}}{d\mathbb{P}}(\omega), \text{where} \quad \hat{y} \quad \text{is the optimizer in} \quad (66).$$

It follows that $\sum_\omega \hat{y}(\omega)\mathbb{P}(\omega) = \hat{y}$, i.e., $\hat{y}$ is the $\mathbb{P}$-average of the prizes $\hat{y}(\omega)$.

(c) The formula

$$v(\hat{y}) = \mathbb{E}_{\mathbb{P}}\Big[V\big(\hat{y}(\omega)\big)\Big] = \mathbb{E}_{\mathbb{P}}\Big[V\Big(\hat{y}\frac{d\hat{Q}}{d\mathbb{P}}(\omega)\Big)\Big]$$

now has the interpretation that the devil gives you (in average) the same amount of gold, namely $v(\hat{y})$ many kilos, independently of whether you do the deal with him at time $t = 0$ or $t = T$.

(d) If you choose any strategy φ we have the inequality

$$\mathbb{E}_{\hat{Q}}[\varphi_T^0] \leqslant \mathbb{E}_{\hat{Q}}[\hat{\varphi}_T^0] = x$$

as $\hat{Q}$ is a λ-consistent price system. Hence

$$\mathbb{E}_{\mathbb{P}}\big[\big(x + \varphi_T^0(\omega)\big)\hat{y}(\omega)\big] \leqslant \mathbb{E}_{\mathbb{P}}\big[\big(x + \hat{\varphi}_T^0(\omega)\big)\hat{y}(\omega)\big] \tag{67}$$

which may be interpreted in the following way: if you accept the devil's offer to get the amount of $V(\hat{y}(\omega))$ kilos of gold at time T, you cannot improve your expected result by changing from $\hat{\varphi}$ to some other trading strategy φ, while the devil remains his choice of prices $\hat{y}(\omega)$ unchanged.

We close this "visualisation" of the duality relations between U, V and u, v by stressing once more that the fictitious posession of a gold mine has, of course, no practical economic relevance and was presented for purely didactic reasons.

3 The Growth-Optimal Portfolio in the Black-Scholes Model

In this chapter we follow the lines of our joint work with St. Gerhold and J. Muhle-Karbe [91] as well as [92] and [90], the latter paper also co-authored with P. Guasoni and analyze the dual optimizer in a Black-Scholes model under transaction costs $\lambda \geqslant 0$. The task is to maximize the expected return (or growth) of a portfolio. This is tantamount to consider utility maximization with respect to logarithmic utility $U(x) = \log(x)$ of terminal wealth at time T,

$$(P_x) \qquad \mathbb{E}\left[\log(V_T)\right] \to \max!, \qquad V_T \in x + \mathcal{C}^\lambda, \tag{68}$$

where $\mathcal{C}^\lambda$ is defined in (13). Our emphasis will be on the limiting behavior for $T \to \infty$.

We take as stock price process $S = (S_t)_{t\geqslant 0}$ the Black-Scholes model

$$S_t = S_0 \exp\left[\sigma W_t + (\mu - \tfrac{\sigma^2}{2})t\right], \tag{69}$$

where $\sigma > 0$ and $\mu \geqslant 0$ are fixed constants.

To keep the notation light, the bond price process will again be assumed to be $B_t \equiv 1$. We remark that the case $B_t = \exp(rt)$ can rather trivially be reduced to the present one, simply by passing to discounted terms.

3.1 The frictionless case

We first recall the situation without transaction costs. This topic is well-known and goes back to the seminal work of R. Merton [181]. For later use we formulate the result in a slightly more general setting: we assume that the volatility σ and the drift μ are arbitrary predictable processes.

We fix the horizon T and assume that $W = (W_t)_{0\leqslant t\leqslant T}$ is a Brownian motion based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ where $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ is the (saturated) filtration generated by W.

Theorem 3.1. *(compare [181]): Suppose that the* $]0,\infty[$*-valued stock price process* $S = (S_t)_{0\leqslant t\leqslant T}$ *satisfies the stochastic differential equation*

$$\tfrac{dS_t}{S_t} = \mu_t dt + \sigma_t \, dW_t, \qquad 0 \leqslant t \leqslant T,$$

where $(\mu_t)_{0\leqslant t\leqslant T}$ *and* $(\sigma_t)_{0\leqslant t\leqslant T}$ *are predictable, real-valued processes such that*

$$\mathbb{E}\left[\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt\right] < \infty. \tag{70}$$

Define the growth optimal process $\hat{V} = (\hat{V}_t)_{0\leqslant t\leqslant T}$, *starting at* $\hat{V}_0 = 1$, *by*

$$\frac{d\hat{V}_t}{\hat{V}_t} = \hat{\pi}_t \cdot \frac{dS_t}{S_t}, \tag{71}$$

where $\hat{\pi}_t$ *equals the mean variance ratio*

$$\hat{\pi}_t = \frac{\mu_t}{\sigma_t^2}. \tag{72}$$

Then $\hat{V}$ *is a well-defined* $]0,\infty[$*-valued process satisfying*

$$\hat{V}_t = \exp\left[\int_0^t \frac{\mu_s}{\sigma_s} dW_s + \int_0^t \frac{\mu_s^2}{2\sigma_s^2} ds\right], \qquad 0 \leqslant t \leqslant T. \tag{73}$$

We then have

$$\mathbb{E}\left[\log(\hat{V}_T)\right] = \mathbb{E}\Big[\int_0^T \frac{\mu_t^2}{2\sigma_t^2} dt\Big]. \tag{74}$$

If $(\pi_t)_{0\leqslant t\leqslant T}$ *is any competing strategy in* (71)*, i.e. an* $\mathbb{R}$*-valued, predictable process such that*

$$\mathbb{E}\left[\int_0^T \pi_t^2 \sigma_t^2 dt\right] < \infty, \text{ and } \int_0^T |\pi_t \mu_t| dt < \infty, \text{ a.s.}, \tag{75}$$

the stochastic differential equation

$$\frac{dV_t}{V_t} = \pi_t \frac{dS_t}{S_t} \tag{76}$$

well-defines a $]0,\infty[$*-valued process*

$$V_t = \exp\left[\int_0^t \pi_s \sigma_s dW_s + \int_0^t \left(\pi_s \mu_s - \frac{\pi_s^2 \sigma_s^2}{2}\right) ds\right], \tag{77}$$

for which we obtain

$$\mathbb{E}\left[\log(V_T)\right] \leqslant \mathbb{E}\left[\log(\hat{V}_T)\right],$$

and, more generally, for stopping times $0 \leqslant \varrho \leqslant \tau \leqslant T$

$$\mathbb{E}\left[\log(\tfrac{V_\tau}{V_\varrho})\right] \leqslant \mathbb{E}\left[\log(\tfrac{\hat{V}_\tau}{\hat{V}_\varrho})\right].$$

Proof. If a strategy $(\pi_t)_{0\leqslant t\leqslant T}$ satisfies (75) we get from Itô's formula and (69) that (77) is the solution to (76) with initial value $V_0 = 1$. Passing to $\hat{\pi}$ defined in (72), the assertion (74) is rather obvious

$$\mathbb{E}\left[\log(\hat{V}_T)\right] = \mathbb{E}\left[\int_0^T \tfrac{\mu_t}{\sigma_t} dW_t + \int_0^T \tfrac{\mu_t^2}{2\sigma_t^2} dt\right] = \mathbb{E}\left[\int_0^T \tfrac{\mu_t^2}{2\sigma_t^2} dt\right]$$

as $(\int_0^t \frac{\mu_s}{\sigma_s} dW_s)_{0\leqslant t\leqslant T}$ is a martingale bounded in $L^2(\mathbb{P})$ by (70).

If $\pi = (\pi_t)_{0\leqslant t\leqslant T}$ is any competing strategy verfying (75), we again obatin

$$\mathbb{E}\left[\log(V_T)\right] = \mathbb{E}\left[\int_0^T \pi_t\sigma_t dW_t + \int_0^T \left(\pi_t\mu_t - \frac{\pi_t^2\sigma_t^2}{2}\right) dt\right] = \mathbb{E}\left[\int_0^T \left(\pi_t\mu_t - \frac{\pi_t^2\sigma_t^2}{2}\right) dt\right].$$

It is obvious that, for fixed $0 \leqslant t \leqslant T$ and $\omega \in \Omega$, the function

$$\pi \to \pi\mu_t(\omega) - \frac{\pi^2\sigma_t^2(\omega)}{2}, \qquad \pi \in \mathbb{R},$$

attains its unique maximum at $\hat{\pi}_t(\omega) = \frac{\mu_t(\omega)}{\sigma_t^2(\omega)}$ so that

$$\mathbb{E}\left[\log(V_T)\right] \leqslant \mathbb{E}\left[\int_0^T \left(\hat{\pi}_t\mu_t - \frac{\hat{\pi}_t^2\sigma_t^2}{2}\right) dt\right] = \mathbb{E}\left[\int_0^T \frac{\mu_t^2}{2\sigma_t^2} dt\right] = \mathbb{E}\left[\log(\hat{V}_T)\right].$$

More generally, for stopping times $0 \leqslant \varrho \leqslant \tau \leqslant T$, we obtain

$$\mathbb{E}\left[\log\left(\frac{V_\tau}{V_\varrho}\right)\right] = \mathbb{E}\left[\int_\varrho^\tau \left(\pi_t\mu_t - \frac{\pi_t^2\sigma_t^2}{2}\right) dt\right] \leqslant \mathbb{E}\left[\int_\varrho^\tau \left(\hat{\pi}_t\mu_t - \frac{\hat{\pi}_t^2\sigma_t^2}{2}\right) dt\right] = \mathbb{E}\left[\log\left(\frac{\hat{V}_\tau}{\hat{V}_\varrho}\right)\right].$$

■

3.2 Passing to transaction costs: some heuristics

Before we pass to a precise formulation of the utility maximization problem for the log-utility maximizer (see Definition 3.9 below) we want to develop the heuristics to find the shadow price process $(\tilde{S}_t)_{t\geqslant 0}$ for the utility maximization problem of optimizing the expected growth of a portfolio. We make two heroic assumptions. In fact, we are allowed to make all kind of heuristic assumptions and bold guesses, as we shall finally pass to verification theorems to justify them.

Assumption 3.2. *When the shadow price $(\tilde{S}_t)_{t\geqslant 0}$ ranges in the interior $](1-\lambda)S_t, S_t[$ of the bid-ask interval $[(1-\lambda)S_t, S_t]$ then the process $\tilde{S}_t$ is a deterministic function of S_t*

$$\tilde{S}_t = g_c(S_t). \tag{78}$$

More precisely, we suppose that there is a family of (deterministic, smooth) functions $g_c(\cdot)$, depending on a real parameter c, such that, whenever we have random times $\varrho \leqslant \tau$ such that $\tilde{S}_t \in](1-\lambda)S_t, S_t[$, for all $t \in]\!]\varrho, \tau[\![$, then there is a parameter c (depending on the stopping time ϱ) such that

$$\tilde{S}_t = g_c(S_t), \qquad \varrho \leqslant t \leqslant \tau.$$

The point is that the parameter c does not change while $\tilde{S}_t$ ranges in the interior $](1-\lambda)S_t, S_t[$ of the bid-ask interval. Only when $\tilde{S}_t$ equals $(1-\lambda)S_t$ or S_t we shall allow the parameter c to vary.

Assumption 3.3. *A log-utility agent, who can invest in* a frictionless way *(i.e. without paying transaction costs) in the market $\tilde{S}$* does not want to change her positions in stock and bond *as long as $\tilde{S}_t$ ranges in the interior $](1-\lambda)S_t, S_t[$ of the bid-ask interval.*

Assumption 3.3 is, of course, motivated by the results on the shadow price process in Chapter 2 (Def. 2.7).

Here are two consequences of the above assumptions. Suppose that $\tilde{S}_t$ satisfies

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t, \tag{79}$$

where $(\tilde{\mu}_t)_{t\geqslant 0}$ and $(\tilde{\sigma}_t)_{t\geqslant 0}$ are predictable processes which we eventually want to determine. Applying Itô to (78) and dropping the subscript c of g_c for the moment (and supposing that g is sufficiently smooth), we obtain $d(g(S_t)) = g'(S_t)dS_t + \frac{g''(S_t)}{2}\, d\langle S\rangle_t$, or, equivalently

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \frac{g'(S_t)}{g(S_t)}\, dS_t + \frac{g''(S_t)}{2g(S_t)}\, d\langle S\rangle_t.$$

Inserting (69) we obtain in (79) above (compare [91])

$$\tilde{\sigma}_t = \frac{\sigma g'(S_t)S_t}{g(S_t)} \tag{80}$$

$$\tilde{\mu}_t = \frac{\mu g'(S_t)S_t + \frac{\sigma^2}{2} g''(S_t)S_t^2}{g(S_t)} \tag{81}$$

and in particular the relation

$$\frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2} = \frac{g(S_t)[\mu g'(S_t)S_t + \frac{\sigma^2}{2} g''(S_t)S_t^2]}{\sigma^2 g'(S_t)^2 S_t^2}. \tag{82}$$

On the other hand, the optimal proportion $\tilde{\pi}$ of the investment $\varphi^1\tilde{S}$ into stock to total wealth $\varphi^0 + \varphi^1\tilde{S}$ for the log-utility optimizer in the frictionless market $\tilde{S}$ is given by

$$\tilde{\pi}_t = \frac{\varphi_t^1\tilde{S}_t}{\varphi_t^0 + \varphi_t^1\tilde{S}_t} = \frac{g(S_t)}{c + g(S_t)}, \tag{83}$$

where

$$c := \frac{\varphi_t^0}{\varphi_t^1} \tag{84}$$

is the ratio of positions φ_t^0 and φ_t^1 in bond and stock respectively. Assumption 3.3 implies that φ_t^0 and φ_t^1, and therefore also the parameter c, should remain constant when $\tilde{S}$ ranges in the interior $](1-\lambda)S_t, S_t[$ of the bid-ask spread.

We have assembled all the ingredients to yield a unifying equation: on the one hand side, the ratio $\tilde{\pi}_t$ of the value of the investment in stock and total wealth (both evaluated by using the shadow price $\tilde{S}$) is given by formula (83). On the other hand, by formula (72) in Theorem 3.1 and Assumption 3.3 we must have $\tilde{\pi}_t = \frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2}$ and the latter ratio is given by (82). Hence

$$\tilde{\pi}_t = \frac{g(S_t)}{c + g(S_t)} = \frac{g(S_t)[\mu g'(S_t)S_t + \frac{\sigma^2}{2} g''(S_t)S_t^2]}{\sigma^2 g'(S_t)^2 S_t^2}.$$

Rearranging this equation and substituting S_t by the variable $s \in \mathbb{R}_+$, we arrive at the ODE

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{2\mu g'(s)}{\sigma^2 s}, \qquad s > 0. \tag{85}$$

Somewhat surprisingly this ODE admits a closed form solution (compare, however, Section 3.9 below for a good reason why we can find a closed form solution). Before spelling out this solution let us pass to a (heuristic) discussion of the initial conditions of the ODE (85). Fix $t_0 \geqslant 0$ and suppose that we have $S_{t_0} = 1$ which is just a matter of normalization. More importantly, suppose also that $\tilde{S}_{t_0} = S_{t_0} = 1$. The economic interpretation is that the economic agent was just buying stock at time t_0 which forces the shadow price $\tilde{S}_{t_0}$ to equal the ask price S_{t_0}. We also suppose (very heuristically!) that $(S_t)_{t\geqslant 0}$ starts a positive excursion at time t_0, i.e. $S_t > S_{t_0}$ for $t > t_0$ such that $t - t_0$ is sufficiently small.

We then are led to the initial conditions for (85)

$$g(1) = 1, \quad g'(1) = 1. \tag{86}$$

The second equation is a "smooth pasting condition" requiring that S_t and $\tilde{S}_t = g(S_t)$ match of first order around $t = t_0$. The necessity of this condition is intuitively rather clear and will become obvious in subsection 3.7 below.

We write $\theta = \frac{\mu}{\sigma^2}$ as (85) only depends on this ratio. As Mathematica tells us, the general form of the solution to (85) satisfying the initial conditions (86) then is given by

$$g(s) = g_c(s) = \frac{-cs + (2\theta - 1 + 2c\theta)s^{2\theta}}{s - (2 - 2\theta - c(2\theta - 1))s^{2\theta}} \tag{87}$$

unless $\theta = \frac{1}{2}$, which is a special case (see (88) below) that can be treated analogously. The parameter c defined in (84) is still free in (87).

As regards the given mean-variance ratio $\theta = \frac{\mu}{\sigma^2} > 0$, we have to distinguish the regimes $\theta \in]0, 1[$, $\theta = 1$, and $\theta > 1$. Let us start by discussing the singular case $\theta = 1$: in this case (see Theorem 3.1) the optimal solution in the frictionless market $S = (S_t)_{t\geqslant 0}$ defined in (69) is given by $\hat{\pi}_t \equiv 1$. Speaking economically, the utility maximizing agent, at time $t = 0$, invests all her wealth into stock and keeps this position unchanged until maturity T. In other words, no dynamic trading takes place in this special case, even without transaction costs. We therefore expect that this case will play a special (degenerate) role when we pass to transaction costs $\lambda > 0$.

The singular case $\theta = 1$ divides the regime $\theta \in]0, 1[$ from the regime $\theta > 1$. In the former the log-utility maximizer holds positive investments in stock as well as in bond, while in the latter case she goes short in bond and invests more than her total wealth into stock. These well-known facts follow immediately from Theorem 3.1 in the frictionless case and we shall see in

Theorem 3.10 below that this basic feature still holds true in the presence of transaction costs, at least for $\lambda > 0$ sufficiently small.

The mathematical analysis reveals that the case $\theta = \frac{1}{2}$ also plays a special role (apart from the singular case $\theta = 1$): in this case the general solution to the ODE (85) under initial conditions (86) involves logarithmic terms rather than powers:

$$g(s) = g_c(s) = \frac{c + 1 + c\ \log(s)}{c + 1 - \log(s)}. \tag{88}$$

But this solution is only special from a mathematical point of view while, from an economic point of view, this case is not special at all and we shall see that the solution (88) nicely interpolates the solution (87), for $\theta \to \frac{1}{2}$.

We now pass to the elementary, but tedious, discussion of the qualitative properties of the functions $g_c(\cdot)$ in (87) and (88) respectively. As this discussion amounts - at least in principle - to an involved version of a high school exercise, we only resume the results and refer for proofs to [91, Appendix A].

3.3 The case $0 < \theta < 1$

In this case we consider the function $g(s) = g_c(s)$ given by (87) and (88) respectively, on the right hand side of $s = 1$, i.e. on the domain $s \in [1, \infty[$. Fix the parameter c in $]\frac{1-\theta}{\theta}, \infty[$ for $\theta \in]0, \frac{1}{2}]$ (resp. in $]\frac{1-\theta}{\theta}, \frac{1-\theta}{\theta - \frac{1}{2}}[$ for $\theta \in]\frac{1}{2}, 1[$). Plugging $s = 1$ into the ODE (85) we observe that the above domains were chosen in such a way to have $g''_c(1) < 0$. Hence for fixed $c \in]\frac{1-\theta}{\theta}, \infty[$ (resp. $c \in]\frac{1-\theta}{\theta}, \frac{1-\theta}{\theta - \frac{1}{2}}[$ for $\theta \in]\frac{1}{2}, 1[$) the function $g_c(\cdot)$ is strictly concave in a neighbourhood of $s = 1$ so that from (86) we obtain

$$g_c(s) < s,$$

for $s \neq 1$ sufficiently close to $s = 1$.

Figure 3 is a picture of the qualitative features of the function $g_c(\cdot)$ on $s \in [1, \hat{s}[$. The point $\hat{s} > 1$ is the pole of $g_c(\cdot)$ where the denominator in (87) (resp. (88)) vanishes.

The function g_c is strictly increasing on $[1, \hat{s}[$; it is concave in a neighborhood of $s = 1$, then has a unique inflection point in $]1, \hat{s}[$, and eventually is convex between the inflection point and the pole $\hat{s}$.

We also observe that, for $\frac{1-\theta}{\theta} < c_1 < c_2$ we have $g_{c_1}(s) > g_{c_2}(s)$, for $s \in [1, \hat{s}[$, where $\hat{s}$ is the pole of the function g_{c_1} as displayed in Figure 4.

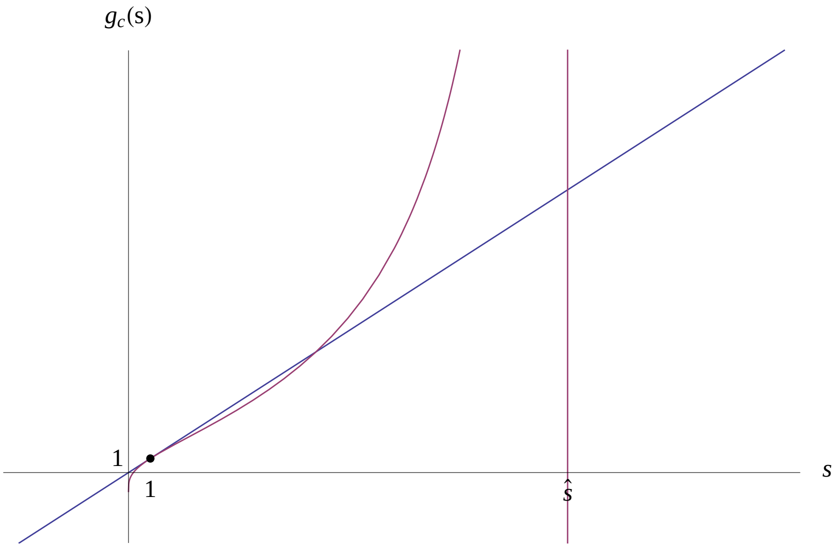


Figure 3: The function $g_c(s)$.

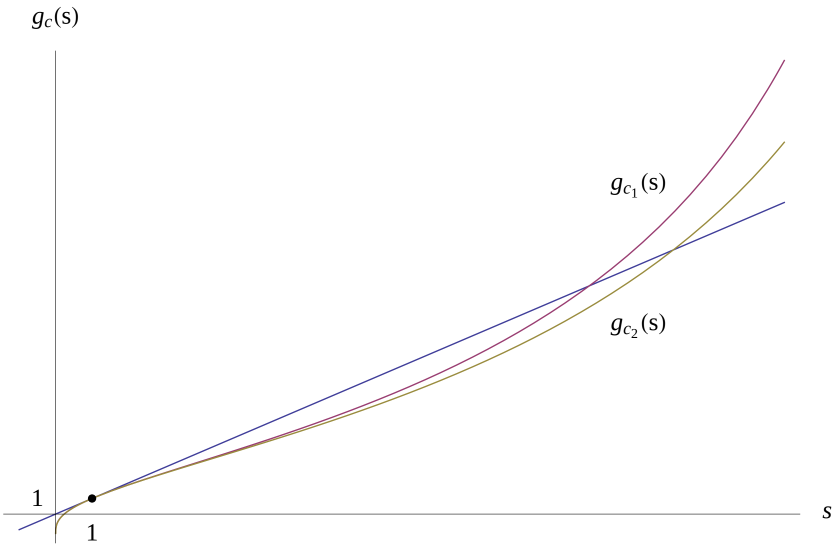


Figure 4: The functions $g_{c_1}(s)$ and $g_{c_2}(s)$, for $c_1 < c_2$.

We still have to complement the boundary conditions (86) for the ODE (85) at the other endpoint, corresponding to the "selling boundary": we want to find a point $\bar{s} = \bar{s}(c) \in]1, \hat{s}[$ and $0 < \lambda < 1$ such that

$$g_c(\bar{s}) = (1-\lambda)\bar{s}, \qquad g_c'(\bar{s}) = (1-\lambda). \tag{89}$$

Geometrically this task corresponds to drawing the unique line through the origin which tangentially touches the graph of $g_c(\cdot)$. See Figure 5.

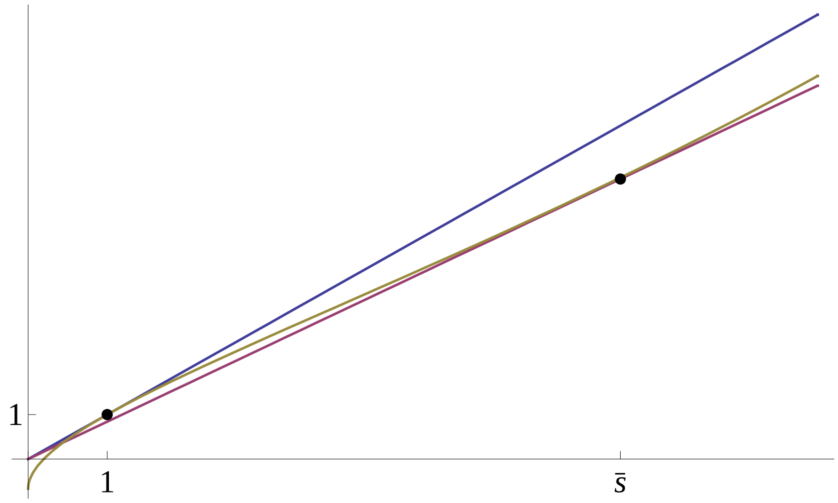


Figure 5: Smooth pasting conditions for the function g.

If we have found this tangent and the touching point $\bar{s}$, then (89) holds true, where $(1-\lambda)$ is the slope of the tangent.

In fact, for fixed $c \in]\frac{1-\theta}{\theta}, \infty[$ and $\theta \in]0, \frac{1}{2}]$ (resp. $c \in]\frac{1-\theta}{\theta}, \frac{1-\theta}{\theta-\frac{1}{2}}[$ and $\theta \in]\frac{1}{2}, 1[$) one may explicitly solve the two equations (89) in the two variables λ and $\bar{s}$ by simply plugging in formula (87) to obtain, for $\frac{1-\theta}{\theta} < c < \infty$,

$$\bar{s} = \bar{s}(c) = \left(\frac{c}{(2\theta - 1 + 2c\theta)(2 - 2\theta - c(2\theta - 1))} \right)^{1/(2\theta-1)}, \tag{90}$$

$$\lambda = \lambda(c) = \frac{(1 - 2(c+1)\theta)\bar{s}(c)^{2\theta} + c\bar{s}(c)}{\bar{s}(c)\left((2(c+1)\theta - c - 2)\bar{s}(c)^{2\theta} + \bar{s}(c)\right)} + 1, \tag{91}$$

$$g(\bar{s}) = \tag{92}$$

$$\frac{(2(c+1)\theta - 1)\left(\left(-\frac{(2(c+1)\theta-1)(2(c+1)\theta-c-2)}{c}\right)^{\frac{1}{1-2\theta}}\right)^{2\theta} - c\left(-\frac{(2(c+1)\theta-1)(2(c+1)\theta-c-2)}{c}\right)^{\frac{1}{1-2\theta}}}{(2(c+1)\theta - c - 2)\left(\left(-\frac{(2(c+1)\theta-1)(2(c+1)\theta-c-2)}{c}\right)^{\frac{1}{1-2\theta}}\right)^{2\theta} + \left(-\frac{(2(c+1)\theta-1)(2(c+1)\theta-c-2)}{c}\right)^{\frac{1}{1-2\theta}}}.$$

In the special case $\theta = \frac{1}{2}$, where $\frac{1-\theta}{\theta} = 1$, we obtain the somewhat simpler formulae

$$\bar{s} = \bar{s}(c) \qquad = \exp\left(\frac{c^2-1}{c}\right), \qquad 1 < c < \infty, \quad (93)$$

$$\lambda = \lambda(c) \qquad = 1 - c^2 \exp\left(\frac{1-c^2}{c}\right), \qquad 1 < c < \infty, \quad (94)$$

$$g(\bar{s}) = g_c(\bar{s}(c)) \quad = c^2, \qquad 1 < c < \infty. \quad (95)$$

We summarize what we have found so far.

Proposition 3.4. *Fix* $\theta \in\,]0,1[$ *and* $c \in\,]\frac{1-\theta}{\theta}, \infty[$ *(resp.* $c \in\,]\frac{1-\theta}{\theta}, \frac{1-\theta}{\theta-\frac{1}{2}}[$ *if* $\theta \in]\frac{1}{2}, 1[$*). Then the function* $g(s) = g_c(s)$ *defined in* (87) *(resp.* (88)*) is strictly increasing in* $[1, \bar{s}]$*, where* $\bar{s} = \bar{s}(c)$ *is defined in* (90) *(resp.* (93)*). In addition,* g *satisfies the boundary conditions*

$$\begin{aligned} g_c(1) &= 1, & g_c'(1) &= 1, \\ g_c(\bar{s}) &= (1-\lambda)\bar{s}, & g_c'(\bar{s}) &= 1-\lambda, \end{aligned}$$

where λ *is given by* (91) *(resp.* (94)*).*

Proof. The energetic reader may verify the above assertions by simply calculating all the above expressions and discussing the function g_c'. Another possibility is to look up the details in [91]. ■

The drawback of the above proposition is that c is the free variable parameterizing the solution. The transaction costs $\lambda = \lambda(c)$ in (91) (resp. (94)) are a function of c. Our original problem, however is stated the other way round: the level $0 < \lambda < 1$ of transaction costs is given and c as well as $\bar{s} = \bar{s}(c)$ and the function $g = g_c$ depend on λ. In other words, we have to invert the formulae (91) and (94). Unfortunately, when we shall do this final step, we will have to leave the pleasant case of closed form solutions which we have luckily encountered so far. We shall only be able to determine the inverse function of (91) (resp. (94)) locally around $\lambda = 0$ as a fractional Taylor series in λ (see (97) below). As this Taylor series only converges in some neighborhood of $\lambda = 0$, from now on, every assertion has to be preceded by the caveat "for $\lambda > 0$ sufficiently small". Hence we are interested in the behavior of the function $\lambda = \lambda(c)$ in (91) (resp. (94)) when c is in a neighborhood of the left limit $\frac{1-\theta}{\theta}$ of its domain: this corresponds to λ being in a neighborhood of zero.

In order to keep the calculations simple we focus on the special case $\theta = \frac{1}{2}$. The arguments carry over to the case of general $0 < \theta < 1$, at the expense of somewhat longer formulae (compare [91]).

Differentiating $\lambda(c)$ in (94) with respect to c we obtain

$$\begin{aligned}
\lambda'(c) &= (c-1)^2 \exp\left(\tfrac{1-c^2}{c}\right), \\
\lambda''(c) &= (c-1)\exp(\tfrac{1-c^2}{c})\tfrac{-3c^3+5c^2+c-1}{c^2}, \\
\lambda'''(c) &= \frac{1+c^2(3+c(-6+(-3+c)^2 c))}{c^4}\exp\left(\frac{1}{c}-c\right)
\end{aligned}$$

so that $\lambda(1) = \lambda'(1) = \lambda''(1) = 0$ while $\lambda'''(1) = 2 \neq 0$. Therefore the Taylor expansion of the analytic function $\lambda(c)$ around $c = 1$ starts as

$$\lambda(c) = \tfrac{1}{3}(c-1)^3 + O(c-1)^4.$$

This implies that the function $c \mapsto \lambda(c)$ given in (94) is locally invertible around $c = 1$ and that the inverse function $\lambda \mapsto c(\lambda)$ has a fractional Taylor expansion in terms of powers of $\lambda^{1/3}$ around $\lambda = 0$, with leading term

$$c(\lambda) = 1 + 3^{1/3}\lambda^{1/3} + O(\lambda^{2/3}). \tag{96}$$

As shown in [91] one may algorithmically determine all the coefficients in the above fractional Taylor expansion (96) of the function $\lambda \mapsto c(\lambda)$ by inverting (94). This not only works for the specially simple case $\theta = \frac{1}{2}$ considered above, but for all $\theta \in \,]0,1[$ and the coefficients are explicit functions of θ, which turn out to be fractional powers of certain rational functions of θ (see Proposition 3.5 below as well as Proposition 6.1 of [91] for the details).

Once we have expanded the parameter c as a function of λ around $\lambda = 0$ we can, for $c = c(\lambda)$, also plug this expansion into all the other quantities depending on c, e.g. $\bar{s} = \bar{s}(c)$ given in (90) (resp. (93)), to again obtain fractional Taylor expansions in λ. We resume our findings in the next proposition and refer to [91] for details and full proofs as well as a discussion of all the higher order coefficients of the series (97) and (98) which can be determined algorithmically.

Proposition 3.5. *Fix $\theta \in]0,1[$. There are fractional Taylor series starting at*

$$\begin{aligned}
c(\lambda) &= \frac{1-\theta}{\theta} + \frac{1-\theta}{2\theta}\left(\frac{6}{\theta(1-\theta)}\right)^{1/3}\lambda^{1/3} \\
&\quad + \frac{(1-\theta)^2}{4\theta}\left(\frac{6}{\theta(1-\theta)}\right)^{2/3}\lambda^{2/3} + O(\lambda)
\end{aligned} \tag{97}$$

$$\bar{s}(\lambda) = 1 + \left(\frac{6}{\theta(1-\theta)}\right)^{1/3}\lambda^{1/3} + \frac{1}{2}\left(\frac{6}{\theta(1-\theta)}\right)^{2/3}\lambda^{2/3} + O(\lambda) \tag{98}$$

such that, for $\lambda \geqslant 0$ sufficiently small, the above series converge. The function $g(s) = g_{c(\lambda)}(s)$, defined on the interval $[1, \bar{s}(\lambda)]$ and given in (87) *(resp.* (88)*), then satisfies the ODE* (85) *as well as the boundary conditions*

$$\begin{aligned} g(1) &= 1, & g'(1) &= 1, \\ g(\bar{s}(\lambda)) &= (1-\lambda)\bar{s}(\lambda), & g'(\bar{s}(\lambda)) &= (1-\lambda). \end{aligned}$$

3.4 Heuristic construction of the shadow price process

Fix $\theta \in]0,1[$ and $\lambda > 0$ as in the previous proposition. We shall continue to do some heuristics in this sub-section to motivate the sub-sequent formal definition. Define

$$\tilde{S}_t = g(S_t), \qquad t \geqslant 0, \tag{99}$$

where $g = g_{c(\lambda)}$ was defined in (87) and $c(\lambda)$ in (97).

Normalize S to satisfy $S_0 = 1$ so that also $\tilde{S}_0 = g(S_0) = 1$, and suppose (again heuristically!) that S starts a positive excursion at time $t = 0$, i.e. that $S_t > 1$ for $t > 0$ sufficiently small. In sub-section 3.2 the function g has been designed in such a way that the log-utility optimizer in the frictionless market $\tilde{S}$ keeps her holdings φ_t^0 and φ_t^1 constant, where the ratio $\frac{\varphi_t^0}{\varphi_t^1} = \frac{\varphi_t^0}{\varphi_t^1 \tilde{S}_0}$ equals the constant $c = c(\lambda)$ (in (97)).

But what happens if S_t hits the boundaries 1 or $\bar{s}$ of the interval $[1, \bar{s}]$? Say, at time $t_0 > 0$ we have for the first time after $t = 0$ that again we have $S_{t_0} = 1$. Consider the Brownian motion $W = (W_t)_{t \geqslant 0}$ during the infinitesimal interval $[t_0, t_0 + dt]$.

Interpreting, following a good tradition applied in physics, W as a random walk on an infinitesimal grid, we have (heuristically!) two possibilities for the increment of W : either $dW_{t_0} := W_{t_0+dt} - W_{t_0} = dt^{1/2}$ or $dW_{t_0} := W_{t_0+dt} - W_{t_0} = -dt^{1/2}$.

Let us start with the former case: we then have $dS_{t_0} = S_{t_0}(\mu\, dt + \sigma\, dt^{1/2})$ so that, continuing to define $\tilde{S}$ by (99)

$$\begin{aligned} d\tilde{S}_{t_0} := g(S_{t_0+dt}) - g(S_{t_0}) &= g'(S_{t_0}) dS_{t_0} + \tfrac{1}{2} g''(S_{t_0}) d\langle S \rangle_{t_0} \\ &= S_{t_0}(\mu\, dt + \sigma\, dt^{1/2}) + \frac{g''(1)}{2} S_{t_0}^2 \sigma^2 dt \\ &= \sigma\, dt^{1/2} + \left(\mu + \frac{g''(1)}{2}\sigma^2\right) dt. \end{aligned} \tag{100}$$

Note that $g''(1) = \frac{2}{c+1} - 2\theta < 0$, as follows from (80).

The case $dW_{t_0} = -dt^{1/2}$ is different from the case $dW_{t_0} = +dt^{1/2}$: in this case we cannot blindly use definition (99) to find $\tilde{S}_{t_0+dt}$, as S_{t_0+dt} is (infinitesimally) outside the domain of definition $[1, \bar{s}]$ of g. In this case we move $\tilde{S}$ identically to S: in Fig. 4 this corresponds geometrically to the fact that $\tilde{S}$ decreases along the identity line. We then get

$$\begin{aligned} d\tilde{S}_{t_0} = dS_{t_0} &= S_{t_0}(\mu\ dt - \sigma\ dt^{1/2}) \\ &= -\sigma\ dt^{1/2} + \mu\ dt. \end{aligned} \tag{101}$$

When S_{t_0} thus has moved out of the domain $[1, \bar{s}]$ of g, the agent also has to rebalance the portfolio $(\varphi_t^0, \varphi_t^1)$ in order to keep the ratio of wealth in bond and wealth in stock

$$c = \frac{\varphi^0_{t_0}}{\varphi^1_{t_0}\tilde{S}_{t_0}} = \frac{\varphi^0_{t_0+dt}}{\varphi^1_{t_0+dt}\tilde{S}_{t_0+dt}} \tag{102}$$

constant. This is achieved by buying an infinitesimal amount (of order $dt^{1/2}$) of stock at ask price $S_{t_0} = \tilde{S}_{t_0} = 1$. In order for (102) to match with (101) we must have

$$d\varphi^1_{t_0} = \varphi^1_{t_0}\frac{c}{c+1}\ \sigma\ dt^{1/2}, \quad d\varphi^0_{t_0} = -\varphi^0_{t_0}\frac{1}{c+1}\ \sigma\ dt^{1/2} \tag{103}$$

as one easily checks by plugging (103) into (102) (neglecting terms of higher order than $dt^{1/2}$). Note in passing that the equality $\tilde{S}_{t_0+dt} = S_{t_0+dt}$ also corresponds to the last fact that the agent is buying stock during the infinitesimal interval $[t_0, t_0 + dt]$.

We continue the discussion of the case $W_{t_0+dt} - W_{t_0} = -dt^{1/2}$ by passing to the next infinitesimal interval $[t_0 + dt, t_0 + 2dt]$: again we have to distinguish the case $W_{t_0+2dt} - W_{t_0+dt} = +dt^{1/2}$ and $W_{t_0+2dt} - W_{t_0+dt} = -dt^{1/2}$. Let us first consider the second case: we then continue to move $\tilde{S}$ in an identical way as S (compare (101)) and to keep buying stock at price S_{t_0+dt} which yields the same formula as in (103) , neglecting again terms of higher order than $dt^{1/2}$.

But what do we do if $W_{t_0+2dt} - W_{t_0+dt} = +dt^{1/2}$? The intuition is that we now move again into the no-trade region, where $\tilde{S}$ should depend on S in a functional way, similarly as in (99). This is indeed the case, *but the function g now has to be rescaled.* The domain of definition $[1, \bar{s}]$ has to be replaced by the interval $[m_t, m_t\bar{s}]$, where $(m_t)_{t\geqslant 0}$ denotes the (local) running minimum of the process $(S_t)_{t\geqslant 0}$: in our present infinitesimal reasoning (neglecting terms of higher order than $dt^{1/2}$) we have $m_{t_0+dt} = S_{t_0+dt} = 1 - \sigma\ dt^{1/2}$. If $(S_t)_{t\geqslant t_0+dt}$ starts a positive excursion at time $t_0 + dt$, which heuristically corresponds to $W_{t_0+2dt} - W_{t_0+dt} = +dt^{1/2}$, we define $\tilde{S}$ by

$$\tilde{S}_t = m_t\ g\left(\frac{S_t}{m_t}\right), \qquad t \geqslant t_0 + dt, \tag{104}$$

where $t \geqslant t_0 + dt$ is sufficiently small so that $(S_t)_{t \geqslant t_0+dt}$ remains above $m_{t_0+dt} = S_{t_0+dt} = 1 - \sigma\ dt^{1/2}$.

We have used the term m_t rather than $1 - \sigma dt^{1/2}$ in order to indicate that the previous formula not only holds true for the infinitesimal reasoning, but also for finite movements by considering the running minimum process $m_t = \inf_{0 \leqslant u \leqslant t} S_u$.

Summing up: during positive excursions of $(S_t)_{t\geqslant 0}$ we expect the process $(\tilde{S}_t)_{t\geqslant 0}$ to be defined by formula (104), while at times t when $(S_t)_{t\geqslant 0}$ hits its running minimum $m_t = \min_{0\leqslant u\leqslant t} S_u$ we simply let $\tilde{S}_t = S_t$ and buy stock similarly as in (103), following the movements of the running minimum $(m_t)_{t\geqslant 0}$.

The behavior of $\tilde{S}$ might remind of a reflected diffusion: by (104), we always have $\tilde{S}_t \leqslant S_t$, with equality happening when S_t equals its running minimum m_t. It is well-known that the set $\{t \in \mathbb{R}_+ : S_t = m_t\}$ is a Cantor-like subset of $\mathbb{R}_+$ of Lebesgue measure zero, related to "local time". There is, however an important difference between the present situation and, say, reflected Brownian motion $(|W_t|)_{t\geqslant 0}$: we shall prove below that $(\tilde{S}_t)_{t\geqslant 0}$ is a *diffusion*, i.e. its semi-martingale characteristics are absolutely continuous with respect to Lebesgue measure. In other words, the process $(\tilde{S}_t)_{t\geqslant 0}$ does not involve a "local time component". The reason for this remarkable feature of $\tilde{S}$ is the smooth pasting condition $g'(1) = 1$ in (86). This condition yielded in the above heuristic calculations that the leading terms of the differentials (100) and (101) are - up to the sign - identical, namely $\sigma\ dt^{1/2}$ and $-\sigma\ dt^{1/2}$. In other words, when $m_t = S_t = \tilde{S}_t$ so that the movement of $\tilde{S}$ is given by the regime (100) or (101), the effect of order $dt^{1/2}$ on the movement of $\tilde{S}_t$ is given by $\sigma\ dW_t$ as the leading terms in (100) and (101) are symmetric. This distinguishes the behavior of the process $\tilde{S}$ from, e.g., reflected Brownian motion where this relation fails to be symmetric when reflection takes place.

