Explicit computation of the q,t-Littlewood–Richardson coefficients

Michael Schlosser*

Abstract. In joint work with Michel Lassalle [C. R. Math. Acad. Sci. Paris **337** (9) (2003), 569–574], we recently presented an explicit expansion formula for Macdonald polynomials. This result was obtained from a recursion for Macdonald polynomials which in turn was derived by inverting the Pieri formula. We use these formulae here to explicitly compute the q,t-Littlewood–Richardson coefficients, thus solving a problem posed by Ian G. Macdonald.

1. Introduction

For notation and basic facts about Macdonald polynomials, we refer to Chapter VI of [**6**].

Let $X = \{x_1, x_2, \dots\}$ be a countable, possibly infinite, set of variables, and let q and t be two independent indeterminates. It is well known that the Macdonald polynomials $P_\lambda(X; q, t)$ form a basis of $\Lambda_{\mathbb{Q}(q,t)}$, the ring of symmetric functions in X with coefficients in $\mathbb{Q}(q, t)$ (the field of rational functions in q and t). Instead of $P_\lambda(X; q, t)$, we write $P_\lambda(q, t)$ or even P_λ for short, as long as there is no confusion.

Given any three partitions λ, μ, ν, the q, t-Littlewood–Richardson coefficients $f^\lambda_{\mu\nu}(q, t) \in \mathbb{Q}(q, t)$ are defined by

$$P_\mu(q,t)\,P_\nu(q,t) = \sum_\lambda f^\lambda_{\mu\nu}(q,t)\,P_\lambda(q,t) \tag{1.1}$$

(see [**6**, p. 343, Eq. (7.1′)]).

In this paper we derive both a recursion and an explicit formula for the q,t-Littlewood–Richardson coefficients $f^\lambda_{\mu\nu}(q, t)$. To establish the recursion formula for these, we utilize our recursion for Macdonald polynomials which was announced in [**4**] (and proved in [**5**]) and the (analytic form of the) Pieri formula for Macdonald polynomials. The recursion for $f^\lambda_{\mu\nu}(q, t)$ is on the number of columns of the partition ν (or of μ, due to the symmetry $f^\lambda_{\mu\nu}(q, t) = f^\lambda_{\nu\mu}(q, t)$). Further, using our explicit expansion formula for Macdonald polynomials from [**4**] (which gives the explicit

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development of any Macdonald polynomial in terms of modified complete symmetric functions) and the bulk version of the Pieri formula for Macdonald polynomials, we explicitly compute $f^{\lambda}_{\mu\nu}(q,t)$ in *analytic* (versus *combinatorial*) terms.

We employ the standard notation $(a;q)_m=\prod_{j=0}^{m-1}(1-aq^j)$, for integer $m\ge 0$, for the q-shifted factorial. Instead of working with the $P_\lambda(q,t)$, we prefer, for convenience, to work with the dual functions $Q_\lambda(q,t)$ (which are also called Macdonald polynomials, see (1.3)). For easy reference, we recall the arguments leading to (1.5). If we apply the automorphism $\omega_{q,t}$ (defined in [**6**, p. 312, Eq. (2.14)]) to each side of (1.1) we obtain (cf. [**6**, p. 327])

$$Q_{\mu'}(t,q)\,Q_{\nu'}(t,q)=\sum_\lambda f^{\lambda}_{\mu\nu}(q,t)\,Q_{\lambda'}(t,q),$$

or equivalently,

$$Q_{\mu}(q,t)\,Q_{\nu}(q,t)=\sum_\lambda f^{\lambda'}_{\mu'\nu'}(t,q)\,Q_{\lambda}(q,t). \tag{1.2}$$

Since

$$Q_\lambda(q,t)=b_\lambda(q,t)\,P_\lambda(q,t), \tag{1.3}$$

where (cf. [**6**, p. 339, Eq. (6.19)])

$$b_\lambda(q,t)=\prod_{1\le i\le k\le l(\lambda)}\frac{(q^{\lambda_i-\lambda_k}t^{k-i+1};q)_{\lambda_k-\lambda_{k+1}}}{(q^{\lambda_i-\lambda_k+1}t^{k-i};q)_{\lambda_k-\lambda_{k+1}}}, \tag{1.4}$$

we conclude, by combining (1.1), (1.2), and (1.3), that

$$f^{\lambda}_{\mu\nu}(q,t)=\frac{b_\lambda(q,t)}{b_\mu(q,t)\,b_\nu(q,t)}\,f^{\lambda'}_{\mu'\nu'}(t,q). \tag{1.5}$$

We can thus equivalently also work with $f^{\lambda'}_{\mu'\nu'}(t,q)$ instead of $f^{\lambda}_{\mu\nu}(q,t)$, while keeping (1.4) and (1.5) in mind.

