Multiple Bernoulli Polynomials and Numbers

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Outline

Two Equivalent Definitions of Bernoulli Polynomials / Numbers

Bernoulli numbers:

By a generating function:

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} b_n \frac{t^n}{n!} .$$

By a recursive formula:

$$\begin{cases} b_0 = 1 , \\ \forall n \in \mathbb{N} , \ \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 . \end{cases}$$

First examples:

$$b_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \cdots$$

Bernoulli polynomials :

By a generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n\geq 0} B_n(x) \frac{t^n}{n!} .$$

By a recursive formula:

$$\begin{cases} B_0(x) = 1 , \\ \forall n \in \mathbb{N} , \ B'_{n+1}(x) = (n+1)B_n(x) , \\ \int_0^1 B_n(x) \, dx = 0 . \end{cases}$$

First examples:

$$\begin{aligned} B_0(x) &= 1 , \\ B_1(x) &= x - \frac{1}{2} , \\ B_2(x) &= x^2 - x + \frac{1}{6} , \\ &\vdots \end{aligned}$$

Some Properties of Bernoulli Polynomials

Property 1: Difference Equation

$\Delta(B_n)(x) = nx^{n-1}$ for all $n \in \mathbb{N}^*$, where $\Delta(f)(z) = f(z+1) - f(z)$.

Property 2: Reflexion Formula

$(-1)^n B_n(1-x) = B_n(x)$ for all $n \in \mathbb{N}$.

Property 3: Expression in the falling factorial basis

$$B_{n+1}(x) - b_{n+1} = \sum_{k=0}^{n} \frac{n+1}{k+1} \left\{ {n \atop k} \right\} (x)_{k+1}$$ for all $n \in \mathbb{N}$, where:

- $(x)_n = x(x-1)\cdots(x-n+1)$.
- $\left\{ {n \atop k} \right\}$ denotes the Stirling number of second kind.

Definition:

The Riemann Zeta Function and Hurwitz Zeta Function are defined, for $\Re e\ s > 1$, and $z \in \mathbb{C} - \mathbb{N}_{<0}$, by:

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{n^s} \quad , \quad \zeta(s,z) = \sum_{n \geq 0} \frac{1}{(n+z)^s} \ .$$

Property:

$s \longmapsto \zeta(s)$ and $s \longmapsto \zeta(s,z)$ can be analytically extend to a meromorphic function on $\mathbb{C}$, with a simple pole located at 1.

Remark:

$$\begin{aligned} \zeta(-n) &= -\frac{b_{n+1}}{n+1} \text{ for all } n \in \mathbb{N} \ . \\ \zeta(-n,z) &= -\frac{B_{n+1}(z)}{n+1} \text{ for all } n \in \mathbb{N} \ . \end{aligned}$$

Multiple Zeta Values and Hurwitz Multiple Zeta Functions

Definition of Hurwitz Multiple Zeta Functions

$\mathcal{H}e^{s_1,\cdots,s_r}(z) = \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}$, if $z \in \mathbb{C} - \mathbb{N}_{<0}$ and $(s_1,\cdots,s_r) \in (\mathbb{N}^*)^r$, such that $s_1 \geq 2$.

Lemma:

For all sequences $(s_1,\cdots,s_r)$, we have:

$$\Delta_-(\mathcal{H}e^{s_1,\cdots,s_r}) = \mathcal{H}e^{s_1,\cdots,s_{r-1}} \cdot J^{s_r} \ ,$$

where $J^s(z) = z^{-s}$ and $\Delta_-(f)(z) = f(z-1) - f(z)$.

Heuristic:

$\mathcal{B}e^{s_1,\cdots,s_r}(z) =$ Multiple (Divided) Bernoulli Polynomials $= \mathcal{H}e^{-s_1,\cdots,-s_r}(z)$.

$be^{s_1,\cdots,s_r} =$ Multiple (Divided) Bernoulli Numbers $= \mathcal{H}e^{-s_1,\cdots,-s_r}(0)$.