A closer look at the differentials (100) and (101) reveals that the terms of order dt do *not coincide* any more. However, this will do no harm, as the set of time instances t where $S_t = \tilde{S}_t$, i.e. S_t equals its running minimum m_t, only is a set of Lebesgue measure zero. Integrating quantities of order dt over such a set will have no effect.

The fact that the terms of order dt do not coincide in (100) and (101) corresponds to the fact that the extended function $G : [0, \bar{s}] \to [0, (1 - \lambda)\bar{s}]$

$$G(s) = \begin{cases} s, & \text{for } 0 \leqslant s \leqslant 1 \\ g(s), & \text{for } 1 \leqslant s \leqslant \bar{s} \end{cases}$$

is once, but *not twice* differentiable: the second derivative is discontinuous at the point $s = 1$ (with finite left and right limits). It is well known that such an isolated discontinuity of the second derivate does not restrict the applicability of Itô's lemma, which is the more formal version of the above heuristics.

Here is another aspect to be heuristically discussed before we turn to the mathematically precise formulation (Theorem 3.6 below) of the present theme. So far we have only dealt with the case when the process $\tilde{S}_t$ equals the ask price S_t or makes some (small) excursion away from it. We still have to discuss the behavior of $\tilde{S}$ when it makes a "large" excursion, so that $\tilde{S}_t$ hits the bid price $(1-\lambda)S_t$. In this case an analogous phenomenon happens, with signs reversed.

To fix ideas, suppose again (heuristically) that the process $(S_t)_{t\geqslant 0}$ starts a positive excursion at $S_0 = 1$ and hits the level $\bar{s} > 1$ at some time $t_1 > 0$. We then have, in accordance with (99),

$$\tilde{S}_t = g(S_t), \qquad 0 \leqslant t \leqslant t_1, \tag{105}$$

and $\tilde{S}_{t_1} = g(\bar{s}) = (1-\lambda)S_{t_1}$, i.e. $\tilde{S}_t$ hits the bid price $(1-\lambda)S_t$ at time $t = t_1$. What happens now? Again we distinguish the cases $dW_{t_1} = W_{t_1+dt} - W_{t_1} = \pm dt^{1/2}$. If $dW_{t_1} = -dt^{1/2}$, we turn back into the no-trade region: we continue to define $\tilde{S}$ via (105) also at time $t_1 + dt$. If, however $dW_{t_1} = +dt^{1/2}$ we define

$$\tilde{S}_{t_1+dt} = (1-\lambda)S_{t_1+dt},$$

i.e., the relation between $\tilde{S}$ and S is given by the straight line through the origin with slope $1-\lambda$ (see Figure 5). We then sell stock at the bid price $\tilde{S}_{t_1} = (1-\lambda)S_{t_1}$ in a similar way as in (103), but now with the signs of $d\varphi^0_{t_1}$ and $d\varphi^1_{t_1}$ reversed, as well as slightly different constants (compare (119) - (122) below).

Instead of considering the running minimum process m, we have to monitor from time t_1 on the (local) *running maximum* process M which is defined by

$$M_t = \max_{t_1\leqslant u\leqslant t} S_u, \qquad t \geqslant t_1.$$

We then define, for $t \geqslant t_1$, similarly as in (104),

$$\tilde{S}_t = \frac{M_t}{\bar{s}}\, g\left(\frac{\bar{s}S_t}{M_t}\right), \tag{106}$$

so that $\tilde{S}_t = (1-\lambda)S_t$ whenever $S_t = M_t$, in which case we sell stock in infinitesimal portions of order $dt^{1/2}$. When $S_t < M_t$ we have $\tilde{S}_t > (1-\lambda)S_t$ in (106) and we do not do any trading. We continue to act according to these rules until the next "large" *negative* excursion happens where we get $S_t = \frac{M_t}{\bar{s}}$ so that $\tilde{S}_t = S_t$ in (106). When this happens we again switch to the regime of buying stock, monitoring (locally) the running minimum process m_t etc etc.

We repeatedly used the word *"locally"* when speaking about the running minimum $(m_t)_{t \geqslant 0}$ (resp. running maximum $(M_t)_{t \geqslant 0}$) of S_t. Let us make precise what we have in mind, thus also starting to translate the above heuristics (e.g., arguing with "immediate" excursions) into proper mathematics. At time $t = 0$, we start by defining $\tilde{S}_0 := S_0$ which corresponds to the fact that we assume that at time $t = 0$ the agent buys stock (which holds true for $\mu > 0$ and λ sufficient small).

Now define sequences of stopping times $(\varrho_n)_{n=0}^{\infty}, (\sigma_n)_{n=1}^{\infty}$ and processes $(m_t)_{t \geqslant 0}$ and $(M_t)_{t \geqslant 0}$ as follows: let $\varrho_0 = 0$ and m the running minimum process of S, i.e.

$$m_t = \inf_{\varrho_0 \leqslant u \leqslant t} S_u, \qquad 0 \leqslant t \leqslant \sigma_1, \tag{107}$$

where the stopping time σ_1 is defined as

$$\sigma_1 = \inf\{t \geqslant \varrho_0 : \tfrac{S_t}{m_t} \geqslant \bar{s}\}.$$

Next define M as the running maximum process of S after time σ_1, i.e.

$$M_t = \sup_{\sigma_1 \leqslant u \leqslant t} S_u, \qquad \sigma_1 \leqslant t \leqslant \varrho_1, \tag{108}$$

where the stopping time ϱ_1 is defined as

$$\varrho_1 = \inf\{t \geqslant \sigma_1 : \tfrac{S_t}{M_t} \leqslant \tfrac{1}{\bar{s}}\}.$$

For $t \geqslant \varrho_1$, we again define

$$m_t = \inf_{\varrho_1 \leqslant u \leqslant t} S_u, \qquad \varrho_1 \leqslant t \leqslant \sigma_2, \tag{109}$$

where

$$\sigma_2 = \inf\{t \geqslant \varrho_1 : \tfrac{S_t}{m_t} \geqslant \bar{s}\},$$

and, for $t \geqslant \sigma_2$, we define

$$M_t = \sup_{\sigma_2 \leqslant u \leqslant t} S_u, \qquad \sigma_2 \leqslant t \leqslant \varrho_2,$$

where

$$\varrho_2 = \inf\{t \geqslant \sigma_2 : \tfrac{S_t}{M_t} \leqslant \tfrac{1}{\bar{s}}\}.$$

Continuing in an obvious way we obtain a.s. finite stopping times $(\varrho_n)_{n=0}^{\infty}$ and $(\sigma_n)_{n=1}^{\infty}$, increasing a.s. to infinity, such that m (resp. M) are the relative running minima (resp. maxima) of S defined on the stochastic intervals $([\![\varrho_{n-1}, \sigma_n]\!])_{n=1}^{\infty}$ (resp. $([\![\sigma_n, \varrho_n]\!])_{n=1}^{\infty}$). Note that

$$\bar{s}m_{\varrho_n} = M_{\varrho_n} = \bar{s}S_{\varrho_n}, \qquad \text{for } n \in \mathbb{N},$$

and

$$\bar{s}m_{\sigma_n} = M_{\sigma_n} = S_{\sigma_n}, \qquad \text{for } n \in \mathbb{N}.$$

We may therefore continuously extend the processes m and M to $\mathbb{R}_+$ by letting

$$M_t := \bar{s}m_t, \qquad \text{for } t \in \bigcup_{n=0}^{\infty} [\![\varrho_n, \sigma_{n+1}]\!], \tag{110}$$

$$m_t := \tfrac{M_t}{\bar{s}}, \qquad \text{for } t \in \bigcup_{n=1}^{\infty} [\![\sigma_n, \varrho_n]\!]. \tag{111}$$

For $t \geqslant 0$, we then have $\bar{s}m_t = M_t$ as well as $m_t \leqslant S_t \leqslant M_t$, and hence

$$m_t \leqslant S_t \leqslant \bar{s}m_t, \qquad \text{for } t \geqslant 0.$$

By construction, the processes m and M are of finite variation and only decrease (respectively increase) on the predictable set $\{m_t = S_t\}$ (resp. $\{M_t = S_t\} = \{m_t = S_t/\bar{s}\}$).

We thus have that the process

$$X_t = \frac{S_t}{m_t} = \frac{\bar{s}S_t}{M_t} \tag{112}$$

takes values in $[1, \bar{s}]$, is reflected at the boundaries and satisfies

$$dX_t = X_t(\mu\ dt + \sigma\ dW_t), \tag{113}$$

when $X_t \in\ ]1, \bar{s}[$.

In other words, $([m_t, M_t])_{t\geqslant 0}$ is an interval-valued process such that $\frac{M_t}{m_t} \equiv \bar{s}$, and such that S_t always lies in $[m_t, M_t]$. The interval $([m_t, M_t])_{t\geqslant 0}$ only changes location when S_t touches m_t or M_t, in which case m_t is driven down (resp. M_t is driven up) whenever S_t hits m_t (resp. M_t).

The full SDE satisfied by the process X therefore is

$$dX_t = X_t(\mu\ dt + \sigma\ dW_t) - \frac{dm_t}{m_t}\left(\mathbb{1}_{\{X_t=1\}} + \bar{s}\mathbb{1}_{\{X_t=\bar{s}\}}\right). \tag{114}$$

3.5 Formulation of the Theorem

Finally, it is time to formulate a mathematically precise theorem.

Theorem 3.6. *Fix* $\theta = \frac{\mu}{\sigma^2} \in \,]0,1[$ *and* $S_0 = 1$ *in the Black-Scholes model* (69)*. Let* $c(\lambda), \bar{s}(\lambda)$*, and* $g(\cdot) = g_{c(\lambda)}(\cdot)$ *be as in Proposition 3.5 where we suppose that the transaction costs* $\lambda > 0$ *are sufficiently small.*

Define the continuous process $\tilde{S} = (\tilde{S}_t)_{t \geqslant 0}$ *by*

$$\tilde{S}_t = m_t \; g\left(\frac{S_t}{m_t}\right), \qquad t \geqslant 0, \tag{115}$$

where the process $(m_t)_{t\geqslant 0}$ *is defined in* (107), (109) *and* (111)*.*

Then $\tilde{S}$ *is an Itô process, starting at* $\tilde{S}_0 = 1$*, and satisfying the stochastic differential equation*

$$d\tilde{S}_t = g'\left(\frac{S_t}{m_t}\right) \; dS_t + \frac{1}{2m_t} \; g''\left(\frac{S_t}{m_t}\right) \; d\langle S\rangle_t. \tag{116}$$

Moreover $\tilde{S}$ *takes values in the bid-ask spread* $[(1-\lambda)S, S]$*.*

Proof. We may apply Itô's formula to (115). Using (112), (114) and keeping in mind that $(m_t)_{t\geqslant 0}$ is of finite variation, we obtain

$$\begin{aligned}
d\tilde{S}_t &= d\left(m_t g(X_t)\right) \\
&= m_t \; d\left(g(X_t)\right) + g(X_t) \; dm_t \\
&= m_t \left(g'(X_t) \; dX_t + \tfrac{g''(X_t)}{2} \; d\langle X\rangle_t\right) + g(X_t) dm_t \\
&= m_t \Big(g'(X_t) \Big(X_t(\mu \; dt + \sigma \; dW_t) - \tfrac{dm_t}{m_t}\left(\mathbb{1}_{\{X_t=1\}} + \bar{s}\mathbb{1}_{\{X_t=\bar{s}\}}\right)\Big) \\
&\qquad\qquad + \tfrac{g''(X_t)}{2} \; X_t^2\sigma^2 \; dt\Big) + g(X_t) \; dm_t \\
&= g'\left(\tfrac{S_t}{m_t}\right) S_t(\mu \; dt + \sigma \; dW_t) + \tfrac{1}{2} g''(\tfrac{S_t}{m_t}) \tfrac{1}{m_t} \; S_t^2 \sigma^2 \; dt \\
&\quad - g'(X_t) \; dm_t \left(\mathbb{1}_{\{X_t=1\}} + \bar{s}\mathbb{1}_{\{X_t=\bar{s}\}}\right) + g(X_t) \; dm_t \\
&= g'\left(\tfrac{S_t}{m_t}\right) \; dS_t + \frac{g''\left(\frac{S_t}{m_t}\right)}{2m_t} \; d\langle S\rangle_t,
\end{aligned}$$

where in the last line we have used that $dm_t \neq 0$ only on $\{X_t = 1\} \cup \{X_t = \bar{s}\}$ and $g(s) = s \; g'(s)$ for $s = 1$ as well as for $s = \bar{s}$. ■

Corollary 3.7. *Under the assumptions of Theorem 3.6, fix a horizon* $T > 0$ *and consider an economic agent with initial endowment* $x > 0$ *who can trade in a frictionless way in the stock* $(\tilde{S}_t)_{0\leqslant t\leqslant T}$ *as defined in* (115)*.*

The unique process $(\hat\varphi^0_t, \hat\varphi^1_t)_{0\leqslant t\leqslant T}$ of holdings in bond and stock respectively which optimizes

$$\mathbb{E}\left[\log\left(x + (\varphi^1\cdot\tilde S)_T\right)\right] \to \max! \tag{117}$$

where φ^1 runs through all predictable, $\tilde S$-integrable, admissible[1] *processes and $\varphi^0_t = x + (\varphi^1\cdot\tilde S)_t - \varphi^1_t\tilde S_t$, is given by the following formulae.*

$$(\hat\varphi^0_{0-}, \hat\varphi^1_{0-}) = (x, 0), \quad (\hat\varphi^0_0, \hat\varphi^1_0) = \left(\frac{c}{c+1}x, \frac{1}{c+1}x\right) \tag{118}$$

and

$$\hat\varphi^0_t = \hat\varphi^0_{\varrho_{k-1}}\left(\frac{m_t}{m_{\varrho_{k-1}}}\right)^{\frac{1}{c+1}} \quad \textit{on} \quad \bigcup_{k=1}^{\infty}[\![\varrho_{k-1}, \sigma_k]\!], \tag{119}$$

$$\hat\varphi^0_t = \hat\varphi^0_{\sigma_k}\left(\frac{m_t}{m_{\sigma_k}}\right)^{\frac{(1-\lambda)\bar s}{c+(1-\lambda)\bar s}} \quad \textit{on} \quad \bigcup_{k=1}^{\infty}[\![\sigma_k, \varrho_k]\!], \tag{120}$$

as well as

$$\hat\varphi^1_t = \hat\varphi^1_{\varrho_{k-1}}\left(\frac{m_t}{m_{\varrho_{k-1}}}\right)^{-\frac{c}{c+1}} \quad \textit{on} \quad \bigcup_{k=1}^{\infty}[\![\varrho_{k-1}, \sigma_k]\!], \tag{121}$$

$$\hat\varphi^1_t = \hat\varphi^1_{\sigma_k}\left(\frac{m_t}{m_{\sigma_k}}\right)^{-\frac{c}{c+(1-\lambda)\bar s}} \quad \textit{on} \quad \bigcup_{k=1}^{\infty}[\![\sigma_k, \varrho_k]\!]. \tag{122}$$

The corresponding fraction of wealth invested into stock is given by

$$\tilde\pi_t = \frac{\hat\varphi^1_t\tilde S_t}{\hat\varphi^0_t + \hat\varphi^1_t\tilde S_t} = \frac{1}{1 + c/g(\frac{S_t}{m_t})}. \tag{123}$$

Proof. By (115), $\tilde S$ is an Itô process with locally bounded coefficients. We may write (116) as

$$\begin{aligned}\frac{d\tilde S_t}{\tilde S_t} &= g'\left(\frac{S_t}{m_t}\right)\frac{dS_t}{m_t g(\frac{S_t}{m_t})} + \frac{1}{2m_t^2}g''\left(\frac{S_t}{m_t}\right)\frac{d\langle S\rangle_t}{g(\frac{S_t}{m_t})}\\ &= \underbrace{\frac{S_t^2\sigma^2 g'(\frac{S_t}{m_t})^2}{m_t^2\left(c + g(\frac{S_t}{m_t})\right)g(\frac{S_t}{m_t})}}_{=:\tilde\mu_t}dt + \underbrace{\frac{S_t\sigma g'(\frac{S_t}{m_t})}{m_t g(\frac{S_t}{m_t})}}_{=:\tilde\sigma_t}dW_t\end{aligned}$$

[1] Admissibility of φ^1 is defined by requiring that the stochastic integral $\varphi^1\cdot\tilde S$ remains uniformly bounded from below.

It follows from the ODE (85) that the mean variance ratio process $\frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2}$ is a bounded process given by

$$\frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2} = \frac{1}{1 + c/g(\frac{S_t}{m_t})}. \tag{124}$$

On the other hand, the adapted process $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t \geqslant 0}$ defined in (119) - (122) is predictable. By definition,

$$\hat{\varphi}_t^0 = c m_t \hat{\varphi}_t^1, \quad t \geqslant 0. \tag{125}$$

For any $k \in \mathbb{N}$, Itô's formula, equation (125), and the fact that $dm_t \neq 0$ only on $\{S_t = m_t\}$ yield

$$d\hat{\varphi}_t^0 + \tilde{S}_t d\hat{\varphi}_t^1 = \left[\left(\frac{m_t}{m_{\varrho_{k-1}}} \right)^{-c/(c+1)} \frac{1}{c+1} \left(\frac{\hat{\varphi}^0_{\varrho_{k-1}}}{m_{\varrho_{k-1}}} - c \hat{\varphi}^1_{\varrho_{k-1}} \right) \right] dm_t = 0,$$

on $[\![\rho_{k-1}, \sigma_k]\!]$ and likewise on $[\![\sigma_k, \rho_k]\!]$ where we use the fact that $dm_t \neq 0$ only on $\{S_t = \bar{s} m_t\}$. Therefore $(\hat{\varphi}^0, \hat{\varphi}^1)$ is self-financing. Again by (125), the fraction

$$\frac{\hat{\varphi}_t^1 \tilde{S}_t}{\hat{\varphi}_t^0 + \hat{\varphi}_t^1 \tilde{S}_t} = \frac{1}{1 + c/g(\frac{S_t}{m_t})}$$

of wealth invested into stocks, when following $(\hat{\varphi}^0, \hat{\varphi}^1)$, coincides with the Merton proportion computed in (124). Hence $(\hat{\varphi}^0, \hat{\varphi}^1)$ is log-optimal and we are done. ■

In order to discuss the economic message of Corollary 3.7, it is instructive to – formally – pass to the limiting case $\lambda = 0$. In this case we have $\tilde{S}_t = S_t = m_t = M_t$, as well as $c = \frac{1-\theta}{\theta}$ and $\bar{s} = 1$, so that the exponents in (119) - (122) equal

$$\frac{1}{c+1} = \theta, \quad -\frac{c}{c+1} = \theta - 1.$$

We thus find after properly passing to the limits in (119) - (122) the well known formulae due to R. Merton [181]

$$\hat{\varphi}_t^0 = (1-\theta) S_t^\theta, \qquad \hat{\varphi}_t^1 = \theta S_t^{\theta - 1} \tag{126}$$

and the fraction of wealth $\tilde{\pi}_t$ invested into stock equals

$$\tilde{\pi}_t = \frac{1}{c+1} = \theta. \tag{127}$$

Passing again to the present case $\lambda > 0$, we have $c > \frac{1-\theta}{\theta}$ and $\bar{s} > 1$. We then find for the exponents in (119), (120)

$$\frac{1}{c+1} < \theta < \frac{(1-\lambda)\bar{s}}{c+(1-\lambda)\bar{s}}. \tag{128}$$

In fact, as was kindly pointed out to us by Paolo Guasoni ([106, Remark after Theorem 5.1]) θ is precisely the arithmetic mean of $\frac{1}{1+c}$ and $\frac{(1-\lambda)\bar{s}}{c+(1-\lambda)\bar{s}}$; this fact can be verified by inserting the formulae (87), (88), (90), and (93) into the identity $g(\bar{s}) = (1-\lambda)\bar{s}$ (compare [91]).

The economic message of (119) - (122) is that we now have to distinguish between the intervals $[\![\varrho_{k-1}, \sigma_k]\!]$ and $[\![\sigma_k, \varrho_k]\!]$. The former are those periods of time when $(m_t)_{0\leqslant t\leqslant T}$ is non-increasing; correspondingly during these intervals the agent only buys stock so that $(\varphi_t^0)_{0\leqslant t\leqslant T}$ is decreasing and $(\varphi_t^1)_{0\leqslant t\leqslant T}$ is increasing. Similarly, the intervals $[\![\sigma_k, \varrho_k]\!]$ are those periods during which $(m_t)_{0\leqslant t\leqslant T}$ is non-decreasing so that the agent only sells stock. The dependence (126) of $(\varphi_t^0)_{0\leqslant t\leqslant T}$ and $(\varphi_t^1)_{0\leqslant t\leqslant T}$ on $\tilde{S}_t = S_t = m_t$ via a power of this process now is replaced by the equations (119) - (122) where the exponents are somewhat different from θ and $(1-\theta)$ respectively, and where we have to distinguish whether we are in the buying or in the selling regime.

As regards the fraction of wealth $\tilde{\pi}_t$ invested into the stock $\tilde{S}$, the message of (123) is that this fraction oscillates between $\frac{1}{1+c}$ and $\frac{1}{1+c/((1-\lambda)\bar{s})}$ as $X_t = \frac{S_t}{m_t}$ oscillates between 1 and $\bar{s}$. Looking again at (128) we obtain — thanks to Paolo Guasoni's observation — that the Merton proportion θ lies precisely in the middle of these two quantities. Economically speaking, this means that the no-trade region is perfectly symmetric around θ, provided that we measure it in terms of the fraction $\tilde{\pi}_t$ of wealth invested into stock where we value the stock by the shadow price $\tilde{S} = g(s)$.

The most important message of Corollary 3.7 is that the optimal strategy $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0\leqslant t\leqslant T}$ only moves when $(m_t)_{t\geqslant 0}$ moves; the buying of stock takes place when $\tilde{S}_t = S_t$ while selling happens only when $\tilde{S}_t = (1-\lambda)S_t$. This property will be crucial when interpreting $\tilde{S}$ as a shadow price process for the bid-ask process $([(1-\lambda)S_t, S_t])_{0\leqslant t\leqslant T}$.

Another important feature of the present situation is time homogeneity. The conclusion of Corollary 3.7 does not depend on the horizon T.

3.6 Formulation of the optimization problem

We now know that Corollary 3.7 is the answer. But we don't know yet precisely, what the question is! To prepare for the precise formulation, let us start with a formal definition of admissible trading strategies in the presence of transaction costs $\lambda > 0$.

Definition 3.8. *Fix a strictly positive stock price process $S = (S_t)_{0 \leqslant t \leqslant T}$ with continuous paths and transaction costs $\lambda > 0$.*

A self-financing trading strategy starting with zero endowment *is a pair of right continuous, adapted finite variation processes $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ such that*
(i) $\varphi_{0-}^0 = \varphi_{0-}^1 = 0$
(ii) $\varphi_t^0 = \varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}$ *and* $\varphi_t^1 = \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow}$, *where* $\varphi_t^{0,\uparrow}, \varphi_t^{0,\downarrow}, \varphi_t^{1,\uparrow}$, *and* $\varphi_t^{1,\downarrow}$ *are the decompositions of φ^0 and φ^1 into the difference of increasing processes, starting at $\varphi_{0-}^{0,\uparrow} = \varphi_{0-}^{0,\downarrow} = \varphi_{0-}^{1,\uparrow} = \varphi_{0-}^{1,\downarrow} = 0$, and satisfying*

$$d\varphi_t^{0,\uparrow} \leqslant (1-\lambda) S_t d\varphi_t^{1,\downarrow}, \quad d\varphi_t^{0,\downarrow} \geqslant S_t d\varphi_t^{1,\uparrow}, \quad 0 \leqslant t \leqslant T. \tag{129}$$

The trading strategy (φ^0, φ^1) is called admissible *if there is $M > 0$ such that*

$$V_t(\varphi^0, \varphi^1) := \varphi_t^0 + (\varphi_t^1)^+ (1-\lambda) S_t - (\varphi_t^1)^- S_t \geqslant -M, \tag{130}$$

holds true a.s., for $0 \leqslant t \leqslant T$.

For example, the process $(\hat{\varphi}_t^0 - x, \hat{\varphi}_t^1)_{0 \leqslant t \leqslant T}$, where $(\hat{\varphi}^0, \hat{\varphi}^1)$ was defined in Corollary 3.7 is an admissible trading strategy with zero endowment. Indeed, the buying of the stock, i.e. $d\varphi_t^{1,\uparrow} \neq 0$, only takes place when $\tilde{S}_t = S_t$ and the selling, i.e. $d\varphi_t^{1,\downarrow} \neq 0$, happens only when $\tilde{S}_t = (1-\lambda)S_t$. In addition, $(\hat{\varphi}_t^0)_{t \geqslant 0}$ and $(\hat{\varphi}_t^1)_{t \geqslant 0}$ are of finite variation and as $0 < \theta < 1$, we have $\hat{\varphi}_t^0 > 0, \hat{\varphi}_t^1 > 0$.

Now we define a convenient version of our optimization problem.

Definition 3.9. *Fix $\theta = \frac{\mu}{\sigma^2} \in \,]0,1[$ in the Black-Scholes model* (69)*, sufficiently small transaction costs $\lambda > 0$, as well as an initial endowment $x > 0$ and a horizon T.*

Let $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ be the process defined in Theorem 3.6. The optimization problem is defined as

$$(P_x) \qquad \mathbb{E}\left[\log(x + \varphi_T^0 + \varphi_T^1 \tilde{S}_T)\right] \to \max! \tag{131}$$

where (φ^0, φ^1) runs through the admissible trading strategies with transaction costs λ starting with zero endowment $(\varphi_{0-}^0, \varphi_{0-}^1) = (0,0)$.

The definition is designed in such a way that the subsequent result holds true.

Theorem 3.10. *Under the hypotheses of Definition 3.9 the unique optimizer in* (131) *is $(\hat{\varphi}^0 - x, \hat{\varphi}^1)$, where $(\hat{\varphi}^0, \hat{\varphi}^1)$ are given by Corollary 3.7.*

Proof. The process $(\hat{\varphi}^0, \hat{\varphi}^1)$ is the unique optimizer to the optimization problem (117) when we optimize over the larger class of admissible trading strategies in the frictionless market $\tilde{S}$.

As $(\hat{\varphi}^0, \hat{\varphi}^1)$ also is an admissible trading strategy in the sense of Definition 3.8 the assertion of the theorem follows *a fortiori*. ∎

Let us have a critical look at the precise features of Definition 3.9. After all, we are slightly cheating: we use the process $\tilde{S}$, which is *part of the solution*, for the *formulation of the problem.* Why do we do this trick? We just have seen that this way of defining the optimization problem allows for the validity of the elegant Theorem 3.10. We also remark that Theorem 3.10 exhibits the same time homogeneity, i.e. non-dependence on the horizon T, as Theorem 3.1 and Corollary 3.7.

But the honest formulation of problem (131) would be

$$(P'_x) \qquad \mathbb{E}\left[\log(x + \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T)\right] \to \max! \tag{132}$$

The economic interpretation of (P'_x) is that at time T the liquidation of the position φ_T^1 in stock has to be done at the ask price S_T or the bid price $(1-\lambda)S_T$, depending on the sign of φ_T^1. On the other hand the problem (P_x) in (131) allows for liquidation at the shadow price $\tilde{S}_T$, which is a random variable taking values in $[(1-\lambda)S_T, S_T]$.

The problem (P'_x) does not allow for a mathematically nice treatment as it lacks time homogeneity (see [91] for a more detailed discussion pertaining to the economic aspects). But (P_x) is a good proxy for (P'_x): the difference between $\tilde{S}_T$ as opposed to $(1-\lambda)S_T$ and S_T is of order λ and only pertains to *one instance* of trading, namely at time T. On the other hand we have seen in Proposition 3.5 (compare also Proposition 3.11 below) that the leading terms of the effects of transaction costs on the *dynamic trading* activities during the interval $[0, T[$ are of order $\lambda^{1/3}$. Hence, for fixed horizon T, the latter effect becomes dominant as $\lambda \to 0$.

The situation becomes even better if we consider the limiting case $T \to \infty$. After proper normalization (see, e.g., (135) below) the difference between (P_x) and (P'_x) completely disappears in the limit $T \to \infty$. For example, in (137) below we find the exact dependence on $\lambda > 0$ (involving *all* the powers of $\lambda^{1/3}$) independently of whether we consider the problem (P_x) or (P'_x). For all these reasons we believe that (P_x) is the "good" definition of the problem.

3.7 The Case $\theta \geqslant 1$

The preceding results pertain to the case $0 < \theta < 1$, where we have seen that the optimal holdings $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)$ in bond as well as in stock are strictly positive, for all $t \geqslant 0$.

The case $\theta = 1$ is degenerate. As is well known and immediately deduced from Theorem 3.1, in the absence of transaction costs the optimal strategy consists in fully investing the initial endowment x into stock at time zero, so that $\hat{\varphi}_t^0 \equiv 0$ and $\hat{\varphi}_t^1 \equiv x$, if S_0 is normalized to 1. In the presence of transaction costs $\lambda > 0$ it is rather obvious, from an economic point of view, that this strategy still is optimal. In fact, if we define the shadow price process $\tilde{S}$ simply by $\tilde{S}_t = S_t$, then the above strategy $(\hat{\varphi}_t^0, \hat{\varphi}_t^1) = (0, x)$, for $0 \leqslant t \leqslant T$ also is the solution to the problem (P_x) in (131) in a formal way.

More challenging is the case $\theta > 1$. In this regime the well-known frictionless optimal strategy involves a *short position* in bond, i.e. $\varphi_t^0 < 0$, and using this leverage to finance a long position φ_t^1 in stock, so that $\varphi_t^1 S_t$ exceeds the current wealth of the agent.

This phenomenon also carries over to the situation under (sufficiently small) transaction costs $\lambda > 0$. In this situation the agent *buys* stock when stock prices are rising and *sells* stock when stock prices are falling, i.e., she has the opposite behavior of the case $0 < \theta < 1$.

Mathematically speaking, this results in the fact that we again look at the function g as defined in (87), satisfying the ODE (85), but now the domain of definition of g is given by an interval $[\bar{s}, 1]$, where $\bar{s} < 1$.

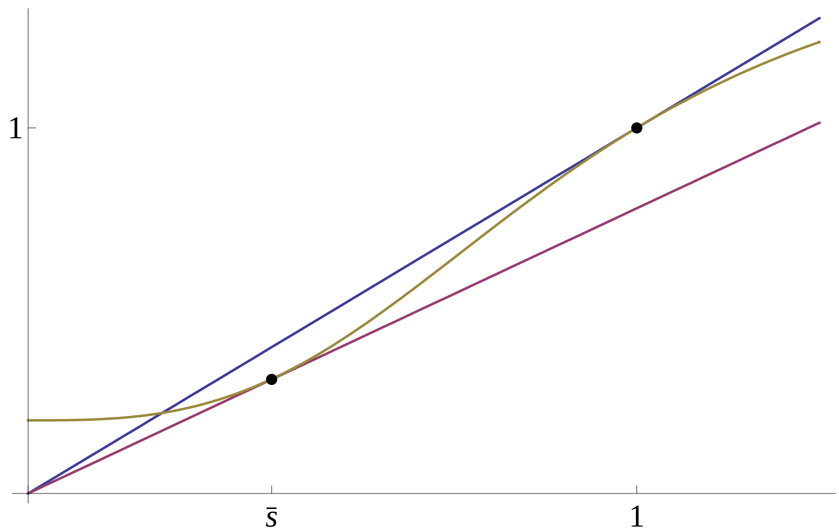


Figure 6: Smooth pasting conditions for the function g, for $\theta > 1$.

The boundary conditions still are given by (86) and (89), and the formula for $c = c(\lambda)$ and $\bar{s} = \bar{s}(\lambda)$ still are given by (97) and (98) (applying the convention $(-x)^{1/3} = -(x^{1/3})$, for $x > 0$; see [91, Proposition 6.1] for details).

Hence, also in the case $\theta > 1$ we find an analogous situation as for $0 < \theta < 1$. The story simply has to be told the other way round: we start again with the normalizing assumption $S_0 = 1$, as well as the definition $\tilde{S}_0 = g(S_0) = 1$, which corresponds to assuming that the agent buys stock at time $t = 0$, just as above.

Now suppose (heuristically) that the stock starts a *negative* excursion at time $t = 0$, i.e. $S_t < 1$, for $t > 0$ small enough. We then define $\tilde{S}$ by

$$\tilde{S}_t = g(S_t), \qquad t \geqslant 0,$$

up to time $t_0 > 0$ when S_t hits again 1, or when S_t hits for the first time $\bar{s}$ (which now is less than 1).

Passing to the general (and generic) case, i.e. dropping the assumption about the negative excursion starting at $t = 0$, we define the running maximum process $(M_t)_{t \geqslant 0}$ locally by

$$M_t = \sup_{0 \leqslant u \leqslant t} S_u, \qquad 0 \leqslant t \leqslant \varrho_1$$

where ϱ_1 is the first time when $S_t/M_t \leqslant \bar{s}$. We define

$$\tilde{S}_t = M_t g\left(\frac{S_t}{M_t}\right), \qquad \text{for } 0 \leqslant t \leqslant \varrho_1.$$

During the stochastic interval $[\![0, \varrho_1]\!]$ the agent *buys* stock whenever $(M_t)_{0 \leqslant t \leqslant \varrho_1}$ moves up, following a similar logic as in (119) - (122) above.

After time ϱ_1 the agent monitors locally the running minimum process $(m_t)_{t \geqslant \varrho_1}$

$$m_t = \min_{\varrho_1 \leqslant u \leqslant t} S_u, \qquad \varrho_1 \leqslant t \leqslant \sigma_1$$

where σ_1 is the first time when $\frac{S_t}{m_t} \geqslant \frac{1}{\bar{s}}$. We define $\tilde{S}_t := \frac{m_t}{\bar{s}} g(\frac{\bar{s} S_t}{m_t})$ for $\varrho_1 \leqslant t \leqslant \sigma_1$. During the stochastic interval $[\![\varrho_1, \sigma_1]\!]$, the agent *sells* stock when m_t moves down.

The reasoning is perfectly analogous to section 3.4 above. We refer to [91] for details and only mention that, for $\theta > 1$, the parameter c in Proposition 3.4 now has to vary in $]\frac{1-\theta}{\theta}, 0[$.

There is still one slightly delicate issue in the case $\theta > 1$ which we have not yet discussed: the *admissibility* of the optimal strategies $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t \geqslant 0}$ which, also in the case $\theta > 1$, are given by formulas (118) - (122). Now the holdings $(\hat{\varphi}_t^1)_{t \geqslant 0}$ in bond are negative so that we have to check more carefully whether the agent is solvent at all times $t \geqslant 0$. As $\hat{\varphi}_t^1 \geqslant 0$, the natural condition is

$$\hat{\varphi}_t^0 + \hat{\varphi}_t^1 S_t (1 - \lambda) \geqslant 0, \qquad t \geqslant 0. \tag{133}$$

We know that

$$\hat{\varphi}_t^0 + \hat{\varphi}_t^1 \tilde{S}_t \geqslant 0, \tag{134}$$

a.s., for each $t \geqslant 0$. Indeed $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t \geqslant 0}$ is the log-optimal portfolio for the frictionless market $(\tilde{S}_t)_{t \geqslant 0}$; it is well-known form the frictionless theory (Theorem 3.1) and rather obvious that (134) has to hold true.

To show that even (133) is satisfied, fix $t_0 \geqslant 0$ and $(\hat{\varphi}_{t_0}^0, \hat{\varphi}_{t_0}^1, \tilde{S}_{t_0})$ such that $\tilde{S}_{t_0} \in \,](1-\lambda)S_{t_0}, S_{t_0}[$. Conditionally on $(\hat{\varphi}_{t_0}^0, \hat{\varphi}_{t_0}^1, \tilde{S}_{t_0})$ define the stopping times ϱ and σ.

$$\begin{aligned} \varrho &= \inf\{t > t_0 : \tilde{S}_t = S_t\}, \\ \sigma &= \inf\{t > t_0 : \tilde{S}_t = (1-\lambda)S_t\}. \end{aligned}$$

Clearly we have, conditionally on $(\hat{\varphi}_{t_0}^0, \hat{\varphi}_{t_0}^1, \tilde{S}_{t_0})$, that $\mathbb{P}[\sigma < \varrho] > 0$. As $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t_0 \leqslant t \leqslant \sigma \wedge \varrho}$ remains constant and using $S_\sigma < S_{t_0}$ on $\{\sigma < \varrho\}$ we deduce from

$$\hat{\varphi}_\sigma^0 + \hat{\varphi}_\sigma^1 \tilde{S}_\sigma \geqslant 0, \qquad \text{on } \{\sigma < \varrho\}$$

that

$$\hat{\varphi}_\sigma^0 + \hat{\varphi}_\sigma^1 (1-\lambda) S_\sigma \geqslant 0 \qquad \text{on } \{\sigma < \varrho\}$$

so that

$$\hat{\varphi}_{t_0}^0 + \hat{\varphi}_{t_0}^1 (1-\lambda) S_{t_0} \geqslant \hat{\varphi}_\sigma^0 + \hat{\varphi}_\sigma^1 (1-\lambda) S_\sigma \geqslant 0.$$

This proves (133).

3.8 The Optimal Growth Rate

We now want to compute the optimal growth rate

$$\delta := \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\log(1 + \hat{\varphi}_T^0 + \tilde{S}_T \hat{\varphi}_T^1)\right] = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2} dt\right], \tag{135}$$

where the initial endowment x is normalized by $x = 1$, and $(\hat{\varphi}^0, \hat{\varphi}^1)$ denotes the log-optimal portfolio for the shadow price $\tilde{S}$ from Corollary 3.7. The second equality follows from Theorem 3.1 and Theorem 3.10 (compare [151, Example 6.4]).

By the construction in (112) the process $X = S/m$ is a geometric Brownian motion with drift which is reflected on the boundaries of the interval $[1, \bar{s}]$ (resp. on $[\bar{s}, 1]$ for the case $\theta > 1$). Therefore, an ergodic theorem for positively recurrent one-dimensional diffusions (cf. e.g. [23, Sections II.36 and II.37]) and elementary integration yield the following result.

Proposition 3.11. *Suppose the conditions of Theorem 3.6 hold true. Then the process $X = S/m$ has the stationary distribution*

$$\nu(ds) = \begin{cases} \dfrac{2\theta - 1}{\bar{s}^{2\theta-1} - 1} s^{2\theta-2} \mathbf{1}_{[1,\bar{s}]}(s) ds & \textit{for } \theta \in (0,1)\backslash\{\frac{1}{2}\}, \\ \dfrac{1}{\log(\bar{s})} s^{-1} \mathbf{1}_{[1,\bar{s}]}(s) ds & \textit{for } \theta = \frac{1}{2}, \\ \dfrac{2\theta - 1}{1 - \bar{s}^{2\theta-1}} s^{2\theta-2} \mathbf{1}_{[\bar{s},1]}(s) ds & \textit{for } \theta \in (1,\infty). \end{cases}$$

Moreover, the optimal growth rate for the frictionless market with price process $\tilde{S}$ as well as for the market with bid-ask process $[(1-\lambda)S, S]$ is given by

$$\begin{aligned} \delta &= \Big| \int_1^{\bar{s}} \frac{\tilde{\mu}^2(s)}{2\tilde{\sigma}^2(s)} \nu(ds) \Big| \\ &= \begin{cases} \dfrac{(2\theta-1)\sigma^2 \bar{s}}{2(1+c)(\bar{s} + (-2 - c + 2\theta(1+c))\bar{s}^{2\theta})} & \textit{for } \theta \in (0,\infty)\backslash\{\frac{1}{2}, 1\}, \\ \dfrac{\sigma^2}{2(1+c)(1 + c - \log \bar{s})} & \textit{for } \theta = \frac{1}{2}, \end{cases} \end{aligned} \tag{136}$$

where c and $\bar{s}$ denote the constants from Proposition 3.5.

As $\lambda \to 0$, the optimal growth rate has the asymptotics

$$\delta = \frac{\mu^2}{2\sigma^2} - \left(\frac{3\sigma^3}{\sqrt{128}} \theta^2 (1-\theta)^2 \right)^{2/3} \lambda^{2/3} + O(\lambda^{4/3}). \tag{137}$$

Proof. The calculation of the invariant distribution ν of the process X is an elementary exercise. The remaining calculations are tedious, but elementary too (see [91, Proposition 5.4 and 6.3]). ■

3.9 Primal versus Dual Approach

In the preceding arguments we have developed the solution to the problem of finding the growth-optimal portfolio under transaction costs by using the "dual" approach, which also sometimes is called the "martingale method" (compare the pioneering paper [43] by Cvitanic and Karatzas). Starting from the Black-Scholes model (69), we have considered the "shadow price process" $\tilde{S} = (\tilde{S}_t)_{t \geqslant 0}$ which in the notation of (36) corresponds to

$$\tilde{S}_t = \frac{\hat{Z}_t^1}{\hat{Z}_t^0}. \tag{138}$$

In the present context the density process $(\hat{Z}_t^0)_{t\geqslant 0}$ is given by Girsanov's formula

$$\hat{Z}_t^0 = \exp\left(-\int_0^t \frac{\tilde{\mu}_s}{\tilde{\sigma}_s} dW_s - \int_0^t \frac{\tilde{\mu}_s^2}{2\tilde{\sigma}_s^2} ds \right). \tag{139}$$

It is the unique $\mathbb{P}$-martingale with respect to the filtration generated by W and starting at $\hat{Z}_0^0 = 1$, such that the process $\hat{Z}_t^1 := \hat{Z}_t^0 \tilde{S}_t$ is a $\mathbb{P}$-martingale too. As we have seen in chapter 2, this solution of the dual problem can be translated into the solution of the primal problem via the first order conditions (37).