The q,t-Littlewood–Richardson coefficients $f^{\lambda}_{\mu\nu}(q,t)$ are usually used to define the skew Macdonald polynomials $P_{\lambda/\mu},Q_{\lambda/\mu}\in\Lambda_{\mathbb{Q}(q,t)}$. Specifically, one writes (cf. [**6**, p. 344, Eq. (7.5)])

$$Q_{\lambda/\mu}(q,t)=\sum_\nu f^{\lambda}_{\mu\nu}(q,t)\,Q_\nu(q,t), \tag{1.6}$$

and (cf. [**6**, p. 351, Eq. (7.8)])

$$Q_{\lambda/\mu}(q,t)=\frac{b_\lambda(q,t)}{b_\mu(q,t)}\,P_{\lambda/\mu}(q,t). \tag{1.7}$$

For $q=t$, the q,t-Littlewood–Richardson coefficients $f^{\lambda}_{\mu\nu}(q,t)$ reduce to the classical Littlewood–Richardson coefficients $c^{\lambda}_{\mu\nu}$. The latter are nonnegative integers which can be combinatorially characterized by the Littlewood–Richardson rule [**6**, Ch. I, Sec. 9]. Other important special cases are listed in [**6**, p. 343, Eq. (7.2)].

In [**6**, p. 347, Remark 4] Ian G. Macdonald raised the question of explicitly computing the coefficients $f^{\lambda}_{\mu\nu}(q,t)$. Our solution in Section 3 is essentially an explicit (though admittedly elaborate) work-out of [**6**, p. 351, Example 5]. The procedure is straightforward and efficient but, unfortunately, does not lead to any combinatorial insight. In particular, we were not able to obtain any q,t-extension of the celebrated Littlewood–Richardson rule. We hope that Theorems 2.1 and

3.4 will prove to be useful. This may concern possible implementation in some computer algebra package. It should be pointed out that all the (multiple) sums appearing in this paper actually have finite support. However, the number of terms in our sums explodes exponentially (with increasing number of rows of the partition involved) due to the determinants appearing in the summand. Thus, our formulae may mainly serve theoretical considerations.

Before we derive a recursion and an explicit formula for $f^{\lambda'}_{\mu'\nu'}(t,q)$ in Sections 2 and 3, respectively, we recall our two main ingredients, namely the Pieri formula and a recursion formula for Macdonald polynomials.

1.1. Pieri formula for Macdonald polynomials. Let $u_1,\dots,u_n$ be n indeterminates and $\mathbb{N}$ the set of nonnegative integers. For $\theta=(\theta_1,\dots,\theta_n)\in\mathbb{N}^n$, let $|\theta|=\sum_{i=1}^n\theta_i$ and define

$$(1.8)\quad D^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)=\prod_{k=1}^n\frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}}\,\frac{(q^{|\theta|+1}u_k;q)_{\theta_k}}{(q^{|\theta|}tu_k;q)_{\theta_k}}\\ \times\prod_{1\le i<j\le n}\frac{(tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}}\,\frac{(q^{-\theta_j+1}u_i/tu_j;q)_{\theta_i}}{(q^{-\theta_j}u_i/u_j;q)_{\theta_i}}.$$

Macdonald symmetric functions satisfy a Pieri formula which generalizes the classical Pieri formula for Schur functions. This generalization was obtained by Macdonald [**6**, p. 331], and independently by Koornwinder [**1**].

Most of the time this Pieri formula is stated in combinatorial terms. Here is its analytic form (cf. [**5**, Th. 4.1]).