Outline

Main Goal

We want to define $\mathcal{B}e^{s_1,\cdots,s_r}(z)$ such that:

- their properties are similar to Hurwitz Multiple Zeta Functions' properties.
- their properties generalize these of Bernoulli polynomials.

Main Goal:

Find some polynomials $\mathcal{B}e^{s_1,\cdots,s_r}$ such that:

$$\begin{cases} \mathcal{B}e^{s}(z) = \dfrac{B_{s+1}(z)}{s+1} \text{ , where } s \geq 0 \text{ ;} \\[2ex] \Delta\left(\mathcal{B}e^{s_1,\cdots,s_r}\right)(z) = \mathcal{B}e^{s_1,\cdots,s_{r-1}}(z)\cdot z^{s_r} \text{ , where } r \geq 2 \text{ , and } \ s_1,\cdots,s_r \geq 0 \text{ ;} \\[2ex] \text{the } Be^{s_1,\cdots,s_r} \text{ are multiplied by the stuffle product.} \end{cases}$$

Lemma:

The familly of difference equations

$$\Delta\left(\mathcal{B}e^{s_1,\cdots,s_r}\right)(z) = \mathcal{B}e^{s_1,\cdots,s_{r-1}}(z)\cdot z^{s_r}$$

has a unique familly of solutions $\mathcal{B}e_0^{s_1,\cdots,s_r} \in z\mathbb{C}[z]$, which are multiplied by the stuffle.

Sketch of proof: $\ker \Delta \cap z\mathbb{C}[z] = \{0\}$. □

Examples:

$$\mathcal{B}eeg_0^X(z) = \sum_{p\geq 0} \mathcal{B}e_0^p(z)\frac{X^p}{p!} = \frac{e^{zX}-1}{e^X-1} .$$

$$\mathcal{B}eeg_0^{X,Y}(z) = \sum_{p,q\geq 0} \mathcal{B}e_0^{p,q}(z)\frac{X^p}{p!}\frac{Y^q}{q!} = \frac{\mathcal{B}eeg_0^{X+Y}(z)-\mathcal{B}eeg_0^{Y}(z)}{e^X-1} .$$

The Form of a Multiple Bernoulli Polynomial

Notations: The previous family $\mathcal{B}e_0^{s_1,\cdots,s_r}$ is denoted by $\mathcal{B}e_0^{\bullet}$.
In general, a family $M^{s_1,\cdots,s_r}$ is denoted by $M^{\bullet}$ and called a **mould** .

Mould product: $M^{\bullet} = A^{\bullet} \times B^{\bullet} \iff M^{s_1,\cdots,s_r} = \sum_{k=0}^{r} A^{s_1,\cdots,s_k} B^{s_{k+1},\cdots,s_r}$.

Theorem:

The solutions $\mathcal{B}e^{\bullet}(z)$, valued in $\mathbb{C}[z]$ and multiplied by the stuffle product, to the difference equation $\Delta\left(\mathcal{B}e^{s_1,\cdots,s_r}\right)(z) = \mathcal{B}e^{s_1,\cdots,s_{r-1}}(z) \cdot z^{s_r}$ satisfying $\mathcal{B}e^{s}(z) = \dfrac{B_{s+1}(z)}{s+1}$ have an exponential generating function $\mathcal{B}eeg^{\bullet}(z)$ of the form

$$\mathcal{B}eeg^{\bullet}(z) = beeg^{\bullet} \times \mathcal{B}eeg_0^{\bullet}(z) \ ,$$

where:

$$\begin{cases} beeg^{\bullet} \text{ is valued in } \mathbb{C} \ . \\ \text{The } beeg^{\bullet}\text{'s are multiplied by the stuffle product.} \\ beeg^{X} = \dfrac{1}{e^X - 1} - \dfrac{1}{X} \ . \end{cases}$$

Outline

A New Expression of the Mould $\mathcal{B}eeg_0^\bullet(z)$

New Goal:

