AN EXTENSION OF A SUPERCONGRUENCE OF LONG AND RAMAKRISHNA

VICTOR J. W. GUO, JI-CAI LIU, AND MICHAEL J. SCHLOSSER

Abstract. We prove two supercongruences for specific truncated hypergeometric series. These include a uniparametric extension of a supercongruence that was recently established by Long and Ramakrishna. Our proofs involve special instances of various hypergeometric identities including Whipple's transformation and the Karlsson–Minton summation.

1. Introduction

Let $(a)_n = a(a+1)\cdots(a+n-1)$ denote the Pochhammer symbol. For complex numbers $a_0, a_1, \ldots, a_r$ and $b_1, \ldots, b_r$, the (generalized) hypergeometric series ${}_{r+1}F_r$ is defined as

$$
{}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_r \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_r)_k} z^k.
$$

Summation and transformation formulas for generalized hypergeometric series play an important part in the investigation of supercongruences. See, for instance, [5, 10, 12, 13, 16–18, 20]. In particular, Long and Ramakrishna [16, Theorems 3 and 2] proved the following two supercongruences:

$$
\sum_{k=0}^{p-1} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \tag{1.1}
$$

and

$$
\sum_{k=0}^{p-1} (6k+1) \frac{\left(\frac{1}{3}\right)_k^6}{k!^6} \equiv \begin{cases} -p\,\Gamma_p\left(\frac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p \equiv 1 \pmod{6}, \\ -\frac{10}{27}p^4\,\Gamma_p\left(\frac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p \equiv 5 \pmod{6}, \end{cases} \tag{1.2}
$$

where p is an odd prime and $\Gamma_p(x)$ is the p-adic Gamma function. The restriction of the supercongruence (1.1) modulo p^2 was earlier established by Van Hamme [19, Equation (H.2)]. The supercongruence (1.2) is even stronger than a conjecture made by Van Hamme

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2010 *Mathematics Subject Classification.* Primary 33C20; Secondary 11A07, 11B65.

Key words and phrases. hypergeometric series; supercongruences; p-adic Gamma function; Whipple's transformation, Karlsson–Minton's summation.

The second author was partially supported by the National Natural Science Foundation of China (grant 12171370).

The third author was partially supported by FWF Austrian Science Fund grant P 32305.

[19, Equation (D.2)] who asserted the corresponding supercongruence modulo p^4 for $p\equiv 1 \pmod 6$. Long and Ramakrishna also mentioned that (1.2) does not hold modulo p^7 in general.

The first purpose of this paper is to prove the following supercongruence. Note that the $r=\pm 1$ cases partially confirm the $d=5$ and $q\to 1$ case of [6, Conjectures 1 and 2].

Theorem 1. *Let $r\leqslant 1$ be an odd integer coprime with 5. Let p be a prime such that $p\equiv -\frac{r}{2} \pmod 5$ and $p\geqslant \frac{5-r}{2}$. Then*

$$\sum_{k=0}^{p-1}(10k+r)\frac{\left(\frac{r}{5}\right)_k^5}{k!^5}\equiv 0 \pmod{p^4}, \tag{1.3}$$

Recently, the second author [14] established the following supercongruence related to (1.2):

$$\sum_{k=0}^{p-1}(6k-1)\frac{\left(-\frac{1}{3}\right)_k^6}{k!^6}\equiv \begin{cases} 140p^4\,\Gamma_p\left(\frac{2}{3}\right)^9 \pmod{p^5}, & \text{if } p\equiv 1 \pmod 6,\\[2ex] 378p\,\Gamma_p\left(\frac{2}{3}\right)^9 \pmod{p^5}, & \text{if } p\equiv 5 \pmod 6, \end{cases} \tag{1.4}$$

where p is a prime.

The second purpose of this paper is to give the following common generalization of the second supercongruence in (1.2), restricted to modulo p^5, and the first supercongruence in (1.4).

Theorem 2. *Let $r\leqslant 1$ be an integer coprime with 3. Let p be a prime such that $p\equiv -r \pmod 3$ and $p\geqslant 3-r$. Then*

$$\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6}\equiv \frac{(-1)^{r+1}\,80rp^4}{81}\cdot\frac{\Gamma_p\left(1+\frac{r}{3}\right)^2}{\Gamma_p\left(1+\frac{2r}{3}\right)^3\Gamma_p\left(1-\frac{r}{3}\right)^4} \times\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3} \pmod{p^5}. \tag{1.5}$$

Letting $r=1$ and $r=-1$ in (1.5) and using (1.9) and (1.11), we arrive at the $p\equiv 5 \pmod 6$ case of (1.2) modulo p^5 and the $p\equiv 1 \pmod 6$ case of (1.4), respectively.