THEOREM 1.1. *Let $\lambda=(\lambda_1,...,\lambda_n)$ be an arbitrary partition with length n and $\lambda_{n+1}\in\mathbb{N}$. For any $1\le k\le n$ define $u_k=q^{\lambda_k-\lambda_{n+1}}t^{n-k}$. We have*

$$Q_{(\lambda_1,\dots,\lambda_n)}\,Q_{(\lambda_{n+1})}=\sum_{\theta\in\mathbb{N}^n}D^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\,Q_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

1.2. A recursion formula for Macdonald polynomials. Let $u_1,\dots,u_n$ be n indeterminates and $\theta=(\theta_1,\dots,\theta_n)\in\mathbb{N}^n$. For convenience, we introduce n auxiliary variables $v_1,\dots,v_n$ defined by $v_k=q^{\theta_k}u_k$. We write

$$(1.9)\quad C^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)=\prod_{k=1}^n t^{\theta_k}\,\frac{(q/t;q)_{\theta_k}}{(q;q)_{\theta_k}}\,\frac{(qu_k;q)_{\theta_k}}{(qtu_k;q)_{\theta_k}}\\ \times\prod_{1\le i<j\le n}\frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}}\,\frac{(tu_i/v_j;q)_{\theta_i}}{(u_i/v_j;q)_{\theta_i}}\\ \times\frac{1}{\Delta(v)}\det_{1\le i,j\le n}\left[v_i^{n-j}\left(1-t^{j-1}\frac{1-tv_i}{1-v_i}\prod_{k=1}^n\frac{u_k-v_i}{tu_k-v_i}\right)\right].$$

The following result of [**5**] gives a recursion of Macdonald polynomials on the length of the indexing partition.

THEOREM 1.2. *Let $\lambda=(\lambda_1,...,\lambda_{n+1})$ be an arbitrary partition with length $n+1$. For any $1\le k\le n$ define $u_k=q^{\lambda_k-\lambda_{n+1}}t^{n-k}$. We have*

$$Q_{(\lambda_1,\dots,\lambda_{n+1})}=\sum_{\theta\in\mathbb{N}^n}C^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\,Q_{(\lambda_{n+1}-|\theta|)}\,Q_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n)}.$$

REMARK 1.3. Theorem 1.2 is not the only known recursion formula for Macdonald polynomials. A completely different recursion (in terms of a q-integral representation) was found by Okounkov [**7**, Th. 1] which expresses Macdonald polynomials of n variables in terms of Macdonald polynomials of $n-1$ variables. While [**7**, Th. 1] extends the determinant ratio formula for Schur functions, Theorem 1.2 above can be shown to extend the Jacobi Trudi determinant formula for Schur functions, see [**5**, Sec. 7].

Theorem 1.2 was proved in [**5**, Th. 5.1] by applying *inverse relations* to the Pieri formula. The corresponding multidimensional matrix inversion was established by using a method developed by Krattenthaler [**2**] and further adapted by the present author [**8**]. A completely different proof of Theorem 1.2 was recently delivered by Michel Lassalle [**3**].

The advantage of Theorem 1.2 (see [**5**, Th. 5.1]) is that it gives rise to explicit expansions of Macdonald polynomials in terms of classical bases of $\Lambda_{\mathbb{Q}(q,t)}$, in particular, in terms of the elementary or the modified complete symmetric functions. We are not aware of any other classical bases expansions for Macdonald polynomials in the general case.

2. A recursion for $f^{\lambda'}_{\mu'\nu'}(t,q)$

Let $\mu=(\mu_1,\dots,\mu_m)$ denote a partition of length m, and $\nu=(\nu_1,\dots,\nu_{n+1})$ denote a partition of length $n+1$. Further, for any $0\le k\le n+1$, let $\nu^{(k)}=(\nu_1,\dots,\nu_k)$ denote the partition consisting of the first k rows of ν. Clearly, $\nu^{(n+1)}=\nu$.

Following [**4**, Sec. 5] we define for any composition $\nu=(\nu_1,\dots,\nu_{n+1})$ of length $n+1$

$$c_{\theta_1,\dots,\theta_n}(\nu):=C^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n), \tag{2.1}$$

and

$$d_{\theta_1,\dots,\theta_n}(\nu):=D^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n), \tag{2.2}$$

with $u_k:=q^{\nu_k-\nu_{n+1}}t^{n-k}$, $k=1,\dots,n$. The rational functions $C^{(q,t)}_{\theta_1,\dots,\theta_n}$ and $D^{(q,t)}_{\theta_1,\dots,\theta_n}$ are defined in (1.9) and (1.8).