It is worthwhile to spell out explicitly the formulation of the dual problem corresponding to (36). The conjugate function $V(y)$ associated to $U(x) = \log(x)$ by (34) is

$$V(y) = -\log(y) - 1, \qquad\qquad y > 0.$$

Under the assumptions of Corollary 3.7 we define for fixed $T > 0$, in analogy to (36) and using (139),

$$\begin{aligned} v(y) &= \mathbb{E}[V(y\hat{Z}_T^0)] \\ &= -\log(y) - 1 + \mathbb{E}\left[\int_0^T \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2} dt\right] \\ &= V(y) + \mathbb{E}\left[\int_0^T \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2} dt\right]. \end{aligned}$$

Hence we find as in Theorem 2.3 that $v(y)$ is the conjugate function to the indirect utility function associated to the shadow price process $\tilde{S}$

$$\begin{aligned} u(x) &= \mathbb{E}[U(x\hat{V}_T)] \\ &= \log(x) + \mathbb{E}\left[U\left(\exp\left(\int_0^T \frac{\tilde{\mu}_t}{\tilde{\sigma}_t} dW_t + \int_0^T \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2} dt\right)\right)\right] \\ &= U(x) + \mathbb{E}\left[\int_0^T \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2} dt\right], \end{aligned}$$

where

$$\hat{V}_T = \exp\left(\int_0^T \frac{\tilde{\mu}_t}{\tilde{\sigma}_t} dW_t + \int_0^T \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2} dt\right)$$

denotes the optimal terminal wealth for the frictionless market $\tilde{S}$.

The above considerations pertain to the frictionless complete market $\tilde{S}$; they carry over verbatim to the bid as process $[(1-\lambda)S, S]$ if we use definition (131) for the formulation of the portfolio optimization problem.

Another approach to finding the growth optimal portfolio is to directly attack the primal problem which leads to a Hamilton-Jacobi-Bellman equation for the *value function* associated to the primal problem; in economic terminology this value function (see (141) below) is called the "indirect utility function".

This strain of literature has a longer history than the "dual approach" [43]. In [234] Taksar, Klass and Assaf give a solution to the present problem of finding the growth optimal portfolio, and in [75] Dumas and Luciano solve the same problem for power utility $U(x) = \frac{x^\gamma}{\gamma}, 0 < \gamma < 1$, rather than for $U(x) = \log(x)$. Let us also mention the work of Davis and Norman [57] and Shreve and Soner [224] on optimal consumption which proceeds by the primal method too. We refer to [124] for an account on the ample literature persuing this "primal" method.

We shall present here the approach of [234] and [75]. Our aim is to relate the "primal" and the "dual" approach, thus gaining additional insight into the problem. While in the preceding subsections the mathematics were finally done in a rigorous way, we now content ourselves to more informal and heuristic considerations. We can afford to do so as we have established things rigorously already above.

Fixing the level $\lambda > 0$ of (sufficiently small) transactions costs, the horizon T, and an initial endowment $(\varphi^0, \varphi^1) \in \mathbb{R}^2_+$ in bond[2] and stock, we define

$$u(\varphi^0, \varphi^1, s, T) = \sup\{\mathbb{E}[\log(\varphi^0_T + \varphi^1_T S_T)|S_0 = s]\} \tag{140}$$

where $(\varphi^0_T, \varphi^1_T)$ runs through all pairs of positive $\mathcal{F}_T$-measurable random variables (modeling the holdings in units of bond and stock at time T) which can be obtained by admissible trading (and paying transaction costs λ) as in (129), starting from initial positions $(\varphi^0_{0-}, \varphi^1_{0-}) = (\varphi^0, \varphi^1)$ in bond and stock.

The term $(\varphi^0_T + \varphi^1_T S_T)$ in (140) above corresponds to the modeling assumption that the position φ^1_T in stock can be liquidated at time T at price S_T. One might also define (140) by using $(\varphi^0_T + \varphi^1_T(1-\lambda)S_T)$. As observed at the end of sub-section 3.6, this difference will play no role when we eventually pass to the (properly scaled) limit $T \to \infty$, hence we may as well use (140) as is done in [75].

Turning back to a fixed horizon $T > 0$, define, for $0 \leqslant t \leqslant T$, the value function

$$u(\varphi^0, \varphi^1, s, t, T) = \sup\{\mathbb{E}[\log(\varphi^0_T + \varphi^1_T S_T)|S_t = s]\}, \tag{141}$$

[2] in [234] and [75] no short-selling is allowed so that $\varphi^0 \geqslant 0, \varphi^1 \geqslant 0$. Hence we assume, as in these papers, that $\theta = \frac{\mu}{\sigma^2} \in \,]0, 1[$.

where now $(\varphi^0_T, \varphi^1_T)$ range in the random variables which can be obtained, similarly as above, by admissible trading during the period $[t, T]$, and starting at time t_- with holdings $(\varphi^0_{t_-}, \varphi^1_{t_-}) = (\varphi^0, \varphi^1)$.

The idea is to pass, for fixed $t > 0$, to the limit $T \to \infty$ in (141) in order to obtain an indirect utility function $u(\varphi^0, \varphi^1, s, t)$ not depending on the horizon T. But, of course, by blindly passing to this limit we shall typically find $u(\varphi^0, \varphi^1, s, t) \equiv \infty$ which yields no information.

The authors of [234] and [75] therefore *assume* that there is a constant $\delta > 0$ such that, by discounting the value of the portfolio $\varphi^0_T + \varphi^1_T S_T$ with the factor $e^{\delta T}$, we *get a finite limit* below.

$$\begin{aligned} u(\varphi^0, \varphi^1, s, t) :&= \lim_{T\to\infty} \sup\{\mathbb{E}[\log(e^{-\delta T}(\varphi^0_T + \varphi^1_T S_T))|S_t = s]\} \\ &= \lim_{T\to\infty} \sup\{\mathbb{E}[\log(\varphi^0_T + \varphi^1_T S_T)|S_t = s]\} - \delta T. \end{aligned} \tag{142}$$

According to our calculations we already *know* that the above $\delta > 0$ must be the optimal growth rate which we have found in (136). But in the primal approach of [234] and [75], the number $\delta > 0$ is a free parameter which eventually has to be determined by analyzing the boundary conditions of the differential equations related to the indirect utility function $u(\varphi^0, \varphi^1, s, t)$.

To analyze the indirect utility function u, we start by making some simplifications. From definition (142) we deduce that

$$u(\varphi^0, \varphi^1, s) := u(\varphi^0, \varphi^1, s, 0) = u(\varphi^0, \varphi^1, s, t) + \delta t, \qquad \text{for } t \geqslant 0 \tag{143}$$

where the left hand side does not depend on t anymore. We also use the scaling property of the logarithm

$$u(c\varphi^0, c\varphi^1, s) = u(\varphi^0, \varphi^1, s) + \log(c),$$

to reduce to the case where we may normalize φ^0 to be one. To eventually reduce the two remaining variables φ^1 and s to simply one dimension, make the economically obvious observation that the variables φ^1 and s only enter into the function u via the product $\varphi^1 s$. Introducing the new variable $y = \frac{\varphi^1 s}{\varphi^0}$, which describes the ratio of the value of the stock investment to the bond investment, we therefore may write u in (142) as

$$\begin{aligned} u(\varphi^0, \varphi^1, s, t) &= \log(\varphi^0) + h\left(\frac{\varphi^1 s}{\varphi^0}\right) - \delta t \\ &= \log(\varphi^0) + h(y) - \delta t \end{aligned} \tag{144}$$

for some function $h : \mathbb{R}_+ \to \mathbb{R}$ to be determined.

Let us find the Hamilton-Jacobi-Bellman equation satisfied by u. According to the basic principle of stochastic optimization [196], we must have that, for any self-financing $\mathbb{R}^2_+$-valued trading strategy $(\varphi^0_t, \varphi^1_t)_{t\geqslant 0}$, the process $(u(\varphi^0_t, \varphi^1_t, S_t, t))_{t\geqslant 0}$ is a super-martingale, which becomes a true (local) martingale if we plug in the optimal strategy $(\hat\varphi^0, \hat\varphi^1)$.

First consider the possible control of keeping $(\varphi^0_t, \varphi^1_t) = (\varphi^0, \varphi^1)$ simply constant: this yields via (69), (142) and (144)

$$\begin{aligned} du(\varphi^0_t, \varphi^1_t, S_t, t) &= u_s dS_t + \frac{u_{ss}}{2} d\langle S\rangle_t - \delta dt \\ &= \frac{\varphi^1_t}{\varphi^0_t} h'(y_t)(S_t\sigma dW_t + S_t\mu dt) + \frac{(\varphi^1_t)^2}{(\varphi^0_t)^2} h''(y_t)\left(\frac{S_t^2\sigma^2}{2} dt\right) - \delta dt, \end{aligned}$$

hence, by taking expectations and using the formal identity $\mathbb{E}[dW_t] = 0$,

$$\mathbb{E}[du(\varphi^0_t, \varphi^1_t, S_t, t)] = \left[S_t\mu \frac{\varphi^1_t}{\varphi^0_t} h'(y_t) + \frac{S_t^2\sigma^2}{2}\frac{(\varphi^1_t)^2}{(\varphi^0_t)^2} h''(y_t) - \delta\right] dt.$$

The term in the bracket has to be non-positive. We know already that, within the no-trade region, it is indeed optimal to keep φ^0_t and φ^1_t constant. Hence, by replacing $y_t = \frac{S_t\varphi^1_t}{\varphi^0_t}$ by the real variable $y > 0$, we expect that the function h will satisfy the ODE

$$h''(y)\frac{y^2\sigma^2}{2} + h'(y)y\mu - \delta = 0, \tag{145}$$

where $y = \frac{\varphi^1 s}{\varphi^0}$ ranges in the no-trade region, which should be a compact interval $[l, r]$ contained in $]0, \infty[$, which we still have to determine.

Equation (145) is an elementary ODE which, by passing to logarithmic coordinates $z = \log(y)$, can be reduced to a linear ODE. In particular, it has a closed form solution. For $\theta = \frac{\mu}{\sigma^2} \in \mathbb{R}_+\backslash\{\frac{1}{2}\}$, the general solution is given by

$$h(y) = \frac{\delta}{\mu - \frac{\sigma^2}{2}}\log(y) + C_1 y^{2\theta-1} + C_2, \tag{146}$$

while for the case $\theta = \frac{\mu}{\sigma^2} = \frac{1}{2}$ we obtain

$$h(y) = \frac{\delta}{\sigma^2}\log(y)^2 + C_1\log(y) + C_2, \tag{147}$$

where the constants C_1, C_2 still are free.

Plugging (146) into the utility function (144) with $t = 0$ we obtain

$$\begin{aligned} u(\varphi^0, \varphi^1, s) &= \log(\varphi^0) + h(y) &(148)\\ &= \log(\varphi^0) + \frac{\delta}{\mu - \frac{\sigma^2}{2}}\log(y) + C_1 y^{2\theta-1}, &(149) \end{aligned}$$

for $\theta \in \mathbb{R}_+\backslash\{\frac{1}{2}\}$, and a similar expression is obtained for $\theta = \frac{1}{2}$. We have set $C_2 = 0$ above, as an additive constant does not matter for the indirect utility. The parameters C_1 and δ are still free.

In [234] and [75] the idea is to analyze the above function and to determine the free boundaries l, r, such that $y \in [l, r]$ is the no-trade region, where the indirect utility function is given by (149) above. We therefore have to deal with 4 free parameters and to find boundary conditions, involving again smooth pasting arguments, to determine them.

We refer to [234] and [75] for the further analysis of this delicate free boundary problem. Eventually these authors achieve numerical solutions of the free boundary problem, but do not try to obtain analytical results, e.g., to develop the quantities in fractional Taylor series in $\lambda^{1/3}$ as we have done above.

Our concern of interest is the relation of the primal approach, in particular the ODE (145), with the dual approach, in particular with the shadow price process $\tilde{S}$.

This link is given by the economic idea of the *marginal rate of substitution.* Fix t and suppose that the triple $(\varphi^0, \varphi^1, s)$ is such that $y = \frac{\varphi^1 s}{\varphi^0}$ lies in the no-trade region. The indirect utility then is given by (144). Changing the position φ^0 of holdings in bond from φ^0 to $\varphi^0 + d\varphi^0$, for some small $d\varphi^0$, the indirect utility changes (of first order) by the quantity $u_{\varphi^0} d\varphi^0$, where u_{φ^0} denotes the partial derivative of $u(\varphi^0, \varphi^1, S)$ with respect to φ^0. By differentiating (144) and using (146) we obtain

$$u_{\varphi^0} = \frac{1}{\varphi^0} - \frac{\delta}{\mu - \frac{\sigma^2}{2}} \frac{1}{y} \frac{y}{\varphi^0} - C_1 \ y^{2\theta - 2} \frac{y}{\varphi^0}.$$

Similarly, changing the position of φ^1 units of stock to $\varphi^1 + d\varphi^1$ units for some small $d\varphi^1$, this change of first order equals $d\varphi^1 \ u_{\varphi^1}$, where

$$u_{\varphi^1} = \frac{\delta}{\mu - \frac{\sigma^2}{2}} \frac{1}{y} \frac{y}{\varphi^1} + C_1 \ y^{2\theta - 2} \frac{y}{\varphi^1}.$$

The natural economic question is the following: what is the price $\tilde{s} = \tilde{s}(\varphi^0, \varphi^1, s)$ for which an economic agent is — of first order — indifferent of buying/selling stock against bond? The obvious answer is that the ratio $\tilde{s} = \frac{d\varphi^0}{d\varphi^1}$ must satisfy the equality $u_{\varphi^0} d\varphi^0 = u_{\varphi^1} d\varphi^1$. In other words, $\tilde{s}$ is given by the "marginal rate of substitution"

$$\tilde{s} = \frac{u_{\varphi^1}(\varphi^0, \varphi^1, s)}{u_{\varphi^0}(\varphi^0, \varphi^1, s)} \tag{150}$$

$$= \frac{\varphi^0}{\varphi^1} \cdot \frac{\frac{\delta}{\mu - \frac{\sigma^2}{2}} + C_1 y^{2\theta-1}}{(1 - \frac{\delta}{\mu - \frac{\sigma^2}{2}}) - C_1 y^{2\theta-1}}. \tag{151}$$

This formula for $\tilde{s}$ looks already reminiscent of the function $\tilde{S} = g(s)$ in (87). To make this relation more explicit, recall that we have made the following normalizations in subsection 3.2 above: the variable s ranges in the interval $[1, \bar{s}]$ and the ratio $\frac{\varphi^0}{\varphi^1}$ of holdings in bond and stock equals the parameter c in formula (104), if we have the normalization $m_t = 1$, so that $\tilde{S}_t = g(S_t)$. Hence $y = \frac{\varphi^1 s}{\varphi^0} = \frac{s}{c}$ so that in (151) we get

$$\tilde{s} = G(s) := \frac{\frac{c\delta}{\mu - \frac{\sigma^2}{2}} + C_1 c^{2-2\theta} s^{2\theta-1}}{(1 - \frac{\delta}{\mu - \frac{\sigma^2}{2}}) - C_1 c^{1-2\theta} s^{2\theta-1}}, \tag{152}$$

Using the relation

$$\delta = \delta(c) = \frac{(2\theta - 1)\sigma^2 \bar{s}(c)}{2(1 + c)(\bar{s} + (-2 - c + 2\theta(1 + c)\bar{s}^{2\theta})}$$

obtained in (136) above, we conclude that the function $G(\cdot)$ defined in (152) above indeed equals the function g in (87) if we choose the free parameter C_1 properly. As the variable s ranges in the interval $[1, \bar{s}]$, we find that the no trade interval $[l, r]$ for the variable y equals $[\frac{1}{c}, \frac{\bar{s}}{c}]$ and we can use the Taylor expansions in powers of $\lambda^{1/3}$ to explicitly determine the values of these boundaries. We thus can provide explicit formulae for all the quantities involved in the solution of the primal problem where the PDE approach only gave numerical solutions.

We now understand better why we found a *closed form solution* for the ODE (85). As regards the function h solving the ODE (145), there is, of course, the closed form solution (146), as this ODE is linear (after passing to logarithmic coordinates). Therefore the indirect utility u in (144) again is given by an explicit formula. Hence the function $G = g$, which is deduced from the "marginal rate of substitution relation" (150), has to be so too. In conclusion, the ODE (85) *must have* a closed from solution.

3.10 Rogers' qualitative argument

We finish this chapter by recalling a lovely "back of an envelope calculation" due to Ch. Rogers [206]. It shows that the leading term for the size $\bar{s}(\lambda)-1$ of the no trade region is of the order $\lambda^{\frac{1}{3}}$ (compare (98)) and that the difference of the growth rate $\delta(\lambda)$ obtained in (137) to the frictionless growth rate $\frac{\mu^2}{2\sigma^2}$ is of the order $\lambda^{\frac{2}{3}}$. In fact, these relations were already obtained in the early work of G. Constantinides [40].

The starting point is the rather intuitive heuristic assumption that, given transaction costs $\lambda > 0$, the log optimal investor will keep the ratio of stock to the total wealth investment in an interval of width w around the Merton proportion $\theta = \frac{\mu}{\sigma^2}$.

Taking the frictionless market as benchmark, what are the (negative) effects of transaction costs λ when choosing the width w? There are two causes. On the one hand side one has to pay transaction costs TRC. From scaling it is rather obvious, at least asymptotically, that these costs are proportional to the size of transaction costs λ and indirectly proportional to the width w, i.e. $TRC \approx c_1\lambda w^{-1}$ for some constant c_1. Indeed, the local time spent at the boundary of the no trade region, where trading takes place, is of the order w^{-1}.

The second negative influence is the cost of misplacement: in comparison to the ideal ratio of the Merton proportion one typically is of the order w away from it. As the utility function attains its optimum at the Merton proportion (and assuming sufficient smoothness), the effect of the misplacement on the performance should be proportional to the square of the misplacement. This is, at least heuristically, rather obvious. Actually, the fact that a function decreases like the square of the misplacement when it is close to its maximum was already observed as early as in 1613 by Johannes Kepler in the context of the volume of wine barrels. Hence the misplacement cost MPC caused by the width w of the no trade region should asymptotically satisfy $MPC \approx c_2 w^2$, for some constant c_2.

The total cost TC of these two causes therefore has an asymptotic bahavior of the form

$$TC = TRC + MPC \approx c_1\lambda w^{-1} + c_2 w^2.$$

We have to minimize this expression as a function of w. Setting the derivative of this function equal to zero gives for the optimal width $\hat{w}$ the asymptotic relation $\hat{w} \approx c\lambda^{1/3}$, where $c = \left(\frac{c_1}{2c_2}\right)^{1/3}$.

As regards the effect of the transaction costs λ on the asymptotic growth rate, we conclude from the above argument that this is the order of the square of the typical misplacement $\hat{w}$ which in turn is of the order $\lambda^{1/3}$. Therefore the difference of the frictionless growth rate $\frac{\mu^2}{2\sigma^2}$ to the rate involving transaction costs is of the oder $\lambda^{2/3}$ (compare (137)).

3.11 Almost sure optimal growth rate and a numerical example

Very recently the preprint [80] has been brought to my attention. It is shown there that the optimality of the above defined strategy with respect to long-term growth rate not only holds true in expectation (as proved above), but also in an almost sure sense ([80], Th. 4.1).

In addition, M. Feodoria and J. Kallsen spell out in [80] a nice and illustrative numerical example. Consider a stock with yearly volatility $\sigma = 20\,\%$ and excess drift rate $\mu = 2\,\%$. The corresponding (discounted) stock price process S starting at $S_0 = 1$ equals

$$S_t = \exp\big[\sigma W_t + \big(\mu - \frac{\sigma^2}{2}\big)t\big] \tag{153}$$

$$= \exp\Big[\frac{1}{5}W_t\Big]. \tag{154}$$

For these (quite realistic) values of μ and σ we therefore find that the long-term growth rate

$$\lim_{T\mapsto\infty} \frac{1}{T}\log S_T$$

of the "buy and hold" strategy of always keeping one stock yields a long-term growth rate equals zero. This holds true in expectation as well as almost surely.

This strategy is, of course, not optimal. In the frictionless setting (Theorem 3.1) it is optimal to hold the fraction $\hat{\pi} = \frac{\mu}{\sigma^2} = \frac{1}{2}$ of the current wealth in stock, and the other half in bond. This yields a long-term excess growth rate equal to $\frac{\mu^2}{2\sigma^2} = 0.5\,\%$. This rate seems surprisingly low as compared to the fact that the expectation of the stock, $\mathbb{E}[\exp(\sigma W_t + (\mu - \frac{\sigma^2}{2})t] = \exp[\mu t]$ grows at an excess rate of $2\,\%$.

If we consider transaction costs of $\lambda = 1\,\%$ we obtain the approximate numerical values $[0.42, 0.58]$ for the no-trade interval. According to the asymptotic formula (137) this lowers the optimal excess growth rate to 0.47. In other words, even in the unfavorable case $\theta = \frac{1}{2}$ (compare (137)) the effect of transaction costs on the optimal long-term growth rate seems rather small.

4 General Duality Theory

In this chapter we continue the line of research of chapter 2 where we have refrained ourselves to the case of finite Ω.

We now consider a stock price process $S = (S_t)_{0 \leqslant t \leqslant T}$ in continuous time with a fixed horizon T. The process is assumed to be based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$, satisfying the usual conditions of completeness and right continuity. We assume that S is adapted and has *continuous*, strictly positive trajectories, i.e. the function $t \to S_t(\omega)$ is continuous, for almost each $\omega \in \Omega$. The extension to the case of càdlàg (right continuous, left limits) processes is more technical and we refer the reader to [48] for a thorough treatment.

To make life easier, we even assume that the filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ is generated by a d-dimensional Brownian motion $(W_t)_{0 \leqslant t \leqslant T}$. This convenient (but not really necessary, see [51]) assumption eases the presentation as it has the following pleasant consequence: if $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ is a local martingale under some measure $Q \sim \mathbb{P}$, then $\tilde{S}$ has $\mathbb{P}$-a.s. continuous paths.

Definition 4.1. *Fix $\lambda > 0$. A process $S = (S_t)_{0 \leqslant t \leqslant T}$ as above satisfies the condition (CPS^λ) of* having a consistent price system under transaction costs $\lambda > 0$, *if there is a process $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$, adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$ such that*

$$(1 - \lambda) S_t \leqslant \tilde{S}_t \leqslant S_t, \qquad 0 \leqslant t \leqslant T,$$

as well as a probability measure Q on $\mathcal{F}$, equivalent to $\mathbb{P}$, such that $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ is a local martingale under Q.

We say that S admits consistent price systems for arbitrarily small transaction costs *if (CPS^λ) is satisfied, for all $\lambda > 0$.*

As in chapter 1 we observe that a λ-consistent price system can also be written as a pair $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$, where now Z^0 is a $\mathbb{P}$-martingale and Z^1 a local $\mathbb{P}$-martingale. The identification again is given by the formulas $Z_T^0 = \frac{dQ}{d\mathbb{P}}$ and $\tilde{S} = \frac{Z^1}{Z^0}$.

In [107] we related the condition of *admitting consistent price systems* for arbitrarily small transaction costs to a *no arbitrage condition* under arbitrarily small transaction costs, thus proving a version of the Fundamental Theorem of Asset Pricing under (small) transaction costs.

It is important to note that we *do not assume* that S is a semi-martingale as one is forced to do in the frictionless theory [64, Theorem 7.2]. However, the process $\tilde{S}$ appearing in Definition 4.1 always is a semi-martingale, as it becomes a local martingale after passing to an equivalent measure Q.

The notion of *self-financing trading strategies* $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$, *starting at* $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (0,0)$ as well as the notion of *admissibility* have been given in Definition 3.8. For the convenience of the reader we recall it.

Definition 4.2. *Fix a stock price process* $S = (S_t)_{0 \leqslant t \leqslant T}$ *with continuous paths, as well as transaction costs* $\lambda > 0$.

A self-financing trading strategy starting with zero endowment *is a pair of right continuous, adapted finite variation processes* $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ *such that*

(*i*) $\varphi^0_{0-} = \varphi^1_{0-} = 0$.

(*ii*) *Denoting by* $\varphi^0_t = \varphi^{0,\uparrow}_t - \varphi^{0,\downarrow}_t$ *and* $\varphi^1_t = \varphi^{1,\uparrow}_t - \varphi^{1,\downarrow}_t$, *the canonical decompositions of* φ^0 *and* φ^1 *into the difference of two increasing processes, starting at* $\varphi^{0,\uparrow}_{0-} = \varphi^{0,\downarrow}_{0-} = \varphi^{1,\uparrow}_{0-} = \varphi^{1,\downarrow}_{0-} = 0$, *these processes satisfy*

$$d\varphi^{0,\uparrow}_t \leqslant (1-\lambda) S_t d\varphi^{1,\downarrow}_t, \quad d\varphi^{0,\downarrow}_t \geqslant S_t d\varphi^{1,\uparrow}_t, \quad 0 \leqslant t \leqslant T. \tag{155}$$

The trading strategy $\varphi = (\varphi^0, \varphi^1)$ *is called* admissible *if there is* $M > 0$ *such that the liquidation value* V^{liq}_t *satisfies*

$$V^{liq}_t(\varphi^0, \varphi^1) := \varphi^0_t + (\varphi^1_t)^+(1-\lambda)S_t - (\varphi^1_t)^- S_t \geqslant -M, \tag{156}$$

a.s., for $0 \leqslant t \leqslant T$.

Remark 4.3. (1) We have chosen to define the trading strategies by explicitly specifying both accounts, the holdings in bond φ^0 as well as the holdings in stock φ^1. It would be sufficient to only specify one of the holdings, e.g. the number of stocks φ^1. Given a (right continuous, adapted) finite variation process $\varphi^1 = (\varphi^1_t)_{0 \leqslant t \leqslant T}$ starting at $\varphi^1_{0_-} = 0$, which we canonically decompose as the difference $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$, we may define the process φ^0 by

$$d\varphi^0_t = (1-\lambda) S_t d\varphi^{1,\downarrow}_t - S_t d\varphi^{1,\uparrow}_t.$$

The resulting pair (φ^0, φ^1) obviously satisfies (155) with equality holding true rather than inequality. However, it is convenient in (155) to consider trading strategies (φ^0, φ^1) which allow for an inequality, i.e. for "throwing away money". But it is clear from the preceding argument that we may always pass to a dominating pair (φ^0, φ^1) where equality holds true in (155).

We still note that we also might start from a (right continuous, adapted) process $(\varphi^0_t)_{0 \leqslant t \leqslant T} = (\varphi^{0,\uparrow}_t - \varphi^{0,\downarrow}_t)_{0 \leqslant t \leqslant T}$ and define φ^1 via

$$d\varphi^1_t = \frac{d\varphi^{0,\downarrow}_t}{S_t} - \frac{d\varphi^{0,\uparrow}_t}{(1-\lambda)S_t}.$$

(2) Now suppose that, in assumption (*ii*) above, the processes $\varphi^{0,\uparrow}, \varphi^{0,\downarrow}$, $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$ are right continuous, adapted, and starting at zero, but not necessarily the canonical decompositions of $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ (resp. $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$). In other words suppose that $\varphi^{0,\uparrow}$ and $\varphi^{0,\downarrow}$ (resp. $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$) may "move simultaneously". If the four processes satisfy the inequalities (155), then these inequalities are also satisfied for the canonical decompositions as one easily checks (and as is economically obvious). Summing up: in (*ii*) above the requirement that $\varphi^{0,\uparrow}, \varphi^{0,\downarrow}, \varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$ are the canonical decompositions could be dropped.

(3) We allow the finite variation process $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ to have jumps which we define to be of right continuous (i.e. càdlàg) type (note that a finite variation process automatically has left and right limits at every point $t \in [0, T]$). Unfortunately, we have a little problem[3] at $t = 0$. In fact, we have already encountered this problem in the discrete time setting in chapter 1 above. In order to model a possible (right continuous) jump at $t = 0$, we have to enlarge the time index set $[0, T]$ by adding the point 0_- which now takes the role of the point $t = -1$ in the discrete time setting of chapter 1. Hence whenever we write $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ we mean, strictly speaking, the process $(\varphi_t^0, \varphi_t^1)_{t \in \{0_-\} \cup [0,T]}$.

We could avoid the problem at $t = 0$ by passing to the *left* continuous modification $(\varphi_{t_-}^0, \varphi_{t_-}^1)_{0 \leqslant t \leqslant T}$ where $(\varphi_{t_-}^0, \varphi_{t_-}^1) = \lim_{u \nearrow t} (\varphi_u^0, \varphi_u^1)$ denotes the left limits, for $0 < t \leqslant T$. In fact, this would be quite natural, as the adapted, càglàd (i.e. left continuous, right limits) process $(\varphi_{t_-}^0, \varphi_{t_-}^1)_{0 \leqslant t \leqslant T}$ is *predictable*, while the càdlàg process $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ may in general fail to be predictable (it only is optional). In the general stochastic integration theory *predictable* processes are the natural class of integrands for general semi-martingales. However, this passage to the càglàd version shifts the "jump" problem at $t = 0$ to a similar problem at the end-point $t = T$, where we would be forced to add a point T_+ to $[0, T]$.

We have therefore decided to choose the càdlàg version $(\varphi_t^0, \varphi_t^1)_{t \in \{0_-\} \cup [0,T]}$ in the above definition for the following reasons:

- (*i*) As long as we restrict ourself to the case of *continuous* processes $S = (S_t)_{0 \leqslant t \leqslant T}$, it does not make a difference whether we consider the integral $\int_0^T \varphi_t^1 dS_t$ or $\int_0^T \varphi_{t_-}^1 dS_t$.
- (*ii*) Most of the preceding literature uses the càdlàg versions $(\varphi_t^0, \varphi_t^1)$.
- (*iii*) The addition of a point T_+ to $[0, T]$ seems even more awkward than

[3] P. A. Meyer once observed that 0_- "plays the role of the devil" in stochastic integration theory.

the addition of a point 0_-. We refer to [48] for a thorough discussion of these issues in the case of a general càdlàg process S.

(4) Finally, we observe for later use that in the definition of admissibility it does not matter whether we require (156), for all deterministic times $0 \leqslant t \leqslant T$, or for all $[0,T]$-valued stopping times τ.

Similarly as in (3) the *simple strategies* are particularly easy cases.

Proposition 4.4. *Fix the continuous process S and $1 > \lambda > 0$. For a right continuous, adapted, finite variation process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ starting at $(\varphi_{0_-}^1, \varphi_{0_-}^0) = (0,0)$ we again denote by $\varphi_t^{0,\uparrow}, \varphi_t^{0,\downarrow}, \varphi_t^{1,\uparrow}, \varphi_t^{1,\downarrow}$ its canonical decomposition into differences of increasing processes.*

The following assertions are equivalent (in an almost sure sense):

(i) The process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ is self-financing, i.e.

$$d\varphi_t^0 \leqslant (1-\lambda)S_t d\varphi_t^{1,\downarrow} - S_t d\varphi_t^{1,\uparrow}, \quad \text{a.s. for } 0 \leqslant t \leqslant T. \tag{157}$$

(ii) For each pair of reals $0 \leqslant a < b \leqslant T$, as well as for $a = 0_-, b = 0$,

$$\varphi_b^{0,\uparrow} - \varphi_a^{0,\uparrow} \leqslant \int_a^b (1-\lambda)S_u d\varphi_u^{1,\downarrow}, \qquad \varphi_b^{0,\downarrow} - \varphi_a^{0,\downarrow} \geqslant \int_a^b S_u d\varphi_u^{1,\uparrow}. \tag{158}$$

(iii) For each pair of rationals $0 \leqslant a < b \leqslant T$, as well as for $a = 0_-$ and $b = 0$

$$\begin{aligned} \varphi_b^{0,\uparrow} - \varphi_a^{0,\uparrow} &\leqslant (\varphi_b^{1,\downarrow} - \varphi_a^{1,\downarrow})(1-\lambda) \max_{a\leqslant u\leqslant b}\{S_u\}, \\ \varphi_b^{0,\downarrow} - \varphi_a^{0,\downarrow} &\geqslant (\varphi_b^{1,\uparrow} - \varphi_a^{1,\uparrow}) \min_{a\leqslant u\leqslant b}\{S_u\}. \end{aligned} \tag{159}$$

Proof. $(i) \Leftrightarrow (ii)$: Inequality (157) states that the process

$$\left(\int_0^t \left[(1-\lambda)S_u d\varphi_u^{1,\downarrow} - S_u d\varphi_u^{1,\uparrow} - d\varphi_u^0\right]\right)_{0\leqslant t\leqslant T}$$

is non-decreasing; this statement is merely reformulated in (158). Note that the integrals in (158) make sense in a pointwise manner as Riemann-Stieltjes integrals.

$(ii) \Leftrightarrow (iii)$: We only have to proof $(iii) \Rightarrow (ii)$. Suppose that (ii) fails to be true, say,

$$\varphi_b^{0,\uparrow} - \varphi_a^{0,\uparrow} > \int_a^b (1-\lambda)S_u d\varphi_u^{1,\downarrow} + \delta(b-a)$$

for some real numbers $0 \leqslant a < b \leqslant T$ and $\delta > 0$ holds true with probability bigger than $\varepsilon > 0$. Then we can approximate a and b by rationals α, β such that the above inequality still holds true. Using the continuity of S we can break the integral $\int_\alpha^\beta$ into a sum of finitely many integrals $\int_{\alpha_i}^{\beta_i}$, with rational endpoints α_i, β_i, such that the oscillation of S on each $[\alpha_i, \beta_i]$ is smaller than $\delta/2$ on a set of probability bigger than $1 - \frac{\varepsilon}{2}$. Then (159) fails to hold true almost surely, for some pair (α_i, β_i). ∎

Proposition 4.5. *Fix $S = (S_t)_{0 \leqslant t \leqslant T}$ and $\lambda > 0$ as above. Let $(\varphi^0, \varphi^1) = (\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ be a self-financing, admissible trading strategy, and $(\tilde{S}, Q)$ be a λ-consistent price system.*

The process

$$\tilde{V}_t = \varphi^0_t + \varphi^1_t \tilde{S}_t, \qquad 0 \leqslant t \leqslant T, \tag{160}$$

is a local Q-super-martingale which is uniformly bounded from below, and therefore a Q-super-martingale.

Proof. As $(\varphi^1_t)_{0 \leqslant t \leqslant T}$ is of bounded variation and $\tilde{S}$ is continuous, the product rule applied to (160) yields

$$d\tilde{V}_t = d\varphi^0_t + \tilde{S}_t d\varphi^1_t + \varphi^1_t d\tilde{S}_t. \tag{161}$$

As $\tilde{S}$ takes values in $[(1-\lambda)S, S]$, we conclude from (157) that the process $(\int_0^t (d\varphi^0_u + \tilde{S}_u d\varphi^1_u))_{0 \leqslant t \leqslant T}$ is non-increasing. The second term in (161) defines the local Q-martingale $(\int_0^t \varphi^1_u d\tilde{S}_u)_{0 \leqslant t \leqslant T} = (\varphi^1 \cdot \tilde{S})_{0 \leqslant t \leqslant T}$. By (156) and the admissibility assumption, the process $\tilde{V}$ is uniformly bounded from below. It therefore is a super-martingale under Q. ∎

Remark 4.6. The interpretation of the process $\tilde{V}$ is the value of the portfolio process (φ^0, φ^1) if we evaluate the position φ^1 in stock at price $\tilde{S}$. Note that $\tilde{V} \geqslant V^{liq}$, where V^{liq} is defined in (156).

Definition 4.7. *Let $S = (S_t)_{0 \leqslant t \leqslant T}$ and $1 > \lambda > 0$ be fixed as above.*

We denote by $\mathcal{A}$ the set of random variables $(\varphi^0_T, \varphi^1_T)$ in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ such that there is an admissible, self-financing, process $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$, as in Definition 4.2 starting at $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (0, 0)$, and ending at $(\varphi^0_T, \varphi^1_T)$.

We denote by $\mathcal{C}$ the set of random variables

$$\begin{aligned} \mathcal{C} &= \{\varphi^0_T \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : (\varphi^0_T, 0) \in \mathcal{A}\} \\ &= \{V^{liq}(\varphi^0_T, \varphi^1_T) : (\varphi^0_T, \varphi^1_T) \in \mathcal{A}\}. \end{aligned}$$

We denote by $\mathcal{A}^M$, resp. $\mathcal{C}^M$ the corresponding subsets of M-admissible elements, i.e. for which there is a process $(\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ satisfying (156)*, for fixed $M > 0$.*

Definition 4.8. *Fix S and $\lambda > 0$ as above, let $\tau : \Omega \to [0,T] \cup \{\infty\}$ be a stopping time, and let f_τ, g_τ be $\mathcal{F}_\tau$-measurable $\mathbb{R}_+$-valued functions. We define the corresponding* ask *and* bid *processes as the $\mathbb{R}^2$-valued processes*

$$a_t = (-S_\tau, 1) \qquad f_\tau\ \mathbb{1}_{[\![\tau,T]\!]}(t), \qquad 0 \leqslant t \leqslant T, \tag{162}$$

$$b_t = ((1-\lambda)S_\tau, -1)\ \ g_\tau\ \mathbb{1}_{[\![\tau,T]\!]}(t), \qquad 0 \leqslant t \leqslant T. \tag{163}$$

We call a process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ a simple, self-financing *process, if it is a finite sum of ask and bid processes as above. Admissibility is defined as in Definition 4.2.*

The interpretation of a_t is the following: an investor does nothing until time τ and then decides to buy f_τ many stocks and to hold them until time T. The resulting holdings in bond and stock are $\varphi_t^0 = -S_\tau f_\tau \mathbb{1}_{[\![\tau,T]\!]}(t)$ and $\varphi_t^1 = f_\tau \mathbb{1}_{[\![\tau,T]\!]}(t)$. The case of b_t is analogous.

In the above definition we also allow for $\tau = 0$ in the above definition: this case models the trading between time $t = 0_-$ and time $t = 0$ at bid ask prices $\{(1-\lambda)S_0, S_0\}$. In this case we interpret the function $\mathbb{1}_{[\![0,T]\!]}$ as $\mathbb{1}_{[\![0,T]\!]}(0_-) = 0$, while $\mathbb{1}_{[\![0,T]\!]}(t) = 1$, for $0 \leqslant t \leqslant T$.

We denote by $\mathcal{A}^s$ the set of $\mathbb{R}^2$-valued random variables (φ^0, φ^1) such that there is a simple (see Definition 4.8), admissible, self-financing, process $(\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ satisfying $(\varphi^0, \varphi^1) \leqslant (\varphi_T^0, \varphi_T^1)$.

Lemma 4.9. *Fix the continuous process S and $\lambda > 0$ as above. The set $\mathcal{A}^s$ is a convex cone in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ which is dense in $\mathcal{A}$ with respect to the topology of convergence in measure.*

More precisely, let $M > 0$ and $(\varphi^0, \varphi^1) = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ be a self-financing process as in Definition 4.7, starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0,0)$ which is M-admissible, i.e.

$$V_t(\varphi^0, \varphi^1) := \varphi_t^0 + (\varphi_t^1)^+(1-\lambda)S_t - (\varphi_t^1)^- S_t \geqslant -M, \qquad 0 \leqslant t \leqslant T.$$

Then there is a sequence $(\varphi^{0,n}, \varphi^{1,n})_{n=1}^\infty$ of simple, self-financing, M-admissible processes starting at $(\varphi_{0_-}^{0,n}, \varphi_{0_-}^{1,n}) = (0,0)$, such that $(\varphi_T^{0,n} \wedge \varphi_T^0, \varphi_T^{1,n} \wedge \varphi_T^1)$ converges to $(\varphi_T^0, \varphi_T^1)$ almost surely.

Proof. The idea of the approximation is simple: the strategy $(\varphi^{0,n}, \varphi^{1,n})$ does the same buying and selling operations as (φ^0, φ^1), but always waits until $(S_t)_{0\leqslant t\leqslant T}$ has moved by some $\delta > 0$; then the $(\varphi^{0,n}, \varphi^{1,n})$-strategy does the same buying/selling in one lump sum, which the strategy (φ^0, φ^1) has done during the preceding time interval. In this way the approximation $(\varphi^{0,n}, \varphi^{1,n})$ still is adapted to the filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ as it only uses past information;

the terms of trade for the strategies (φ^0, φ^1) and $(\varphi^{0,n}, \varphi^{1,n})$ are close to each other, as the continuous process S has only moved by at most δ during the preceding (stochastic) time interval.

Here are the more formal details: fix the self-financing, M-admissible strategy (φ^0, φ^1) and $1 > \varepsilon > 0$. As (φ^0, φ^1) is of finite variation we may find a constant $V_\varepsilon > 1$ such that the probability of $(\varphi^0_t)_{0 \leqslant t \leqslant T}$ having total variation $\mathrm{Var}_T(\varphi^0)$ bigger than V_ε, has probability less than $\varepsilon > 0$.

Let σ be the stopping time

$$\sigma = \inf\{t \in [0,T] : \ \mathrm{Var}_t(\varphi^0) \geqslant V_\varepsilon\}, \tag{164}$$

so that $\mathbb{P}[\sigma < \infty] < \varepsilon$, and let $\delta = \min(\frac{\varepsilon}{V_\varepsilon}, \frac{\lambda}{3})$. Define a sequence of stopping times $(\tau_k)_{k=0}^\infty$ by $\tau_0 = 0$ and, for $k \geqslant 0$,

$$\tau_{k+1} = \inf \left\{ t \in [\![\tau_k, T]\!] : \frac{S_t}{S_{\tau_k}} = (1+\delta) \text{ or } (1-\delta) \right\} \wedge \sigma, \tag{165}$$

where, as in (164), the inf over the empty set is infinity.

As the trajectories of $S = (S_t)_{0 \leqslant t \leqslant T}$ are continuous and strictly positive, the sequence $(\tau_k)_{k=0}^\infty$ increases to infinity a.s. on $\{\sigma = \infty\}$. Fix $K \in \mathbb{N}$ such that $\mathbb{P}[\tau_K < \infty] < 2\varepsilon$. Now construct inductively the approximating simple process $(\varphi^{0,n}, \varphi^{1,n})$, where $n \in \mathbb{N}$ will correspond to some $\varepsilon_n > 0$ and $\delta_n \leqslant \frac{\varepsilon_n}{V_{\varepsilon_n}}$ to be specified below.

At time $t = 0$ we observe that $(\varphi^0_0, \varphi^1_0)\mathbb{1}_{[\![0,T]\!]}(t)$ is the sum of the terms (162) and (163),i.e.

$$\begin{aligned}(\varphi^0_0, \varphi^1_0)\mathbb{1}_{[\![0,T]\!]}(t) &= a^0_t + b^0_t \\ &= ((-S_0, 1) f_{\tau_0} + ((1-\lambda)S_0, -1)\, g_{\tau_0})\, \mathbb{1}_{[\![0,T]\!]}(t),\end{aligned}$$

where $f_{\tau_0} = (\varphi^1_0 - \varphi^1_{0\_})^+ = (\varphi^1_0)^+$ and $g_{\tau_0} = (\varphi^1_0 - \varphi^1_{0\_})^- = (\varphi^1_0)^-$.