Our proof of Theorem 1 will require Whipple's well-poised ${}_7F_6$ transformation formula (see [2, p. 28]):

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & -n \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix};1\right] \\ &= \frac{(a+1)_n(a-d-e+1)_n}{(a-d+1)_n(a-e+1)_n}\,{}_4F_3\left[\begin{matrix} 1+a-b-c, & d, & e, & -n \\ d+e-a-n, & 1+a-b, & 1+a-c \end{matrix};1\right], \end{aligned} \tag{1.6}$$

where n is a non-negative integer, and Karlsson–Minton's summation formula (see, for example, [4, Equation (1.9.2)]):

$$ {}_{r+1}F_r\left[\begin{matrix} -n, & b_1+m_1, & \ldots, & b_r+m_r \\ & b_1, & \ldots, & b_r \end{matrix} ;1\right]=0, \tag{1.7}$$

where $n, m_1, \ldots, m_r$ are non-negative integers and $n > m_1 + \cdots + m_r$. Our proof of Theorem 2 relies on a ${}_7F_6$ transformation formula slightly different from Whipple's ${}_7F_6$ transformation formula (1.6), obtained as a result from combining (1.6) with a ${}_4F_3$ transformation formula. The transformation was already utilized by the second author to prove (1.4).

Furthermore, in order to prove Theorem 2, we require some properties of the p-adic Gamma function, collected in the following two lemmas.

Lemma 1. [3, Section 11.6] *Suppose p is an odd prime and $x \in \mathbb{Z}_p$. Then*

$$\Gamma_p(0)=1, \quad \Gamma_p(1)=-1, \tag{1.8}$$

$$\Gamma_p(x)\Gamma_p(1-x)=(-1)^{a_p(x)}, \tag{1.9}$$

$$\Gamma_p(x)\equiv\Gamma_p(y) \pmod p \quad \textit{for } x\equiv y \pmod p, \tag{1.10}$$

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)}=\begin{cases}-x & \textit{if } v_p(x)=0,\\ -1 & \textit{if } v_p(x)>0,\end{cases} \tag{1.11}$$

where $a_p(x) \in \{1,2,\ldots,p\}$ with $x \equiv a_p(x) \pmod p$ and $v_p(\cdot)$ denotes the p-order.

Lemma 2. [16, Lemma 17, (4)] *Let p be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a+1, \ldots, a+n-1$ are in $p\mathbb{Z}_p$, then*

$$(a)_n=(-1)^n\frac{\Gamma_p(a+n)}{\Gamma_p(a)}. \tag{1.12}$$

In the following Sections 2 and 3, we give proofs of Theorems 1 and 2, respectively. The final Section 4 is devoted to a discussion and includes two conjectures.

2. Proof of Theorem 1

Motivated by the work of McCarthy and Osburn [17] and Mortenson [18], we take the following choice of parameters in (1.6). Let $a=\frac{r}{5}$, $b=\frac{r+5}{10}$, $c=\frac{r+3p}{5}$, $d=\frac{r+3ip}{5}$, $e=\frac{r-3ip}{5}$, and $n=\frac{3p-r}{5}$, where $i^2=-1$. Then we conclude that

$$ {}_6F_5\left[\begin{matrix} \frac{r}{5}, & 1+\frac{r}{10}, & \frac{r+3p}{5}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5} \\ & \frac{r}{10}, & 1-\frac{3p}{5}, & 1-\frac{3ip}{5}, & 1+\frac{3ip}{5}, & 1+\frac{3p}{5} \end{matrix};1\right]$$

$$=\frac{\left(1+\frac{r}{5}\right)_{\frac{3p-r}{5}}\left(1-\frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1-\frac{3ip}{5}\right)_{\frac{3p-r}{5}}\left(1+\frac{3ip}{5}\right)_{\frac{3p-r}{5}}}\,{}_4F_3\left[\begin{matrix} \frac{5-r-6p}{10}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix};1\right], \tag{2.1}$$