Let $Q_\lambda=Q_\lambda(q,t)$. We are interested in the coefficient of Q_λ in the product $Q_\mu Q_\nu$. We have

$$Q_\mu\,Q_\nu=Q_\mu\sum_{\theta\in\mathbb{N}^n}c_\theta(\nu)\,Q_{\nu^{(n)}+\theta}\,Q_{(\nu_{n+1}-|\theta|)},$$

by application of Theorem 1.2. Now, we apply (1.2) and obtain

$$Q_\mu\,Q_\nu=\sum_{\theta\in\mathbb{N}^n}c_\theta(\nu)\sum_\rho f^{\rho'}_{\mu'(\nu^{(n)}+\theta)'}(t,q)\,Q_\rho\,Q_{(\nu_{n+1}-|\theta|)},$$

which after application of the Pieri formula in Theorem 1.1 is

$$\begin{aligned}Q_\mu Q_\nu=&\sum_{\theta\in\mathbb{N}^n}c_\theta(\nu)\sum_\rho f^{\rho'}_{\mu'(\nu^{(n)}+\theta)'}(t,q)\\&\times\sum_{\phi\in\mathbb{N}^{m+n}}d_\phi(\rho_1,\dots,\rho_{m+n},\nu_{n+1}-|\theta|)\,Q_{(\rho_1+\phi_1,\dots,\rho_{m+n}+\phi_{m+n},\nu_{n+1}-|\theta|-|\phi|)}.\end{aligned}$$

Now, by extracting the coefficient of Q_λ, where $l(\lambda) \le m+n+1$, on both sides of this equation, we obtain the following result.

THEOREM 2.1. *Let λ, μ, ν be three partitions with $l(\mu) = m$, and $l(\nu) = n+1$. Then, if $|\mu| + |\nu| \neq |\lambda|$, we have $f^{\lambda'}_{\mu'\nu'}(t,q) = 0$. Otherwise, if $|\mu| + |\nu| = |\lambda|$, we have the recursion*

$$f^{\lambda'}_{\mu'\nu'}(t,q) = \sum_{\theta \in \mathbb{N}^n} c_\theta(\nu) \sum_\rho f^{\rho'}_{\mu'(\nu^{(n)}+\theta)'}(t,q)\; d_{\lambda^{(m+n)}-\rho}(\rho_1, \dots, \rho_{m+n}, \nu_{n+1} - |\theta|).$$

3. An explicit formula for $f^{\lambda'}_{\mu'\nu'}(t,q)$

Assume the definitions of the previous section. Further, let $\mathbb{M}^{(n)}$ be the set of lower triangular $n \times n$ matrices with nonnegative integers. Let $\nu = (\nu_1, \dots, \nu_{n+1})$ be a partition of length $n+1$. For any $\theta = (\theta(i,j))_{1 \le i,j, \le n} \in \mathbb{M}^{(n)}$, we define a set of n partitions $\{\sigma(\nu,\theta,k), 1 \le k \le n\}$ where $\sigma(\nu,\theta,k)$ has length $k+1$ and is defined by

$$\sigma(\nu,\theta,k)_i = \nu_i + \sum_{j=k+1}^{n} \theta(j,i), \quad \text{for } 1 \le i \le k+1.$$

Note that $\{\sigma(\nu,\theta,k), 1 \le k \le n\}$ essentially corresponds to $\{\mu(\theta,k), 1 \le k \le n\}$ of [**4**, Th. 5.1].

REMARK 3.1. Observe that the notation we are using here follows that of [**4**] but is slightly different from the one used in [**5**]. In particular, the term $\theta(j,i)$ used in this paper corresponds to $\theta_{i,j+1}$ of [**5**]. Also note that $\mathbb{M}^{(n)}$ is defined in [**5**] differently, where it denotes the set of *upper* triangular $n \times n$ matrices with nonnegative integers and 0 on the diagonal.

By iteration we readily deduce from Theorem 1.2 the following explicit expansion of any Macdonald polynomial in terms of one row Macdonald polynomials, see [**4**, Th. 5.1].