At time τ_1 we want to adjust our holdings in bond and stock to have $\varphi^{1,n}_{\tau_1} = \varphi^1_{\tau_1}$, i.e. that the holding in stock at time τ_1 are the same, for the strategy (φ^0, φ^1) and $(\varphi^{0,n}, \varphi^{1,n})$. This can be done by defining

$$a^1_t + b^1_t = [(-S_{\tau_1}, 1) f_{\tau_1} + ((1-\lambda)S_{\tau_1}, -1) g_{\tau_1}]\, \mathbb{1}_{[\![\tau_1,T]\!]}(t), \qquad 0 \leqslant t \leqslant T, \tag{166}$$

where $f_{\tau_1} = (\varphi^1_{\tau_1} - \varphi^1_{\tau_0})^+$ and $g_{\tau_1} = (\varphi^1_{\tau_1} - \varphi^1_{\tau_0})^-$, where $\tau_0 = 0$ so that $\varphi^1_{\tau_0} = \varphi^1_0$ (as opposed to $\varphi^1_{0\_}$). We add this process to $a^0_t + b^0_t$, i.e. we define

$$(\varphi^{0,n,1}_t, \varphi^{1,n,1}_t) = (a^0_t + b^0_t) + (a^1_t + b^1_t), \qquad 0 \leqslant t \leqslant T.$$

We then have that the process $(\varphi^{0,n,1}, \varphi^{1,n,1})$ jumps at times 0 and τ_1 only, and satisfies

$$\varphi^{1,n,1}_{\tau_1} = \varphi^1_{\tau_1}.$$

As regards the holdings $\varphi^{0,n,1}_{\tau_1}$ in bond at time τ_1, we cannot assert that $\varphi^{0,n,1}_{\tau_1} = \varphi^0_{\tau_1}$, but we are not far off the mark: speaking economically, the strategy (φ^0, φ^1) has changed the position in bond during the interval $[\![\tau_0, \tau_1]\!]$ from φ^0_0 to $\varphi^0_{\tau_1}$ by buying $(\varphi^1_{\tau_1} - \varphi^1_{\tau_0})_+$, resp. selling $(\varphi^1_{\tau_1} - \varphi^1_{\tau_0})_-$, numbers of stock. These are figures accumulated over the interval $[\![\tau_0, \tau_1]\!]$. As the stock price S_t is in the interval $[(1-\delta)S_0, (1+\delta)S_0]$ for $t \in]\!]\tau_0, \tau_1]\!]$ and $\delta < \frac{\lambda}{3}$, we may estimate by (157)

$$(\varphi^{0,n,1}_{\tau_1} - \varphi^{0,n,1}_{\tau_0}) - (\varphi^0_{\tau_1} - \varphi^0_{\tau_0}) = \varphi^{0,n,1}_{\tau_1} - \varphi^0_{\tau_1} \geqslant -3\delta\, |\varphi^0_{\tau_1} - \varphi^0_{\tau_0}|. \qquad (167)$$

Now continue in an analogous way on the intervals $]\!]\tau_{k-1}, \tau_k]\!]$, for $k = 1, \dots, K$, to find $a^k_t + b^k_t$ as in (166)

$$a^k_t + b^k_t = [(-S_{\tau_k}, 1) f_{\tau_k} + ((1-\lambda)S_{\tau_k}, -1) g_{\tau_k}]\, \mathbb{1}_{[\![\tau_k, T]\!]}(t), \quad 0 \leqslant t \leqslant T, \quad (168)$$

so that the process

$$(\varphi^{0,n,k}_t, \varphi^{1,n,k}_t) = \sum_{j=0}^{k} (a^j_t + b^j_t), \qquad 0 \leqslant t \leqslant T,$$

satisfies $\varphi^{1,n,k}_{\tau_j} = \varphi^1_{\tau_j}$, for $j = 0, \dots, k$, and

$$\varphi^{0,n,k}_{\tau_k} - \varphi^0_{\tau_k} \geqslant -3\delta \sum_{j=1}^{k} |\varphi^0_{\tau_j} - \varphi^0_{\tau_{j-1}}|. \qquad (169)$$

Finally define the process $(\varphi^{0,n}, \varphi^{1,n}) := (\varphi^{0,n,K}, \varphi^{1,n,K})$.

We have not yet made precise what we do, when, for the first time $k = 1, \dots, K$, we have $\tau_k = \infty$. In this case we interpret (168) by letting $\tau_k := T$ rather than $\tau_k = \infty$, i.e. as a final trade at time T, to make sure that $\varphi^{1,n,k}_T = \varphi^1_T$ on $\{\tau_k = \infty\}$.

Hence the process $(\varphi^{0,n}, \varphi^{1,n})$ is such that, on the set $\{\tau_K = \infty\}$, we have $\varphi^{1,n}_T = \varphi^1_T$ so that

$$\mathbb{P}[\varphi^{1,n}_T = \varphi^1_T] > 1 - 2\varepsilon. \qquad (170)$$

By (169) we may also estimate on $\{\tau_K < \infty\} \subseteq \{\sigma < \infty\}$

$$\begin{aligned} \varphi^{0,n}_{\tau_K} - \varphi^0_{\tau_K} &\geqslant -3\delta \sum_{j=1}^{K} |\varphi^0_{\tau_j} - \varphi^0_{\tau_{j-1}}| \\ &\geqslant -3\delta\, [V_\varepsilon + 2\delta], \end{aligned}$$

so that

$$\mathbb{P}[\varphi^{0,n}_T \geqslant \varphi^0_T - 4\varepsilon] \geqslant 1 - 2\varepsilon. \qquad (171)$$

As regards the admissibility of $(\varphi^{0,n}, \varphi^{1,n})$: this process is not yet M-admissible, but it is straightforward to check that it is $M + 3\delta(V_\varepsilon + 2\delta)$-admissible. Hence by multiplying $(\varphi^{0,n}, \varphi^{1,n})$ by the factor $c := \frac{M}{M+3\delta(V_\varepsilon+2\delta)}$ we obtain an M-admissible process $(c\varphi^{0,n}, c\varphi^{1,n})$ such that $(c\varphi_T^{0,n} \wedge \varphi_T^1, c\varphi_T^{1,n})$ is close to $(\varphi_T^0, \varphi_T^1)$ in probability.

Finally, we have to specify $\varepsilon = \varepsilon_n$: it now is clear that it will be sufficient to choose $\varepsilon_n = 2^{-n}$ in the above construction to obtain the a.s. convergence of $(\varphi_T^{0,n} \wedge \varphi_T^0, \varphi_T^{1,n})$ to $(\varphi_T^0, \varphi_T^1)$. ∎

The following lemma was proved by L. Campi and the author [32] in the more general framework of Kabanov's modeling of d-dimensional currency markets. Here we spell out the proof for a single risky asset model. In Definition 4.2 we postulated as a qualitative — a priori — assumption that the strategies (φ^0, φ^1) have *finite variation.* The next lemma provides an — a posteriori — quantitative control on the size of the finite variation.

Lemma 4.10. *Let S and $\lambda > 0$ be as above, and suppose that $(CPS^{\lambda'})$ is satisfied, for some $0 < \lambda' < \lambda$, i.e., there is a consistent price system for transaction costs λ'. Fix $M > 0$. Then the total variation of the process $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ remains bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, when (φ^0, φ^1) runs through all M-admissible, λ-self-financing strategies.*

More explicitly: for $M > 0$ and $\varepsilon > 0$, there is $C > 0$ such that, for all M-admissible, self-financing strategies (φ^0, φ^1), starting at $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (0, 0)$, and all partitions $0_- = t_0 < t_1 < \ldots < t_K = T$ we have

$$\mathbb{P}\left[\sum_{k=1}^{K} |\varphi^0_{t_k} - \varphi^0_{t_{k-1}}| \ \geqslant C\right] < \varepsilon, \tag{172}$$

$$\mathbb{P}\left[\sum_{k=1}^{K} |\varphi^1_{t_k} - \varphi^1_{t_{k-1}}| \ \geqslant C\right] < \varepsilon. \tag{173}$$

Proof. Fix $0 < \lambda' < \lambda$ as above. By the hypothesis $(CPS^{\lambda'})$ there is a probability measure $Q \sim \mathbb{P}$, and a local Q-martingale $(\tilde{S}_t)_{0 \leqslant t \leqslant T}$ such that $\tilde{S}_t \in [(1 - \lambda')S_t, S_t]$.

Fix $M > 0$ and a self-financing (with respect to transaction costs λ), M-admissible process $(\varphi_t^0, \varphi_t^1)_{t \geqslant 0}$, starting at $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (0, 0)$. Write $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ as the canonical differences of increasing processes, as in Definition 4.2. We shall show that

$$\mathbb{E}_Q\left[\varphi_T^{0,\uparrow}\right] \leqslant \frac{M}{\lambda - \lambda'}. \tag{174}$$

Define the process $((\varphi^0)', (\varphi^1)')$ by

$$\left((\varphi^0)'_t, (\varphi^1)'_t\right) = \left(\varphi^0_t + \frac{\lambda - \lambda'}{1-\lambda}\varphi^{0,\uparrow}_t, \varphi^1_t\right), \qquad 0 \leqslant t \leqslant T.$$

This is a self-financing process under transaction costs λ': indeed, whenever $d\varphi^0_t > 0$ so that $d\varphi^0_t = d\varphi^{0,\uparrow}_t$, the agent sells stock and receives $d\varphi^{0,\uparrow}_t = (1-\lambda)S_t d\varphi^{1,\downarrow}_t$ (resp. $(1-\lambda')S_t d\varphi^{1,\downarrow}_t = \frac{1-\lambda'}{1-\lambda} d\varphi^{0,\uparrow}_t$) under transaction costs λ (resp. λ'). The difference between these two terms is $\frac{\lambda-\lambda'}{1-\lambda} d\varphi^{0,\uparrow}_t$; this is the amount by which the λ'-agent does better than the λ-agent. It is also clear that $((\varphi^0)', (\varphi^1)')$ under transaction costs λ' still is M-admissible.

By Proposition 4.5 the process $((\varphi^0)'_t + \varphi^1_t \tilde{S}_t)_{0 \leqslant t \leqslant T} = (\varphi^0_t + \frac{\lambda-\lambda'}{1-\lambda}\varphi^{0,\uparrow}_t + \varphi^1_t \tilde{S}_t)_{0 \leqslant t \leqslant T}$ is a Q-super-martingale. Hence $\mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T] + \frac{\lambda-\lambda'}{1-\lambda}\mathbb{E}_Q[\varphi^{0,\uparrow}_T] \leqslant 0$. As $\varphi^0_T + \varphi^1_T \tilde{S}_T \geqslant -M$ we have shown (174).

To obtain a control on $\varphi^{0,\downarrow}_T$ too, we may assume w.l.g. in the above reasoning that the strategy (φ^0, φ^1) is such that $\varphi^1_T = 0$, i.e. the position in stock is liquidated at time T. We then must have $\varphi^0_T \geqslant -M$ so that $\varphi^{0,\downarrow}_T \leqslant \varphi^{0,\uparrow}_T + M$. Therefore we obtain the following estimate for the total variation $\varphi^{0,\uparrow}_T + \varphi^{0,\downarrow}_T$ of φ^0

$$\mathbb{E}_Q\left[\varphi^{0,\uparrow}_T + \varphi^{0,\downarrow}_T\right] \leqslant M\left(\frac{2}{\lambda - \lambda'} + 1\right). \tag{175}$$

The passage from the $L^1(Q)$-estimate (175) to the $L^0(\mathbb{P})$-estimate (172) is standard: for $\varepsilon > 0$ there is $\delta > 0$ such that for a subset $a \in \mathcal{F}$ with $Q[A] < \delta$ we have $\mathbb{P}[A] < \varepsilon$. Letting $C = \frac{M}{\delta}(\frac{2}{\lambda-\lambda'} + 1)$ and applying Tschebyschoff to (175) we get

$$\mathbb{P}\left[\varphi^{0,\uparrow}_T + \varphi^{0,\downarrow}_T \geqslant C\right] < \varepsilon,$$

which implies (172).

As regards (173) we note that, by the continuity and strict positivity assumption on S, for $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\mathbb{P}\left[\inf_{0 \leqslant t \leqslant T} S_t < \delta\right] < \frac{\varepsilon}{2}.$$

Hence we may control $\varphi^{1,\uparrow}_T$ by using the second inequality in (159); then we can control $\varphi^{1,\downarrow}_T$ by a similar reasoning as above so that we obtain (173) for a suitably adapted constant C. ■

Remark 4.11. In the above proof we have shown that the elements $\varphi^{0,\uparrow}_T, \varphi^{0,\downarrow}_T, \varphi^{1,\uparrow}_T, \varphi^{1,\downarrow}_T$ remain bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, when (φ^0, φ^1) runs through the M-admissible self-financing process and $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$

denote the canonical decompositions. For later use we remark that the proof shows, in fact, that also the convex combinations of these functions $\varphi_T^{0,\uparrow}$ etc. remain bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Indeed the estimate (174) shows that the convex hull of the functions $\varphi_T^{0,\uparrow}$ is bounded in $L^1(Q)$ and (175) yields the same for $\varphi_T^{0,\downarrow}$. For $\varphi_T^{1,\uparrow}$ and $\varphi_T^{1,\downarrow}$ the argument is similar.

In order to prove the subsequent Theorem 4.13 we still need one more preparation (compare [211]).

Proposition 4.12. *Fix S and $1 > \lambda > 0$ as above, and suppose that S satisfies $(CPS^{\lambda'})$, for* each $\lambda' > 0$.

Let $(\varphi_t)_{0 \leqslant t \leqslant T} = (\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ be a self-financing and admissible process under transaction costs λ, and suppose that there is $M > 0$ s.t. for the terminal value V_T^{liq} we have

$$V_T^{liq}(\varphi^0, \varphi^1) = \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \geqslant -M. \tag{176}$$

Then we also have

$$V_\tau^{liq}(\varphi_\tau^0, \varphi_\tau^1) = \varphi_\tau^0 + (\varphi_\tau^1)^+(1-\lambda)S_\tau - (\varphi_\tau^1)^- S_\tau \geqslant -M, \tag{177}$$

a.s., for every stopping time $0 \leqslant \tau \leqslant T$, i.e. φ is M-admissible.

Proof. We start with the observation, that by liquidating the stock position at time T, we may assume in (176) w.l.g. that $\varphi_T^1 = 0$, so that $\varphi_T^0 \geqslant -M$.

Supposing that (177) fails, we may find $\frac{\lambda}{2} > \alpha > 0$, a stopping time $0 \leqslant \tau \leqslant T$, such that either $A = A_+$ or $A = A_-$ satisfies $\mathbb{P}[A] > 0$, where

$$A_+ = \{\varphi_\tau^1 \geqslant 0,\, \varphi_\tau^0 + \varphi_\tau^1 \tfrac{1-\lambda}{1-\alpha} S_\tau < -M\}, \tag{178}$$

$$A_- = \{\varphi_\tau^1 \leqslant 0,\, \varphi_\tau^0 + \varphi_\tau^1 (1-\alpha)^2 S_\tau < -M\}. \tag{179}$$

Choose $0 < \lambda' < \alpha$ and a λ'-consistent price system $(\tilde{S}, Q)$. As $\tilde{S}$ takes values in $[(1-\lambda')S, S]$, we have that $(1-\alpha)\tilde{S}$ as well as $\frac{1-\lambda}{1-\alpha}\tilde{S}$ take values in $[(1-\lambda)S, S]$ so that $((1-\alpha)\tilde{S}, Q)$ as well as $(\frac{1-\lambda}{1-\alpha}\tilde{S}, Q)$ are consistent price systems under transaction costs λ. By Proposition 4.5 we obtain that

$$\Big(\varphi_t^0 + \varphi_t^1 (1-\alpha)\tilde{S}_t\Big)_{0 \leqslant t \leqslant T}, \text{ and } (\varphi_t^0 + \varphi_t^1 \tfrac{1-\lambda}{1-\alpha}\tilde{S}_t)_{0 \leqslant t \leqslant T}$$

are Q-supermartingales. Arguing with the second process and using that $\tilde{S} \leqslant S$ we obtain from (178) the inequality

$$\mathbb{E}_Q\left[\varphi_T^0 + \varphi_T^1 \frac{1-\lambda}{1-\alpha}\tilde{S}_T | A_+\right] \leqslant \mathbb{E}_Q\left[\varphi_\tau^0 + \varphi_\tau^1 \frac{1-\lambda}{1-\alpha}\tilde{S}_\tau | A_+\right] < -M.$$

Arguing with the first process and using that $\tilde{S} \geqslant (1-\lambda')S \geqslant (1-\alpha)S$ (which implies that $\varphi^1_\tau(1-\alpha)\tilde{S}_\tau \leqslant \varphi^1_\tau(1-\alpha)^2 S_\tau$ on A_-) we obtain from (179) the inequality

$$\mathbb{E}_Q\left[\varphi^0_T + \varphi^1_T(1-\alpha)\tilde{S}_T|A_-\right] \leqslant \mathbb{E}_Q\left[\varphi^0_\tau + \varphi^1_\tau(1-\alpha)\tilde{S}_\tau|A_-\right] < -M.$$

Either A_+ or A_- has strictly positive probability; hence we arrive at a contradiction, as $\varphi^1_T = 0$ and $\varphi^0_T \geqslant -M$. ■

The assumption $CPS^{\lambda'}$, for *each* $\lambda' > 0$, cannot be dropped in Proposition 4.12 as shown by an explicit example in [211].

We now can state the central result from [32] in the present framework. Recall Definition 4.7 of the sets $\mathcal{A}^M$ and $\mathcal{C}^M$. Proposition 4.12 has the following important consequence concerning these definitions. We may equivalently define $\mathcal{A}^M$ as the set of random variables $(\varphi^0_T, \varphi^1_T)$ in $\mathcal{A}$ such that $V^{liq}(\varphi^0_T, \varphi^1_T) \geqslant -M$. The point is that the requirement $\varphi = (\varphi^0_T, \varphi^1_T) \in \mathcal{A}$ only implies that φ is the terminal value of an $\bar{M}$-admissible strategy, for some $\bar{M} > 0$ which – a priori – has nothing to do with M. But Proposition 4.12 tells us that $V^{liq}(\varphi^0_T, \varphi^1_T) \geqslant -M$ already implies that we may replace the a priori constant $\bar{M}$ by the constant M. In other words, if the liquidation value of an admissible φ is above the threshold $-M$ at the terminal time T, it also is so at all previous times $0 \leqslant t \leqslant T$.

Theorem 4.13. *Fix $S = (S_t)_{0\leqslant t\leqslant T}$ and $\lambda > 0$ as above, and suppose that $(CPS^{\lambda'})$ is satisfied, for each $0 < \lambda' < \lambda$. For $M > 0$, the convex set $\mathcal{A}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ as well as the convex set $\mathcal{C}^M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ are closed with respect to the topology of convergence in measure.*

For the proof we use the following well-known variant of Komlos' theorem. This result ([64, Lemma A 1.1]) turned out to be very useful in the applications to Mathematical Finance.

For the convenience of the reader we reproduce the proof.

Lemma 4.14. *Let $(f_n)_{n=1}^\infty$ be a sequence of $\mathbb{R}_+$-valued, measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$.*

There is a sequence $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$ of convex combinations which converges a.s. to some $[0, \infty]$-valued function g_0.

If $(f_n)_{n=1}^\infty$ is such that the convex hull $\operatorname{conv}(f_1, f_2, \ldots)$ is bounded in the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the function g_0 takes a.s. finite values.

Proof. Choose $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$ such that

$$\lim_{n\to\infty} \mathbb{E}[\exp(-g_n)] = \lim_{n\to\infty} \inf_{g\in\operatorname{conv}(f_n, f_{n+1}, \ldots)} \mathbb{E}[\exp(-g)]. \tag{180}$$

For fixed $1 > \varepsilon > 0$ we claim that

$$\lim_{n,m\to\infty} \mathbb{P}[(A_n \cup A_m) \cap B_{n,m}] = 0, \tag{181}$$

where

$$\begin{aligned} A_n &= \{g_n \in [0, \tfrac{1}{\varepsilon}]\} \\ A_m &= \{g_m \in [0, \tfrac{1}{\varepsilon}]\} \\ B_{n,m} &= \{|g_n - g_m| \geqslant \tfrac{\varepsilon}{2}\}. \end{aligned}$$

Indeed, the function $x \to e^{-x}$ is strictly convex on $[0, \frac{1}{\varepsilon} + \frac{\varepsilon}{2}]$ so that, for given $\varepsilon > 0$, there is $\delta > 0$ such that, for $x, y \in [0, \frac{1}{\varepsilon} + \frac{\varepsilon}{2}]$ satisfying $(x-y) \geqslant \frac{\varepsilon}{2}$ we have

$$\exp\left(-\frac{x+y}{2}\right) \leqslant \frac{\exp(-x) + \exp(-y)}{2} - \delta.$$

For $\omega \in (A_n \cup A_m) \cap B_{n,m}$ we therefore have

$$\exp\left(-\frac{g_n(\omega) + g_m(\omega)}{2}\right) \leqslant \frac{\exp(-g_n(\omega)) + \exp(-g_m(\omega))}{2} - \delta.$$

Using the convexity of $x \to e^{-x}$ on $[0, \infty[$ (this time without strictness) we get

$$\mathbb{E}\left[\exp\left(-\frac{g_n + g_m}{2}\right)\right] \leqslant \mathbb{E}\left[\frac{\exp(-g_n) + \exp(-g_m)}{2}\right] - \delta\mathbb{P}[(A_n \cup A_m) \cap B_{n,m}].$$

The negation of (181) reads as

$$\limsup_{n,m\to\infty} \mathbb{P}[(A_n \cup A_m) \cap B_{n,m}] = \alpha > 0.$$

This would imply that

$$\liminf_{n,m\to\infty} \mathbb{E}\left[\exp\left(-\frac{g_n + g_m}{2}\right)\right] \leqslant \lim_{n\to\infty} \inf_{g\in\mathrm{conv}(f_n, f_{n+1},\ldots)} \mathbb{E}[\exp(-g)] - \alpha\delta,$$

in contradiction to (180), which shows (181).

By passing to a subsequence, still denoted by $(g_n)_{n=1}^{\infty}$, we may suppose that, for fixed $1 > \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}[(A_n \cup A_{n+1}) \cap B_{n,n+1}] < \infty, \tag{182}$$

and, by passing to a diagonal sequence, that this holds true for each $1 > \varepsilon > 0$. Taking a subsequence once more and applying Borel-Cantelli we get that, for almost each $\omega \in \Omega$, either $g_n(\omega) \to \infty$ or $(g_n(\omega))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}_+$.

As regards the second assertion, the condition on the L^0-boundedness states that, for $\eta > 0$, we may find $M > 0$ such that $\mathbb{P}[g \geqslant M] < \eta$, for each $g \in \text{conv}\,(f_n, f_{n+1}, \ldots)$. This L^0-boundedness condition prevents $(g_n)_{n=1}^{\infty}$ from converging to $+\infty$ with positive probability. ■

Convex combinations work very much like subsequences. For example, one may form sequences of convex combinations of sequences of convex combinations: if $g_n \in \text{conv}(f_n, f_{n+1}, \ldots)$ and $h_n \in \text{conv}(g_n, g_{n+1}, \ldots)$, then h_n is a sequence of convex combinations of the original sequence $(f_n)_{n=1}^{\infty}$, i.e. $h_n \in \text{conv}(f_n, f_{n+1}, \ldots)$. Similarly, the concept of a diagonal subsequence carries over in an obvious way. This will repeatedly used in the subsequent proof.

Proof of Theorem 4.13. Fix $M > 0$ and let $(\varphi_T^n)_{n=1}^{\infty} = (\varphi_T^{0,n}, \varphi_T^{1,n})_{n=1}^{\infty}$ be a sequence in $\mathcal{A}^M$. We may find self-financing, M-admissible strategies $(\varphi_t^{0,n}, \varphi_t^{1,n})_{0 \leqslant t \leqslant T}$, starting at $(\varphi_{0_-}^{0,n}, \varphi_{0_-}^{1,n}) = (0,0)$, with given terminal values $(\varphi_T^{0,n}, \varphi_T^{1,n})$. As above, decompose canonically these processes as $\varphi_t^{0,n} = \varphi_t^{0,n,\uparrow} - \varphi_t^{0,n,\downarrow}$, and $\varphi_t^{1,n} = \varphi_t^{1,n,\uparrow} - \varphi_t^{1,n,\downarrow}$. By Lemma 4.10 and the subsequent remark we know that $(\varphi_T^{0,n,\uparrow})_{n=1}^{\infty}, (\varphi_T^{0,n,\downarrow})_{n=1}^{\infty}, (\varphi_T^{1,n,\uparrow})_{n=1}^{\infty}$, and $(\varphi_T^{1,n,\downarrow})_{n=1}^{\infty}$ as well as their convex combinations are bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ too, so that by Lemma 4.14 we may find convex combinations converging a.s. to some elements $\varphi_T^{0,\uparrow}, \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow}$, and $\varphi_T^{1,\downarrow} \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. To alleviate notation we denote the sequences of convex combinations still by the original sequences. We claim that $(\varphi_T^0, \varphi_T^1) = (\varphi_T^{0,\uparrow} - \varphi_T^{0,\downarrow}, \varphi_T^{1,\uparrow} - \varphi_T^{1,\downarrow})$ is in $\mathcal{A}^M$ which will readily show the closedness of $\mathcal{A}^M$ with respect to the topology of convergence in measure.

By inductively passing to convex combinations, still denoted by the original sequences, we may, for each rational number $r \in [0, T[$, assume that $(\varphi_r^{0,n,\uparrow})_{n=1}^{\infty}, (\varphi_r^{0,n,\downarrow})_{n=1}^{\infty}, (\varphi_r^{1,n,\uparrow})_{n=1}^{\infty}$, and $(\varphi_r^{1,n,\downarrow})_{n=1}^{\infty}$ converge a.s. to some elements $\bar{\varphi}_r^{0,\uparrow}, \bar{\varphi}_r^{0,\downarrow}, \bar{\varphi}_r^{1,\uparrow}$, and $\bar{\varphi}_r^{1,\downarrow}$ in $L^0(\Omega, \mathcal{F}, \mathbb{P})$. By passing to a diagonal subsequence, we may suppose that this convergence holds true for all rationals $r \in [0, T[$.

Clearly the four processes $\bar{\varphi}_{r \in \mathbb{Q} \cap [0,T[}^{0,\uparrow}$ etc, indexed by the rationals r in $[0, T[$, still are increasing and define an M-admissible process, indexed by $[0, T[\cap \mathbb{Q}$, in the sense of (156). They also satisfy (159), where we define $\bar{\varphi}_{0_-}^{0,\uparrow} = 0$ and $\bar{\varphi}_T^{0,\uparrow} = \varphi_T^{0,\uparrow}$ (etc. for the other three cases).

We still have to pass to a right continuous version and to extend the processes to all real numbers $t \in [0,T]$. This is done by letting

$$\varphi_t^{0,\uparrow} = \lim_{\substack{r \searrow t \\ r \in \mathbb{Q}}} \bar{\varphi}_r^{0,\uparrow}, \qquad 0 \leqslant t < T, \tag{183}$$

and $\varphi_{0_-}^{0,\uparrow} = 0$. Note that the terminal value $\varphi_T^{0,\uparrow}$ is still given by the first step of the construction. The three other cases, $\varphi^{0,\downarrow}$, $\varphi^{1,\uparrow}$, and $\varphi^{1,\downarrow}$ are, of course, defined in an analogous way. These continuous time processes again satisfy the self-financing conditions (159).

Finally, define the process $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ as $(\varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}, \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow})_{0 \leqslant t \leqslant T}$. From Proposition 4.4 *(iii)* we obtain that this defines a self-financing trading strategy in the sense of Definition 4.2 with the desired terminal value $(\varphi_T^0, \varphi_T^1)$. The M-admissibility follows from Proposition 4.12.

We thus have shown that $\mathcal{A}^M$ is closed. The closedness of $\mathcal{C}^M$ is an immediate consequence. ■

In fact we have not only proved a *closedness* of $\mathcal{A}^M$ with respect to the topology of convergence in measure. Rather we have shown a *convex compactness* property (compare [156], [245]). Indeed, we have shown that, for any sequence $(\varphi_T^n)_{n=1}^\infty \in \mathcal{A}^M$, we can find a sequence of convex combinations which converges a.s., and therefore in measure, to an element $\varphi_T \in \mathcal{A}^M$.

4.1 Passage from L^0 to appropriate Banach spaces

The message of Theorem 4.13 is stated in terms of the topological vector space $L^0(\mathbb{R}^2)$ and with respect to convergence in measure. We now translate it into the setting of appropriately defined Banach spaces. This needs some preparation. For a fixed, positive number $S > 0$ we define the norm $|\cdot|_S$ on $\mathbb{R}^2$ by

$$|(x^0, x^1)|_S = \max\{|x^0 + x^1 S|, |x^0 + x^1(1-\lambda)S|\}. \tag{184}$$

Its unit ball is the convex hull of the four points $\{(1,0), (-1,0), (\frac{2-\lambda}{\lambda}, -\frac{2}{\lambda S}), (-\frac{2-\lambda}{\lambda}, \frac{2}{\lambda S})\}$.

To motivate this definition we consider for a fixed number $S > 0$, similarly as in (1), the solvency cone $K_S = \{(x^0, x^1) : x^0 \geqslant \max(-x^1 S, -x^1(1-\lambda)S)\}$. For $\xi \in \mathbb{R}$, let $K_S(\xi)$ be the shifted solvency cone $K_S(\xi) = K_S - \xi = \{(x^0, x^1) : (x^0 + \xi, x^1) \in K_S)\}$. With this notation, the unit ball of $(\mathbb{R}^2, |\cdot|_S)$ is the biggest set which is symmetric around 0 and contained in $K_S(1)$.

The dual norm $|\cdot|_S^*$ is given, for $(Z^0, Z^1) \in \mathbb{R}^2$, by

$$|(Z^0, Z^1)|_S^* = \max\{|Z^0|, |\tfrac{2-\lambda}{\lambda} Z^0 - \tfrac{2}{\lambda S} Z^1|\}, \tag{185}$$

as one readily verifies by looking at the extreme points of the unit ball of $(\mathbb{R}^2, |\cdot|_S)$. The unit ball of $(\mathbb{R}^2, |\cdot|_S^*)$ is the convex hull of the four points $\{(1,S),(-1,-S),(1,(1-\lambda)S),(-1,-(1-\lambda)S)\}$.

These norms on $\mathbb{R}^2$ are tailor-made to define Banach spaces L^1_S and L^∞_S in isometric duality where S will depend on $\omega \in \Omega$. Let $S = (S_t)_{0 \leqslant t \leqslant T}$ now denote an $\mathbb{R}_+$-valued process. We define the Banach space L^1_S as

$$L^1_S = L^1_S(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) = \tag{186}$$
$$\left\{ Z_T = (Z^0_T, Z^1_T) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) : \|Z_T\|_{L^1_S} = \mathbb{E}\left[|(Z^0_T, Z^1_T)|^*_{S_T} \right] < \infty \right\}$$

Its dual L^∞_S then is given by

$$L^\infty_S = L^\infty_S(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) = \tag{187}$$
$$\left\{ \varphi_T = (\varphi^0_T, \varphi^1_T) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) : \|\varphi_T\|_{L^\infty_S} = \operatorname{ess\,sup}\left[|(\varphi^0_T, \varphi^1_T)|_{S_T} \right] < \infty \right\}.$$

These spaces are designed in such a way that $\mathcal{A} \cap L^\infty_S$ is "Fatou dense" in $\mathcal{A}$. We do not elaborate in detail on the notion of "Fatou closedness" which was introduced in [218] but only present the idea which is relevant in the present context.

For $\varphi_T = (\varphi^0_T, \varphi^1_T) \in \mathcal{A}^M$ we have (156)

$$V^{liq}_T = \varphi^0_T + (\varphi^1_T)^+(1-\lambda)S_T - (\varphi^1_T)^- S_T \geqslant -M, \tag{188}$$

which may be written as

$$\min\left\{ (\varphi^0_T + \varphi^1_T(1-\lambda)S_T), (\varphi^0_T + \varphi^1_T S_T) \right\} \geqslant -M \tag{189}$$

or

$$\max\left\{ -(\varphi^0_T + \varphi^1_T(1-\lambda)S_T), -(\varphi^0_T + \varphi^1_T S_T) \right\} \leqslant M. \tag{190}$$

In order to obtain $|(\varphi^0_T, \varphi^1_T)|_{S_T} \leqslant M$ we still need the inequality

$$\max\left\{ (\varphi^0_T + \varphi^1_T(1-\lambda)S_T), (\varphi^0_T + \varphi^1_T S_T) \right\} \leqslant M. \tag{191}$$

In general, there is little reason why (191) should be satisfied, for an element $\varphi_T = (\varphi^0_T, \varphi^1_T) \in \mathcal{A}^M$. Indeed, the agent may have become "very rich" which may cause (191) to fail to hold true. But there is an easy remedy: just "get rid of the superfluous assets"

More formally: fix $M > 0$, and $\varphi_T = (\varphi^0_T, \varphi^1_T) \in \mathcal{A}^M$, as well as a number $C \geqslant M$. We shall define the C-truncation φ^C_T of φ_T in a pointwise way: if $|\varphi_T(\omega)|_{S_T(\omega)} \leqslant C$ we simply let

$$\varphi^C_T(\omega) = \varphi_T(\omega).$$

If $|\varphi_T(\omega)|_{S_T(\omega)} > C$ we define

$$\varphi_T^C = \mu(\varphi_T^0(\omega), \varphi_T^1(\omega)) + (1-\mu)(-M, 0) \tag{192}$$

which is a convex combination of $\varphi_T(\omega)$ and the lower left corner $(-M, 0)$ of the $M-$ ball of $(\mathbb{R}^2, |\cdot|_{S_T(\omega)})$; for $\mu \in [0,1]$ above we choose the biggest number in $[0,1]$ such that $|\varphi_T^C(\omega)|_{S_T(\omega)} \leqslant C$. Note that, for $C' \geqslant C \geqslant M$ we have $\varphi_T^{C'}(\omega) - \varphi_T^C(\omega) \in K_{S_T(\omega)}$, i.e. we can obtain φ_T^C from $\varphi_T^{C'}$ (as well as from φ_T) by a self-financing trade at time T.

By construction φ_T^C lies in the Banach space L_S^∞, its norm being bounded by C. Sending C to infinity the random variables $\varphi_T^{0,C}$ increase (with respect to the order induced by the cone K_T) a.s. to φ_T^0.

Summing up: the intersection $\mathcal{A} \cap L_S^\infty$ is dense in $\mathcal{A}$ in the sense that, for $\varphi_T \in \mathcal{A}$ there is an *increasing* sequence $(\varphi_T^k)_{k \geqslant 0}$ in $\mathcal{A} \cap L_S^\infty$ converging a.s. to φ_T. This is what we mean by "Fatou-dense".

Following a well-known line of argument (compare [64]), Theorem 4.13 thus translates into the following result.

Theorem 4.15. *Fix S and $\lambda > 0$, and suppose that $(CPS^{\lambda'})$ is satisfied, for each $0 < \lambda' < \lambda$. The convex cone $\mathcal{A} \cap L_S^\infty \subseteq L_S^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, as well as the convex cone $\mathcal{C} \cap L^\infty \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ are closed with respect to the weak-star topology induced by L_S^1 (resp. L^1).*

Proof. By the Krein-Smulian theorem [220] the cone $\mathcal{A} \cap L_S^\infty$ is σ^*-closed iff its intersection with the unit ball of L_S^∞ is σ^*-closed. Hence it suffices to show that $\mathcal{A} \cap$ (ball (L_S^∞)) is σ^*-compact. By a result of A. Grothendieck ([97], see also the version [69, Prop.5.2.4] which easily extends to the present 2-dimensional setting), the σ^*-compactness of a bounded, convex subset of L^∞ is equivalent to the following property: for every sequence $(\varphi_T^n)_{n=1}^\infty \in \mathcal{A} \cap$ (ball (L_S^∞)) converging a.s. to φ_T, we have that the limit again is in $\mathcal{A} \cap$ ball (L_S^∞). By the definition of the norm of L_S^∞ and using Proposition 4.12 we have that $\varphi_T^n \in \mathcal{A}^1$, for each n, so that Theorem 4.13 implies that the limit φ_T again is in $\mathcal{A}^1$. As the inequalities (190) and (191) clearly remain valid by passing from $(\varphi_T^n)_{n=1}^\infty$ to the limit φ_T we obtain that $\varphi_T \in \mathcal{A} \cap$ (ball L_S^∞). This shows the σ^*-closedness of $\mathcal{A} \cap L_S^\infty$.

The σ^*-closedness of $\mathcal{C}$ follows from the σ^*-closedness of $\mathcal{A}$ and the fact that L^∞ is a σ^*-closed subset of L_S^∞. ■

Theorem 4.15 allows us to apply the duality theory to the dual pairs $\langle L_S^1, L_S^\infty \rangle$ and $\langle L^1, L^\infty \rangle$ respectively. Denoting as above by $(\mathcal{A} \cap L_S^\infty)^\circ$ (resp. $(\mathcal{C} \cap L^\infty)^\circ$) the polar of $\mathcal{A} \cap L^\infty$ in L_S^1 (resp. of $\mathcal{C} \cap L^\infty$ in L^1), the bipolar theorem ([220]; see also Proposition A.1 in the appendix) as well as Theorem

4.15 imply that $(\mathcal{A} \cap L_S^\infty)^{\circ\circ} = \mathcal{A} \cap L_S^\infty$ and $(\mathcal{C} \cap L^\infty)^{\circ\circ} = \mathcal{C} \cap L^\infty$. In fact, we shall be able to characterize the polars $(\mathcal{A} \cap L_S^\infty)^\circ$ and $(\mathcal{C} \cap L_S^\infty)^\circ$ in terms of consistent price systems.

We remark that the distinction between $\mathcal{A}$ and $\mathcal{A} \cap L_S^\infty$ (resp. $\mathcal{C}$ and $\mathcal{C} \cap L^\infty$) is rather a formality; the passage to these intersections only serves to put us into the well-established framework of the duality theory of Banach spaces. For example, we shall consider the polar set

$$(\mathcal{C} \cap L^\infty)^\circ = \{Z_T^0 \in L^1 : \langle \varphi_T^0, Z_T^0 \rangle = \mathbb{E}[\varphi_T^0 Z_T^0] \leqslant 0, \quad \text{for every } \varphi_T^0 \in \mathcal{C} \cap L^\infty\} \tag{193}$$

and an analogous definition for $(\mathcal{A} \cap L_S^\infty)^\circ \subseteq L_S^1$. We note that we could equivalently define

$$\mathcal{C}^\circ = \{Z_T^0 \in L^1 : \langle \varphi_T^0, Z_T^0 \rangle = \mathbb{E}[\varphi_T^0 Z_T^0] \leqslant 0 \quad \text{for every } \varphi_T^0 \in \mathcal{C}\}$$

Indeed, as each $\varphi_T^0 \in \mathcal{C}$ is uniformly bounded from below, the expectation appearing above is well-defined (possibly assuming the value infinity) and it follows from monotone convergence that

$$\mathbb{E}[\varphi_T^0 Z_T^0] \leqslant 0 \quad \text{iff} \quad \mathbb{E}[(\varphi_T^0 \wedge n) Z_T^0] \leqslant 0,$$

for every $n \geqslant 0$. A similar remark applies to $(\mathcal{A} \cap L_S^\infty)^\circ$. To alleviate notation we shall therefore write $\mathcal{C}^\circ$ and $\mathcal{A}^\circ$ instead of $(\mathcal{C} \cap L^\infty)^\circ$ and $(\mathcal{A} \cap L_S^\infty)^\circ$.

4.2 The dual variables

To characterize the polars of $\mathcal{A}$ and $\mathcal{C}$, let $(\tilde{S}, Q)$ be a consistent price system (Def. 4.1) for the process S under transaction costs λ. As usual, we denote by $(Z_t^0)_{0 \leqslant t \leqslant T}$ the density process $Z_t^0 = \mathbb{E}[\frac{dQ}{d\mathbb{P}} | \mathcal{F}_t]$ and by $(Z_t^1)_{0 \leqslant t \leqslant T}$ the process $(Z_t^0 \tilde{S}_t)_{0 \leqslant t \leqslant T}$, so that Z^0 (resp. Z^1) is a martingale (resp. a local martingale) under $\mathbb{P}$.

Definition 4.16. *Given S and $\lambda > 0$ as above, we denote by $B(1)$ the convex, bounded set of non-negative random variables $\{Z_T = (Z_T^0, Z_T^1)\}$ such that Z_T is the terminal value of a consistent price process as above. Denote by $\mathcal{B}(1)$ the norm closure of $B(1)$ in L_S^1, and by $\mathcal{B}$ the cone generated by $\mathcal{B}(1)$, i.e.*

$$\mathcal{B} = \bigcup_{y \geqslant 0} \mathcal{B}(y),$$

where $\mathcal{B}(y) = y\mathcal{B}(1)$.

We denote by $D(1)$ the projection of $B(1)$ onto $L^1(\mathbb{R})$ (via the canonical projection of L_S^1 onto its first coordinate), and by $\mathcal{D}(1)$ and $\mathcal{D}$ its norm closure and the cone generated by $\mathcal{D}(1)$, respectively.

Proposition 4.17. *Let S and $\lambda > 0$ be as in Theorem 4.13, and suppose again that $(CPS^{\lambda'})$ holds true, for all $0 < \lambda' < \lambda$.*

Then $\mathcal{B}$ (resp. $\mathcal{D}$) is a closed set in L^1_S (resp. L^1) and $\mathcal{B}$ (resp. $\mathcal{D}$) equals the polar cone $\mathcal{A}^\circ$ of $\mathcal{A}$ (resp. $\mathcal{C}^\circ$ of $\mathcal{C}$) in L^1_S (resp. in L^1).

Proof. To obtain the inclusion $\mathcal{B} \subseteq \mathcal{A}^\circ$, we shall show that

$$\langle(\varphi^0_T, \varphi^1_T), (Z^0_T, Z^1_T)\rangle = \mathbb{E}[\varphi^0_T Z^0_T + \varphi^1_T Z^1_T] \leqslant 0, \tag{194}$$

for all $\varphi_T = (\varphi^0_T, \varphi^1_T) \in \mathcal{A}$ and for all $Z_T = (Z^0_T, Z^1_T) \in B(1)$.