It is easy to see that, for $k \geqslant 0$ and any p-adic integer b,

$$(a+bp)_k\,(a-bp)_k\,(a+bip)_k\,(a-bip)_k \equiv (a)_k^4 \pmod{p^4}. \tag{2.2}$$

Hence, the left-hand side of (2.1) is congruent to

$$\begin{aligned}\sum_{k=0}^{\frac{3p-r}{5}} \frac{\left(1+\frac{r}{10}\right)_k \left(\frac{r}{5}\right)_k^5}{\left(\frac{r}{10}\right)_k (1)_k^5} &= \frac{1}{r}\sum_{k=0}^{\frac{3p-r}{5}}(10k+r)\frac{\left(\frac{r}{5}\right)_k^5}{k!^5} \\ &\equiv \frac{1}{r}\sum_{k=0}^{p-1}(10k+r)\frac{\left(\frac{r}{5}\right)_k^5}{k!^5} \pmod{p^4},\end{aligned}$$

where we have used the fact that $\frac{\left(\frac{r}{5}\right)_k}{k!} \equiv 0 \pmod{p}$ for $\frac{3p-r}{5} < k \leqslant p-1$ (the condition $p \geqslant \frac{5-r}{2}$ in the theorem is to guarantee $\frac{3p-r}{5} \leqslant p-1$). Since $\frac{3p-r}{5} \geqslant \frac{2p+r}{5}$, we have

$$\begin{aligned}\frac{\left(1+\frac{r}{5}\right)_{\frac{3p-r}{5}} \left(1-\frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1-\frac{3ip}{5}\right)_{\frac{3p-r}{5}} \left(1+\frac{3ip}{5}\right)_{\frac{3p-r}{5}}} &= \frac{\left(1+\frac{r}{5}\right)_{\frac{3p-r}{5}} \left(1-\frac{r}{5}\right)_{\frac{3p-r}{5}}}{\prod_{j=1}^{\frac{3p-r}{5}}\left(j^2+\frac{9p^2}{25}\right)} \\ &\equiv 0 \pmod{p^2}.\end{aligned}$$

Finally, by the congruences

$$(a+bip)_k\,(a-bip)_k \equiv (a+bp)_k\,(a-bp)_k \equiv (a)_k^2 \pmod{p^2} \tag{2.3}$$

for any p-adic integer b, we obtain

$$\begin{aligned}{}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix};1\right] &\equiv {}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r}{5}, & \frac{r}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix};1\right] \\ &\equiv {}_4F_3\left[\begin{matrix}\frac{5-r-6p}{10}, & \frac{r+p}{5}, & \frac{r-p}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix};1\right] \\ &= 0 \pmod{p^2},\end{aligned}$$

where we have utilized Karlsson–Minton's summation (1.7) with $n = \frac{3p-r}{5}$, $b_1 = \frac{2r-3p}{5}$, $b_2 = \frac{r+5}{10}$, $b_3 = \frac{5-3p}{5}$, $m_1 = \frac{1-r}{2}$, $m_2 = \frac{2p+r-5}{10}$, and $m_3 = \frac{2p+r-5}{5}$ in the last step.

3. Proof of Theorem 2

We can verify (1.5) for $r=1$ and $p=2$ by hand. In what follows, we assume that p is an odd prime. Recall the following transformation formula [14, Equation (4.2)]:

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} t, & 1+\frac{t}{2}, & -n, & t-a, & t-b, & t-c, & 1-t-m+n+a+b+c \\ & \frac{t}{2}, & 1+t+n, & 1+a, & 1+b, & 1+c, & 2t+m-n-a-b-c \end{matrix}; 1\right] \\
&= \frac{(1+t)_n(a+b+2-m-t)_n(a+c+2-m-t)_n(b+c+2-m-t)_n}{(1+a)_n(1+b)_n(1+c)_n(a+b+c+1-m-2t)_n} \\
&\quad \times \frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)} \\
&\quad \times {}_4F_3\left[\begin{matrix} -m, & -n, & a+b+c+1-m-2t, & a+b+c+1+n-m-t \\ a+b+1-m-t, & a+c+1-m-t, & b+c+1-m-t \end{matrix}; 1\right]. \qquad (3.1)
\end{aligned}$$

Let ζ be a fifth primitive root of unity. Setting $m=1-r$, $t=\frac{r}{3}$, $n=\frac{2p-r}{3}$, $a=\frac{2p\zeta}{3}$, $b=\frac{2p\zeta^2}{3}$ and $c=\frac{2p\zeta^3}{3}$ in (3.1) and using $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$, the left-hand side of (3.1) becomes