THEOREM 3.2 (Expansion formula). *Let $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be an arbitrary partition with length $n+1$. For any $\theta = (\theta(i,j))_{i,j=1}^n \in \mathbb{M}^{(n)}$, let us consider a sequence of n partitions $\{\mu(\theta,k), 1 \le k \le n\}$ where $\mu(\theta,k)$ has length $k+1$ and is defined by*

$$\mu(\theta,k)_i = \lambda_i + \sum_{j=k+1}^{n} \theta(j,i) \qquad (1 \le i \le k+1).$$

We have

$$Q_\lambda = \sum_{\theta \in \mathbb{M}^{(n)}} \prod_{k=1}^{n} c_{\theta(k,1)\dots\theta(k,k)}(\mu(\theta,k)) \prod_{k=0}^{n} Q_{\left(\lambda_{k+1} + \sum_{j=k+1}^n \theta(j,k+1) - \sum_{j=1}^k \theta(k,j)\right)}.$$

Let $\mathbb{M}^{(n,m+n)}$ be the set of all "m-shifted lower-triangular" $n \times (m+n)$ matrices with nonnegative integers. By "m-shifted lower-triangular" we mean that $\theta(i,j) = 0$, if $\theta \in \mathbb{M}^{(n,m+n)}$ and $i + m < j$. Note that $\mathbb{M}^{(n,n)} = \mathbb{M}^{(n)}$.

Let $\xi = (\xi_1, \dots, \xi_{m+n+1})$ be a composition of length $m+n+1$. For any $\varphi = (\varphi(i,j))_{\substack{1 \le i \le n \\ 1 \le j \le m+n}} \in \mathbb{M}^{(n+1,m+n)}$, we define a sequence of $n+1$ compositions

$\{\tau(\xi,\varphi,k), 1\le k\le n+1\}$ where $\tau(\xi,\varphi,k)$ has length $m+k$ and is defined by

$$\tau(\xi,\varphi,k)_i=\xi_i+\sum_{j=i-m+1}^{k}\varphi(j,i)-\sum_{j=1}^{i-1}\varphi(i-m,j),\quad \text{for } 1\le i\le m+k. \tag{3.1}$$

In the above definition the value of $\varphi(i,j)$ is understood to be 0 if either i or j is out of range, in particular, if i or j is not a positive integer.

By iterating the Pieri formula we immediately deduce the following result.

Theorem 3.3 (Bulk version of the Pieri formula). *Let $\mu=(\mu_1,\dots,\mu_m)$ be a partition of length m. Then*

$$Q_\mu\, Q_{(\mu_{m+1})}\dots Q_{(\mu_{m+n+1})} = \sum_{\varphi\in\mathbb{M}^{(n+1,m+n)}}\prod_{k=1}^{n+1} d_{\varphi(k,1),\dots,\varphi(k,m+k-1)}\big(\tau(\mu,\varphi,k)\big)\, Q_{(\omega_1(\mu,\varphi),\dots,\omega_{m+n+1}(\mu,\varphi))},$$

where

$$\omega_k(\mu,\varphi)=\mu_k+\sum_{j=k+1}^{m+n+1}\varphi(j-m,k)-\sum_{j=1}^{k-1}\varphi(k-m,j),\quad \textit{for } 1\le k\le m+n+1.$$

With Theorems 3.2 and 3.3 at hand, we are now ready for our derivation of an explicit formula for $f^{\lambda'}_{\mu'\nu'}(t,q)$.

Let $\mu=(\mu_1,\dots,\mu_m)$ be a partition of length m, and let $\nu=(\nu_1,\dots,\nu_{n+1})$ be a partition of length $n+1$. By Theorem 3.2 we have

$$Q_\mu\, Q_\nu = Q_\mu \sum_{\theta\in\mathbb{M}^{(n)}}\prod_{k=1}^{n} c_{\theta(k,1),\dots,\theta(k,k)}\big(\sigma(\nu,\theta,k)\big) \times\prod_{k=1}^{n+1} Q_{\left(\nu_k+\sum_{j=k}^{n}\theta(j,k)-\sum_{j=1}^{k-1}\theta(k-1,j)\right)}.$$