From $\mathcal{B}eeg^\bullet(z) = beeg^\bullet \times \mathcal{B}eeg_0^\bullet(z)$, determine a suitable mould $be^\bullet$, or $beeg^\bullet$ such that the reflexion formula is a nice generalization of

$$(-1)^n B_n(1-z) = B_n(z) \ , n \in \mathbb{N} \ .$$

Definition:

$$\mathcal{S}ing^{X_1,\cdots,X_r}(z) = \frac{e^{z(X_1+\cdots+X_r)}}{(e^{X_1}-1)(e^{X_1+X_2}-1)\cdots(e^{X_1+\cdots+X_r}-1)} \text{ , for all } z \in \mathbb{C}.$$

Property:

$$\mathcal{B}eeg_0^\bullet(z) = \left(\mathcal{S}ing^\bullet(0)\right)^{\times -1} \times \mathcal{S}ing^\bullet(z) \ .$$

Key Point of the Reflexion Formula of $\mathcal{S}ing^{\bullet}(z)$

Notations: $$M^{-\bullet} : (\omega_1, \cdots, \omega_r) \longmapsto M^{-\omega_1, \cdots, -\omega_r} \ .$$

$$M^{\overleftarrow{\bullet}} : (\omega_1, \cdots, \omega_r) \longmapsto M^{\omega_r, \cdots, \omega_1} \ .$$

Lemma:

Let $\mathcal{S}g^{\bullet}$ be the mould defined by: $\mathcal{S}g^{X_1, \cdots, X_r} = (-1)^r$.
For all $z \in \mathbb{C}$, we have:

$$\mathcal{S}ing^{-\bullet}(0) = \left(\mathcal{S}ing^{\overleftarrow{\bullet}}(0)\right)^{\times -1} \times \mathcal{S}g^{\bullet} \quad , \quad \mathcal{S}ing^{-\bullet}(1-z) = \left(\mathcal{S}ing^{\overleftarrow{\bullet}}(z)\right)^{\times -1} .$$

Sketch of proof: Use $\dfrac{1}{e^X - 1} + \dfrac{1}{e^{-X} - 1} + 1 = 0$. □

Examples:

$$\begin{aligned} \mathcal{S}ing^{-X,-Y}(0) &= \mathcal{S}ing^{X,Y}(0) + \mathcal{S}ing^{X+Y}(0) + \mathcal{S}ing^{X}(0) + 1 \ . \\ \mathcal{S}ing^{-X,-Y}(1-z) &= \mathcal{S}ing^{X,Y}(z) + \mathcal{S}ing^{X+Y}(z) \ . \end{aligned}$$

Reflexion Formula for Multiple Bernoulli Polynomials

Corollary:

For all $z \in \mathbb{C}$, we have

$$\mathcal{B}eeg_0^{-\bullet}(1-z) \times \mathcal{B}eeg_0^{\overleftarrow{\bullet}}(z) = 1^{\bullet} + I^{\bullet} \ ,$$

where $1^{x_1,\cdots,x_r} = \begin{cases} 1 & \text{if } r = 0 \ . \\ 0 & \text{if } r > 0 \ . \end{cases}$ and $I^{x_1,\cdots,x_r} = \begin{cases} 1 & \text{if } r = 1 \ . \\ 0 & \text{if } r \neq 1 \ . \end{cases}$

Theorem:

For all $z \in \mathbb{C}$, we have:

$$\mathcal{B}eeg^{-\bullet}(1-z) \times \mathcal{B}eeg^{\overleftarrow{\bullet}}(z) = beeg^{-\bullet} \times (1^{\bullet} + I^{\bullet}) \times beeg^{\overleftarrow{\bullet}} \ .$$

Outline

A Mould Equation

Goal: Solve the mould equation $\mathcal{X}eeg^\bullet \times (1^\bullet + I^\bullet) \times \overleftarrow{\mathcal{X}eeg^{-\bullet}} = 1^\bullet$.