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} 1+\frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3} \\ & \frac{r}{6}, & 1+\frac{2p}{3}, & 1+\frac{2p\zeta}{3}, & 1+\frac{2p\zeta^2}{3}, & 1+\frac{2p\zeta^3}{3}, & 1+\frac{2p\zeta^4}{3} \end{matrix}; 1\right] \\
&\equiv \frac{1}{r}\sum_{k=0}^{\frac{2p-r}{3}} (6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^5},
\end{aligned}$$

where we have used the facts that none of the denominators in ${}_7F_6$ contain a multiple of p (the condition $p \geqslant 3-r$ in the theorem is to guarantee $\frac{2p-r}{3} \leqslant p-1$) and

$$(u+vp)_k\,(u+vp\zeta)_k\left(u+vp\zeta^2\right)_k\left(u+vp\zeta^3\right)_k\left(u+vp\zeta^4\right)_k \equiv (u)_k^5 \pmod{p^5}.$$

Furthermore, for $\frac{2p-r}{3} < k \leqslant p-1$ we have $\left(\frac{r}{3}\right)_k \equiv 0 \pmod{p}$. Thus,

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} 1+\frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3} \\ & \frac{r}{6}, & 1+\frac{2p}{3}, & 1+\frac{2p\zeta}{3}, & 1+\frac{2p\zeta^2}{3}, & 1+\frac{2p\zeta^3}{3}, & 1+\frac{2p\zeta^4}{3} \end{matrix}; 1\right] \\
&\equiv \frac{1}{r}\sum_{k=0}^{p-1} (6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^5}. \qquad (3.2)
\end{aligned}$$

On the other hand, we determine the terminating hypergeometric series on the right-hand side of (3.1) modulo p:

$$\frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)}$$
$$\times {}_4F_3\left[\begin{matrix}-m, & -n, & a+b+c+1-m-2t, & a+b+c+1+n-m-t\\ a+b+1-m-t, & a+c+1-m-t, & & b+c+1-m-t\end{matrix};1\right]$$
$$\equiv 8\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3} \pmod{p}. \tag{3.3}$$

Moreover,

$$\frac{(1+t)_n(a+b+2-m-t)_n(a+c+2-m-t)_n(b+c+2-m-t)_n}{(1+a)_n(1+b)_n(1+c)_n(a+b+c+1-m-2t)_n}$$
$$=\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}}{(-1)^{\frac{2p-r}{3}}\left(1+\frac{2p\zeta}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^2}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^3}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^4}{3}\right)_{\frac{2p-r}{3}}}. \tag{3.4}$$

Note that

$$\left(1+\frac{r}{3}\right)_{\frac{2p-r}{3}}=\frac{2p}{3}\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}, \tag{3.5}$$

and

$$\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}$$
$$=\frac{5p^3}{27}\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{p-2r-3}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{p-2r-3}{3}}$$
$$\times\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{p-2r-3}{3}}\left(\frac{3+p(2\zeta+2\zeta^2+1)}{3}\right)_{\frac{p+r}{3}}$$
$$\times\left(\frac{3+p(2\zeta+2\zeta^3+1)}{3}\right)_{\frac{p+r}{3}}\left(\frac{3+p(2\zeta^2+2\zeta^3+1)}{3}\right)_{\frac{p+r}{3}}. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at

$$\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}}{(-1)^{\frac{2p-r}{3}}\left(1+\frac{2p\zeta}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^2}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^3}{3}\right)_{\frac{2p-r}{3}}\left(1+\frac{2p\zeta^4}{3}\right)_{\frac{2p-r}{3}}}$$

$$\equiv \frac{(-1)^{\frac{2p-r}{3}}10p^4}{81}\cdot\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}\left(1+\frac{2r}{3}\right)^3_{\frac{p-2r-3}{3}}(1)^3_{\frac{p+r}{3}}}{(1)^4_{\frac{2p-r}{3}}} \pmod{p^5}. \tag{3.7}$$

It follows from (3.2)–(3.4) and (3.7) that

$$\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \equiv \frac{(-1)^{\frac{2p-r}{3}}80rp^4}{81}\cdot\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}\left(1+\frac{2r}{3}\right)^3_{\frac{p-2r-3}{3}}(1)^3_{\frac{p+r}{3}}}{(1)^4_{\frac{2p-r}{3}}}$$
$$\times\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3} \pmod{p^5}. \tag{3.8}$$