We apply Theorem 3.3 and obtain for the above expression

$$\sum_{\theta\in\mathbb{M}^{(n)}}\prod_{k=1}^{n} c_{\theta(k,1),\dots,\theta(k,k)}\big(\sigma(\nu,\theta,k)\big) \times\sum_{\varphi\in\mathbb{M}^{(n+1,m+n)}}\prod_{k=1}^{n+1} d_{\varphi(k,1),\dots,\varphi(k,m+k-1)}\big(\tau(\xi(\mu,\nu,\theta),\varphi,k)\big) \times Q_{\left(\eta(\xi(\mu,\nu,\theta),\varphi)_1,\dots,\eta(\xi(\mu,\nu,\theta),\varphi)_{m+n+1}\right)}, \tag{3.2}$$

where

$$\xi(\mu,\nu,\theta)_k=\begin{cases}\mu_k & \text{if } 1\le k\le m\\ \omega(\nu,\theta)_{k-m}, & \text{if } m+1\le k\le m+n+1\end{cases}, \tag{3.3}$$

where

$$\omega(\nu,\theta)_k=\nu_k+\sum_{j=k}^{n}\theta(j,k)-\sum_{j=1}^{k-1}\theta(k-1,j),\quad \text{for } 1\le k\le n+1,$$

and where

$$\eta(\xi(\mu,\nu,\theta),\varphi)_k = \xi(\mu,\nu,\theta)_k + \sum_{j=k+1}^{m+n+1} \varphi(j-m,k) - \sum_{j=1}^{k-1} \varphi(k-m,j),$$

for $1 \le k \le m+n+1$.

Extracting the coefficient of Q_λ in (3.2), for a partition λ with $l(\lambda) \le m+n+1$, we need to have

$$\lambda_k = \eta(\xi(\mu,\nu,\theta),\varphi)_k = \xi(\mu,\nu,\theta)_k + \varphi(n+1,k) + \sum_{j=k+1}^{m+n} \varphi(j-m,k) - \sum_{j=1}^{k-1} \varphi(k-m,j),$$

for $1 \le k \le m+n+1$. Consequently, we need

$$\varphi(n+1,k) = \lambda_k - \xi(\mu,\nu,\theta)_k - \sum_{j=k+1}^{m+n} \varphi(j-m,k) + \sum_{j=1}^{k-1} \varphi(k-m,j), \tag{3.4}$$

for $1 \le k \le m+n$, and furthermore,

$$\lambda_{m+n+1} = \omega(\nu,\theta)_{n+1} - \sum_{j=1}^{m+n} \varphi(n+1,j), \tag{3.5}$$

which, by using (3.4), is

$$\begin{aligned}
\lambda_{m+n+1} &= \omega(\nu,\theta)_{n+1} - (\lambda_1 + \dots + \lambda_{m+n}) + \xi(\mu,\nu,\theta)_1 + \dots + \xi(\mu,\nu,\theta)_{m+n} \\
&\qquad + \sum_{k=1}^{m+n} \sum_{j=k+1}^{m+n} \varphi(j-m,k) - \sum_{k=1}^{m+n} \sum_{j=1}^{k-1} \varphi(k-m,j) \\
&= \omega(\nu,\theta)_{n+1} - (\lambda_1 + \dots + \lambda_{m+n}) + (\mu_1 + \dots + \mu_m) + \omega(\nu,\theta)_1 + \dots + \omega(\nu,\theta)_n \\
&\qquad + \sum_{1 \le k < j \le m+n} \varphi(j-m,k) - \sum_{1 \le j < k \le m+n} \varphi(k-m,j) \\
&= -(\lambda_1 + \dots + \lambda_{m+n}) + (\mu_1 + \dots + \mu_m) + \omega(\nu,\theta)_1 + \dots + \omega(\nu,\theta)_{n+1} \\
&= -(\lambda_1 + \dots + \lambda_{m+n}) + (\mu_1 + \dots + \mu_m) + (\nu_1 + \dots + \nu_{n+1}) \\
&\qquad + \sum_{k=1}^{n+1} \sum_{j=k}^{n} \theta(j,k) - \sum_{k=1}^{n+1} \sum_{j=1}^{k-1} \theta(k-1,j) \\
&= -(\lambda_1 + \dots + \lambda_{m+n}) + (\mu_1 + \dots + \mu_m) + (\nu_1 + \dots + \nu_{n+1}) \\
&\qquad + \sum_{1 \le k \le j \le n} \theta(j,k) - \sum_{1 \le j < k \le n+1} \theta(k-1,j) \\
&= -(\lambda_1 + \dots + \lambda_{m+n}) + (\mu_1 + \dots + \mu_m) + (\nu_1 + \dots + \nu_{n+1}).
\end{aligned}$$

Hence, there are only terms appearing in (3.2) when $|\mu| + |\nu| = |\lambda|$.