Indeed, associate to φ_T an admissible trading strategy $(\varphi_t)_{0\leqslant t\leqslant T} = (\varphi^0_t, \varphi^1_t)_{0\leqslant t\leqslant T}$ and to Z_T a consistent price system $(\tilde{S}, Q) = ((\frac{Z^1_t}{Z^0_t})_{0\leqslant t\leqslant T}, Z^0_T)$. By Proposition 4.5 the process

$$\tilde{V}_t = \varphi^0_t + \varphi^1_t \tilde{S}_t, \qquad\qquad 0 \leqslant t \leqslant T,$$

is a Q-supermartingale, starting at $\tilde{V}_{0_-} = 0$, so that

$$\mathbb{E}_{\mathbb{P}}[\varphi^0_T Z^0_T + \varphi^1_T Z^1_T] = \mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T] \leqslant 0.$$

This shows (194) which, by continuity and positive homogeneity, also holds true, for all $Z_T = (Z^0_T, Z^1_T) \in \mathcal{B}$. We therefore have shown that $\mathcal{B}$ is contained in the polar $\mathcal{A}^\circ$ of $\mathcal{A}$.

As regards the reverse inclusion $\mathcal{A}^\circ \subseteq \mathcal{B}$, we have to show that, for $\varphi_T = (\varphi^0_T, \varphi^1_T) \in L^\infty_S$, such that (194) is satisfied, for all $Z_T = (Z^0_T, Z^1_T) \in B(1)$, we have that $\varphi_T \in \mathcal{A}$.

Fix $\bar{\varphi}_T = (\bar{\varphi}^0_T, \bar{\varphi}^1_T) \notin \mathcal{A}$. By Theorem 4.15 and the Hahn-Banach theorem we may find an element $\bar{Z}_T = (\bar{Z}^0_T, \bar{Z}^1_T) \in L^1_S$ such that (194) holds true, for $\bar{Z}_T$ and all $\varphi_T \in \mathcal{A}$ while

$$\Big\langle(\bar{\varphi}^0_T, \bar{\varphi}^1_T), (\bar{Z}^0_T, \bar{Z}^1_T)\Big\rangle > 0. \tag{195}$$

As $\mathcal{A}$ contains the non-positive functions, we have that $(\bar{Z}^0_T, \bar{Z}^1_T)$ takes values a.s. in $\mathbb{R}^2_+$. In fact, we may suppose that $\bar{Z}^0_T$ and $\bar{Z}^1_T$ are a.s. strictly positive. Indeed, by the assumption CPS^λ there is a λ-consistent price system $\hat{Z} = (\hat{Z}^0, \hat{Z}^1)$. For $\varepsilon > 0$, the convex combination $(1-\varepsilon)\bar{Z}_T + \varepsilon \hat{Z}_T$ still satisfies (194), for each $\varphi^1_T \in \mathcal{A}$. For $\varepsilon > 0$ sufficiently small, (195) is satisfied too. Hence, by choosing $\varepsilon > 0$ sufficiently small, we may assume that $\bar{Z}^0_T$ and $\bar{Z}^1_T$ are strictly positive.

We also may assume that $\mathbb{E}[\bar{Z}^0_T] = 1$ so that $\frac{d\bar{Q}}{d\mathbb{P}} := \bar{Z}^0_T$ defines a probability measure $\bar{Q}$ which is equivalent to $\mathbb{P}$. We now have to work towards a contradiction.

To focus on the essence of the argument, let us assume for a moment that $S = (S_t)_{0\leqslant t\leqslant T}$ is uniformly bounded. We then may define the $\mathbb{R}^2_+$-valued martingale $\bar{Z} = (\bar{Z}^0, \bar{Z}^1)$ by

$$\bar{Z}_t = (\bar{Z}^0_t, \bar{Z}^1_t) = \mathbb{E}[(\bar{Z}^0_T, \bar{Z}^1_T)|\mathcal{F}_t], \qquad 0 \leqslant t \leqslant T. \tag{196}$$

Indeed by (185) and (186), we have $\bar{Z}^1_T \leqslant C\bar{Z}^0_T \leqslant C^*|\bar{Z}_T|^*_{S_T}$ almost surely, for some constants C, C^*, depending on the uniform bound of S. Hence $\bar{Z}_T$ is integrable so that $\bar{Z}_t$ in (196) is well-defined. We shall verify that $\bar{Z} = (\bar{Z}_t)_{0\leqslant t\leqslant T}$ indeed defines a consistent price system. To do so, we have to show that, for $0 \leqslant t \leqslant T$,

$$\tilde{S}_t := \frac{\bar{Z}^1_t}{\bar{Z}^0_t} \in [(1-\lambda)S_t, S_t]\,, \qquad \text{a.s.} \tag{197}$$

Negating (197) we may find some $0 \leqslant u \leqslant T$ such that one of the following two sets has strictly positive measure

$$A_+ = \left\{\frac{\bar{Z}^1_u}{\bar{Z}^0_u} > S_u\right\}, \quad A_- = \left\{\frac{\bar{Z}^1_u}{\bar{Z}^0_u} < (1-\lambda)S_u\right\}.$$

In the former case, define the process $\varphi^1 = (\varphi^0, \varphi^1)$ as in (162) by

$$(\varphi^0_t, \varphi^1_t) = (-S_u, 1)\mathbb{1}_{A_+}\mathbb{1}_{[\![u,T]\!]}(t), \qquad 0 \leqslant t \leqslant T.$$

Using the boundedness of S, we conclude that $(\varphi^0_T, \varphi^1_T) = (\varphi^0_u, \varphi^1_u) = (-S_u, 1)\mathbb{1}_{A_+}$ is an element of $\mathcal{A}$ for which we get

$$\begin{aligned}\mathbb{E}\left[\varphi^0_T\bar{Z}^0_T + \varphi^1_T\bar{Z}^1_T\right] =& \mathbb{E}\left[\mathbb{E}\left[\varphi^0_u\bar{Z}^0_T + \varphi^1_u\bar{Z}^1_T|\mathcal{F}_u\right]\right]\\ =& \mathbb{E}\left[\varphi^0_u\bar{Z}^0_u + \varphi^1_u\bar{Z}^1_u\right]\\ =& \mathbb{E}\left[\bar{Z}^0_u\left(-S_u + \tfrac{\bar{Z}^1_u}{\bar{Z}^0_u}\right)\mathbb{1}_{A_+}\right] > 0,\end{aligned}$$

a contradiction to (194).

If $\mathbb{P}[A_-] > 0$ we apply a similar argument to (163).

Summing up: we have arrived at the desired contradiction proving the inclusion $\mathcal{A}^\circ \subseteq \mathcal{B}$, under the additional assumption that S is uniformly bounded.

Now we drop the boundedness assumption on S. By the continuity of S we may find a localizing sequence $(\tau_n)^\infty_{n=1}$ of $[0,T] \cup \{\infty\}$-valued stopping times, increasing a.s. to ∞, such that each stopped processes $S^{\tau_n} = (S_{t\wedge\tau_n})_{0\leqslant t\leqslant T}$ is bounded. Indeed, it suffices to take $\tau_n = \inf\{t : S_t > n\}$.

Denote by $\mathcal{A}_{\tau_n} = \mathcal{A} \cap L^\infty_S(\Omega, \mathcal{F}_{\tau_n}, \mathbb{P})$ the subset of $\mathcal{A} \cap L^\infty_S$ formed by the elements $\varphi_T = (\varphi^0_T, \varphi^1_T)$ which are $\mathcal{F}_{\tau_n}$-measurable. We then have that $\mathcal{A}_{\tau_n}$ is the cone corresponding to the stopped process S^{τ_n} via Definition 4.7. By stopping, we also have that $\bigcup_{n=1}^\infty \mathcal{A}_{\tau_n} \cap L^\infty_S(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ is weak-star dense in $\mathcal{A} \cap L^\infty_S(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$.

Denote by $\bar{Z}_{\tau_n}$ the restriction of the functional $\bar{Z}_T = (\bar{Z}^0_T, \bar{Z}^1_T)$ to $L^\infty_S(\mathcal{F}_{\tau_n})$ which we may identify with a pair $(\bar{Z}^0_{\tau_n}, \bar{Z}^1_{\tau_n})$ of $\mathcal{F}_{\tau_n}$-measurable functions.

By taking conditional expectations as in (196), we may associate to the random variables $(\bar{Z}^0_{\tau_n}, \bar{Z}^1_{\tau_n})$ the corresponding martingales, denoted by $\bar{Z}^n = (\bar{Z}^{0,n}_t, \bar{Z}^{1,n}_t)_{0\leqslant t\leqslant \tau_n\wedge T}$.

Of course, this sequence of processes is consistent, i.e., for $n \leqslant m$, the process $\bar{Z}^m$, stopped at τ_n, equals the process $\bar{Z}^n$. As regards the first coordinate, it is clear that $(\bar{Z}^0_{\tau_n\wedge T})_{n=1}^\infty$ converges in the norm of $L^1(\mathbb{P})$ to $\bar{Z}^0_T$, which is the density of the probability measure $\bar{Q}$. The associated density process is $\bar{Z}^0_t = \mathbb{E}[\bar{Z}^0_T|\mathcal{F}_t]$. The slightly delicate issue is the second coordinate of $\bar{Z}$. The sequence $(\bar{Z}^1_{\tau_n\wedge T})$ only converges a.s. to $\bar{Z}^1_T$, but not necessarily with respect to the norm of $L^1(\mathbb{P})$. In other words, by pasting together the processes $(\bar{Z}^{1,n}_t)_{0\leqslant t\leqslant\tau_n\wedge T}$, and letting

$$\bar{Z}^1_t = \lim_{n\to\infty} \bar{Z}^{1,n}_t,$$

the limit holding true a.s., for each $0 \leqslant t \leqslant T$, we well-define a *local* $\mathbb{P}$-martingale $(\bar{Z}^1_t)_{0\leqslant t\leqslant T}$. This process may fail to be a true $\mathbb{P}$-martingale. But this does not really do harm: the process $(\bar{Z}^0_t, \bar{Z}^1_t)_{0\leqslant t\leqslant T}$ still is a consistent price system under transaction costs λ in the sense of Definition 4.1. Indeed, by the first part of the proof we have that, for $t \in [0, T]$ and $n \in \mathbb{N}$,

$$\frac{\bar{Z}^1_t}{\bar{Z}^0_t} \in [(1-\lambda)S_t, S_t], \qquad \text{a.s. on } \{t \leqslant \tau_n\}.$$

As $\bigcup_{n=1}^\infty \{t \leqslant \tau_n\} = \Omega$ a.s., for each fixed $0 \leqslant t \leqslant T$, we have obtained (197). We note in passing that Definition 4.1 was designed in a way that we allow for *local* martingales in the second coordinate $(\bar{Z}^1_t)_{0\leqslant t\leqslant T}$.

Summing up: we have found a consistent price system $\bar{Z} = (\bar{Z}^0_t, \bar{Z}^1_t)_{0\leqslant t\leqslant T}$ in the sense of Definition 4.1 such that the terminal value $(\bar{Z}^0_T, \bar{Z}^1_T)$ satisfies (195). This contradiction shows that the cones $\mathcal{A}$ and $\mathcal{B}$ are in polar duality and finishes the proof of the first assertion of the theorem.

The corresponding assertion for the cones $\mathcal{C} \cap L^\infty$ and $\mathcal{D}$ now follows. For $\varphi^0_T \in \mathcal{C}$ we have, by definition, that $(\varphi^0_T, 0) \in \mathcal{A}$ so that $\langle(\varphi^0_T, 0), (Z^0_T, Z^1_T)\rangle =$

$\langle \varphi^0_T, Z^0_T \rangle \leqslant 0$, for each consistent price system $Z = (Z^0, Z^1)$. This yields the inclusion $\mathcal{D} \subseteq (\mathcal{C} \cap L^\infty)^\circ$. Conversely, if $(\varphi^0_T, 0) \notin \mathcal{A}$ we may find by the above argument a consistent price system $\bar{Z}$ such that $\langle (\varphi^0_T, 0), (\bar{Z}^0_T, \bar{Z}^1_T) \rangle > 0$, which yields the inclusion $(\mathcal{C} \cap L^\infty)^\circ \subseteq \mathcal{D}$.

The proof of Proposition 4.17 now is complete. ■

We now are in a position to state and prove the central result of this chapter, the super-replication theorem (compare Corollary 1.11).

Theorem 4.18. *Suppose that the continuous, adapted process $S = (S_t)_{0 \leqslant t \leqslant T}$ satisfies $(CPS^{\lambda'})$, for each $0 < \lambda' < 1$, and fix $0 < \lambda < 1$.*

Suppose that the $\mathbb{R}^2$-valued random variable $\varphi_T = (\varphi^0_T, \varphi^1_T)$ satisfies

$$V^{liq}(\varphi^0_T, \varphi^1_T) = \varphi^0_T + (\varphi^1_T)^+(1-\lambda)S_T - (\varphi^1_T)^- S_T \geqslant -M. \tag{198}$$

For a constant $x^0 \in \mathbb{R}$ the following assertions then are equivalent:

(i) $\varphi_T = (\varphi^0_T, \varphi^1_T)$ is the terminal value of some self-financing, admissible trading strategy $(\varphi_t)_{0 \leqslant t \leqslant T} = (\varphi^0_t, \varphi^1_t)_{0 \leqslant t \leqslant T}$ under transaction costs λ, starting at $(\varphi^0_{0_-}, \varphi^1_{0_-}) = (x^0, 0)$.

(ii) $\mathbb{E}_Q[\varphi^0_T + \varphi^1_T \tilde{S}_T] \leqslant x^0$, for every λ-consistent price system $(\tilde{S}, Q)$.

Proof. First suppose that $\varphi_T = (\varphi^0_T, \varphi^1_T) \in L^\infty_S$. Then (i) is tantamount to $(\varphi^0_T - x^0, \varphi^1_T)$ being an element of $\mathcal{A} \cap L^\infty_S$. By Proposition 4.17, Theorem 4.15, and the Bipolar Theorem (Proposition A.1 in the Appendix), this is equivalent to

$$\mathbb{E}_Q[\varphi^0_T - x^0 + \varphi^1_T \tilde{S}_T] \leqslant 0,$$

holding true for all λ-consistent price systems $(\tilde{S}, Q)$ which amounts to (ii).

Dropping the assumption $\varphi_T \in L^\infty_S$, we consider, for $C \geqslant M$, the C-truncations φ^C_T defined after (192) which are well-defined in view of (198). Recall that $\varphi^C_T \in L^\infty_S$ and $(\varphi^C_T)_{C \geqslant M}$ increases to φ_T, as $C \to \infty$. We may apply the first part of the argument to each φ^C_T and then send C to infinity: assume that (i) (and therefore, equivalently, (ii)) holds true, for each φ^C_T, where C is sufficiently large. Then (ii) also holds true for φ_T by monotone convergence, and (i) also holds true for φ_T by Theorem 4.13. ■

Corollary 4.19. *Under the assumptions of Theorem 4.18, let $\varphi^0_T \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ be a random variable bounded from below, i.e.*

$$\varphi^0_T \geqslant -M, \qquad a.s.$$

for some real number M. For a real constant x^0 the following are equivalent.

(i) $\varphi_T = (\varphi_T^0, 0)$ is the terminal value of some self-financing, admissible trading strategy $(\varphi_t)_{0\leqslant t\leqslant T} = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$ under transaction costs λ, starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (x^0, 0)$.

(ii) $\mathbb{E}_Q[\varphi_T^0] \leqslant x^0$, for every λ-consistent price system $(\tilde{S}, Q)$.

Proof. Apply Theorem 4.18 to $(\varphi_T^0, 0)$. ■

4.3 Non-negative Claims

We shall need the following generalisation of the notion of λ-consistent price systems (compare Def. 5.1 below).

Definition 4.20. *Fix the continuous, adapted, strictly positive process $S = (S_t)_{0\leqslant t\leqslant T}$, and $\lambda > 0$. The* λ-consistent equivalent super-martingale deflators *are defined as the set $\mathcal{Z}^e = \mathcal{Z}^e(1)$ of strictly positive processes $Z = (Z_t^0, Z_t^1)_{0\leqslant t\leqslant T}$, starting at $Z_0^0 = 1$, such that, for every x-admissible, λ-self-financing process $\varphi = (\varphi_t^0, \varphi_t^1)_{0\leqslant t\leqslant T}$, starting at $(\varphi_{0_-}^0, \varphi_{0_-}^1) = (0, 0)$, we have that the process*

$$(x + \varphi_t^0)Z_t^0 + \varphi_t^1 Z_t^1, \qquad 0 \leqslant t \leqslant T,$$

is a non-negative supermartingale.

If, in addition, Z is a local martingale, we call Z a local martingale deflator and denote the corresponding set by $\mathcal{Z}^{loc,e}$.

By dropping the super-script e we define the sets $\mathcal{Z}$ (resp $\mathcal{Z}^{loc}$) of λ-consistent super-martingale deflators (resp. local martingale deflators)*, where we only impose the non-negativity of the elements Z.*

We note that Proposition 4.5 implies that $\mathcal{Z}^{loc,e}$ contains the λ-consistent price systems, where we identify $(\tilde{S}, Q)$ with the process $(Z_t^0, Z_t^1)_{0\leqslant t\leqslant T}$ given by $Z_t^0 = \mathbb{E}[\frac{dQ}{d\mathbb{P}}|\mathcal{F}_t]$ and $Z_t^1 = \tilde{S}_t Z_t^0$.

For the applications in the next chapter, which concerns utility maximization, we shall deal with *positive* elements φ_T^0 only. For this setting we now develop a similar duality theory as in Theorem 4.18 and Corollary 4.19. We start with a definition relating the cones $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ defined in 4.7 and Definition 4.16 above to bounded subsets of $L^0(\mathbb{R}^2)$ and $L_+^0(\mathbb{R})$, respectively.

Definition 4.21. *For $x > 0$, we define*

$$\mathfrak{A}(x) = \{(\varphi_T^0, \varphi_T^1) : \varphi_T^0 + (\varphi_T^1)^+(1-\lambda)S_T - (\varphi_T^1)^- S_T \geqslant 0, \\ \text{and } (\varphi_T^0 - x, \varphi_T^1) \in \mathcal{A}\},$$
$$\mathfrak{C}(x) = \{\varphi_T^0 \geqslant 0 : \varphi_T^0 - x \in \mathcal{C}\} = \{\varphi_T^0 : (\varphi_T^0, 0) \in \mathfrak{A}(x)\}.$$

For $y > 0$, we define

$$\mathfrak{B}(y) = \{(Z_T^0, Z_T^1) : \text{there is } Z \in \mathcal{Z}(y) \text{ with terminal value } (Z_T^0, Z_T^1)\},$$
$$\mathfrak{D}(y) = \{Z_T^0 : \text{there is } Z \in \mathcal{Z}(y) \text{ with a terminal value } (Z_T^0, Z_T^1), \\ \text{for some } Z_T^1\}.$$

We denote by $\mathfrak{B}^{loc}(y)$ and $\mathfrak{D}^{loc}(y)$ the corresponding sets when the supermartingale deflator $Z \in \mathcal{Z}(y)$ is required to be a local martingale, i.e. $Z \in \mathcal{Z}^{loc}(y)$.

Theorem 4.22. *Suppose that the continuous, strictly positive process $S = (S_t)_{0\leqslant t\leqslant T}$ satisfies condition $(CPS^{\lambda'})$, for each $0 < \lambda' < 1$. Fix $0 < \lambda < 1$.*

(i) The sets $\mathfrak{A}(x), \mathfrak{C}(x), \mathfrak{B}(y), \mathfrak{D}(y)$ defined in Definition 4.21 are convex, closed (w.r to convergence in measure) subsets of $L^0(\mathbb{R}^2)$ and $L^0_+(\mathbb{R})$ respectively. The sets $\mathfrak{A}(x), \mathfrak{C}(x)$ and $\mathfrak{D}(y)$ are also solid.

(ii) Fix $x > 0, y > 0$ and $\varphi_T^0 \in L^0_+(\mathbb{R})$. We have $\varphi_T^0 \in \mathfrak{C}(x)$ iff

$$\langle \varphi_T^0, Z_T^0 \rangle \leqslant xy, \tag{199}$$

for all $Z_T^0 \in \mathfrak{D}(y)$ and iff, for all λ-consistent price systems $(\tilde{S}, Q)$ we have

$$\mathbb{E}_Q[\varphi_T^0] \leqslant x. \tag{200}$$

(ii') We have $Z_T^0 \in \mathfrak{D}(y)$ iff

$$\langle \varphi_T^0, Z_T^0 \rangle \leqslant xy \tag{201}$$

for all $\varphi_T^0 \in \mathfrak{C}(x)$.

(iii) The sets $\mathfrak{A}(1)$ and $\mathfrak{C}(1)$ are bounded in $L^0(\mathbb{R}^2)$ and $L^0(\mathbb{R})$ respectively and contain the constant functions $(\mathbb{1}, 0)$ (resp. $\mathbb{1}$).

Proof. (i) The convexity of the four sets is obvious. As regards the solidity recall that a set $C \subseteq L^0_+(\mathbb{R})$ is solid if $0 \leqslant \psi_T^0 \leqslant \varphi_T^0 \in C$ implies that $\psi_T^0 \in C$. As regards $\mathfrak{C}(x)$, this property clearly holds true as one is allowed to "throw away bonds" at terminal time T. As regards the solidity of $\mathfrak{D}(y)$: if there is $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T} \in \mathcal{Z}(y)$, and Y_T^0 satisfies $0 \leqslant Y_T^0 \leqslant Z_T^0$, we may define an element $Y = (Y_t^0, Y_t^1)_{0 \leqslant t \leqslant T} \in \mathcal{Z}(y)$ by letting

$$(Y_t^0, Y_t^1) = \begin{cases} (Z_t^0, Z_t^1), & 0 \leqslant t < T, \\ (Y_T^0, Z_T^1 \frac{Y_T^0}{Z_T^0}), & t = T, \end{cases}$$

which shows the solidity of $\mathfrak{D}(y)$.

The L^0-closedness of $\mathfrak{A}(x)$ and $\mathfrak{C}(x)$, follows from Theorem 4.13. Indeed $x > 0$ corresponds to the admissibility constant $M > 0$ in Theorem 4.13 and the operations of shifting these sets by the constant vector $(x\mathbb{1}, 0)$ and intersecting them with the positive orthant preserves the L^0-closedness.

Let us now pass to the closedness of $\mathfrak{B}(y)$ and $\mathfrak{D}(y)$. Fix a Cauchy sequence $Z_T^n = (Z_T^{0,n}, Z_T^{1,n})$ in $\mathfrak{B}(y)$ and associate to it the supermartingales $Z^n = (Z_t^{0,n}, Z_t^{1,n})_{0 \leqslant t \leqslant T}$ as in Def 4.21. Applying Lemma 4.14 and passing to convex combinations similarly in the proof of Theorem 4.13 we may pass to a limiting càdlàg process $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$ in the following way (the "Fatou Limit" construction from [50]).

First pass to pointwise limits of convex combinations of $(Z_r^{0,n}, Z_r^{1,n})_{n=1}^{\infty}$, where r ranges in the rational numbers in $[0, T]$ and then pass to the càdlàg versions, which exist as the limiting process $(Z_r^0, Z_r^1)_{r \in [0,T] \cap \mathbb{Q}}$ is a super-martingale (we suppose w.l.g. that T is rational). The fact that, for every 1-admissible λ-self-financing $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ the process

$$V_t = (1 + \varphi_t^0) Z_t^0 + \varphi_t^1 Z_t^1, \qquad 0 \leqslant t \leqslant T,$$

is a super-martingale, now follows from Fatou's lemma. The argument for $\mathfrak{D}(y)$ is similar.

We thus have proved assertion (i).

(ii) Let $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ be an admissible, self-financing process starting at $\varphi_{0_-} = (x, 0)$ and ending at $(\varphi_T^0, 0)$. Let $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$ be a super-martingale deflator starting at $Z_0 = (y, Z_0^1)$, for some $Z_0^1 \in [(1-\lambda) y S_0, y S_0]$. By definition

$$\varphi_t^0 Z_t^0 + \varphi_t^1 Z_t^1, \qquad 0 \leqslant t \leqslant T,$$

is a super-martingale starting at xy so that inequality (199) holds true.

Conversely, assertion (200) follows from Theorem 4.18 and Corollary 4.19.

(ii') If $Z_T^0 \in \mathfrak{D}(y)$ and $\varphi_T^0 \in \mathfrak{C}(x)$, we have already shown the inequality (199). As regards the "only if" assertion, condition (199) may be rephrased abstractly as the assertion that $\mathfrak{D}(1) = \frac{1}{y}\mathfrak{D}(y)$ equals the polar set of $\mathfrak{C}(1) =$

$\frac{1}{x}\mathfrak{C}(x)$ as defined in (202) below. On the other hand it follows from Proposition 4.17 and Corollary 4.19 that the polar of the set

$$D(1) = \{Z_T^0 \in L_+^0 : \tfrac{dQ}{d\mathbb{P}} = Z_T^0 \text{ for a consistent price system } (\tilde{S}, Q)\}$$

equals $\mathfrak{C}(1)$. Hence by the subsequent version of the bipolar theorem we have that, if Z_T^0 satisfies (201), it is an element of the closed, convex, and solid hull of $D(1)$. As $D(1) \subseteq \mathfrak{D}(1)$ we conclude from (i) that this implies $Z_T^0 \in \mathfrak{D}(1)$.

(iii) By hypothesis (CPS^λ) there is a λ-consistent price system $(\tilde{S}, Q)$. We denote by $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$ the corresponding density process in $\mathcal{Z}^e$. For each $\varepsilon > 0$ there is $\delta > 0$ such that, for a subset $A \in \mathcal{F}$ with $\mathbb{P}[A] \geqslant \varepsilon$ we have $\mathbb{E}[\mathbb{1}_A Z_T^0] \geqslant \delta$ and $\mathbb{E}[\mathbb{1}_A Z_T^1] \geqslant \delta$. This shows that $\mathfrak{A}(1)$ is bounded in L^0. The L^0-boundedness of $\mathfrak{C}(1)$ follows and the final assertion is obvious. ■

We have used in the proof of (ii') above the subsequent version of the bipolar theorem pertaining to subsets *of the positive orthant* L_+^0 of L^0.

Proposition 4.23. *([30], compare also [245]) For a subset* $D \subseteq L_+^0(\Omega, \mathcal{F}, \mathbb{P})$ *we define its polar in* L_+^0 *as*

$$D^\circ = \{g \in L_+^0 : \mathbb{E}[gh] \leqslant 1, \text{ for all } h \in D\}. \tag{202}$$

Then the bipolar $D^{\circ\circ}$ *equals the closed (with respect to convergence in measure), convex, solid hull of* D.

5 The local duality theory

In this chapter we extend the duality theory to the setting where the corresponding concepts such as no arbitrage in its many variants, existence of consistent price systems etc. only hold true *locally*. For example, this situation arises naturally in the stochastic portfolio theory as promoted by R. Fernholz and I. Karatzas. We refer to the paper [149] by I. Karatzas and K. Kardaras (compare also [154] and [235]) where the local duality theory is developed in the classical frictionless setting.

Recall that a property (P) of a stochastic process $S = (S_t)_{0 \leqslant t \leqslant T}$ holds true locally if there is a sequence of stopping times $(\tau_n)_{n=1}^{\infty}$ increasing to infinity such that each of the stopped processes $S^{\tau_n} = (S_{t \wedge \tau_n})_{0 \leqslant t \leqslant T}$ has property (P).

We say that (P) is a local property if the fact that S has property (P) locally implies that S has property (P).

The frictionless case

In the subsequent definition we formulate the notion of a super-martingale deflator in the frictionless setting. The tilde super-scripts indicate that we are in the semi-martingale setting.

Definition 5.1. *(see [149] and [235]) Let $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ be a semi-martingale based on and adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$. The set of* equivalent super-martingale deflators *$\tilde{\mathcal{Z}}^e$ is defined as the $]0, \infty[$-valued processes $(\tilde{Z}_t)_{0 \leqslant t \leqslant T}$, starting at $\tilde{Z}_0 = 1$, such that, for every $\tilde{S}$-integrable predictable process $\tilde{H} = (\tilde{H}_t)_{0 \leqslant t \leqslant T}$ verifying*

$$1 + (\tilde{H} \cdot \tilde{S})_t \geqslant 0, \qquad 0 \leqslant t \leqslant T, \tag{203}$$

the process

$$\tilde{Z}_t(1 + (\tilde{H} \cdot \tilde{S})_t), \qquad 0 \leqslant t \leqslant T, \tag{204}$$

is a super-martingale under $\mathbb{P}$. Dropping the super-script e we obtain the corresponding class $\tilde{\mathcal{Z}}$ of $[0, \infty[$-valued super-martingale deflators.

We call $\tilde{Z} \in \tilde{\mathcal{Z}}$ a local martingale deflator *if, in addition, $\tilde{Z}$ is a local martingale. We denote by $\tilde{\mathcal{Z}}^{loc}$ (resp. $\tilde{\mathcal{Z}}^{e,loc}$) the set of local (resp. equivalent local) martingale deflators.*

We say that $\tilde{S}$ satisfies the property (ESD) (resp. (ELD)) of existence of an equivalent super-martingale (resp. local martingale) deflator *if $\tilde{\mathcal{Z}}^e \neq \varnothing$ (resp. $\tilde{\mathcal{Z}}^{e,loc} \neq \varnothing$).*

We remark that, for a probability measure Q equivalent to $\mathbb{P}$ under which $\tilde{S}$ is a local martingale, we have that the density process $\tilde{Z}_t = \mathbb{E}[\frac{dQ}{d\mathbb{P}}|\mathcal{F}_t]$ defines an equivalent local martingale deflator.

We first give an easy example of a process $\tilde{S}$, for which (NFLVR) fails while there does exist a local-martingale deflator (see [149, Ex. 4.6] for a more sophisticated example, involving the three-dimensional Bessel process). In fact, we formulate this example in such a way that it also highlights the persistence of this phenomenon under transaction costs.

Proposition 5.2. *There is a continuous semi-martingale $S = (S_t)_{0 \leqslant t \leqslant 1}$, based on a Brownian filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant 1}$, such that there is an equivalent local-martingale deflator $(Z_t)_{0 \leqslant t \leqslant 1}$ for S. On the other hand, for $0 \leqslant \lambda < \frac{1}{2}$, there does not exist a λ-consistent price system $(\tilde{S}, Q)$ associated to S.*

Proof. Let $W = (W_t)_{t \geqslant 0}$ be an $(\mathcal{F}_t)_{t \geqslant 0}$-Brownian motion, where $(\mathcal{F}_t)_{t \geqslant 0}$ is the natural (right-continuous, saturated) filtration generated by W.

Define the martingale $Z = \mathcal{E}(-W)$

$$Z_t = \exp(-W_t - \tfrac{t}{2}), \qquad t \geqslant 0,$$

and let $N = Z^{-1}$, i.e.

$$N_t = \exp(W_t + \tfrac{t}{2}), \qquad t \geqslant 0,$$

so that N satisfies the SDE

$$\frac{dN_t}{N_t} = dW_t + dt.$$

Define the stopping time τ as

$$\tau = \inf\{t : \ Z_t = \tfrac{1}{2}\} = \inf\{t : \ N_t = 2\},$$

and note that τ is a.s. finite. Define the stock price process S as the time-changed restriction of N to the stochastic interval $[\![0, \tau]\!]$, i.e.

$$S_t = N_{\tan\left(\frac{\pi}{2}t\right) \wedge \tau}, \qquad 0 \leqslant t \leqslant 1.$$

By Girsanov there is only one candidate for the density process of an equivalent martingale measure, namely $\left(Z_{\tan\left(\frac{\pi}{2}t\right) \wedge \tau}\right)_{0 \leqslant t \leqslant 1}$. But the example is cooked up in such a way that $\left(Z_{\tan\left(\frac{\pi}{2}t\right) \wedge \tau}\right)_{0 \leqslant t \leqslant 1}$ only is a *local martingale*. Of course, $\left(Z_{\tan\left(\frac{\pi}{2}t\right) \wedge \tau}\right)_{0 \leqslant t \leqslant 1}$ is an equivalent local martingale deflator.

As regards the final assertion, fix $0 \leqslant \lambda < \frac{1}{2}$, and suppose that there is a λ-consistent price system $(\tilde{S}, Q)$. As $\tilde{S} \in [(1-\lambda)S, S]$ we have $\tilde{S}_0 \leqslant 1$ and $\tilde{S}_1 \geqslant 2(1-\lambda) > 1$, almost surely. On the other hand, assuming that $\tilde{S}$ is a Q-super-martingale implies that $\mathbb{E}_Q[\tilde{S}_1] \leqslant \mathbb{E}_Q[\tilde{S}_0]$, and we arrive at a contradiction. ∎

Remark 5.3. For later use we note that $S_t = N_{\tan\left(\frac{\pi}{2}t\right)\wedge\tau}$ is the so-called numéraire portfolio (see, e.g. [149]), i.e., the unique process of the form $1 + H \cdot S$ verifying $1 + (H \cdot S) \geqslant 0$, and maximizing the logarithmic utility

$$u(1) = \sup\{\mathbb{E}[\log(1 + (H \cdot S)_1)]\},$$

where H runs through the 1-admissible predictable strategies. Indeed, this assertion follows from Theorem 3.1.

The value function u above is finite, namely $u(1) = \log(2)$, and, more generally, $u(x) = \log(2) + \log(x)$, although the process S does not admit an equivalent martingale measure. In other words, log-utility optimization does make sense although the process S obviously allows for an arbitrage as $S_0 = 1$ while $S_1 = 2$.

We next resume two notions from [161]. The tilde indicates again that we are in the frictionless setting.

Definition 5.4. *Let $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ be a semi-martingale.*

For $x > 0, y > 0$, define the sets

$$\tilde{\mathcal{C}}(x) = \{\tilde{X}_T : 0 \leqslant \tilde{X}_T \leqslant x + (\tilde{H} \cdot \tilde{S})_T\}$$

of non-negative claims attainable at price x, where $\tilde{H}$ runs through the predictable, $\tilde{S}$-integrable processes such that $(\tilde{H} \cdot \tilde{S})_t \geqslant -x$, for all $0 \leqslant t \leqslant T$.

Dually let

$$\tilde{\mathcal{D}}(y) = \{y\tilde{Z}_T\}$$

where $\tilde{Z}_T$ now runs through the terminal values of super-martingale deflators $(\tilde{Z}_t)_{0 \leqslant t \leqslant T} \in \tilde{\mathcal{Z}}$.

Let us comment on the issue of non-negativity versus strict positivity in the definition of $\tilde{\mathcal{D}}(y)$. This corresponds to the difference between equivalent local martingale measures Q for the process $\tilde{S}$ versus local martingale measures which only are absolutely continuous with respect to $\mathbb{P}$. It is well-known in this more classical context that the norm *closure* of the set $\mathcal{M}^e(\tilde{S})$ of equivalent local martingale measures Q equals the set of absolutely continuous martingale measures. Similarly, to obtain the norm *closedness* of $\tilde{\mathcal{D}}(1)$

in $L^1(\mathbb{P})$ in the above theorem we have to allow for non-negative processes $(\tilde{Z}_t)_{0\leqslant t\leqslant T} \in \tilde{\mathcal{Z}}$ rather than strictly positive processes $(\tilde{Z}_t)_{0\leqslant t\leqslant T} \in \tilde{\mathcal{Z}}^e$.

It was shown in [161], Proposition 3.1 that the condition (EMM) of existence of an equivalent local martingale measure is sufficient to imply the crucial polarity relations between $\tilde{\mathcal{C}}(x)$ and $\tilde{\mathcal{D}}(y)$ similarly as in Theorem 4.22 above (compare 5.8 below.) At the time of the writing of [161] the condition (EMM) seemed to be the natural assumption in this context. But, as mentioned at the beginning of this chapter, it was observed notably by I. Karatzas and C. Kardaras that the polarity between $\tilde{\mathcal{C}}(x)$ and $\tilde{\mathcal{D}}(y)$ still holds true if one only imposes the *local version* of the condition (EMM) which is the condition $(NUPBR)$ defined below.

Definition 5.5. *[149, Def. 4.1] Let $\tilde{S} = (\tilde{S}_t)_{0\leqslant t\leqslant T}$ be a semi-martingale. We say that $\tilde{S}$ allows for an* unbounded profit with bounded risk *if there is $\alpha > 0$ such that, for every $C > 0$, there is a predictable, $\tilde{S}$-integrable process $\tilde{H}$ such that*

$$(\tilde{H} \cdot \tilde{S})_t \geqslant -1, \qquad 0 \leqslant t \leqslant T,$$

while

$$\mathbb{P}\Big[(\tilde{H} \cdot \tilde{S})_T \geqslant C\Big] \geqslant \alpha.$$

If $\tilde{S}$ does not allow for such profits, we say that $\tilde{S}$ satisfies the condition $(NUPBR)$ of no unbounded profit with bounded risk.

While the name $(NUPBR)$ was only introduced in 2007 in the above quoted paper [149], the concept appears already much earlier in the literature. In [67] the equivalent condition stated in Theorem 5.6 (i'') below and its relation to no arbitrage was extensively studied. It also appears in [132] under the name of "no asymptotic arbitrage of first kind" in the more general setting of large financial markets.

We now turn to a result form the paper [149] of I. Karatzas and C. Kardaras (Theorem 5.6). While these authors deal with the more complicated case of general semi-martingales (even allowing for convex constraints) we only deal with the case of continuous semi-martingales $\tilde{S}$. This simplifies things considerably as the problem boils down to a careful inspection of Girsanov's formula.

Fix the continuous semi-martingale $\tilde{S}$. By the Bichteler-Dellacherie theorem (see, e.g., [199] or [12]), $\tilde{S}$ uniquely decomposes into

$$\tilde{S} = \tilde{M} + \tilde{A} \tag{205}$$

where $\tilde{M}$ is a local martingale starting at $\tilde{M}_0 = \tilde{S}_0$, and $\tilde{A}$ is predictable and of bounded variation starting at $\tilde{A}_0 = 0$. These processes $\tilde{M}$ and $\tilde{A}$ are continuous too and the quadratic variation process $\langle \tilde{M} \rangle_t$ is well-defined and a.s. finite. The so-called "structure condition" introduced by M. Schweizer [222] states that $d\tilde{A}_t$ is a.s. absolutely continuous with respect to $d\langle \tilde{M} \rangle_t$. If $\tilde{S}$ fails to have this property, it is well-known and easy to prove that $\tilde{S}$ allows for arbitrage (in a very strong sense made precise, e.g., in [149, Def. 3.8]). The underlying idea goes as follows: if $d\tilde{A}_t$ fails to be absolutely continuous with respect to $d\langle \tilde{M} \rangle_t$ then one can well-define a predictable trading strategy $H = (H_t)_{0 \leqslant t \leqslant T}$ which equals $+1$ where $d\tilde{A}_t > 0$ and $d\langle \tilde{M} \rangle_t = 0$ while it equals -1 where $d\tilde{A}_t < 0$ and $d\langle \tilde{M} \rangle_t = 0$. The strategy H clearly yields an arbitrage.

As $\tilde{S}$ is strictly positive we may therefore write, by slight abuse of notion,

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = dM_t + \varrho_t d\langle M \rangle_t \tag{206}$$

where M is a local martingale and ϱ_t a predictable process. The reader who is not so comfortable with the formalities of general continuous semi-martingales may very well think of the example of an SDE

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \sigma_t dW_t + \varrho_t(\sigma_t^2 dt), \tag{207}$$

where W is a Brownian motion and σ, ϱ are predictable processes, without missing anything essential in the subsequent arguments.

Assuming in (206) the integrability condition

$$\int_0^T \varrho_t^2 d\langle M \rangle_t < \infty, \quad \text{a.s.} \tag{208}$$

we may well-define the Girsanov density process

$$\tilde{Z}_t = \exp\left\{ -\int_0^t \varrho_u dM_u - \tfrac{1}{2} \int_0^t \varrho_u^2 d\langle M \rangle_u \right\} \qquad 0 \leqslant t \leqslant T. \tag{209}$$

By Itô this is a strictly positive local martingale, such that $\tilde{Z}\tilde{S}$ is a local martingale too. In particular (209) yields an *equivalent local-martingale deflator.*

The reciprocal $\tilde{N} = \tilde{Z}^{-1}$ is called the numéraire portfolio, i.e.

$$\tilde{N}_t = \exp\left\{ \int_0^t \varrho_u dM_u + \tfrac{1}{2} \int_0^t \varrho_u^2 d\langle M \rangle_u \right\}. \tag{210}$$

By Itô's formula $\tilde{N}$ is a stochastic integral on $\tilde{S}$, given by $\frac{d\tilde{N}_t}{\tilde{N}_t} = \varrho_t \frac{d\tilde{S}_t}{\tilde{S}_t}$, and enjoys the property of being the optimal portfolio for the log-utility maximizer (compare Theorem 3.1). For much more on this issue we refer, e.g., to [11] and [149].

Our aim is to characterize condition (208) in terms of the condition $(NUPBR)$ of Definition 5.5. Essentially (208) can fail in two different ways. We shall illustrate this with two prototypical examples (compare [65]) of processes $\tilde{S}$, starting at $\tilde{S}_0 = 1$. First consider

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = dW_t + (1-t)^{-\frac{1}{2}} dt, \qquad 0 \leqslant t \leqslant 1, \tag{211}$$

so that $\int_0^{1-\varepsilon} \varrho_t^2 dt < \infty$, for all $\varepsilon > 0$, while $\int_0^1 \varrho_t^2 dt = \infty$ almost surely. In this case it is straightforward to check directly that the sequence $(\tilde{N}_{1-\frac{1}{n}})_{n=1}^{\infty}$, where $\tilde{N}$ is defined in (210), yields an unbounded profit with bounded risk, as $\tilde{N} > 0$ and $\lim_{t\to 1} \tilde{N}_t = \infty$, a.s.