By Lemmas 1 and 2, we have

$$\frac{\left(1+\frac{r}{3}\right)_{\frac{2p-r-3}{3}}\left(1+\frac{2r}{3}\right)^3_{\frac{p-2r-3}{3}}(1)^3_{\frac{p+r}{3}}}{(1)^4_{\frac{2p-r}{3}}}$$

$$\overset{(1.12)}{=}\frac{(-1)^{\frac{2p-r}{3}+r}\Gamma_p\left(\frac{2p}{3}\right)\Gamma_p\left(\frac{p}{3}\right)^3\Gamma_p\left(1+\frac{p+r}{3}\right)^3\Gamma_p(1)}{\Gamma_p\left(1+\frac{r}{3}\right)\Gamma_p\left(1+\frac{2r}{3}\right)^3\Gamma_p\left(1+\frac{2p-r}{3}\right)^4}$$

$$\overset{(1.10)}{\equiv}\frac{(-1)^{\frac{2p-r}{3}+r}\Gamma_p(0)^4\,\Gamma_p\left(1+\frac{r}{3}\right)^2\Gamma_p(1)}{\Gamma_p\left(1+\frac{2r}{3}\right)^3\Gamma_p\left(1-\frac{r}{3}\right)^4} \pmod{p}$$

$$\overset{(1.8)}{=}\frac{(-1)^{\frac{2p-r}{3}+r+1}\Gamma_p\left(1+\frac{r}{3}\right)^2}{\Gamma_p\left(1+\frac{2r}{3}\right)^3\Gamma_p\left(1-\frac{r}{3}\right)^4}. \tag{3.9}$$

The proof of (1.5) then follows from (3.8) and (3.9).

4. Discussion

We know that many supercongruences have nice q-analogues (see [1, 6–9, 11, 15]). For example, we have the following conjectural q-analogue of (1.3): for the same p and r as in Theorem 1,

$$\sum_{k=0}^{p-1}[10k+r]\frac{(q^r;q^5)_k^5}{(q^5;q^5)_k^5}q^{\frac{5(3-r)k}{2}} \equiv 0 \pmod{[p]^4}, \tag{4.1}$$

where $[n]=1+q+\cdots+q^{n-1}$ is the q-integer and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the q-shifted factorial.

Although there are q-analogues of Whipple's well-poised ${}_7F_6$ transformation and of Karlsson–Minton's summation (see [4, Appendix (II.27) and (III.18)]), we are unable to give a proof of (4.1). This is because we only know a q-analogue of (2.3) (see [7, Lemma 1]) but do not know any q-analogues of (2.2). Besides, we do not know how to prove (4.1) by using the method of 'creative microscoping' devised in [9] either.

While in Theorem 2 we were able to provide a common generalization of the second supercongruence in (1.2) (restricted to modulo p^5) and the first supercongruence in (1.4), it appears to be rather difficult to extend Theorem 1 to a higher supercongruence involving the p-adic Gamma function in the spirit of Theorem 2, even in the special cases $r=1$ or $r=-1$.

We end our paper with two further conjectures for future research. Conjecture 1 concerns a stronger version of Theorem 2 and includes the second supercongruence in (1.2) as a special case. Conjecture 2 concerns a common generalization of the first supercongruence in (1.2) and the second supercongruence in (1.4).

Conjecture 1. *The supercongruence* (1.5) *holds modulo* p^6 *for any prime* $p>3$.

Conjecture 2. *Let* $r\leqslant 1$ *be an integer coprime with* 3. *Let* $p\geqslant 7$ *be a prime such that* $p\equiv r \pmod 3$ *and* $p\geqslant 3-2r$. *Then*

$$\sum_{k=0}^{p-1}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6}\equiv\frac{(-1)^r\,8rp}{3}\cdot\frac{\Gamma_p\left(1+\frac{r}{3}\right)^2}{\Gamma_p\left(1+\frac{2r}{3}\right)^3\Gamma_p\left(1-\frac{r}{3}\right)^4}\times\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3}\pmod{p^6}. \tag{4.2}$$

The $r=1$ case of Conjecture 1 was proved by Long and Ramakrishna [16].