Now, in the multisum in (3.2), for each matrix $\varphi \in \mathbb{M}^{(n+1,m+n)}$, we replace the entries of the last row, $\varphi(n+1,k)$, $1 \le k \le m+n$, according to (3.4). After having performed these substitutions, we reduce $(\varphi(i,j)) \in \mathbb{M}^{(n+1,m+n)}$ by removing its last row and column (since $(\varphi(i,m+n)) = 0$, for $i = 1,\dots,n$) so that the inner summation in (3.2) now runs over all $(\varphi(i,j)) \in \mathbb{M}^{(n,m+n-1)}$. Also, note that due to (3.4), $\tau(\xi(\mu,\nu,\theta),\varphi,n+1)$ simply reduces to $(\lambda_1,\dots,\lambda_{m+n+1})$, since

$$\begin{aligned}\tau(\xi(\mu,\nu,\theta),\varphi,n+1)_i &= \xi(\mu,\nu,\theta)_i+\varphi(n+1,i)+\sum_{j=i-m+1}^{n}\varphi(j,i)-\sum_{j=1}^{i-1}\varphi(i-m,j)\\ &= \xi(\mu,\nu,\theta)_i+\Big(\lambda_i-\xi(\mu,\nu,\theta)_i-\sum_{j=i+1}^{m+n}\varphi(j-m,i)+\sum_{j=1}^{i-1}\varphi(i-m,j)\Big)\\ &\qquad+\sum_{j=i-m+1}^{n}\varphi(j,i)-\sum_{j=1}^{i-1}\varphi(i-m,j)=\lambda_i,\qquad \text{for } 1\le i\le m+n,\end{aligned}$$

and

$$\tau(\xi(\mu,\nu,\theta),\varphi,n+1)_{m+n+1}=\xi(\mu,\nu,\theta)_{m+n+1}-\sum_{j=1}^{m+n}\varphi(n+1,j)=\lambda_{m+n+1},$$

by (3.1), (3.3) and (3.5).

Thus, we have derived the following result.

Theorem 3.4. *Let λ,μ,ν be three partitions with $l(\mu)=m$, and $l(\nu)=n+1$. Then, if $|\mu|+|\nu|\neq|\lambda|$, we have $f^{\lambda'}_{\mu'\nu'}(t,q)=0$. Otherwise, if $|\mu|+|\nu|=|\lambda|$, we have*

$$\begin{aligned}f^{\lambda'}_{\mu'\nu'}(t,q)&=\sum_{\theta\in\mathbb{M}^{(n)}}\prod_{k=1}^{n}c_{\theta(k,1),\dots,\theta(k,k)}\big(\sigma(\nu,\theta,k)\big)\\ &\quad\times\sum_{\varphi\in\mathbb{M}^{(n,m+n-1)}}\prod_{k=1}^{n}d_{\varphi(k,1),\dots,\varphi(k,m+k-1)}\big(\tau(\xi(\mu,\nu,\theta),\varphi,k)\big)\\ &\qquad\times d_{\lambda_1-\psi(\xi(\mu,\nu,\theta),\varphi)_1,\dots,\lambda_{m+n}-\psi(\xi(\mu,\nu,\theta),\varphi)_{m+n}}(\lambda_1,\dots,\lambda_{m+n+1}),\end{aligned}$$

where

$$\psi(\xi(\mu,\nu,\theta),\varphi)_k=\xi(\mu,\nu,\theta)_k+\sum_{j=k+1}^{m+n}\varphi(j-m,k)-\sum_{j=1}^{k-1}\varphi(k-m,j),$$

for $1\le k\le m+n$, where

$$\xi(\mu,\nu,\theta)_k=\begin{cases}\mu_k & \text{if } 1\le k\le m\\ \omega(\nu,\theta)_{k-m} & \text{if } m+1\le k\le m+n\end{cases},$$

and where

$$\omega(\nu,\theta)_k=\nu_k+\sum_{j=k}^{n}\theta(j,k)-\sum_{j=1}^{k-1}\theta(k-1,j),\quad \text{for } 1\le k\le n.$$

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Fakultät für Mathematik der Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: schlosse@ap.univie.ac.at

URL: http://www.mat.univie.ac.at/~schlosse