The second example is

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = dW_t + t^{-\frac{1}{2}} dt, \qquad 0 \leqslant t \leqslant 1, \tag{212}$$

so that $\int_0^{\varepsilon} \varrho_t^2 dt = \infty$, for all $\varepsilon > 0$. This case is trickier as now the singularity is at the beginning of the interval $[0,1]$, and not at the end. This leads to the concept of *immediate arbitrage* as anlayzed in [65]. Using the law of the iterated logarithm, it is shown there (Example 3.4) that in this case, one may find an $\tilde{S}$-integrand $\tilde{H}$ such that $\tilde{H} \cdot \tilde{S} \geqslant 0$ and $\mathbb{P}[(\tilde{H} \cdot \tilde{S})_t > 0] = 1$, for each $t > 0$. For the explicit construction of $\tilde{H}$ we refer to [65]. As one may multiply $\tilde{H}$ with an arbitrary constant $C > 0$ this again yields an unbounded profit with bounded risk.

Summing up, in both of the examples (211) and (212) we obtain an *unbounded profit with bounded risk.* These two examples essentially cover the general case.

We have thus motivated the following *local* version of the Fundamental Theorem of Asset Pricing (see [149, Th. 4.12] for a more general result).

Theorem 5.6. *Let* $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ *be a* continuous semi-martingale *of the form*

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = dM_t + dA_t,$$

where $(M_t)_{0\leqslant t\leqslant T}$ *is a local martingale and* $(A_t)_{0\leqslant t\leqslant T}$ *a predictable process of finite variation. The following assertions are equivalent.*

(i) *The condition* ($NUPBR$) *of* no unbounded profit with bounded risk *holds true (Def. 5.5).*

(i') *Locally,* $\tilde{S}$ *satisfies the condition* ($NFLVR$) *of* no free lunch with vanishing risk.

(i'') *The set* $\tilde{\mathcal{C}}(1)$ *is bounded in* $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) *We have* $dA_t = \varrho_t d\langle M\rangle_t$ *and the process* ϱ *satisfies* $\int_0^T \varrho_t^2 d\langle M\rangle_t < \infty$, *a.s.*

(ii') *We have* $dA_t = \varrho_t d\langle M\rangle_t$ *and the* Girsanov density process $\tilde{Z}$

$$\tilde{Z}_t = \exp\left\{-\int_0^t \varrho_u dM_u - \tfrac{1}{2}\int_0^t \varrho_u^2 d\langle M\rangle_u\right\}, \qquad 0 \leqslant t \leqslant T,$$

is a well-defined strictly positive local martingale.

(ii'') *We have* $dA_t = \varrho_t d\langle M\rangle_t$ *and the* numéraire portfolio $\tilde{N} = \tilde{Z}^{-1}$

$$\tilde{N}_t = \exp\left\{\int_0^t \varrho_u dM_u + \tfrac{1}{2}\int_0^t \varrho_u^2 d\langle M\rangle_u\right\}, \qquad 0 \leqslant t \leqslant T, \tag{213}$$

is a well-defined process (and therefore a.s. finite).

(iii) *The set of* equivalent super-martingale deflators $\tilde{\mathcal{Z}}^e$ *is non-empty (ESD).*

(iii') *The set of* equivalent local martingale deflators in $\tilde{\mathcal{Z}}^{e,loc}$, *is non-empty* ($ELD$).

(iii'') *Locally, the set of equivalent martingale measures for the process* $\tilde{S}$ *is non-empty.*

Proof. $(ii) \Leftrightarrow (ii'') \Leftrightarrow (ii') \Rightarrow (iii') \Leftrightarrow (iii'') \Rightarrow (iii)$ is obvious, and $(i'') \Leftrightarrow (i)$ holds true by Definition 5.5.

$(iii) \Rightarrow (i'')$: By definition, $\tilde{\mathcal{C}}(1)$ fails to be bounded in L^0 if there is $\alpha > 0$ such that, for each $M > 0$, there is $\tilde{X}_T = 1 + (\tilde{H} \cdot \tilde{S})_T \in \tilde{\mathcal{C}}(1)$ such that

$$\mathbb{P}[\tilde{X}_T \geqslant M] \geqslant \alpha. \tag{214}$$

Fix $\tilde{Z} \in \tilde{\mathcal{Z}}^e$. The strict positivity of $\tilde{Z}_T$, implies that

$$\beta := \inf\{\mathbb{E}[\tilde{Z}_T \mathbb{1}_A] : \mathbb{P}[A] \geqslant \alpha\}$$

is strictly positive. Letting $M > \frac{1}{\beta}$ in (214) we arrive at a contradiction to the super-martingale assumption

$$1 = \mathbb{E}[\tilde{Z}_0 \tilde{X}_0] \geqslant \mathbb{E}[\tilde{Z}_T \tilde{X}_T] \geqslant \beta M > 1.$$

$(i) \Rightarrow (ii)$ This is the non-trivial implication. It is straightforward to deduce from (i) that there is a predictable process ϱ satisfying (206) (compare [222] and the discussion preceding Theorem 5.6). We have to show that (208) is satisfied. The reader might keep the examples (211) and (212) in mind. Define the stopping time

$$\tau = \inf \left\{ t \in [0, T] : \int_0^t \varrho_u^2 d\langle M \rangle_u = \infty \right\}.$$

Condition (ii) states that $\mathbb{P}[\tau < \infty] = 0$. Assuming the contrary, the set $\{\tau < \infty\}$ then splits into the two $\mathcal{F}_\tau$-measurable sets

$$A^c = \{\tau < \infty\} \cap \left\{ \lim_{t \nearrow \tau} \int_0^t \varrho_u^2 d\langle M \rangle_u = \infty \right\},$$
$$A^d = \{\tau < \infty\} \cap \left\{ \lim_{t \nearrow \tau} \int_0^t \varrho_u^2 d\langle M \rangle_u < \infty \right\},$$

where c refers to "continuous" and d to "discontinuous".

If $\mathbb{P}[A^c] > 0$ it suffices to define the stopping times

$$\tau_n = \inf \left\{ t : \int_0^t \varrho_u^2 d\langle M \rangle_u \geqslant 2^n \right\}.$$

For each $n \in \mathbb{N}$, the numéraire portfolio $\tilde{N}_{\tau_n}$ at time τ_n is well-defined and given by

$$\tilde{N}_{\tau_n} = \exp \left\{ \int_0^{\tau_n} \varrho_u dM_u + \tfrac{1}{2} \int_0^{\tau_n} \varrho_u^2 d\langle M \rangle_u \right\}.$$

It is straightforward to check that $\tilde{N}_{\tau_n}$ tends to $+\infty$ a.s. on A^c, which gives a contradiction to (i).

We still have to deal with the case $\mathbb{P}[A^c] = 0$ so that we have $\mathbb{P}[A^d] > 0$. This is the situation of the "Immediate Arbitrage Theorem". We refer to [65, Th. 3.7] for a proof that in this case we may find an $\tilde{S}$-integrable, predictable process $\tilde{H}$ such that $(\tilde{H} \cdot \tilde{S})_t > 0$, for all $\tau < t \leqslant T$ almost surely on A^d. This contradicts assumption (i).

$(ii') \Rightarrow (i')$: Suppose that the Girsanov density process $\tilde{Z}$ is well-defined and strictly positive. We may define, for $\varepsilon > 0$, the stopping time

$$\tau_\varepsilon = \inf \left\{ t \in [0, T] : \ \tilde{Z}_t \geqslant \varepsilon^{-1} \quad \text{or} \quad \tilde{S}_t \geqslant \varepsilon^{-1} \right\},$$

so that τ_ε increases to infinity. The stopped process $\tilde{S}^{\tau_\varepsilon}$ then admits an equivalent martingale measure, namely $\frac{dQ}{d\mathbb{P}} = \tilde{Z}_{\tau_\varepsilon}$.

$(i') \Rightarrow (i)$ obvious as $(NUPBR)$ is a local property. ■

Remark 5.7. Kostas Kardaras kindly pointed out that there is a more direct way of showing the implication $(i) \Rightarrow (ii)$ above (see [154]). While our above argument, i.e. reducing to the case of the examples (211) and (212), allows for some additional insight, there is an easier proof of the implication $(i) \Rightarrow (ii)$ available.

Assuming (i) and using the above notion, define, similarly as in (213), for $n \in \mathbb{N}$, the process

$$N_t^n = \exp\left\{\int_0^t \varrho_u^{(n)} dM_u + \tfrac{1}{2}\int_0^t \varrho_u^{(n)} d\langle M\rangle_u\right\}, \qquad 0 \leqslant t \leqslant T, \tag{215}$$

where

$$\varrho_t^{(n)} = \varrho_t \mathbb{1}_{\{|\varrho_t| \leqslant n\}}. \tag{216}$$

Clearly $(N_t^n)_{0\leqslant t\leqslant T}$ is a well-defined local martingale. Suppose that (ii) fails, i.e. that $\int_0^T \varrho_t^2 d\langle M\rangle_t = \infty$ on a set $A \in \mathcal{F}_T$ with $\mathbb{P}[A] > 0$. Then it again is straightforward to check that the sequence of random variables $(N_T^n)_{n=1}^\infty$ tends to infinity, almost surely on the set A. In other words, the sequence of processes $(N_t^n)_{0\leqslant t\leqslant T}$ defines an unbounded profit with bounded risk, a contradiction to (i).

We now can show the polarity between the sets $\tilde{\mathcal{C}}(x)$ and $\tilde{\mathcal{D}}(y)$ as defined in Definition 5.4 similarly as in Theorem 4.22 above.

Theorem 5.8. *Let the strictly positive, continuous semi-martingale $\tilde{S}$ satisfy one of the equivalent conditions listed in Theorem 5.6 and let $x, y > 0$.*

(i) $\tilde{\mathcal{C}}(x)$ and $\tilde{\mathcal{D}}(y)$ are convex, closed (w.r. to convergence in measure), solid subsets of $L_+^0(\mathbb{R})$.

(ii) For $g, h \in L_+^0(\mathbb{R})$ we have

$$\begin{aligned} g \in \tilde{\mathcal{C}}(x) \quad &\textit{iff} \quad \mathbb{E}[gh] \leqslant xy, \quad \textit{for all} \quad h \in \tilde{\mathcal{D}}(y), \quad \textit{and} \\ h \in \tilde{\mathcal{D}}(y) \quad &\textit{iff} \quad \mathbb{E}[gh] \leqslant xy, \quad \textit{for all} \quad g \in \tilde{\mathcal{C}}(x). \end{aligned}$$

(iii) The constant function $x\mathbb{1}$ is in $\tilde{\mathcal{C}}(x)$.

Proof. Let $(\tau_k)_{k=1}^\infty$ be a localizing sequence such that each stopped process $\tilde{S}^k := \tilde{S}^{\tau_k}$ admits an equivalent martingale measure Q^k. We also let $x = y = 1$ to simplify notation and write $\tilde{\mathcal{C}}^k$ and $\tilde{\mathcal{D}}^k$ for the sets corresponding to the process $\tilde{S}^k$.

(i) We have to show the closedness of $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$. As regards $\tilde{\mathcal{C}}$ let $(g^n)_{n=1}^\infty$ be a sequence in $\tilde{\mathcal{C}}$, converging a.s. to $g \in L^0_+$. For each $k \in \mathbb{N}$ the sequence $(g^n \mathbb{1}_{\{\tau_k=\infty\}})_{n=1}^\infty$ is in $\tilde{\mathcal{C}}^k$ and converges a.s. to $g\mathbb{1}_{\{\tau_k=\infty\}}$ which also is in $\tilde{\mathcal{C}}^k$ by the closedness of $\tilde{\mathcal{C}}^k$ established in [63]. Let H^k be a predictable, S^k-integrable, admissible integrand such that

$$x + (H^k \cdot S^k)_T \geqslant g\mathbb{1}_{\{\tau_k=\infty\}}$$

By [64] and passing to a sequence of convex combinations of the $(H^m)_{m=1}^\infty$ we may suppose that, for each k, the sequence of integrands $(H^m \mathbb{1}_{[\![0,\tau_k\wedge T]\!]})_{m=k}^\infty$ converges to a limit, which we temporarily denote by $\bar{H}^k$. For $k_1 \leqslant k_2$ we have that the restriction of $\bar{H}^{k_2}$ to $[\![0,\tau_{k_1}\wedge T]\!]$ equals $\bar{H}^{k_1}$ if we have done the construction of $(H^m)_{m=1}^\infty$ in a "diagonal" way. Hence we may well-define the limit H of the sequence $(\bar{H}^k)_{k=1}^\infty$ by pasting things together. This limit is a predictable, admissible, $\tilde{S}$-integrable process H such that $x + (H \cdot \tilde{S})_T \geqslant g$. This shows the closedness of $\tilde{\mathcal{C}}$.

The closedness of $\tilde{\mathcal{D}}$ follows by the same argument of a "Fatou-limit" as in the proof of Theorem 4.22. The remaining properties of the sets $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ are obvious.

(ii) Let $g \in L^0_+(\mathbb{R})$ and denote by g^k the random variable $g^k = g\mathbb{1}_{\{\tau_k=\infty\}}$. For $h \in L^0_+(\mathbb{R})$, we have $\mathbb{E}[gh] \leqslant 1$ iff $\mathbb{E}[g^k h] \leqslant 1$, for all k. Also by (i) we have that $g \in \tilde{\mathcal{C}}$ iff $g^k \in \tilde{\mathcal{C}}$, for all k. This shows the first line of (ii) and the second line follows by the same token.

(iii) is obvious. ■

Remark 5.9. The above polarity between $\tilde{\mathcal{C}}(x)$ and $\tilde{\mathcal{D}}(y)$ is sufficient to prove the basic duality Theorem 2.2 of [161]. Indeed, the abstract version of this theorem, which is Theorem 3.2 in [161] was precisely formulated in terms involving only the validity of the polarity relations as listed in Theorem 5.8.

The case of proportional transaction costs

We now give a similar local version of the Fundamental Theorem of Asset Pricing in the context of transaction costs. We shall use the subsequent variants of the concept of no arbitrage. The first notion of the subsequent definition is from [107] with an additional boundedness condition in the case (b) below. We need this additional condition here as we now use a different notion of admissibility than in [107].

Definition 5.10. *Let $S = (S_t)_{0\leqslant t\leqslant T}$ be a strictly positive, continuous process. We say that S allows for an* obvious arbitrage *if there are $\alpha > 0$ and $[0,T] \cup$*

$\{\infty\}$*-valued stopping times* $\sigma \leqslant \tau$ *with* $\mathbb{P}[\sigma < \infty] = \mathbb{P}[\tau < \infty] > 0$ *such that either*

$$(a)\ S_\tau \geqslant (1+\alpha) S_\sigma, \qquad a.s.\ on\ \{\sigma < \infty\},$$

or

$$(b)\ S_\tau \leqslant \tfrac{1}{1+\alpha} S_\sigma, \qquad a.s.\ on\ \{\sigma < \infty\}.$$

In the case of (b) we also impose that $(S_t)_{\sigma \leqslant t \leqslant \tau}$ *is uniformly bounded.*

We say that S *allows for an* obvious immediate arbitrage *if, in addition, we have*

$$(a)\ S_t \geqslant S_\sigma, \qquad for\ t \in [\![\sigma, \tau]\!],\ a.s.\ on\ \{\sigma < \infty\},$$

or

$$(b)\ S_t \leqslant S_\sigma, \qquad for\ t \in [\![\sigma, \tau]\!],\ a.s.\ on\ \{\sigma < \infty\}.$$

We say that S *satisfies the condition* (NOA) *(resp.* $(NOIA)$*) of* no obvious arbitrage *(resp.* no obvious immediate arbitrage*) if no such opportunity exists.*

It is indeed rather obvious how to make an arbitrage if (NOA) fails, provided the transaction costs $0 < \lambda < 1$ are sufficiently small (compare [107]). Assuming e.g. condition (a), one goes long in the asset S at time σ and closes the position at time τ. In case of an *obvious immediate arbitrage* one is in addition assured that during such an operation the stock price will never fall under the initial value S_σ.

In the case of condition (b) one does a similar operation by going short in the asset S. The boundedness condition in the case (b) of (NOA) makes sure that such a strategy is admissible.

Next we formulate an analogue of Theorem 5.6 in the setting of transaction costs.

Theorem 5.11. *Let* $S = (S_t)_{0 \leqslant t \leqslant T}$ *be a strictly positive, continuous process. The following assertions are equivalent.*

(i) *Locally, for each* $0 < \lambda < 1$, *there is no obvious immediate arbitrage* $(NOIA)$.

(i') *Locally, for each* $0 < \lambda < 1$, *there is no obvious arbitrage* (NOA).

(i'') *Locally, for each* $0 < \lambda < 1$, *the process* S *does not allow for an arbitrage under transaction costs* λ, *i.e.*

$$\mathcal{C}^k \cap L^0_+ = \{0\}, \tag{217}$$

for each k, where $\mathcal{C}^k$ is the cone given by Definition 4.7 for the stopped processes S^{τ_k}, and $(\tau_k)_{k=1}^{\infty}$ is a suitable localizing sequence.

(i''') Locally, for each $0 < \lambda < 1$, the process S does not allow for a free lunch with vanishing risk under transaction costs λ, i.e.

$$\overline{\mathcal{C}^k \cap L^{\infty}} \cap L^{\infty}_{+} = \{0\}, \tag{218}$$

for each k, where the bar denotes the closure with respect to the norm topology of L^{∞}.

(i'''') Locally, for each $0 < \lambda < 1$, the process S does not allow for a free lunch under transaction costs λ, i.e.

$$\overline{\mathcal{C}^k \cap L^{\infty}} \cap L^{\infty}_{+} = \{0\}, \tag{219}$$

for each k, where now the bar denotes the closure with respect to the weak star topology of L^{∞}.

(ii) Locally, for each $0 < \lambda < 1$, the condition (CPS^{λ}) of existence of a λ-consistent price system *holds true.*

(ii') For each $0 < \lambda < 1$ the set $\mathcal{Z}^{loc,e}$ of λ-consistent equivalent local martingale deflators is non-empty.

Proof. $(i'''') \Rightarrow (i''') \Rightarrow (i'') \Rightarrow (i') \Rightarrow (i)$ is straight-forward, as well as $(ii) \Leftrightarrow (ii')$.

$(i) \Rightarrow (ii)$: As assumption (ii) is a local property we may assume that S satisfies $(NOIA)$.

To prove (ii) we do a similar construction as in ([107], Proposition 2.1): we suppose in the sequel that the reader is familiar with the proof of [107], Proposition 2.1 and define the – preliminary – stopping time $\bar{\varrho}_1$ by

$$\bar{\varrho}_1 = \inf\left\{t > 0 : \tfrac{S_t}{S_0} \geqslant 1 + \lambda \text{ or } \tfrac{S_t}{S_0} \leqslant \tfrac{1}{1+\lambda}\right\}.$$

In fact, in [107] we wrote $\frac{\varepsilon}{3}$ instead of λ which does not matter as both quantities are arbitrarily small.

Define the sets $\bar{A}_1^+, \bar{A}_1^-$, and $\bar{A}_1^0$ as

$$\bar{A}_1^+ = \{\bar{\varrho}_1 < \infty, S_{\bar{\varrho}_1} = (1+\lambda)S_0\}, \tag{220}$$

$$\bar{A}_1^- = \left\{\bar{\varrho}_1 < \infty, S_{\bar{\varrho}_1} = \tfrac{1}{1+\lambda}S_0\right\}, \tag{221}$$

$$\bar{A}_1^0 = \{\bar{\varrho}_1 = \infty\}. \tag{222}$$

It was observed in [107] that assumption (NOA) rules out the cases $\mathbb{P}[\bar{A}_1^+] = 1$ and $\mathbb{P}[\bar{A}_1^-] = 1$. But under the present weaker assumption $(NOIA)$ we cannot a priori exclude the possibilities $\mathbb{P}[\bar{A}_1^+] = 1$ and $\mathbb{P}[\bar{A}_1^-] = 1$. To refine the argument from [107] in order to apply to the present setting, we distinguish two cases. Either we have $\mathbb{P}[\bar{A}_1^+] < 1$ and $\mathbb{P}[\bar{A}_1^-] < 1$; in this case we let $\varrho_1 = \bar{\varrho}_1$ and proceed exactly as in the proof of ([107], Proposition 2.1) to complete the first inductive step.

The second case is that one of the probabilities $\mathbb{P}[\bar{A}_1^+]$ or $\mathbb{P}[\bar{A}_1^-]$ equals one. We assume w.l.g. $\mathbb{P}[\bar{A}_1^+] = 1$, the other case being similar.

Define the real number $\beta \leqslant 1$ as the essential infimum of the random variable $\min_{0 \leqslant t \leqslant \bar{\varrho}_1} \frac{S_t}{S_0}$. We must have $\beta < 1$, otherwise the pair $(0, \bar{\varrho}_1)$ would define an *immediate obvious arbitrage.* We also have the obvious inequality $\beta \geqslant \frac{1}{1+\lambda}$.

We define, for $1 > \gamma \geqslant \beta$ the stopping time

$$\bar{\varrho}_1^{\gamma} = \inf \left\{ t > 0 \ : \ \tfrac{S_t}{S_0} \geqslant 1 + \lambda \text{ or } \tfrac{S_t}{S_0} \leqslant \gamma \right\}.$$

Defining $\bar{A}_1^{\gamma,+} = \{S_{\bar{\varrho}_1^{\gamma}} = (1+\lambda)S_0\}$ and $\bar{A}_1^{\gamma,-} = \left\{S_{\bar{\varrho}_1^{\gamma}} = \gamma S_0\right\}$ we find an a.s. partition of $\bar{A}_1^+$ into the sets $\bar{A}_1^{\gamma,+}$ and $\bar{A}_1^{\gamma,-}$. Clearly $\mathbb{P}[\bar{A}_1^{\gamma,-}] > 0$, for $1 > \gamma > \beta$. We claim that $\lim\limits_{\gamma \searrow \beta} \mathbb{P}[\bar{A}_1^{\gamma,-}] = 0$. Indeed, supposing that this limit were positive, we again could find an *obvious immediate arbitrage* as in this case we have that $\mathbb{P}[\bar{A}_1^{\beta,-}] > 0$. Hence the pair of stopping times

$$\sigma = \bar{\varrho}_1^{\beta}.\mathbb{1}_{\{S_{\bar{\varrho}_1^{\beta}} = \beta S_0\}} + \infty \mathbb{1}_{\{S_{\bar{\varrho}_1^{\beta}} = (1+\lambda) S_0\}}$$

and

$$\tau = \bar{\varrho}_1.\mathbb{1}_{\{S_{\bar{\varrho}_1^{\beta}} = \beta S_0\}} + \infty \mathbb{1}_{\{S_{\bar{\varrho}_1^{\beta}} = (1+\lambda) S_0\}}$$

would define an *obvious immediate arbitrage.*

We thus may find $1 > \gamma > \beta$ such that $\mathbb{P}[\bar{A}_1^{\gamma,-}] < \frac{1}{2}$. After having found this value of γ we can define the stopping time ϱ_1 in its final form as

$$\varrho_1 := \bar{\varrho}_1^{\gamma}.$$

Next we define, similarly as in (220) and (221) the sets

$$A_1^+ = \{\varrho_1 < \infty, S_{\varrho_1} = (1+\lambda)S_0\}$$

$$A_1^- = \{\varrho_1 < \infty, S_{\varrho_1} = \gamma S_0\}$$

to obtain a partition of Ω into two sets of positive measure.

As in [107] we define a probability measure Q_1 on $\mathcal{F}_{\varrho_1}$ by letting $\frac{dQ_1}{d\mathbb{P}}$ be constant on these two sets, where the constants are chosen such that $Q_1[A_1^+] = \frac{1-\beta}{1+\lambda-\beta}$ and $Q_1[A_1^-] = \frac{\lambda}{1+\lambda-\beta}$. We then may define the Q_1-martingale $(\tilde{S}_t)_{0 \leqslant t \leqslant \varrho_1}$ by

$$\tilde{S}_t = \mathbb{E}_{Q_1}[S_{\varrho_1}|\mathcal{F}_t], \qquad 0 \leqslant t \leqslant \varrho_1,$$

to obtain a process remaining in the interval $[\gamma S_0, (1+\lambda)S_0]$.

The above weights for Q_1 were chosen in such a way to obtain

$$\tilde{S}_0 = \mathbb{E}_{Q_1}[S_{\varrho_1}] = S_0.$$

This completes the first inductive step similarly as in [107]. Summing up, we obtained ϱ_1, Q_1 and $(\tilde{S}_t)_{0 \leqslant t \leqslant \varrho_1}$ precisely as in the proof of ([107], Proposition 2.1) with the following additional possibility: it may happen that ϱ_1 does not stop when S_t first hits $(1+\lambda)S_0$ or $\frac{S_0}{1+\lambda}$, but rather when S_t first hits $(1+\lambda)S_0$ or xS_0, for some $\frac{1}{1+\lambda} < x < 1$. In this case we have $\mathbb{P}[A_1^0] = 0$ and we made sure that $\mathbb{P}[A_1^-] < \frac{1}{2}$, i.e., we have a control on the probability of $\{S_{\varrho_1} = \beta S_0\}$.

We now proceed as in [107] with the inductive construction of ϱ_n, Q_n and $(\tilde{S}_t)_{0 \leqslant t \leqslant \varrho_n}$. The new ingredient is that again we have to take care (conditionally on $\mathcal{F}_{\varrho_{n-1}}$) of the additional possibility $\mathbb{P}[A_n^+] = 1$ or $\mathbb{P}[A_n^-] = 1$. Supposing again w.lg. that we have the first case, we deal with this possibility precisely as for $n = 1$ above, but now we make sure that $\mathbb{P}[A_n^-] < 2^{-n}$.

This completes the inductive step and we obtain, for each $n \in \mathbb{N}$, an equivalent probability measure Q_n on $\mathcal{F}_{\varrho_n}$ and a Q_n-martingale $(\tilde{S}_t)_{0 \leqslant t \leqslant \varrho_n}$ taking values in the bid ask spread $([\frac{1}{1+\lambda}S_t, (1+\lambda)S_t])_{0 \leqslant t \leqslant \varrho_n}$. We note in passing that there is no loss of generality in having chosen this normalization of the bid ask spread instead of the usual normalization $[(1-\lambda')S', S']$ by passing from S to $S' = (1-\frac{\lambda}{2})S$ and from λ to $\lambda' = \frac{\lambda}{2}$.

There is one more thing to check to complete the proof of (ii) : we have to show that the stopping times $(\varrho_n)_{n=1}^{\infty}$ increase almost surely to infinity. This is verified in the following way: suppose that $(\varrho_n)_{n=1}^{\infty}$ remains bounded on a set of positive probability. On this set we must have that $\frac{S_{\varrho_{n+1}}}{S_{\varrho_n}}$ equals $(1+\lambda)$ or $\frac{1}{1+\lambda}$, except for possibly finitely many $n's$. Indeed, the above requirement $\mathbb{P}[A_n^-] < 2^{-n}$ makes sure that a.s. the novel possibility of moving by a value different from $(1+\lambda)$ or $\frac{1}{1+\lambda}$ can only happen finitely many times. Therefore we may, as in [107], conclude from the continuity and strict positivity of the trajectories of S that ϱ_n increases a.s. to infinity which completes the proof of (ii).

$(ii) \Rightarrow (i'''')$ As (ii) as well as (i'''') are local properties holding true for each $0 < \lambda < 1$, it will suffice to show that (CPS^{λ}) implies (219), for fixed $0 < \lambda < 1$.

Let $(\tilde{S}, Q)$ be a λ-consistent price system and define the half-space H of $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$

$$H = \left\{ \varphi_T^0 \in L^{\infty} : \mathbb{E}_Q[\varphi_T^0] \leqslant 0 \right\},$$

which is σ^*-closed and satisfies $H \cap L_+^{\infty} = \{0\}$. It follows from Proposition 4.5 that, for all self-financing, admissible trading strategies $(\varphi_t^0, \varphi_t^1)_{0 \leqslant t \leqslant T}$ we have that $(\varphi_t^0 Z_t^0 + \varphi_t^1 Z_t^1)_{0 \leqslant t \leqslant T}$ is a super-martingale under $\mathbb{P}$, which implies that $\overline{\mathcal{C} \cap L^{\infty}} \cap L_+^{\infty} = \{0\}$. Hence (219) holds true. ■

Recall Theorem 4.22 from the previous chapter. It states the polarity between the sets $\mathfrak{C}(x)$ and $\mathfrak{D}(y)$ in L_+^0 which is required in Proposition 3.1 of [161]. This result which will turn out to be the basis of the duality theory of portfolio optimization in the next.

The crucial hypothesis in Theorem 4.22 is the assumption of $(CPS^{\lambda'})$, for each $0 < \lambda' < 1$. In fact, it is sufficient to impose this hypothesis only *locally* i.e. under one of the conditions listed in Theorem 5.11. For a more general version of the subsequent result which also pertains to càdlàg processes we refer to [48] and [51].

Theorem 5.12. *Suppose that the continuous, strictly positive process* $S = (S_t)_{0 \leqslant t \leqslant T}$ *satisfies condition* $(CPS^{\lambda'})$ **locally**, *for each* $0 < \lambda' < 1$, *and let* $x, y > 0$. *Fix* $0 < \lambda < 1$, *as well as* $x > 0, y > 0$.

(i) The sets $\mathfrak{C}(x), \mathfrak{D}(y)$ *defined in Definition 4.21 are convex, closed (w.r to convergence in measure), solid subsets of* $L_+^0(\mathbb{R})$. *Fix* $x > 0, y > 0$ *and* $\varphi_T^0 \in L_+^0(\mathbb{R})$.

(ii) For $\varphi_T^0 \in L_+^0(\mathbb{R})$, *we have* $\varphi_T^0 \in \mathfrak{C}(x)$ *iff*

$$\langle \varphi_T^0, Z_T^0 \rangle \leqslant xy, \tag{223}$$

for all $Z_T^0 \in \mathfrak{D}(y)$, *and iff this holds true, for all* $Z_T^0 \in \mathcal{D}^{loc}(y)$, *i.e., which are the terminal values of a local martingale deflator* $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$.

(ii′) For $Z_T^0 \in L_+^0(\mathbb{R})$ *we have* $Z_T^0 \in \mathfrak{D}(y)$ *iff*

$$\langle \varphi_T^0, Z_T^0 \rangle \leqslant xy \tag{224}$$

for all $\varphi_T^0 \in \mathfrak{C}(x)$.

(iii) The set $\mathfrak{C}(x)$ *is bounded in* $L_+^0(\mathbb{R})$ *and contains the constant function* $x\mathbb{1}$.

Proof. (i) As in the proof of Theorem 5.8 let $(\tau_k)_{k=1}^{\infty}$ be a localizing sequence such that each stopped process $S^k = S^{\tau_k}$ admits a λ'-consistent price system, for each $\lambda' > 0$. Again we let $x = y = 1$ and write $\mathfrak{C}^k$ and $\mathfrak{D}^k$ for the sets corresponding to S^k. Also fix $0 < \lambda < 1$.

To show that $\mathfrak{C}$ is closed with respect to the topology of convergence in measure, fix a sequence $(\varphi_T^n)_{n=1}^{\infty} \in \mathfrak{C}$ which converges a.s. to some $\varphi_T \in L_+^0(\mathbb{R})$. As in the proof of Theorem 5.8, consider, for fixed $k \geqslant 1$, the random variables $\varphi_T^n \mathbb{1}_{\{\tau_k = \infty\}}$ which are elements of $\mathfrak{C}^k$. The limit $\varphi_T \mathbb{1}_{\{\tau_k=\infty\}}$ then is in $\mathfrak{C}^k$ too. Using Lemma 4.10 and and repeating the proof of Theorem 4.13 we can conclude that $\varphi_T \in \mathfrak{C}$ by letting k tend to infinity.

As regards the closedness of $\mathfrak{D}$ let $(Z^{0,n}, Z^{1,n})_{n=1}^{\infty}$ be a sequence of super-martingale deflators for S in the sense of Definition 4.20 such that $(Z_T^{0,n})_{n=1}^{\infty}$ converges a.s. to some $Z_T^0 \in L_+^0(\mathbb{R})$. Repeating once more the construction of a Fatou-limit as in the proof of Theorem 4.22 we conclude that Z_T^0 is the terminal value of a super-martingale deflator $(Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$.

The remaining assertions of (i) are rather obvious.

(ii) By (i) and Theorem 4.22 we have that $\varphi_T^0 \in \mathfrak{C}$ iff $\varphi_T^0 \mathbb{1}_{\{\tau_k=\infty\}}$ is in $\mathfrak{C}^k$, for each $k \in \mathbb{N}$. This is the case iff, for each $k \in \mathbb{N}$, and each λ-consistent price system $Z^k = (Z_t^{0,k}, Z_t^{1,k})_{0 \leqslant t \leqslant T}$ for the process S^k, we have

$$\mathbb{E}\big[Z_T^{0,k} \varphi_T^0 \mathbb{1}_{\{\tau_k=\infty\}}\big] \leqslant 1 \tag{225}$$

We claim that (225) holds true, for each $Z_T^{0,k} \in \mathfrak{D}^k$, iff it holds true for each $Z_T^0 \in \mathfrak{D}$ and iff it holds true, for every Z_T^0 which is the terminal value of the first coordinate of a local martingale deflator $(Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$. This is a slightly delicate point as, in the case of transaction costs λ, the consistent price systems do not enjoy the concatenation property which one usually applies for density processes of equivalent martingale measures in the frictionless setting (compare the discussion after Corollary 1.12).

Here is the way to overcome this difficulty. Suppose that there is some k and $\bar{Z}_T^{0,k} \in \mathfrak{D}^k$ such that (225) fails. We have to construct a λ-consistent local martingale deflator Z such that (225) also fails for Z_T^0 in place of $Z_T^{0,k}$. By Theorem 4.22 we may suppose that $\bar{Z}_T^{0,k}$ is, in fact, the terminal value of the first coordinate of a λ-consistent price system $\bar{Z}^k = (\bar{Z}_t^{0,k}, \bar{Z}_t^{1,k})_{0 \leqslant t \leqslant T}$. In fact, we may suppose that $\bar{Z}$ is a $\bar{\lambda}$-consistent price system, for some $\bar{\lambda} > 0$ which is strictly smaller than λ. Indeed, by hypothesis, there is some $\frac{\lambda}{2}$-consistent price system $\check{Z}^k = (\check{Z}_t^{0,k}, \check{Z}_t^{1,k})_{0 \leqslant t \leqslant T}$ for the process S^k. We have

$$\mathbb{E}[\check{Z}_T^{0,k} \varphi_T^0 \mathbb{1}_{\{\tau_k=\infty\}}] \in [0, \infty[.$$

If (225) fails for $\bar{Z}_T^{0,k}$ then it also fails for a convex combination $(1 - \mu)\bar{Z}^k + \mu \check{Z}^k$, for some $\mu > 0$ small enough. This convex combination then is a $\bar{\lambda}$-consistent price system of S^k, for some $0 < \bar{\lambda} < \lambda$. Summing up, we may and do assume that the random variable $\bar{Z}_T^{0,k}$, for which (225) fails, pertains to a $\bar{\lambda}$- consistent price system $\bar{Z}^k$, for some $0 < \bar{\lambda} < \lambda$.

After this preparation we have some room to manouvre in the construction of a concatenation. For $\epsilon > 0$ to be specified below, find an ϵ-consistent local martingale deflator $(Z_t^\epsilon)_{0 \leqslant t \leqslant T}$ for the process S. Define

$$Z_t^0 = \begin{cases} \bar{Z}_t^{0,k}, & \text{for} \quad 0 \leqslant t \leqslant \tau_k \wedge T, \\ \bar{Z}_{\tau_k}^{0,k} \frac{Z_t^{0,\epsilon}}{Z_{\tau_k}^{0,\epsilon}}, & \text{for} \quad \tau_k \leqslant t \leqslant T, \end{cases}$$

$$Z_t^1 = \begin{cases} (1-\epsilon)\bar{Z}^{1,k}, & \text{for} \quad 0 \leqslant t \leqslant \tau_k \wedge T, \\ (1-\epsilon)\bar{Z}_{\tau_k}^{1,k} \frac{Z_t^{1,\epsilon}}{Z_{\tau_k}^{1,\epsilon}}, & \text{for} \quad \tau_k \leqslant t \leqslant T. \end{cases}$$

One checks that $Z = (Z_t^0, Z_t^1)_{0 \leqslant t \leqslant T}$ is a local martingale such that, for $\epsilon > 0$ sufficiently small, the quotient $\frac{Z_t^1}{Z_t^0}$ remains in $[(1-\lambda)S_t, S_t]$, for $0 \leqslant t \leqslant T$. In other words, we constructed a λ-consistent local martingale deflator Z. As Z coincides with $\bar{Z}^k$ on $\{\tau_k = \infty\}$ we obtain

$$\mathbb{E}[Z_T^0 \varphi_T^0 \mathbb{1}_{\{\tau_k = \infty\}}] > 1.$$

This contradiction finishes the proof of (ii).

(ii') Fix $Z_T^0 \in L_+^0(\mathbb{R})$ such that (224) holds true, for all $\varphi_T^0 \in \mathfrak{C}$. By Theorem 4.22 we have that $Z_T^0 \mathbb{1}_{\{\tau_k = \infty\}}$ is in $\mathfrak{D}^k$, for each k. Therefore there is a λ-consistent price system $\bar{Z}^{0,k} = (\bar{Z}_t^{0,k}, \bar{Z}_t^{1,k})_{0 \leqslant t \leqslant T}$ for the process S^k such that

$$\bar{Z}_T^{0,k} \geqslant Z_T^0 \mathbb{1}_{\{\tau_k = \infty\}}.$$

By the argument in the proof of (ii) above we may find a λ-consistent local martingale deflator $Z^k = (Z_t^{0,k}, Z_t^{1,k})_{0 \leqslant t \leqslant T}$ for S such that we still have

$$Z_T^{0,k} \geqslant (1 - k^{-1}) Z_T^0 \mathbb{1}_{\{\tau_k = \infty\}}.$$

By repeating once more the construction in the proof of Theorem 4.22 we may pass to a Fatou limit $\bar{Z}$ of the sequence $(Z^k)_{k=1}^\infty$. Hence $\bar{Z}$ is a λ-consistent super-martingale deflator verifying $\bar{Z}_T^0 \geqslant Z_T^0$. We thus have shown that (224) implies $Z_T^0 \in \mathfrak{D}$. The reverse implication follows from the definition of a super-martingale deflator.

(iii) is obvious. ■

6 Portfolio Optimization under Proportional Transaction Costs

As in Chapter 2 we fix a strictly concave, differentiable utility function $U : \,]0,\infty[\, \to \mathbb{R}$, satisfying the Inada conditions (29). As usual we have to impose the following additional regularity condition in order to obtain satisfactory duality results.

Definition 6.1. *([161]): The asymptotic elasticity of the utility function* $U : \,]0,\infty[\, \to \mathbb{R}$ *is defined as*

$$AE(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}. \tag{226}$$

We say that U *has* reasonable asymptotic elasticity *if* $AE(U) < 1$.

For example, for $U(x) = \frac{x^\gamma}{\gamma}$, where $\gamma \in \,]-\infty, 1[\setminus\{0\}$, we have $AE(U) = \gamma < 1$. We note that, for an increasing concave function U we always have $AE(U) \leqslant 1$. A typical example of a function U for which $AE(U) = 1$ is $U(x) = \frac{x}{\log(x)}$, for x sufficiently large, as one verifies by calculating (226).

We again denote by V the conjugate function

$$V(y) = \sup\{U(x) - xy : x > 0\}, \qquad y > 0,$$

and refer to [161, Corollary 6.1] for equivalent reformulations of the asymptotic elasticity condition (226) in terms of V.

We adopt the setting of Chapter 5 where we considered a continuous price process $(S_t)_{0\leqslant t\leqslant T}$ which locally satisfies condition (CPS^λ) of existence of λ-consistent price systems, for all $0 < \lambda < 1$. We again assume throughout this chapter that the underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$ is Brownian so that every $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$-martingale is continuous.

We have established in Theorem 5.12 that the sets $\mathfrak{C}(1)$ and $\mathfrak{D}(1)$ satisfy the requirements of Proposition 3.1 in [161]. We therefore are verbatim in the setting of ([161], Th 2.2 and 3.2). For the convenience of the reader we restate the aspects of these theorems which are relevant in the present setting.

Theorem 6.2. *Suppose that the continuous, strictly positive process* $S = (S_t)_{0\leqslant t\leqslant T}$ *satisfies condition* $(CPS^{\lambda'})$ *locally, for each* $0 < \lambda' < 1$. *Fix* $\lambda > 0$, *and define the primal and dual value function as*

$$u(x) = \sup_{\varphi_T^0 \in \mathfrak{C}(x)} \mathbb{E}[U(\varphi_T^0)], \tag{227}$$

$$v(y) = \inf_{Z_T^0 \in \mathfrak{D}(y)} \mathbb{E}[V(Z_T^0)]. \tag{228}$$

Suppose that the utility function U has reasonable asymptotic elasticity (226) *and that $u(x) < \infty$, for some $x > 0$. The following assertions hold true.*

(i) *The functions $u(x)$ and $v(y)$ are finitely valued, for all $x > 0, y > 0$, and mutually conjugate*

$$v(y) = \sup_{x>0}[u(x) - xy], \qquad u(x) = \inf_{y>0}[v(y) + xy].$$

The functions u and v are continuously differentiable and strictly concave (resp. convex) and satisfy

$$u'(0) = -v'(0) = \infty, \qquad u'(\infty) = v'(\infty) = 0.$$

(ii) *The optimizers $\hat{\varphi}_T^0(x)$ in* (227) *(resp. $\hat{Z}_T^0(y)$ in* (228)*) exist, are unique, and take their values a.s. in $]0, \infty[$. If $x > 0$ and $y > 0$ are related by $u'(x) = y$ (or, equivalently, $x = -v'(y)$), then $\hat{\varphi}_T^0(x)$ and $\hat{Z}_T^0(y)$ are related by*

$$\hat{Z}_T^0(y) = U'(\hat{\varphi}_T^0(x)), \qquad \hat{\varphi}_T^0(x) = -V'(\hat{Z}_T^0(y)).$$

(iii) *For $x > 0$ and $y > 0$ such that $u'(x) = y$ we have*

$$xy = \mathbb{E}[\hat{\varphi}_T^0(x)\hat{Z}_T^0(y)].$$

Hence, for every pair $(\widehat{\varphi}, \hat{Z})$ of primal and dual optimizers, i.e. an admissible, self-financing $\widehat{\varphi}$ and a supermartingale deflator Z with terminal values $\widehat{\varphi}_T^0$ and $\hat{Z}_T^0$, the process $(\hat{Z}_t^0\hat{\varphi}_t^0 + \hat{Z}_t^1\hat{\varphi}_t^1)_{0\leqslant t\leqslant T} = (\hat{Z}_t^0(\hat{\varphi}_t^0 + \hat{\varphi}_t^1\tilde{S}_t))_{0\leqslant t\leqslant T}$ is a uniformly integrable martingale, where $\tilde{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$.