Consequence: Define $\mathcal{B}eeg^\bullet = \mathcal{X}eeg^\bullet \times \mathcal{B}eeg_0^\bullet$, so that:

$$\mathcal{B}eeg^{-\bullet}(1-z) \times \overleftarrow{\mathcal{B}eeg^{\bullet}}(z) = 1^\bullet \ .$$

Definition:

The mould $\sqrt{\mathcal{S}g^\bullet}$ is defined by:

$$\sqrt{\mathcal{S}g^{x_1,\cdots,x_r}} = \frac{(-1)^r}{2^{2r}}\binom{2r}{r}$$

Resolution of the Mould Equation

Property:

Any solution $\mathcal{X}eeg^\bullet$ of $\mathcal{X}eeg^\bullet \times (1^\bullet + I^\bullet) \times \overleftarrow{\mathcal{X}eeg}^{-\bullet} = 1^\bullet$ comes from a mould $\mathcal{Z}eg^\bullet$ satisfying:

$$\begin{cases} \mathcal{Z}eg^\bullet + \overleftarrow{\mathcal{Z}eg}^{-\bullet} = 0^\bullet \ , \\ \mathcal{Z}eg^\bullet \text{ is “primitive”} \ , \end{cases}$$

and is given by: $\mathcal{X}eeg^\bullet = Exp(\mathcal{Z}eg^\bullet) \times \sqrt{\mathcal{S}g^\bullet}$.

Sketch of proof: group-like $=$ Exp(primitive) . □

Definition of Multiple Bernoulli Polynomials and Numbers

The mould $\mathcal{Z}eg^{\bullet}$ defined by

$$\begin{cases} \mathcal{Z}eg^{\emptyset} := 0 \ , \\ \mathcal{Z}eg^{X_1,\cdots,X_r} := \dfrac{(-1)^{r-1}}{r} \mathcal{Z}eg^{X_1+\cdots+X_r} \ , \end{cases}$$

where $\mathcal{Z}eg^{X} = \dfrac{1}{e^X - 1} - \dfrac{1}{X} + \dfrac{1}{2}$, satifies the required conditions:

$$\begin{cases} \mathcal{Z}eg^{\bullet} + \mathcal{Z}eg^{\overleftarrow{-\bullet}} = 0 \ , \\ \mathcal{Z}eg^{\bullet} \text{ is ``primitive''} \ , \end{cases}$$

Definition:

The moulds $\mathcal{B}eeg^{\bullet}(z)$ and $beeg^{\bullet}$ are defined by:

$$\mathcal{B}eeg^{\bullet}(z) = Exp(\mathcal{Z}eg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}} \times (\mathcal{S}ing^{\bullet}(0))^{\times -1} \times \mathcal{S}ing^{\bullet}(z) \ .$$

$$beeg^{\bullet} = Exp(\mathcal{Z}eg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}}$$

The coefficients $\mathcal{B}e^{\bullet}(z)$ and $\mathcal{B}e^{\bullet}$ of these exponential generating series are called respectively the **Multiple Bernoulli Polynomials and Numbers**.

Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

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$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.

Table of Multiple Bernoulli Numbers in length 2

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$q=1$	$\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two diagonals is “constant”.

Table of Multiple Bernoulli Numbers in length 2

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$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
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$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$

- one out of four Multiple Bernoulli Numbers is null.
- one out of two diagonals is “constant”.
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$.

Table of Multiple Bernoulli Numbers in length 2

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$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
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$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
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- one out of four Multiple Bernoulli Numbers is null.
- one out of two diagonals is “constant”.
- cross product around the zeros are equals : $28800 \cdot 127008 = 60480^2$.
- “symmetrie” relatively to $p = q$.