We now explain the difficulties we encountered trying to prove Conjecture 1 for arbitrary integer r satisfying the conditions stated in Theorem 2. It is reasonable to follow the method successfully used by Long and Ramakrishna [16] to establish the desired modulo p^6 supercongruence in the $r=1$ case.

Let ζ be a fifth primitive root of unity. Numerical calculation suggests

$${}_7F_6\left[\begin{matrix}1+\frac{r}{6}, & \frac{r}{3}, & \frac{r}{3}-x, & \frac{r}{3}-\zeta x, & \frac{r}{3}-\zeta^2 x, & \frac{r}{3}-\zeta^3 x, & \frac{r}{3}-\zeta^4 x\\ & \frac{r}{6}, & 1+x, & 1+\zeta x, & 1+\zeta^2 x, & 1+\zeta^3 x, & 1+\zeta^4 x\end{matrix};1\right]_{\frac{2p-r}{3}}\in p\mathbb{Z}_p[[x^5]] \tag{4.3}$$

(the $\frac{2p-r}{3}$ as a subindex of the ${}_7F_6$ series means that the respective series is truncated and contains only its first $\frac{2p-r}{3}+1$ terms, just as on the right-hand side of (4.4)). The case $r=1$ of this assertion was proved by Long and Ramakrishna in [16] using Bailey's ${}_9F_8$ transformation. Letting $x=\frac{2p}{3}$ in (4.3) (in which case the series gets naturally truncated from the top, due to the appearance of $\frac{r-2p}{3}$, a negative integer, as an upper parameter), we obtain

$$
{}_7F_6\left[\begin{matrix}1+\frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3} \\ & \frac{r}{6}, & 1+\frac{2p}{3}, & 1+\frac{2p\zeta}{3}, & 1+\frac{2p\zeta^2}{3}, & 1+\frac{2p\zeta^3}{3}, & 1+\frac{2p\zeta^4}{3}\end{matrix};1\right]
$$

$$
\equiv \frac{1}{r}\sum_{k=0}^{\frac{2p-r}{3}}(6k+r)\frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^6}. \tag{4.4}
$$

For $r=1$, the left-hand side of (4.4) is summable and reduces to a closed form (see [16, (4.6)]) which was shown by Long and Ramakrishna to be congruent to a product of p-adic Gamma functions modulo p^6.

For an arbitrary integer r satisfying the conditions in Theorem 2, the left-hand side of (4.4) is a multiple of a ${}_4F_3$ series (see Equation (3.3) and (3.4)) instead of a product. Since, in our proof of Theorem 2, we show the right-hand side of (3.4) is a multiple of p^4, in order to determine the left-hand side of (4.4) modulo p^6, it would suffice to evaluate the left-hand side of (3.3) modulo p^2 and show that it agrees with the right-has side of (3.3) (modulo p^2). However, putting $m=1-r$, $t=\frac{r}{3}$, $n=\frac{2p-r}{3}$, $a=\frac{2p\zeta}{3}$, $b=\frac{2p\zeta^2}{3}$ and $c=\frac{2p\zeta^3}{3}$ in (3.3), numerical calculation suggests that (3.3) (valid as a congruence modulo p) is in general invalid as a congruence modulo p^2. (A counterexample is, for instance, $r=-1$ and $p=13$.) We can only deduce that the left-hand side of (3.3) is congruent to the following form modulo p^2:

$$
8\sum_{k=0}^{1-r}\frac{(r-1)_k\left(\frac{r}{3}\right)_k^3}{(1)_k\left(\frac{2r}{3}\right)_k^3}+pf(r),
$$

where $f(r)$ is not always divisible by p.

The proof of (4.3) in the case $r=1$ given by Long and Ramakrishna in [16] is rather involved. While it is feasible that one can establish its r-extension (4.3) using Bailey's ${}_9F_8$ transformation as well, we find it hard to determine $f(r)$, quite in contrast to the case $r=1$. To conclude, we do not see how Long and Ramakrishna's method would extend to prove Conjecture 1.

Acknowledgment. The authors would like to thank Ling Long for helpful comments on this paper.

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School of Mathematics and Statistics, Huaiyin Normal University, Huai'an 223300, Jiangsu, People's Republic of China

E-mail address: `jwguo@hytc.edu.cn`

Department of Mathematics, Wenzhou University, Wenzhou 325035, People's Republic of China

E-mail address: `jcliu2016@gmail.com`

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

E-mail address: `michael.schlosser@univie.ac.at`