In order to find a candidate for the shadow price process as in chapter 2 above we also need the following result which again is taken from previous work of D. Kramkov and the present author [161].

Proposition 6.3. *Under the assumptions of Theorem 6.2 the subset $\mathfrak{D}^{loc}(1)$ of $\mathfrak{D}(1)$ (see Definition 4.21 and 4.20) for the notion of a local martingale deflator) satisfies*

$$\sup_{Z_T^0 \in \mathfrak{D}^{loc}(1)} \mathbb{E}[gZ_T^0] = \sup_{Z_T^0 \in \mathfrak{D}(1)} \mathbb{E}[gZ_T^0], \tag{229}$$

for each $g \in \mathfrak{C}(1)$, and is closed under countable convex combinations. Hence

$$v(y) = \inf_{Z_T^0 \in \mathfrak{D}^{loc}(1)} \mathbb{E}[V(yZ_T^0)], \qquad y > 0.$$

We thus may find a sequence $(Z^n)_{n=1}^\infty = ((Z_t^{0,n}, Z_t^{1,n})_{0 \leqslant t \leqslant T})_{n=1}^\infty$ of local martingale deflators such that

$$v(y) = \lim_{n \to \infty} \mathbb{E}[V(yZ_T^{0,n})]. \tag{230}$$

Proof. The equality (229) was shown in Theorem 5.12 and the remaining assertion follows from Proposition 3.2 in [161]. ■

In order to formulate the main result of this chapter (Theorem 6.5 below) in proper generality we still need the following notion which was introduced by C. Bender [14]. Note that this property is satisfied for continuous processes which are a local martingale with respect to a measure Q equivalent to $\mathbb{P}$. We shall see in chapter 8 below that also fractional Brownian motion enjoys this property. This is remarkable as fractional Brownian motion is far from being a martingale.

Definition 6.4. *Let $X = (X_t)_{t \geqslant 0}$ be a real-valued continuous stochastic process and σ a finite stopping time. Set $\sigma_+ = \inf\{t > \sigma \mid X_t - X_\sigma > 0\}$ and $\sigma_- = \inf\{t > \sigma \mid X_t - X_\sigma < 0\}$. Then we say that X satisfies the condition (TWC) of "two way crossing", if $\sigma_+ = \sigma_-$ $\mathbb{P}$-a.s.*

We can now formulate a theorem which, jointly with Theorem 7.3 and Theorem 8.4, is a central result of the present lectures.

Theorem 6.5. *As in Theorem 6.2 suppose that the continuous, strictly positive process S satisfies condition $(CPS^{\lambda'})$ locally, for each $0 < \lambda' < 1$. Suppose in addition that S satisfies the two way crossing property (TWC).*

Consider again transaction costs $0 < \lambda < 1$, as well as a utility function U having reasonable asymptotic elasticity, and suppose that the value function $u(x)$ in (227) is finite, for some $x > 0$. Fix $x > 0$ and let $y = u'(x)$.

Then the optimizer $\hat{Z}_T^0(y) \in \mathfrak{D}(y)$ in (228) is the terminal value of the first coordinate of a **local** *martingale deflator $(\hat{Z}_t^0(y), \hat{Z}_t^1(y))_{0 \leqslant t \leqslant T}$.*

In fact, we could have dropped the condition $(CPS^{\lambda'})$ locally, for all $\lambda' > 0$, as it follows already from the assumption (TWC). Indeed, the two way crossing property clearly implies the no obvious immediate arbitrage condition $(NOIA)$ locally (Def. 5.10). We have seen in Theorem 5.11 that the latter condition is equivalent to the condition $(CPS^{\lambda'})$ locally, for each $0 < \lambda' < 1$.

We shall split the message of Theorem 6.5 into the two subsequent results 6.6 and 6.7 which serve to clarify the role of the assumption (TWC). Clearly the two subsequent results imply Theorem 6.5.

Theorem 6.6. *Under the assumptions of Theorem 6.2 suppose in addition that the liquidation value process* V^{liq} *associated to an optimizer* $\hat{\varphi}$ *of* (227), *defined as*

$$V^{liq}(\hat{\varphi}_t) := \hat{\varphi}^0_t + (1-\lambda)(\hat{\varphi}^1_t)^+ S_t - (\hat{\varphi}^1_t)^- S_t, \qquad 0 \leqslant t \leqslant T, \tag{231}$$

is almost surely strictly positive, i.e. $\inf_{0\leqslant t\leqslant T} V^{liq}(\hat{\varphi}_t) > 0$, *a.s.*

Then the assertion of Theorem 6.5 holds true, i.e. the dual optimizer $\hat{Z}^0_T(y) \in \mathfrak{D}(y)$ *is the terminal value of the first coordinate of a* local martingale deflator $(\hat{Z}^0_t(y), \hat{Z}^1_t(y))_{0\leqslant t\leqslant T} \in \mathcal{Z}^{loc,e}$.

Proposition 6.7. *Under the assumptions of Theorem 6.5, i.e. assuming (TWC), the liquidation value process* $V^{liq}(\hat{\varphi}_t)_{0\leqslant t\leqslant T}$ *in* (231) *is strictly positive.*

We need some preparation for the proof of these two results. We let $y = 1$ and drop it in the sequel. Applying Proposition 6.3 we associate to the unique dual optimizer $\hat{Z}^0_T$ of (228) an approximating sequence $((Z^n_t)_{0\leqslant t\leqslant T})_{n=1}^\infty$ of local martingale deflators satisfying (230). As in the proof of Theorem 4.22 we may suppose that this sequence Fatou-converges to a càdlàg supermartingale $(\hat{Z}_t)_{0\leqslant t\leqslant T}$. Its terminal value $\hat{Z}^0_T$ is the unique dual optimizer in (228).

Our aim is to show that the process $\hat{Z}$ is a local martingale. We define its Doob-Meyer decomposition

$$d\hat{Z}^0_t = d\hat{M}^0_t - d\hat{A}^0_t, \tag{232}$$

$$d\hat{Z}^1_t = d\hat{M}^1_t - d\hat{A}^1_t, \tag{233}$$

where $\hat{M}$ is a local martingale and the predictable processes $\hat{A}^0$ and $\hat{A}^1$ are non-decreasing. To prove the conclusion of Theorem 6.6 we have to show that they vanish. We start by showing that $\hat{A}^0$ and $\hat{A}^1$ are aligned in a way described by (235).

Lemma 6.8. *Under the assumption of Theorem 6.2, let* $\varepsilon > 0$ *and* $\sigma \leqslant \tau$ *be* $[0,T]$*-valued stopping times such that* $1-\varepsilon \leqslant \frac{S_\tau}{S_\sigma} \leqslant 1+\varepsilon$. *Then*

$$\begin{aligned}(1-\varepsilon)(1-\lambda)S_\sigma \mathbb{E}\left[\hat{A}^0_\tau - \hat{A}^0_\sigma | \mathcal{F}_\sigma\right] &\leqslant \mathbb{E}\left[\hat{A}^1_\tau - \hat{A}^1_\sigma | \mathcal{F}_\sigma\right] \\ &\leqslant (1+\varepsilon)S_\sigma \mathbb{E}\left[\hat{A}^0_\tau - \hat{A}^0_\sigma | \mathcal{F}_\sigma\right].\end{aligned} \tag{234}$$

Hence

$$(1-\lambda)S_t d\hat{A}^0_t \leqslant d\hat{A}^1_t \leqslant S_t d\hat{A}^0_t \tag{235}$$

which is the symbolic differential notation for the integral inequality

$$\int_0^T (1-\lambda)S_t \mathbb{1}_D d\hat{A}^0_t \leqslant \int_0^T \mathbb{1}_D d\hat{A}^1_t \leqslant \int_0^T S_t \mathbb{1}_D d\hat{A}^0_t \tag{236}$$

which we require to hold true for every optional subset $D \subseteq \Omega \times [0,T]$.

Hence we have for the process $\widehat{\varphi}$ *in* (231) *that*

$$\int_0^T (\widehat{\varphi}^0_{t-} d\hat{A}^0_t + \widehat{\varphi}^1_{t-} d\hat{A}^1_t) \geqslant \int_0^T V^{liq}(\widehat{\varphi}_{t-}) d\hat{A}^0_t. \tag{237}$$

Proof. Before abording the proof we observe that (235) may intuitively be interpreted as the assertion that the ratio $\frac{d\hat{A}^1_t}{d\hat{A}^0_t}$ takes values in the bid ask spread $[(1-\lambda)S_t, S_t]$.

To show (234) we may and do assume that the stopping times σ and τ are such that we have

$$\hat{Z}_\sigma = \mathbb{P}\text{-}\lim_{n\to\infty} Z^n_\sigma \quad \text{and} \quad \hat{Z}_\tau = \mathbb{P}\text{-}\lim_{n\to\infty} Z^n_\tau, \tag{238}$$

the limits taken in probability.

Indeed (234) holds true iff it holds true for $\sigma^h = (\sigma + h) \wedge (T-h)$ and $\tau^h = (\tau + h) \wedge (T-h)$ instead of σ and τ, for $h > 0$ arbitrarily close to zero. This follows from the right-continuity and the uniform integrability of the process $\hat{A}$.

For all but at most countably many $h > 0$ we must have that the process $\hat{Z}$ is a.s. continuous at time σ^h and τ^h. This implies that the Fatou-limit $\hat{Z}$ of $(Z^n)_{n=1}^\infty$ then satisfies

$$\hat{Z}_{\sigma^h} = \mathbb{P} - \lim_{n\to\infty} Z^n_{\sigma^h} \quad \text{and} \quad \hat{Z}_{\tau^h} = \mathbb{P} - \lim_{n\to\infty} Z^n_{\tau^h}. \tag{239}$$

In [50] and [51] the interested reader can find much more on this topic.

Summing up, there is no loss of generality in assuming (238). By passing to a subsequence of $(Z^n)_{n=1}^\infty$ we may assume that the convergence in (238) takes place almost surely. By stopping there is also no loss of generality in assuming that all stopped local martingales $(Z^n)^\tau$ are actually true martingales. We therefore obtain a.s.

$$\lim_{n\to\infty} \left(Z^{0,n}_\tau - Z^{0,n}_\sigma\right) = \hat{Z}^0_\tau - \hat{Z}^0_\sigma = \left(\hat{M}^0_\tau - \hat{M}^0_\sigma\right) - \left(\hat{A}^0_\tau - \hat{A}^0_\sigma\right), \tag{240}$$

$$\lim_{n\to\infty}\left(Z^{1,n}_\tau - Z^{1,n}_\sigma\right) = \hat{Z}^1_\tau - \hat{Z}^1_\sigma = \left(\hat{M}^1_\tau - \hat{M}^1_\sigma\right) - \left(\hat{A}^1_\tau - \hat{A}^1_\sigma\right). \tag{241}$$

We then have that

$$\lim_{C\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\left(Z^{0,n}_\tau - Z^{0,n}_\sigma\right)\mathbb{1}_{\{Z^{0,n}_\tau - Z^{0,n}_\sigma \geqslant C\}}|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\hat{A}^0_\tau - \hat{A}^0_\sigma|\mathcal{F}_\sigma\right], \tag{242}$$

holds true a.s., and similarly

$$\lim_{C\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\left(Z^{1,n}_\tau - Z^{1,n}_\sigma\right)\mathbb{1}_{\{Z^{1,n}_\tau - Z^{1,n}_\sigma \geqslant C\}}|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\hat{A}^1_\tau - \hat{A}^1_\sigma|\mathcal{F}_\sigma\right]. \tag{243}$$

Indeed, we have

$$\begin{aligned} 0 &= \mathbb{E}\left[Z^{0,n}_\tau - Z^{0,n}_\sigma|\mathcal{F}_\sigma\right] \\ &= \mathbb{E}\left[\left(Z^{0,n}_\tau - Z^{0,n}_\sigma\right)\mathbb{1}_{\{Z^{0,n}_\tau - Z^{0,n}_\sigma \geqslant C\}}|\mathcal{F}_\sigma\right] + \mathbb{E}\left[\left(Z^{0,n}_\tau - Z^{0,n}_\sigma\right)\mathbb{1}_{\{Z^{0,n}_\tau - Z^{0,n}_\sigma < C\}}|\mathcal{F}_\sigma\right]. \end{aligned}$$

Note that

$$\begin{aligned} &\lim_{C\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\left(Z^{0,n}_\tau - Z^{0,n}_\sigma\right)\mathbb{1}_{\{Z^{0,n}_\tau - Z^{0,n}_\sigma < C\}}|\mathcal{F}_\sigma\right] \\ &= \mathbb{E}\left[\hat{Z}^0_\tau - \hat{Z}^0_\sigma|\mathcal{F}_\sigma\right] = -\mathbb{E}\left[\hat{A}^0_\tau - \hat{A}^0_\sigma|\mathcal{F}_\sigma\right], \end{aligned}$$

where the last equality follows from (240). We thus have shown (242) and (243) follows analogously.

We even obtain from (242) and (243) that

$$\lim_{C\to\infty}\lim_{n\to\infty}\mathbb{E}\left[Z^{0,n}_\tau\mathbb{1}_{\{Z^{0,n}_\tau \geqslant C\}}|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\hat{A}^0_\tau - \hat{A}^0_\sigma|\mathcal{F}_\sigma\right] \tag{244}$$

and

$$\lim_{C\to\infty}\lim_{n\to\infty}\mathbb{E}\left[Z^{1,n}_\tau\mathbb{1}_{\{Z^{1,n}_\tau \geqslant C\}}|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\hat{A}^1_\tau - \hat{A}^1_\sigma|\mathcal{F}_\sigma\right] \tag{245}$$

Indeed, the sequence $(Z^{0,n}_\sigma)_{n=1}^\infty$ converges a.s. to $\hat{Z}^0_\sigma$ so that by Egoroff's theorem it converges uniformly on sets of measure bigger than $1-\varepsilon$. Therefore the terms involving $Z^{0,n}_\sigma$ in (242) (resp. $Z^{1,n}_\sigma$ in (243)) disappear in the limit $C \to \infty$.

Finally, observe that

$$\tfrac{Z^{1,n}_\tau}{Z^{0,n}_\tau} \in \left[(1-\lambda)S_\tau, S_\tau\right] \subseteq \left[(1-\varepsilon)(1-\lambda)S_\sigma, (1+\varepsilon)S_\sigma\right].$$

Conditioning again on $\mathcal{F}_\sigma$ this implies on the one hand

$$\lim_{C\to\infty}\lim_{n\to\infty}\mathbb{E}\left[Z^{1,n}_\tau\mathbb{1}_{\{Z^{0,n}_\tau \geqslant C\}}|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\hat{A}^1_\tau - \hat{A}^1_\sigma|\mathcal{F}_\sigma\right]$$

and on the other hand

$$\frac{\mathbb{E}[\hat{A}^1_\tau - \hat{A}^1_\sigma|\mathcal{F}_\sigma]}{\mathbb{E}[\hat{A}^0_\tau - \hat{A}^0_\sigma|\mathcal{F}_\sigma]} = \lim_{C\to\infty}\lim_{n\to\infty} \frac{\mathbb{E}[Z^{1,n}_\tau \mathbb{1}_{\{Z^{0,n}_\tau \geqslant C\}}|\mathcal{F}_\sigma]}{\mathbb{E}[Z^{0,n}_\tau \mathbb{1}_{\{Z^{0,n}_\tau \geqslant C\}}|\mathcal{F}_\sigma]}$$
$$\in \big[(1-\varepsilon)(1-\lambda)S_\sigma, (1+\varepsilon)S_\sigma\big],$$

which is assertion (234).

As regards (236) it is routine to deduce it from (234) by approximation.

Finally, inequality (237) follows from (236) and the definition of the liquidation value

$$\begin{aligned}
&\int_0^t \Big(\widehat{\varphi}^0_{u-} d\widehat{A}^0_u + \widehat{\varphi}^1_{u-} d\widehat{A}^1_u\Big) \\
&= \int_0^t \Big(\widehat{\varphi}^0_{u-} d\widehat{A}^0_u + \widehat{\varphi}^1_{u-} d\widehat{A}^1_u\Big)\mathbf{1}_{\{\widehat{\varphi}^1_{u-}\leqslant 0\}} + \int_0^t \Big(\widehat{\varphi}^0_{u-} d\widehat{A}^0_u + \widehat{\varphi}^1_{u-} d\widehat{A}^1_u\Big)\mathbf{1}_{\{\widehat{\varphi}^1_{u-}>0\}} \\
&\geqslant \int_0^t \big(\widehat{\varphi}^0_{u-} - \widehat{\varphi}^1_{u-} S_u\big)\mathbf{1}_{\{\widehat{\varphi}^1_{u-}\leqslant 0\}} d\widehat{A}^0_u + \int_0^t \big(\widehat{\varphi}^0_{u-} + \widehat{\varphi}^1_{u-}(1-\lambda) S_u\big)\mathbf{1}_{\{\widehat{\varphi}^1_{u-}>0\}} d\widehat{A}^0_u \\
&= \int_0^t V^{liq}(\widehat{\varphi}_{u-}) d\hat{A}^0_u.
\end{aligned}$$

■

Turning back to the proofs of Theorem 6.6 and Proposition 6.7, we shall use the fact established in Theorem 6.2 (*iii*) that the process

$$X = \big(\widehat{\varphi}^0_t \hat{Z}^0_t + \widehat{\varphi}^1_t \hat{Z}^1_t\big)_{0\leqslant t\leqslant T} \tag{246}$$

is a uniformly integrable P-martingale satisfying $X_T > 0$ almost surely. Applying Itô and keeping in mind that φ has finite variation we obtain

$$\begin{aligned}
dX_t &= \hat{Z}^0_t d\widehat{\varphi}^0_t + \hat{Z}^1_t d\widehat{\varphi}^1_t + \widehat{\varphi}^0_t d\hat{Z}^0_t + \widehat{\varphi}^1_t d\hat{Z}^1_t &(247)\\
&= \hat{Z}^0_t\Big(d\widehat{\varphi}^0_t + \frac{\hat{Z}^1_t}{\hat{Z}^0_t} d\widehat{\varphi}^1_t\Big) + \big(\widehat{\varphi}^0_t d\hat{M}^0_t + \widehat{\varphi}^1_t d\hat{M}^1_t\big) - \big(\widehat{\varphi}^0_t d\hat{A}^0_t + \widehat{\varphi}^1_t d\hat{A}^1_t\big). &(248)
\end{aligned}$$

The second term is the increment of a local martingale.

The first and the third term are the increments of non increasing processes, hence both have to vanish. This allows for two interesting conclusions.

Let us start with the first term. As the process $\hat{Z}^0$ is strictly positive we conclude that $d\widehat{\varphi}^0_t + \tilde{S}_t d\widehat{\varphi}^1_t$ vanishes a.s. for all $0 \leqslant t \leqslant T$, where $\tilde{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$. This amounts precisely to the relation

$$\{d\widehat{\varphi}^1_t > 0\} \subseteq \{\tilde{S}_t = S_t\} \quad \text{and} \quad \{d\widehat{\varphi}^1_t < 0\} \subseteq \{\tilde{S}_t = (1-\lambda)S_t\}. \tag{249}$$

Anticipating that $\tilde{S}$ will be interpreted as a *shadow price process* this relation states that the optimizing agent only buys stock (i.e. $d\widehat{\varphi}^1_t > 0$) when $\tilde{S}_t = S_t$ and only sells stock (i.e. $d\widehat{\varphi}^1_t < 0$) when $\tilde{S}_t = (1-\lambda)S_t$. Compare Proposition 7.2 below.

The fact that the third term of (248) must vanish amounts to the subsequent proof of Theorem 6.6.

Proof of Theorem 6.6. It follows from the above discussion and (237) that

$$\int_0^T V^{liq}(\widehat{\varphi}_{t-})d\hat{A}^0_t = 0, \qquad a.s.$$

If $V^{liq}(\widehat{\varphi})$ is a.s. strictly positive this implies that the process $\hat{A}^0$ vanishes and therefore by (235) also the process $\hat{A}^1$ vanishes.

This amounts to the assertion that $\hat{Z}$ is a local martingale. ■

Proof of Proposition 6.7. To show that the liquidation value (231) remains almost surely positive, we argue by contradiction. Define

$$\sigma_n = \inf\{t \in [0,T] \mid V^{liq}(\widehat{\varphi}_t) \leqslant n^{-1}\}, \tag{250}$$

and

$$\sigma = \lim_{n\to\infty} \sigma_n.$$

Suppose that $\mathbb{P}[\sigma < \infty] > 0$ and let us work towards a contradiction.

First observe that $V^{liq}(\widehat{\varphi}_\sigma) = 0$ on $\{\sigma < \infty\}$. Indeed, applying the product rule to (231) and noting that $\widehat{\varphi}$ has finite variation we obtain

$$dV^{liq}(\widehat{\varphi}_t) = \big((\widehat{\varphi}^1_t)^+(1-\lambda) - (\widehat{\varphi}^1_t)^-\big)dS_t + \big(d\widehat{\varphi}^0_t + (1-\lambda)S_t d(\widehat{\varphi}^1_t)^+ - S_t d(\widehat{\varphi}^1_t)^-\big). \tag{251}$$

The first term is the increment of a continuous process while the second term is, by the self-financing condition under transaction costs, the increment of a non-increasing right continuous process. Hence $V^{liq}(\widehat{\varphi}_\sigma) = 0$ on the set $\{\sigma < \infty\}$.

So suppose that $P[\sigma < \infty] > 0$. We may and do assume that S "moves immediately after σ", i.e. $\sigma = \inf\{t > \sigma \mid S_t \neq S_\sigma\}$. Indeed, we may replace σ on the set $\{\sigma < \infty\}$ by the stopping time $\sigma_+ = \sigma_-$ defined in Def. 6.4. Note that $\sigma_+ < T$ on $\{\sigma < \infty\}$ as $V^{liq}(\widehat{\varphi}_T) > 0$, almost surely.

We shall again use the fact that the process

$$X = \big(\widehat{\varphi}^0_t \hat{Z}^0_t + \widehat{\varphi}^1_t \hat{Z}^1_t\big)_{0\leqslant t\leqslant T} \tag{252}$$

is a uniformly integrable P-martingale satisfying $X_T > 0$ almost surely.

Firstly, this implies that $\widehat{\varphi}^1_\sigma \neq 0$ a.s. on $\{\sigma < \infty\}$. Indeed, otherwise we have $\widehat{\varphi}_\sigma = (0,0)$ with positive probability. As X is a uniformly integrable martingale with strictly positive terminal value $X_T > 0$ we arrive at a contradiction.

Hence we have $\widehat{\varphi}^1_\sigma \neq 0$. Let us first suppose that $\widehat{\varphi}^1_\sigma > 0$ on a subset of positive measure of $\{\sigma < \infty\}$ which we assume w.l.g. to equal $\{\sigma < \infty\}$. We then cannot have $\tilde{S}_\sigma := \frac{\hat{Z}^1_\sigma}{\hat{Z}^0_\sigma} = (1-\lambda)S_\sigma$ with strictly positive probability. Indeed, this would imply that $V^{liq}(\widehat{\varphi}_\sigma) = X_\sigma = 0$ on this set which yields a contradiction as in the previous paragraph.

Hence we have that $\tilde{S}_\sigma > (1-\lambda)S_\sigma$ on $\{\sigma < \infty\}$. This implies by (249) that the utility-optimizing agent applying the strategy $\widehat{\varphi}$ cannot sell stock at time σ as well as for some time after σ by the continuity of S and $\tilde{S}$.

Note, however, that – a priori – the optimizing agent may very well buy stock during this period. But we shall see that this is not to her advantage.

Define the stopping time ϱ_n as the first time t after σ when one of the following events happens

(i) $\min_{\sigma\leqslant u\leqslant t}\{\tilde{S}_u\} - (1-\lambda)S_t \leqslant (\tilde{S}_\sigma - (1-\lambda)S_\sigma)/2$ or

(ii) $S_t \leqslant S_\sigma - \frac{1}{n}$

(iii) $\min_{\sigma\leqslant u\leqslant t}\{S_u\} \geqslant (1-\lambda)\max_{\sigma\leqslant u\leqslant t}\{S_u\}$

By the hypothesis of (TWC) of "two way crossing", we conclude that, a.s. on $\{\sigma < \infty\}$, we have that ϱ_n decreases to σ and that we have $S_{\varrho_n} = S_\sigma - \frac{1}{n}$, for n large enough. Choose n large enough such that $S_{\varrho_n} = S_\sigma - \frac{1}{n}$ on a subset of $\{\sigma < \infty\}$ of positive measure. Then $V^{liq}(\widehat{\varphi}_{\varrho_n})$ is strictly negative on this set which will give the desired contradiction. Indeed, the assumption $\widehat{\varphi}^1_\sigma > 0$ implies that the agent suffers a strict loss from her holdings in stock because $S_{\varrho_n} < S_\sigma$. Condition (i) makes sure that the agent cannot have sold stock between σ and ϱ_n. The agent may have bought additional stock during the interval $[\![\sigma, \varrho_n]\!]$. However, this cannot result in a positive effect either as condition (iii) makes sure that the last term below is non positive

$$V^{liq}_{\varrho_n}(\widehat{\varphi}) \leqslant V^{liq}_\sigma(\widehat{\varphi}) + \widehat{\varphi}^1_\sigma(1-\lambda)(S_{\varrho_n} - S_\sigma) - \int_\sigma^{\varrho_n}\big(S_u - (1-\lambda)S_{\varrho_n}\big)d\widehat{\varphi}^{1,\uparrow}_u < 0,$$

almost surely on $\{S_{\varrho_n} = S_\sigma - \frac{1}{n}\}$. This yields the desired contradiction.

As regards the remaining case that $\widehat{\varphi}^1_\sigma < 0$ on $\{\sigma < \infty\}$ the argument goes, with signs reversed, in an analogous way. ■

Remark 6.9. Finally let us discuss the uniqueness of the process $\hat{Z}^0 = (\hat{Z}^0_t)_{0\leqslant t\leqslant T}$ (to be distinguished from the uniqueness of the terminal value $\hat{Z}^0_T$ which is guaranteed by Theorem 6.2). It turns out that this issue is quite subtle. I thank Lingqi Gu, Yiqing Lin and Junjian Yang for pertinent discussions on this topic which were crucial to clarify the question.

Under the assumptions of Theorem 6.2 and *assuming* that $\hat{Z}^0$ happens to be a true martingale, the uniqueness of the process $(\hat{Z}^0_t)_{0\leqslant t\leqslant T}$ is an immediate consequence of the uniqueness of its terminal value $\hat{Z}^0_T$.

In general, however, even under the assumptions of Theorem 6.5, the local martingale $(\hat{Z}^0_t)_{0\leqslant t\leqslant T}$, i.e. the first coordinate of an optimal local martingale deflator $\hat{Z}$, is not unique although its starting value $\hat{Z}^0_0 = 1$ as well as its terminal value $\hat{Z}^0_T$ are unique.

We sketch a counter-example. The construction will be somewhat indirect. We start by first defining a shadow price process $\hat{S}$ (which will eventually turn out to be non-unique) while the stock price process S will only be defined later.

We place ourselves into the setting of Proposition 5.2 and fix transaction costs $0 < \lambda < \frac{1}{2}$, initial endowment $x = 1$, and logarithmic utility $U(x) = \log(x)$. With the notation of Proposition 5.2 we define τ^λ as

$$\tau^\lambda = \inf\{t : Z_t = \frac{1}{2(1-\lambda)}\}.$$

and

$$\hat{S}_t = N_{\tan(\frac{\pi}{2}t)\wedge\tau^\lambda}, \qquad 0 \leqslant t \leqslant 1.$$

Considering $\hat{S}$ as a frictionless financial market it follows from Prop. 5.2 that the optimal strategy consists in constantly holding one unit of stock. To be precise: the optimal strategy consists of buying one stock at time $t = 0$ at price $\hat{S}_0 = 1$, and selling it at time $t = 1$ at price $\hat{S}_1 = 2(1-\lambda)$.

Turning to the definition of the stock price process S under transaction costs λ make the following observation: If $S = (S_t)_{0\leqslant t\leqslant 1}$ is *any* continuous, adapted process, starting at $S_0 = 1$ and ending at $S_1 = 2$, and such that $\hat{S}$ remains in the bid-ask spread $[(1-\lambda)S, S]$, this trading strategy is still optimal for the process S under transaction costs λ. Indeed, it is obvious that the above assumptions on S guarantee that there is no better strategy. Otherwise we would obtain a contradiction to the optimality of the above strategy in the frictionless market $\hat{S}$.

We write $\hat{Z}^0 = (\hat{Z}^0_t)_{0\leqslant t\leqslant 1}$ for the process $\hat{Z}^0 = (\hat{S})^{-1}$ which is a local $\mathbb{P}$-martingale. For any S satisfying the above properties, we then have that the process $\hat{Z} = (\hat{Z}^0_t, 1)_{0\leqslant t\leqslant 1}$ defines a local martingale deflator inducing the dual optimizer (228) under transaction costs λ.

Next we define a perturbation of the process $\hat{S}$ which will be denoted by $\check{S}$. To do so we shall define a perturbation of $\hat{Z}^0 = (\hat{S})^{-1}$. First decompose this local martingale into a true martingale $\hat{M}$ plus a potential $\hat{P}$, so that $\hat{Z}^0 = \hat{M} + \hat{P}$, by

$$\hat{M}_t = \mathbb{E}[\hat{Z}^0_1|\mathcal{F}_t], \qquad \hat{P}_t = \hat{Z}^0_t - \hat{M}_t, \qquad 0 \leqslant t \leqslant 1.$$

The process $\hat{P}$ is a non-negative local martingale starting at $\hat{P}_0 = \frac{1-2\lambda}{2(1-\lambda)}$ and ending at $\hat{P}_1 = 0$. Let σ be the first moment when $\hat{P}$ hits the level 1. Note that $\mathbb{P}[\sigma < \infty] = \frac{1-2\lambda}{2(1-\lambda)}$ as the stopped process $\hat{P}^\sigma$ is a martingale. For $\delta > 0$ choose an arbitrary $\mathcal{F}_\sigma$-measurable function f taking values in $[1-\delta, 1+\delta]$ such that

$$\mathbb{E}[f \mathbb{1}_{\{\sigma<\infty\}}] = \frac{1-2\lambda}{2(1-\lambda)}$$

and such that f is not identically equal to 1 on $\{\sigma < \infty\}$.

Define the potential $\check{P}$ by

$$\check{P}_t = \begin{cases} \mathbb{E}[f \mathbb{1}_{\{\sigma<\infty\}} \,|\, \mathcal{F}_{t\wedge\sigma}], & 0 \leqslant t \leqslant \sigma, \\ f\hat{P}_t, & \sigma \leqslant t \leqslant 1. \end{cases} \tag{253}$$

The process $\check{P}$ again is a local martingale starting at $\check{P}_0 = \frac{1-2\lambda}{2(1-\lambda)}$ and ending at $\check{P}_1 = 0$. Note that the ratio $\frac{\check{P}_t}{\hat{P}_t}$ remains in $[1-\delta, 1+\delta]$, a.s. for $0 \leqslant t < 1$.

Define $\check{Z}^0 := \hat{M} + \check{P}$ and $\check{S} := (\check{Z}^0)^{-1}$. The ratio $\frac{\check{Z}_t}{\hat{Z}_t}$ also remains in $[1-\delta, 1+\delta]$ and therefore the ratio $\frac{\check{S}_t}{\hat{S}_t}$ remains in $[(1+\delta)^{-1}, (1-\delta)^{-1}]$. As before, consider $\check{S}$ as a frictionless price process. It again has the property that the log-optimal strategy (without transaction costs) consists in buying one stock at time $t = 0$ and selling it at time $t = 1$. This follows from the well-known fact that $\check{S}$ equals the numéraire portfolio of the frictionless market defined by $\check{S}$ (compare Proposition 5.2).

Again, for any continuous, adapted stock price process S (considered under transaction costs λ), starting at $S_0 = 1$, ending at $S_1 = 2$, and such that $\check{S}$ remains in the bid-ask spread $[(1-\lambda)S, S]$, this strategy is still optimal and $(\check{Z}_t)_{0\leqslant t\leqslant 1} = (\check{Z}^0_t, 1)_{0\leqslant t\leqslant 1}$ induces a dual optimizer in (228).

Finally, we look for such a stock price S for which $\hat{S}$ as well as $\check{S}$ are shadow prices: we can find plenty of continuous, adapted processes S with $S_0 = 1, S_1 = 2$ and such that $\hat{S}$ as well as $\check{S}$ remain in the bid-ask spread $[(1-\lambda)S, S]$. For example, define

$$m_t = \max(\hat{S}_t, \check{S}_t), \quad M_t = (1-\lambda)^{-1} \min(\hat{S}_t, \check{S}_t).$$

If δ satisfies $(1-\lambda)(1+\delta) < (1-\delta)$, we have a.s. that $m_t < M_t$, for $0 \leqslant t \leqslant 1$.

Define S by

$$S_t = (1-t)m_t + tM_t, \qquad 0 \leqslant t \leqslant 1.$$

The process S is continuous, adapted, starts at $S_0 = 1$ and ends at $S_1 = 2$. It takes values in $[\hat{S}, (1-\lambda)^{-1}\hat{S}]$ as well as $[\check{S}, (1-\lambda)^{-1}\check{S}]$ so that S satisfies the above properties. For such a process S we therefore have found two different local martingale deflators, namely $\hat{Z}$ and $\check{Z}$, such that the terminal values $\hat{Z}^0_T$ and $\check{Z}^0_T$ both induce the dual optimizer in (228).

We finish this remark with two easy observations concerning the non-uniqueness of the other optimal processes appearing in Theorem 6.2.

The non-uniqueness of the second coordinate $\hat{Z}^{1,\lambda}$ of an optimal super-martingale deflator $\hat{Z}^\lambda$ is an easy observation. It may already occur in the setting of finite Ω as was observed in Example 2.4.

As regards the non-uniqueness of the portfolio process $\widehat{\varphi} = (\widehat{\varphi}^0_t, \widehat{\varphi}^1_t)_{0 \leqslant t \leqslant T}$ this is a cheap shot too. While its starting value $\widehat{\varphi}_{0-} = (x, 0)$ and its terminal value $\widehat{\varphi}_t = (\widehat{\varphi}^0_T, 0)$ are unique by Theorem 6.2 the process $\widehat{\varphi}$ is non-unique for rather silly reasons. For example let $S = (S_t)_{0 \leqslant t \leqslant T}$ be the Black-Scholes model considered in chapter 3 where we choose the parameters such that, under fixed transactions costs $\lambda > 0$, the optimal trading strategy $(\widehat{\varphi}_t)_{0 \leqslant t \leqslant T}$ (which is unique in this setting) is such that the agent buys or sells stock at time $t = 0$, i.e. $\widehat{\varphi}^1_0 \mp \widehat{\varphi}^1_{0-} = 0$. Of course, this is the generic case when we choose the parameters.

Now shift the process by one unit of time: define $(\bar{S}_t)_{0 \leqslant t \leqslant T+1}$ by $\bar{S}_t = S_0$, for $0 \leqslant t \leqslant 1$ and $\bar{S}_t = S_{t-1}$, for $1 \leqslant t \leqslant T+1$. We also have to shift the corresponding filtration $\bar{\mathcal{F}}$ by one unit of time, i.e., $\bar{\mathcal{F}}_t = \mathcal{F}_{(t-1)\vee 0.}$ Clearly the optimal trading strategy $\widehat{\varphi}$ now also simply has to be shifted by 1, with one small difference. The trade which had to be done for the original process S at time $t = 0$, can now be done for $\bar{S}$ at time $t = 0$, or $t = 1$, or at any time inbetween.

7 The Shadow Price Process

In this chapter we analyse in more detail the notion of a shadow price process. In the setting of finite Ω this concept was introduced in Definition 2.7. There the existence of a shadow price process resulted from the solution of the dual problem and was rather straightforward to prove. The assumption on the stock price process $S = (S_t)_{t=0}^T$ which we had to impose in chapter 2 was just the no arbitrage condition (NA^λ).

We now want to identify sufficient conditions for a continuous price process $S = (S_t)_{0 \leqslant t \leqslant T}$ which guarantee the existence of a shadow price process. We shall find in Theorem 7.3 below conditions which are sufficiently weak to apply to models based on fractional Brownian motion which we shall consider in the next chapter.

We start by shaping the definition of a shadow price process which is appropriate for the present setting. Fix again the continuous, strictly positive process S, transaction costs $\lambda > 0$ and the utility function U as in Theorem 6.2. Again we consider the portfolio optimization problem

$$\mathbb{E}[U(\varphi_T^0)] \mapsto \max! \tag{254}$$

where φ_T^0 ranges in $\mathfrak{C}(x)$ as in Definition 4.21. To recapitulate: we optimize over the set of contingent claims φ_T^0 which are attainable from initial wealth $(x, 0)$ and subsequent trading in S under transaction costs λ in an admissible way.

Passing temporarily to the frictionless setting, fix a continuous, strictly positive semi-martingale $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ and consider the analogous frictionless optimization problem

$$\mathbb{E}[U(x + (H \cdot \tilde{S})_T)] \mapsto \max! \tag{255}$$

where H runs through the set of predictable, $\tilde{S}$-integrable, admissible trading strategies.

We shall consider the situation when $\tilde{S}_t$ takes values in the bid-ask spread $[(1-\lambda)S_t, S_t]$. Similarly as in chapter 2 it is straightforward to check the economically obvious fact that the value of the frictionless problem (255) is at least as big as the value of the problem (254) under transaction costs (Proposition 7.2 below). The relevant question is: can we find a process $\tilde{S}$ taking values in $[(1-\lambda)S, S]$ such that (254) and (255) are equal?

Definition 7.1. *Fix the continuous, strictly positive process S, the continuous semi-martingale $\tilde{S}$ taking values in the bid-ask spread $[(1-\lambda)S, S]$, as well as a utility function U satisfying the reasonable asymptotic elasticity* *condition of Definition 6.1. Also fix $\lambda > 0$, initial wealth $x > 0$, and the horizon T. Suppose that S satisfies the assumptions of Theorem 6.2 and that the value of the problem* (254) *is finite, for some $x > 0$.*

We then say that $\tilde{S}$ is a shadow price process if the values of the problems (254) *and* (255) *coincide.*

Proposition 7.2. *Under the assumptions of Theorem 6.2 as well as those of the above definition, suppose that $\tilde{S}$ is a shadow price process. Let $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leqslant t \leqslant T}$ be an optimizer for* (254) *and define $\hat{H}_t := \widehat{\varphi}_t^1$ and the process*

$$\hat{V}_t := x + (\hat{H} \cdot \tilde{S})_t, \quad \textit{for} \quad 0 \leqslant t \leqslant T.$$

Then $\hat{H}$ is a well-defined optimizer for the frictionless problem (255) *and we have almost surely*

$$\{d\widehat{\varphi}_t^1 > 0\} \subseteq \{\tilde{S}_t = S_t\} \quad \textit{and} \quad \{d\widehat{\varphi}_t^1 < 0\} \subseteq \{\tilde{S}_t = (1-\lambda)S_t\}. \tag{256}$$

A small technical remark seems in order. The above defined process $\hat{H}$ is $\tilde{S}$-integrable (it is of finite variation), admissible (we have $\hat{V}_t \geqslant V^{liq}(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1) \geqslant 0$ as we shall presently see) and predictable. Indeed, the càdlàg process φ is optional and therefore predictable as we have assumed the filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ to be Brownian. But the Brownian assumption is not really relevant here. By the continuity of $\tilde{S}$ we can simply pass from the càdlàg finite variation process $\hat{H}$ to its càglàd version without changing the stochastic integral $\hat{H} \cdot \tilde{S}$. Also note that the special values $\widehat{\varphi}_{0_-} = (x, 0)$ and $\widehat{\varphi}_T = (\widehat{\varphi}_T^0, 0)$ do not matter when defining the process $\hat{H}$ and the resulting stochastic integral.

Proof of Proposition 7.2. By Theorem 6.2 we find an optimizer $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leqslant t \leqslant T}$ for the problem (254). For the process $(\widehat{\varphi}_t^0 + \widehat{\varphi}_t^1 \tilde{S}_t)$ we obtain from Itô

$$d(\widehat{\varphi}_t^0 + \widehat{\varphi}_t^1 \tilde{S}_t) = \widehat{\varphi}_t^1 d\tilde{S}_t + (d\widehat{\varphi}_t^0 + \tilde{S}_t d\widehat{\varphi}_t^1)$$

so that

$$\widehat{\varphi}_T^0 = x + (\hat{H} \cdot \tilde{S})_T + \int_0^T (d\widehat{\varphi}_t^0 + \tilde{S}_t d\widehat{\varphi}_t^1). \tag{257}$$

Hence $\widehat{\varphi}_T^0 \leqslant x + (\hat{H} \cdot \tilde{S})_T$ and equality holds true a.s. iff the above integral vanishes a.s. Our hypothesis of the equality between (254) and (255) implies that this indeed must be the case. Hence $\hat{H}$ is an optimizer for the frictionless problem (255) and we also obtain the inclusions (256). ∎

The analysis of chapter 2 told us that the obvious candidate for a shadow price process $\tilde{S}$ is the quotient $\frac{\hat{Z}^1}{\hat{Z}^0}$ of the dual optimizer $\hat{Z} = (\hat{Z}_t^0, \hat{Z}_t^1)_{0 \leqslant t \leqslant T}$ of problem (254). Cvitanić and Karatzas have shown in [43] that this candidate is indeed a shadow price process, provided that the a dual optimizer $\hat{Z}$ in Theorem 6.2 is induced by a local martingale. The subsequent – amazingly short – proof of this remarkable result is one more demonstration of the power of the duality methods in the context of portfolio optimization.

Theorem 7.3. *Under the assumptions of Theorem 6.2 fix $x > 0$ and $y = u'(x)$ and suppose that there is an equivalent local martingale deflator $\hat{Z}(y) = (\hat{Z}^0(y), \hat{Z}^1(y)) \in \mathcal{Z}^{loc,e}(y)$ such that $\hat{Z}^0_T(y)$ is the dual optimizer in (228). Then the process $\tilde{S}$ defined by*

$$\tilde{S} = \frac{\hat{Z}^1(y)}{\hat{Z}^0(y)}$$

is a shadow price process in the sense of Definition 7.1 and the optimizers $\widehat{\varphi}^0_T$ of (254) and $x + (\hat{H} \cdot \tilde{S})_T$ of (255) coincide.