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$							
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$q=6$							

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
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$q=2$	0		0		0		0
$q=3$							
$q=4$	0		0		0		0
$q=5$							
$q=6$	0		0		0		0

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$q=0$							
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$q=2$	0	$\frac{1}{240}$	0		0		0
$q=3$		$-\frac{1}{2880}$					
$q=4$	0	$-\frac{1}{504}$	0		0		0
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$q=6$	0	$\frac{1}{480}$	0		0		0

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$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
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$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$		$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$							
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Construction Table of Multiple Bernoulli Numbers in length 2

$be^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$q=0$	$\frac{3}{8}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$	0
$q=1$	$-\frac{1}{24}$	$\frac{1}{288}$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$
$q=2$	0	$\frac{1}{240}$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0
$q=3$	$\frac{1}{240}$	$-\frac{1}{2880}$	$-\frac{1}{504}$	$\frac{1}{28800}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$
$q=4$	0	$-\frac{1}{504}$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0
$q=5$	$-\frac{1}{504}$	$\frac{1}{6048}$	$\frac{1}{480}$	$-\frac{1}{60480}$	$-\frac{1}{264}$	$\frac{1}{127008}$	$\frac{691}{65520}$
$q=6$	0	$\frac{1}{480}$	0	$-\frac{1}{264}$	0	$\frac{691}{65520}$	0

Definition

The **Stirling number of the second kind**, denoted by $\left\{ n \atop k \right\}$, is the number of ways to partition a set of n objects into k non-empty subsets.

Recurence relation:

$$\begin{cases} \left\{ 0 \atop 0 \right\} = 1 \ , \ \left\{ n \atop 0 \right\} = 0 \text{ if } n > 0. \\ \left\{ n+1 \atop k \right\} = k \left\{ n \atop k \right\} + \left\{ n \atop k-1 \right\} \end{cases}$$

Triangle:

n	0	1	2	3	4	5	6
$k=0$	1						
$k=1$	0	1					
$k=2$	0	1	1				
$k=3$	0	1	3	1			
$k=4$	0	1	7	6	1		
$k=5$	0	1	15	25	10	1	
$k=6$	0	1	31	90	65	15	1

A Recurrence Simillar to the Stirling Number of 2nd Kind

Reminder: The falling factorial is defined by: $(z)_n = z(z-1)\cdots(z-n+1)$.

Coefficient in the falling factorial $(z)_n$ of $\mathcal{B}e^{1,1,k}(z) - \mathcal{B}e^{1,1,k}(0)$:

n	1	2	3	4	5	6	7	8	9	10	11
$k=0$	1	0	12	96	36						
$k=1$	0	1	24	300	240	36					
$k=2$	0	1	49	924	1260	420	36				
$k=3$	0	1	99	2821	5964	3360	636	36			
$k=4$	0	1	199	8562	26677	22764	7176	888	36		
$k=5$	0	1	399	25885	115270	140497	65820	13392	1176	36	
$k=6$	0	1	799	78054	486965	817755	535417	159564	22800	1500	36

Recurrence relation: $u_{n+1,k} = k u_{n,k} + u_{n,k-1}$ for all $k > 0$.

Conclusion

1. We have respectively defined the Multiple Bernoulli Polynomials and Multiple Bernoulli Numbers by:

$$\begin{cases} \mathcal{B}eeg^{\bullet}(z) & = & Exp(\mathcal{Z}eg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}} \times (\mathcal{S}ing^{\bullet}(0))^{\times -1} \times \mathcal{S}ing^{\bullet}(z) \ . \\ beeg^{\bullet} & = & Exp(\mathcal{Z}eg^{\bullet}) \times \sqrt{\mathcal{S}g^{\bullet}} \end{cases}$$

2. The Multiple Bernoulli Polynomials satisfy a nice generalization of:
   - the difference equation $\Delta(B_n)(x) = nx^{n-1}$.
   - the reflection formula $(-1)^n B_n(1-x) = B_n(x)$.
   - the expression of Bernoulli polynomials in the falling factorial basis.

3. The Multiple Bernoulli Polynomials/Numbers can be multiplied using the stuffle product.

THANK YOU FOR YOUR ATTENTION !