We remark that, by the assumption that the filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ is Brownian, we are sure that $\hat{Z}$ and therefore $\tilde{S}$ are continuous so that we are in the setting of Definition 7.1. But again this continuity of $\tilde{S}$ is merely for convenience. In ([48], Proposition 3.7) a more general version of the theorem, pertaining also to discontinuous processes S and $\tilde{S}$, was proved.

Proof. For expository reasons suppose first that $\hat{Z}(y)$ is a true martingale so that $\frac{d\hat{Q}}{d\mathbb{P}} = \frac{\hat{Z}^0_T(y)}{y}$ defines an equivalent martingale measure for the process $\tilde{S}$.

Considering the portfolio optimization problem (255) for the price process $\tilde{S}$ we are precisely in the situation of Theorem 3.2 of [161]. Indeed, $\tilde{S}$ is a semi-martingale admitting an equivalent martingale measure, namely $\hat{Q}$, and such that the dual problem, and therefore also the primal problem, has a finite value.

We have that $\hat{Z}^0_T(y)$ must be the dual optimizer for this frictionless problem. Indeed, in the dual problem of (255) we optimize $\mathbb{E}[V(y\frac{dQ}{d\mathbb{P}})]$ over all equivalent martingale measures Q for the fixed process $\tilde{S}$. This is a subset of the set $\mathfrak{D}(y)$ considered on (228) where we consider all processes $\tilde{S}$ taking values in the bid-ask spread $[(1-\lambda)S, S]$. Hence, a fortiori, $\hat{Z}^0_T(y)$ must also be the dual optimizer for the frictionless problem (255).

It follows that also the primal optimizer $\widehat{\varphi}^0_T(x)$ of (254) and $x+(\hat{H}\cdot\tilde{S})_T$ of (255) must coincide. Indeed, by the first order condition (Theorem 6.2 (ii)) both random variables must a.s. equal $-V'(\hat{Z}^0_T(y))$. In particular $\tilde{S}$ satisfies the requirements of Definition 7.1 and therefore is a shadow price.

Finally we drop the assumption that $\hat{Z}$ is a martingale and only assume that it is a local martingale. We then are in the setting of Theorem 5.8 above.

Applying again Theorem 3.2 of [161] we may conclude exactly as in the first part of the proof. ■

Finally we may combine Theorem 7.3 with Theorem 6.5 to obtain the central result of this chapter.

Theorem 7.4. *Fix U satisfying* (226)*, the horizon T, and transaction costs $\lambda > 0$.*

Suppose that the strictly positive continuous process $S = (S_t)_{0 \leqslant t \leqslant T}$ satisfies the "two way crossing property" (TWC) *(Def. 6.4) and that the value function $u(x)$ in* (227) *is finite, for some $x > 0$.*

Then there is a shadow price process $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ as defined in Def. 7.1.

Proof. As mentioned after Theorem 6.5 it is not necessary to explicitly assume the local property $(CPS^{\lambda'})$, for all $0 < \lambda' < \lambda$, as this property follows from the assumption (TWC). Theorem 6.5 and Theorem 7.3 therefore apply. ■

8 A case study: Fractional Brownian Motion

We resume here the theme of (exponential) fractional Brownian motion which was briefly discussed in the introduction. In fact, the challenge posed by this example was an important motivation for the present research.

Fractional Brownian motion has been proposed by B. Mandelbrot [178] as a model for stock price processes more than 50 years ago. It is defined as a stationary centered Gaussian process $(B_t)_{t\geqslant 0}$ such that $\operatorname{Var}(B_t) = t^{2H}$ where the Hurst parameter H is in $]0,1[$. Until today this idea poses a number of open problems. From a mathematical point of view a major difficulty arises from the fact that fractional Brownian motion fails to be a semimartingale (except for the classical Brownian case $H = \frac{1}{2}$). Tools from stochastic calculus are therefore hard to apply and it is difficult to reconcile this model with the usual no arbitrage theory of mathematical finance. Indeed, it was shown in ([64], Theorem 7.2) that a stochastic process which fails to be a semimartingale automatically allows for arbitrage (in a sense which was made precise there). In the special case of fractional Brownian motion this was also shown directly in a very convincing way by C. Rogers [205] (compare also [33]).

One way to avoid this deadlock arising from the violation of the no-arbitrage paradigm is the consideration of proportional transaction costs. The introduction of proportional transaction costs λ, for arbitrarily small $\lambda > 0$, makes the arbitrage opportunities disappear. As we shall see, Theorem 7.4 applies perfectly to the case of fractional Brownian motion, for any Hurst index $H \in (0,1)$. As utility we may take any function $U : \mathbb{R}_+ \mapsto \mathbb{R}$ having the reasonable asymptotic elasticity condition (Def. 6.1).

Let us define the setting more formally. As driver of our model S we fix a standard Brownian motion $(W_t)_{-\infty<t<\infty}$, indexed by the entire real line and normalized by $W_0 ==$, in its natural (right continuous, saturated) filtration $(\mathcal{F}_t)_{-\infty<t<\infty}$. We let the Brownian motion W run from $-\infty$ on in order to apply the elegant integral representation below (258) due to Mandelbrot and van Ness; see [189].

We note for later use that the Brownian motion $(W_t)_{0\leqslant t\leqslant T}$, now indexed by $[0,T]$, has the integral representation property with respect to the filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$. The only difference to the more classical setting, where we consider the filtration $(\mathcal{G}_t)_{0\leqslant t\leqslant T}$ generated by $(W_t)_{0\leqslant t\leqslant T}$ is that $\mathcal{F}_0$ is not trivial anymore. But this causes little trouble. We simply have to do all the arguments conditionally on $\mathcal{F}_0$.

Fix a Hurst parameter $H \in]0,1[\setminus\{\frac{1}{2}\}$. We may define the fractional Brownian motion $(B_t)_{t\geqslant 0} = (B_t^H)_{t\geqslant 0}$ as

$$B_t = C(H) \int_{-\infty}^{t} \left((t-s)^{H-\frac{1}{2}} - \left(|s|^{H-\frac{1}{2}} \mathbb{1}_{(-\infty,0)} \right) \right) dW_s, \quad t \geqslant 0, \tag{258}$$

where $C(H)$ is a normalizing constant which is not relevant in the sequel (see [189], section 1.1 or [205], formula (1.1)). For $H = \frac{1}{2}$ we simply have $B_t = W_t$, for $t > 0$.

We may further define a non-negative stock price process $S = (S_t)_{0 \leqslant t \leqslant T}$ by letting

$$S_t = \exp(B_t), \qquad 0 \leqslant t \leqslant T, \tag{259}$$

or, slightly more generally,

$$S_t = \exp(\sigma B_t + \mu t), \qquad 0 \leqslant t \leqslant T, \tag{260}$$

for some $\sigma > 0$ and $\mu \in \mathbb{R}$. For the sake of concreteness we stick to (259) but the subsequent results also hold true for (260).

We want to apply Theorem 7.4 to this model of a stock price. To do so, we have to show two properties: the condition (TWC) as well as the finiteness of the value function $u(x)$ for some $x > 0$.

The first issue was recently settled in a positive – and highly impressive – way by Rémi Peyre [195].

Theorem 8.1 (R. Peyre)**.** *For each $H \in]0,1[$ fractional Brownian motion $(B_t)_{t \geqslant 0}$ has the "two way crossing property" (TWC).*

The proof of this theorem is demanding and goes beyond the scope of the present lecture notes.

As regards the finiteness of $u(x)$ in Theorem 7.4 this is a consequence of the subsequent Proposition 8.2 obtained in [52]. We need some notation. Fix $\delta > 0$ and define inductively the stopping times $(\tau_j)_{j=0}^{\infty}$ by $\tau_0 = 0$ and

$$\tau_j = \inf\{t > \tau_{j-1} | |B_t - B_{\tau_{j-1}}| \geqslant \delta\}.$$

We define the *number of δ-fluctuations up to time T* as the random variable

$$F_T^{(\delta)} := \sup\{j \geqslant 0 | \tau_j \leqslant T\}$$

We then have the following estimate.

Proposition 8.2. *[52] With the notation above, there exist finite positive constants $C = C(H), C' = C'(H)$ only depending on H such that*

$$\mathbb{P}[F_T^{(\delta)} \geqslant n] \leqslant C' \exp(-C^{-1} \delta^2 T^{-2H} n^{1+(2H \wedge 1)}), \qquad \textit{for} \quad n \in \mathbb{N}. \tag{261}$$

The proof of the above estimate is substantially easier than the proof of Theorem 8.1 but still too technical to reproduce it here. The message is that, for each $H \in]0,1[$, the probability of the sets $\{F_T^{(\delta)} \geqslant n\}$ is decaying in a super-exponential way as n tends to infinity. From this fact it is easy to obtain bounds on the tail behaviour of $\varphi_T^0 \in \mathfrak{C}(1)$.

Lemma 8.3. *Fix $H \in]0,1[$, the model* (259)*, as well as $\lambda > 0, T > 0$ and $\delta > 0$ such that $(1-\lambda)e^{2\delta} < 1$.*

There exists a constant K, depending only on δ and λ, such that, for each $\varphi_T^0 \in \mathfrak{C}(1)$ we have

$$\varphi_T^0 \leqslant K^n \qquad \text{on} \quad \{F_T^{(\delta)} \leqslant n\}. \tag{262}$$

In particular $\{\mathbb{E}[U(\varphi_T^0)] : \varphi_T^0 \in \mathfrak{C}(1)\}$ remains bounded from above, for each concave function $U : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$.

Proof. As regards the final sentence it follows from (262) and (261) that $\{\mathbb{E}[\varphi_T^0] : \varphi_T^0 \in \mathfrak{C}(1)\}$ remains bounded. This implies the final assertion as any concave function U is dominated by an affine function.

It remains to show (262). Fix an admissible trading strategy φ starting at $\varphi_{0_-} = (1,0)$ and ending at $\varphi_T = (\varphi_T^0, 0)$. Define the "optimistic value" process $(V^{opt}(\varphi_t))_{0 \leqslant t \leqslant T}$ by

$$V^{opt}(\varphi_t) = \varphi_t^0 + (\varphi_t^1)^+ S_t - (\varphi_t^1)^-(1-\lambda)S_t.$$

The difference to the liquidation value V^{liq} as defined in (156) is that we interchanged the roles of S and $(1-\lambda)S$. Clearly $V^{opt} \geqslant V^{liq}$.

Fix a trajectory $(B_t(\omega))_{0 \leqslant t \leqslant T}$ of (258) as well as $j \in \mathbb{N}$ such that $\tau_j(\omega) < T$. We claim that there is a constant $K = K(\lambda, \delta)$ such that, for every $\tau_j(\omega) \leqslant t \leqslant \tau_{j+1}(\omega) \wedge T$,

$$V^{opt}(\varphi_t(\omega)) \leqslant K V^{opt}(\varphi_{\tau_j}(\omega)). \tag{263}$$

To prove this claim we have to do some rough estimates. Fix t as above. Note that S_t is in the interval $[e^{-\delta}S_{\tau_j}(\omega), e^{\delta}S_{\tau_j}(\omega)]$ as $\tau_j(\omega) \leqslant t \leqslant \tau_{j+1}(\omega) \wedge T$. To fix ideas suppose that $S_t(\omega) = e^{\delta}S_{\tau_j}(\omega)$. We try to determine the trajectory $(\varphi_u)_{\tau_j(\omega) \leqslant u \leqslant t}$ which maximises the value on the left hand side for given $V := V^{opt}(\varphi_{\tau_j}(\omega))$ on the right hand side. As we are only interested in an upper bound we may suppose that the agent is clairvoyant and knows the entire trajectory $(S_u(\omega))_{0 \leqslant u \leqslant T}$.

In the present case where $S_t(\omega)$ is assumed to be at the upper end of the interval $[e^{-\delta}S_{\tau_j}(\omega), e^{\delta}S_{\tau_j}(\omega)]$ the agent who is trying to maximize $V^{opt}(\varphi_t(\omega))$

wants to exploit this up-movement by investing into the stock S as much as possible. But she cannot make $\varphi_u^1 \in \mathbb{R}_+$ arbitrarily large as she is restricted by the admissibility condition $V_u^{liq} \geqslant 0$ which implies that $\varphi_u^0 + \varphi_u^1(1-\lambda)S_u(\omega) \geqslant 0$, for all $\tau_j(\omega) \leqslant u \leqslant t$. As for these u we have $S_u(\omega) \leqslant e^{\delta} S_{\tau_j}(\omega)$ this implies the inequality

$$\varphi_u^0 + \varphi_u^1(1-\lambda)e^{\delta} S_{\tau_j}(\omega) \geqslant 0, \quad \tau_j(\omega) \leqslant u \leqslant t. \tag{264}$$

As regards the starting condition $V^{opt}(\varphi_{\tau_j}(\omega))$ we may assume w.l.o.g. that $\varphi_{\tau_j}(\omega) = (V, 0)$ for some number $V > 0$. Indeed, any other value of $\varphi_{\tau_j}(\omega) = (\varphi_{\tau_j}^0(\omega), \varphi_{\tau_j}^1(\omega))$ with $V^{opt}(\varphi_{\tau_j}(\omega)) = V$ may be reached from $(V, 0)$ by either buying stock at time $\tau_j(\omega)$ at price $S_{\tau_j}(\omega)$ or selling it at price $(1-\lambda)S_{\tau_j}(\omega)$. Hence we face the elementary deterministic optimization problem of finding the trajectory $(\varphi_u^0, \varphi_u^1)_{\tau_j(\omega) \leqslant u \leqslant t}$ starting at $\varphi_{\tau_j}(\omega) = (V, 0)$ and respecting the self-financing condition (155) as well as inequality (264), such that it maximizes $V^{opt}(\varphi_t)$. Keeping in mind that $(1-\lambda) < e^{-2\delta}$, a moment's reflection reveals that the best (clairvoyant) strategy is to wait until the moment $\tau_j(\omega) \leqslant \bar{t} \leqslant t$ when $S_{\bar{t}}(\omega)$ is minimal in the interval $[\tau_j(\omega), t]$, then to buy at time $\bar{t}$ as much stock as is allowed by the inequality (264), and then keeping the positions in bond and stock constant until time t. Assuming the most favourable (limiting) case $S_{\bar{t}}(\omega) = e^{-\delta} S_{\tau_j}(\omega)$, simple algebra gives $\varphi_u = (V, 0)$, for $\tau_j(\omega) \leqslant u < \bar{t}$ and

$$\varphi_u = \Big(V - x, \frac{x e^{\delta}}{S_{\tau_j}(\omega)}\Big), \quad \bar{t} \leqslant u \leqslant t,$$

where

$$x = \frac{V}{1-(1-\lambda)e^{2\delta}}.$$

Using $S_t(\omega) = e^{\delta} S_{\tau_j}(\omega)$ we therefore may estimate in (263)

$$V^{opt}(\varphi_t(\omega)) \leqslant V\Big[\Big(1 - \frac{1}{1-(1-\lambda)e^{2\delta}}\Big) + \frac{e^{2\delta}}{1-(1-\lambda)e^{2\delta}}\Big]. \tag{265}$$

Due to the hypothesis $(1-\lambda)e^{2\delta} < 1$ the term in the bracket is a finite constant K, depending only on λ and δ.

We have assumed a maximal up-movement $S_t(\omega) = e^{\delta} S_{\tau_j}(\omega)$. The case of a maximal down-movement $S_t(\omega) = e^{-\delta} S_{\tau_j}(\omega)$ as well as any intermediate case follow by the same token yielding again the estimate (263) with the same constant given by (265). Observing that $V^{opt} \geqslant V^{liq}$ and $V^{liq}(\widehat{\varphi}_T) = \widehat{\varphi}_T^0$ we obtain inductively (262) thus finishing the proof. ■

We thus have assembled all the ingredients to formulate the main result of these lectures.

Theorem 8.4. *Fix U satisfying* (226)*, the horizon T, and transaction costs $\lambda > 0$. Let $H \in]0,1[$ and $S = (S_t)_{0 \leqslant t \leqslant T}$ exponential fractional Brownian motion as in* (259)*.*

Then there is a shadow price process $\tilde{S} = (\tilde{S}_t)_{0 \leqslant t \leqslant T}$ as defined in Definition 7.1.

Proof. The above discussion has shown that the assumptions of Theorem 7.4 are satisfied. ∎

We now can formulate a consequence of the above results on portfolio optimization which seems remarkable, independently of the above financial applications. It is a general result on the pathwise behaviour of fractional Brownian motion: they may touch Itô processes in a non-trivial way without involving local time or related concepts pertaining to the reflection of a Brownian motion.

Theorem 8.5. *Let $(B_t)_{0 \leqslant t \leqslant T}$ be fractional Brownian motion* (258) *with Hurst index $H \in]0,1[$, and $\alpha > 0$ (which corresponds to $\alpha = -\log(1-\lambda)$ in Theorem 8.4 above).*

There is an Itô process $(X_t)_{0 \leqslant t \leqslant T}$ such that

$$B_t - \alpha \leqslant X_t \leqslant B_t, \qquad 0 \leqslant t \leqslant T, \tag{266}$$

holds true almost surely.

For $\varepsilon > 0$, we may choose $\alpha > 0$ sufficiently small so that the trajectory $(X_t)_{0 \leqslant t \leqslant T}$ touches the trajectories $(B_t)_{0 \leqslant t \leqslant T}$ as well as the trajectories $(B_t - \alpha)_{0 \leqslant t \leqslant T}$ with probability bigger than $1 - \varepsilon$, i.e.

$$\mathbb{P}\left[\min_{0 \leqslant t \leqslant T}(X_t - B_t) = -\alpha, \max_{0 \leqslant t \leqslant T}(X_t - B_t) = 0\right] > 1 - \epsilon. \tag{267}$$

Proof. The theorem is a consequence of Theorem 8.4 where we simply take $X = \log(\tilde{S})$.

We only have to show the last assertion. We follow the argument of Lemma 5.2 in [49]. Fix a (strictly increasing) *bounded* utility function $U : \mathbb{R}_+ \mapsto \mathbb{R}$ as in Def. 6.1, e.g. $U(x) = \frac{x^\gamma}{\gamma}$, for some $\gamma < 0$.

We claim that, for $\lambda = 1 - e^{-\alpha}$ small enough, the optimal strategy $\widehat{\varphi}(1)$ in Theorem 8.4 is non-trivial with probability bigger than $1 - \epsilon$. By the trivial strategy we mean that no trading takes place. In other words we claim that

$$\mathbb{P}[(\widehat{\varphi}_t)_{0 \leqslant t \leqslant T} \not\equiv (1,0)] > 1 - \epsilon. \tag{268}$$

It follows from [205] (or the proof of Theorem 7.2 in [64]) that we may find, for $\varepsilon > 0$ and $M > 0$, a simple predictable process ϑ of the form

$$\vartheta_t = \sum_{i=0}^{N-1} g_i \mathbb{1}_{]\!]\tau_i,\tau_{i+1}]\!]}(t)$$

where $g_i \in L^\infty(\Omega, \mathcal{F}_{\tau_i}, P)$ and $0 = \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_N = T$ are stopping times such that, S^{τ_N} is bounded and

$$(\vartheta \bullet S)_T = \sum_{i=0}^{N} g_i (S_{\tau_{i+1}} - S_{\tau_i}) \tag{269}$$

satisfies $\vartheta \bullet S \geqslant -\epsilon$ almost surely and $\mathbb{P}[(\vartheta \bullet S)_T \geqslant M] > 1 - \varepsilon$.

For $0 < \lambda < 1$, we may interpret ϑ also in the setting of transaction costs. More formally: associate to ϑ a λ-self-financing process $\varphi = (\varphi^0, \varphi^1)$ as above starting at $(\varphi^0_0, \varphi^1_0) = (0, 0)$, such that $\varphi^1 = \vartheta \mathbb{1}_{(0,T)}$ and φ^0 is defined by having equality in (155). Choosing $\lambda > 0$ sufficiently small we obtain $\varphi^0_T \geqslant -2\epsilon$ almost surely as well as $P[\varphi^0_T \geqslant M - 1] > 1 - 2\varepsilon$.

It follows that the value $u(1)$ of (227) increases to $U(\infty) = \lim_{x \nearrow \infty} U(x) < \infty$ as λ goes to zero. This implies that $\widehat{\varphi}^0_T \nearrow \infty$ in probability as λ goes to zero which yields (268).

Fix a trajectory such that $(\widehat{\varphi}_t(1))_{0 \leqslant t \leqslant T} \not\equiv 0$. Then there must be some buying *as well as* some selling of the stock. Indeed $\widehat{\varphi}$ starts at $(1, 0)$ and ends at $(\widehat{\varphi}^0_T(1), 0)$. In view of (256) we obtain (267). ■

Let us comment on the interpretation of the above theorem. Define σ and τ to be the stopping times

$$\sigma = \inf\{t \in [0, T] : X_t = B_t - \alpha\}, \quad \tau = \inf\{t \in [0, T] : X_t = B_t\}.$$

Note that $\{\sigma < \infty\} = \{\tau < \infty\}$ by the preceding argument. For sufficiently small $\alpha > 0$, we have $\mathbb{P}[\sigma < \infty] = \mathbb{P}[\tau < \infty] > 1 - \varepsilon$. We may suppose w.l.o.g. that $\tau < \sigma$ (the case $\sigma < \tau$ is analogous). Consider the difference process

$$D_t = B_t - X_t, \qquad 0 \leqslant t \leqslant T, \tag{270}$$

which, is non-negative and vanishes for $t = \tau$. We formulate a consequence of the above considerations.

Corollary 8.6. *On the set $\{\tau < \sigma\}$ we have that $\sigma \leqslant T$ almost surely, and that the process $(D_t)_{\tau \leqslant t \leqslant \sigma}$ starts at zero, remains non-negative and ends at $D_\sigma = \alpha$.*

This statement should be compared to the well-known fact, that there are *no* stopping times $\tau < \sigma$ such that $P[\tau < T] = P[\sigma \leqslant T] > 0$ and such that $B_\sigma - B_\tau > \alpha$, almost surely on $\{\tau < T\}$. Indeed, this follows from the stickiness property of fractional Brownian motion as proved by P. Guasoni ([102]; compare also [107]). Adding to B the Itô process X somewhat miraculously changes this behaviour of B drastically as formulated in the above corollary.

A Appendix

In chapter 1 we have used a number of elementary results from linear algebra. In particular, this includes the following facts:

- The bipolar set of a closed, convex set in $\mathbb{R}^d$ containing the origin is the set itself.
- A set containing the origin is polyhedral iff its polar is polyhedral.
- The projection of a polyhedral cone is again a polyhedral cone.

For the convenience of the reader we provide proofs and present the underlying theory in a rather self-contained way in this appendix.

Let E be a vector space over the real numbers with finite dimension d and E' its dual. The space E then is isomorphic to $\mathbb{R}^d$ and we will use this fact in some of the discussion below, in which case we will denote the origin by $0 \in \mathbb{R}^d$ and the canonical basis by $\{e_1, \ldots, e_d\}$.

A.1 Polar sets

We start with some basic definitions following [243] and [88]; shorter introductions to the geometry of convex sets can be found in [86] and [93]. For any set $A \subseteq E$, the smallest closed convex set containing A is called the *closed convex hull* of A, i.e. $\overline{\text{conv}}(A)$ is the intersection of all closed convex sets containing A. A closed convex set $C \subseteq E$ is called a *closed convex cone* if $\lambda a \in C$ for every a in C and $\lambda \geqslant 0$. The *closed convex cone generated by a set* $W \subseteq E$ is the closure of the convex cone

$$\text{cone}(W) := \left\{ \sum_{i \in I} \mu_i w_i : \; w_i \in W, \; \mu_i \geqslant 0 \right\},$$

where I is finite. It is the smallest closed convex cone containing W. We define $\text{cone}(\varnothing) := \{0\}$. The following properties of cones can be checked easily:

- Every closed convex cone contains the origin.
- The intersection of two closed convex cones is again a closed convex cone.

For a set $A \subseteq E$ we define the *polar* A° of E as

$$A^\circ := \{y \in E' : \langle x, y \rangle \leqslant 1, \text{ for all } x \in A\}.$$

If A is a cone, we may equivalently define A° as

$$A^\circ = \{y \in E' : \langle x, y\rangle \leqslant 0, \text{ for all } x \in A\}.$$

If A is a linear space, we even may equivalently define A° as the annihilator

$$A^\circ = \{y \in E' : \langle x, y\rangle = 0, \text{ for all } x \in A\}.$$

The *Minkowski sum* of two sets $A, B \subseteq E$ is defined as the set

$$A + B := \{a + b,\ a \in A,\ b \in B\}.$$

It is easy to verify that, for any two sets $A \subseteq B \subseteq E$, we have $A^\circ \supseteq B^\circ$. If $C_1, C_2 \subseteq E$ are cones, then $(C_1 + C_2)^\circ = C_1^\circ \cap C_2^\circ$. Note that the polar of a cone is a closed convex cone.

The following theorem is a version of the celebrated Hahn-Banach theorem. The proof presented here can be found in [220]; for a more general discussion see for example [202].

Proposition A.1 (Bipolar Theorem)**.** *For a set $A \subseteq E$ the* bipolar $A^{\circ\circ} = (A^\circ)^\circ$ *equals the closed convex hull of $A \cup \{0\}$.*

Proof. Let $B = \operatorname{conv}(A \cup \{0\})$. Since $B \supseteq A$ we have $B^\circ \subseteq A^\circ$.

On the other hand, let $y \in A^\circ$ and $M \in \mathbb{N}$ and pick $\lambda_i \in [0,1]$, for $1 \leqslant i \leqslant M$, such that $\sum_{i=1}^M \lambda_i = 1$. Then we have, for any $a_i \in A \cup \{0\}$:

$$1 \geqslant \sum_{i=1}^M \lambda_i \langle y, a_i\rangle = \sum_{i=1}^M \langle y, \lambda_i a_i\rangle = \langle \sum_{i=1}^M \lambda_i a_i, y\rangle.$$

Every $x \in B$ can be written as $x = \sum_{i=1}^M \lambda_i a_i$. It follows that $B^\circ \supseteq A^\circ$ and hence $A^\circ = B^\circ$.

We will now prove that $B^{\circ\circ} = \overline{B}$ which finishes the proof. Let $x \in \overline{B}$. For any $y \in B^\circ$ we have $\langle x, y\rangle \leqslant 1$ by definition and continuity, from which it follows that $x \in B^{\circ\circ}$ and therefore $\overline{B} \subseteq B^{\circ\circ}$. Conversely, assume $x_1 \notin \overline{B}$. Then there exists an $y \in E'$ and a constant c such that $\langle x, y\rangle \leqslant c$, for $x \in B$, and $\langle x_1, y\rangle > c$ (this follows from the Hahn-Banach theorem in its version as separating hyperplanes theorem, see for example [220]).

Because $0 \in B$ we have $c \geqslant 0$. We can even assume $c > 0$. It follows that $\langle x, y/c\rangle \leqslant 1$, for $x \in B$, and thus $y/c \in B^\circ$. But from $\langle x_1, y/c\rangle > 1$ we see that $x_1 \notin B^{\circ\circ}$. ■

Corollary A.2. *If $C \subseteq E$ is a closed convex cone then $C^{\circ\circ} = C$.*

A.2 Polyhedral sets

We will now introduce the concept of *polyhedral sets*, which can be defined in two distinct ways. The first definition builds a polyhedron "from inside": Let V and W be two finite sets in E. The Minkowski sum of $\mathrm{conv}(V)$ and the cone generated by W

$$P = \mathrm{conv}(V) + \mathrm{cone}(W)$$

is called a *V-polyhedron*, where the name comes from the fact that such a polyhedron is defined using its vertices. Note that P is *closed*.

Polyhedral sets can also be built "from outside". A set $P \subseteq E$ is called an *H-polyhedron*, if it can be expressed as the finite intersection of closed halfspaces, that is

$$P = \bigcap_{i=1}^{N} \{x \in E : \langle x, y_i \rangle \leqslant c_i\},$$

for some elements $y_i \in E'$, and some constants c_i, $i \in \{1, \dots N\}$. As a subset of $\mathbb{R}^d$ such a polyhedron can be written as

$$P = P(A, z) := \left\{x \in \mathbb{R}^d : Ax \leqslant z\right\} \qquad \text{for some } A \in \mathbb{R}^{N \times d}, z \in \mathbb{R}^N.$$

Note that an H-polyhedron with all $c_i = 0$, i.e. of the form $P(A, 0)$, is in fact a closed convex cone: we shall encounter such *polyhedral cones* quite often.

These two distinct characterizations for polyhedral sets are useful for calculations and will play an important part in the following discussion. As we will verify below, the notions of V- and H-polyhedral sets are equivalent.

Our first Lemma deals with the projection of H-cones. The proof and a more thorough discussion can be found in [243].

Proposition A.3. *A projection of an H-cone along any coordinate directions e_k, $1 \leqslant k \leqslant d$, is again an H-cone. More specifically, if C is an H-cone in $\mathbb{R}^d$, then so is its elimination cone $\mathrm{elim}_k(C) := \{x + \mu e_k : x \in C,\ \mu \in \mathbb{R}\}$ and its projection cone $\mathrm{proj}_k(C) := \mathrm{elim}_k(C) \cap \{x \in \mathbb{R}^d : \langle x, e_k \rangle = 0\}$.*

Proof. Note that it suffices to show that the set $\mathrm{elim}_k(C)$ is an H-cone, for any k, because the projection cone is the intersection of the elimination cone with the two halfspaces $\{x \in \mathbb{R}^d : \langle x, e_k \rangle \leqslant 0\}$ and $\{x \in \mathbb{R}^d : \langle x, -e_k \rangle \leqslant 0\}$.

Suppose that $C = P(A, 0)$ and $a_1, a_2, \dots, a_N$ are the row vectors of A. We will construct a new matrix A^k such that $\mathrm{elim}_k(C) = P(A^k, 0)$.
Claim: $A^k = \{a_i : a_{ik} = 0\} \cup \{a_{ik} a_j - a_{jk} a_i : a_{ik} > 0, a_{jk} < 0\}$
If $x \in C$ then $Ax \leqslant 0$. But then we also have $A^k x \leqslant 0$, because A^k consists of nonnegative linear combinations of rows of A. Therefore $C \subseteq P(A^k, 0)$. As

the k^{th} component of A^k is zero by construction, we even have $\text{elim}_k(C) \subseteq P(A^k, 0)$.

On the other hand, let $x \in P(A^k, 0)$. We want to show that there is a $\mu \in \mathbb{R}$ such that $x - \mu e_k \in C$, i.e. $A(x - \mu e_k) \leqslant 0$. Writing these equations out, we obtain the inequalities $a_j x - a_{jk}\mu \leqslant 0$, or

$$\begin{array}{ll} \mu \geqslant \frac{a_i x}{a_{ik}}, & \text{if } a_{ik} > 0, \\ \mu \leqslant \frac{a_j x}{a_{jk}}, & \text{if } a_{jk} < 0. \end{array}$$

Such a μ exists, because if $a_{ik} > 0$ and $a_{jk} < 0$, then $(a_{ik}a_j - a_{jk}a_i)x \leqslant 0$, since $x \in P(A^k, 0)$, which can be written as

$$\frac{a_i x}{a_{ik}} \leqslant \frac{a_j x}{a_{jk}}.$$

It follows that $P(A^k, 0) \subseteq \text{elim}_k(C)$, finishing the proof. ∎

Proposition A.4. *Every V-polyhedron is an H-polyhedron and vice versa.*

We split the proposition into two claims for the two directions, which we prove independently.

Claim: Every V-polyhedron is an H-polyhedron.

Remark A.5. Proving the claim directly turns out to be rather tedious, due to the difficulty of manipulating the necessary sets. There is, however, an elegant proof using *homogenization*: Every polyhedron in d-dimensional space can be regarded as a polyhedral cone in dimension $d + 1$. The equivalence between V-cones and H-cones is easier to show. The direct proof uses Fourier-Motzkin elimination to calculate the sets explicitly. It can be found, together with the indirect proof given here, in [243].

Proof. By mapping a point $x \in \mathbb{R}^d$ to $\left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right) \in \mathbb{R}^{d+1}$ we associate with every polyhedral set P in $\mathbb{R}^d$ a cone in $\mathbb{R}^{d+1}$ in the following way: If $P = P(A, z)$ is a H-polyhedral set, define

$$C(P) := P\left(\left(\begin{smallmatrix} -1 & 0 \\ -z & A \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right).$$

Conversely, if $C \in \mathbb{R}^{d+1}$ is an arbitrary H-cone, then $\{x \in \mathbb{R}^d : \left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right) \in C\}$ is a (possibly empty) H-polyhedral set.

On the other hand, if $P = \text{conv}(V) + \text{cone}(W)$ is a V-polyhedral set for some finite sets V and W, we define

$$C(P) := \text{cone}\left(\begin{smallmatrix} 1 & 0 \\ V & W \end{smallmatrix}\right),$$

that is, we add a zeroth coordinate to the vectors in V and W before generating the cone, namely 1 and 0, respectively. As before, a straightforward calculation shows that if C is a V-cone in $\mathbb{R}^{d+1}$, then $\{x \in \mathbb{R}^d : \binom{1}{x} \in C\}$ is a V-polyhedral set in $\mathbb{R}^d$.

If we can now show that every V-cone is an H-cone we are done, since every V-polyhedral set in $\mathbb{R}^d$ can be identified with a V-cone in $\mathbb{R}^{d+1}$ and the H-cone then translated back to the H-polyhedral set. Consider thus a V-cone, which can be written as

$$C = \left\{ x \in \mathbb{R}^d : \exists \lambda_i \geqslant 0 : x = \sum_i \lambda_i w_i,\ w_i \in W \right\},$$

or equivalently as

$$C = \left\{ (x, \lambda) \in \mathbb{R}^{d+n} : \lambda_i \geqslant 0, x = \sum_i \lambda_i w_i,\ w_i \in W \right\},$$

the latter set being an H-cone in $\mathbb{R}^{d+n}$. By successsively projecting the cone onto the hyperplanes for which the k^{th} coordinate equals zero, for $d < k \leqslant d + n$, we obtain a cone in $\mathbb{R}^d$ since we already showed that such a projection of an H-cone is again an H-cone. This finishes the proof of the claim. ■

The second part of the equivalence can also be shown directly or via homogenization, but we will give a third proof, which makes use of an elegant induction argument. For a thorough discussion of these concepts (and the proof of the following claim) see also [88].

Claim: Every H-polyhedron is a V-polyhedron.

Proof. Let P be an intersection of finitely many closed halfspaces in $\mathbb{R}^d$. We may assume w.l.g. that the dimension of P is d and will prove the claim by induction on d. If $d = 1$, then P is a halfline or a closed interval and the claim is clear. For $d \geqslant 2$ we will show that every point in P can be represented as the convex combination $a = (1 - \lambda)b + \lambda c,\ \ 0 \leqslant \lambda \leqslant 1$, where b and c belong to two distinct facets F and G of P respectively, i.e.

$$F = \operatorname{conv}(V_F) + \operatorname{cone}(W_F) \quad \text{and} \quad G = \operatorname{conv}(V_G) + \operatorname{cone}(W_G),$$

for some finite sets V_F, W_F, V_G, W_G. This suffices to prove the claim since the Minkowski sum of two V-polyhedral sets is again a V-polyhedral set.

Since every facet has dimension $d - 1$, we know that the boundaries of P are polyhedral sets. Let a be any point in the interior of P. Then there is some line l through a that intersects two facets of P, which is not parallel to any of the generating hyperplanes and intersects them in distinct points. Since a must lie between two such intersection points it is the linear combination of finitely many elements of V-polyhedral sets and because a was an arbitrary point in the interior of P, it follows that P itself is V-polyhedral. ■

The next proof can also be found in [88], along with other constructive results regarding polyhedra.

Proposition A.6. *Let $A \subseteq E$ be a polyhedral set. Then its polar A° also is a polyhedral set.*

Proof. We show that the polar of a V-polyhedron is an H-polyhedron, which we calculate explicitly. Let therefore A be of the form

$$A = \mathrm{conv}(V) + \mathrm{cone}(W) = \mathrm{conv}(\{v_1, \ldots, v_N\}) + \mathrm{cone}(\{w_1, \ldots, w_K\}),$$

for some finite sets V and W. By definition, we have

$$\begin{aligned} A^\circ &= \left\{ y \in E' : \left\langle \sum_{i=1}^{N} \lambda_i v_i + \sum_{j=1}^{K} \mu_j w_j, y \right\rangle \leqslant 1, \lambda_i \geqslant 0, \mu_j \geqslant 0, \sum \lambda_i = 1 \right\} \\ &= \left\{ y \in E' : \sum_i \lambda_i \langle v_i, y \rangle + \sum_j \mu_j \langle w_j, y \rangle \leqslant 1, \lambda_i \geqslant 0, \mu_j \geqslant 0, \sum \lambda_i = 1 \right\}. \end{aligned}$$

We therefore find that

$$A^\circ = \bigcap_{i=1}^{N} \{y \in E' : \langle v_i, y \rangle \leqslant 1\} \cap \bigcap_{j=1}^{K} \{y \in E' : \langle w_j, y \rangle \leqslant 0\},$$

which is an H-polyhedron. ■

Corollary A.7. *A convex, closed set containing the origin is polyhedral iff its polar is so too.*

Proof. This follows immediately from the previous proposition and the bipolar theorem, since then A° is polyhedral and $A = A^{\circ\circ}$. ■

A.3 The Legendre Transformation

Definition A.8. *Let $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a concave upper semi-continuous function and $D = \mathrm{int}\{u > -\infty\} \neq \varnothing$ its domain, which we assume to be non-empty. The* conjugate *v of u is the function*

$$v(y) := \sup\{u(x) - xy,\ x \in \mathbb{R}\}.$$

The function v is the Legendre transform of $-u(-x)$ and is therefore convex rather than concave.[4] Given the conjugate function v, the original function u can be recovered via the transformation

$$u(x) := \inf\{v(y) + xy,\ y \in \mathbb{R}\}.$$

From these definitions it is immediately clear that for every $(x, y) \in \mathbb{R}^2$ we have *Fenchel's inequality*:

$$u(x) - v(y) \leqslant xy. \tag{271}$$

Note that equality holds when the supremum (respectively the infimum) in the above definitions is attained for the corresponding values of x and y.

Definition A.9. *The* subdifferential $\partial v(y_0)$ *of a convex function* v *at* y_0 *is the set of* $x \in \mathbb{R}$ *such that*

$$v(y) \geqslant v(y_0) + x \cdot (y - y_0), \quad \textit{for all } y \in \mathbb{R}.$$

For a concave function u *we define the* superdifferential $\partial u(x_0)$ *of at* x_0 *equivalently as the set of* $y \in \mathbb{R}$ *satisfying*

$$u(x) \leqslant u(x_0) + y \cdot (x - x_0), \quad \textit{for all } x \in \mathbb{R}.$$

If $\partial u(x_0)$ *consists of one single element* y*, then* u *is* differentiable at x_0 *and* $\nabla u(x_0) = y$*. Equivalently if* $\partial v(y_0)$ *consists of one single element* x*, then* v *is* differentiable at y_0 *and* $\nabla v(y_0) = x$*.*

Our first duality result links the super- and subdifferential of the conjugate functions u and v:

Proposition A.10. *The superdifferential* $\partial u(x_0)$ *contains* y_0 *iff* $-x_0 \in \partial v(y_0)$*.*

Proof. Let y_0 be in $\partial u(x_0)$. Then we have, for every x,

$$\begin{aligned} u(x) &\leqslant u(x_0) + y_0(x - x_0) \\ u(x) - y_0 x &\leqslant u(x_0) - y_0 x_0. \end{aligned}$$

Since this also holds for the supremum and using Fenchel's inequality on the right hand side, we obtain for every y in $\mathbb{R}$

$$\begin{aligned} v(y_0) &\leqslant u(x_0) - x_0 y_0 \leqslant v(y) + x_0 y - x_0 y_0 \\ v(y_0) &\leqslant v(y) + x_0(y - y_0), \end{aligned}$$

which is exactly the requirement for $-x_0$ to be in the subdifferential $\partial v(y_0)$. The other direction can be proved analogously. ■

[4] In fact, the classical duality theory considers the (convex) function $-u(-x)$ to obtain a perfectly symmetric setting.

There is an important duality regarding the smoothness and the strict concavity of the dual functions u and v. The following proof can be found in [117].

Proposition A.11. *The following are equivalent:*

1. $u : D \to \mathbb{R}$ *is strictly concave.*

2. v *is differentiable on the interior of its domain.*

Proof. $(i) \Rightarrow (ii)$. Suppose that (ii) fails, i.e. there is some y such that $\partial v(y)$ contains two distinct points, and call them $-x_1$ and $-x_2$. We may suppose that $x_1 < x_2$. This is equivalent to the requirement that $y \in \partial u(x_1) \cap \partial u(x_2)$. For $i = 1, 2$ we have

$$u(x_i) - v(y) = x_i y$$

and using Fenchel's inequality, we get (for $0 \leqslant \lambda \leqslant 1$):

$$\lambda u(x_1) + (1-\lambda)u(x_2) - v(y) = y \cdot (\lambda x_1 + (1-\lambda)x_2) \tag{272}$$

$$\geqslant u(\lambda x_1 + (1-\lambda)x_2) - v(y). \tag{273}$$

But this implies that u is affine on $[x_1, x_2]$, a contradiction since u is strictly concave. Therefore $\partial v(y)$ must be single-valued for all $y \in \text{int dom }(v)$, i.e. v is continuously differentiable.

$(ii) \Rightarrow (i)$. Suppose that there are two distinct points x_1 and x_2 such that u is affine on the line segment $[x_1, x_2]$. If we set $x := \frac{1}{2}(x_1 + x_2)$, there is an y such that $\nabla v(y) = x$, i.e. $y \in \partial u(x)$. We can write

$$0 = u(x) - v(y) - xy = \frac{1}{2}\sum_{i=1}^{2}\left[u(x_i) - v(y) - yx_i\right],$$

which implies (using Fenchel's inequality), that each of the terms in the bracket on the right hand side must vanish. We therefore have $y \in \partial u(x_1) \cap \partial u(x_2)$, i.e. $\partial v(y)$ contains more than one point, which contradicts the assumption that v is differentiable. ■

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