Non commutative Gandhi polynomials and surjectives pistols

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Gandhi polynomials

- Definition of $(C_n)_{n\geq 1}$:

$$\begin{cases} C_1(x) &= 1 \\ C_{n+1}(x) &= \Delta(C_n(x)\cdot x^2) \quad \text{if } n \geq 1, \end{cases}$$

where $\Delta(f(x)) = f(x+1) - f(x)$, for any function f.

- Examples:

$$\begin{aligned} C_1(x) &= 1 \\ C_2(x) &= 2x+1 \\ C_3(x) &= 6x^2+8x+3 \\ &\dots \end{aligned}$$

Their coefficients are positive integers. Is their a "natural" combinatorial interpretation of these polynomials ?

Surjective pistols of Dumont

Yes, and it is the surjective pistols of Dumont.

- A *surjective pistol* p of size $2n$ is a surjective map from $\{1, \cdots, 2n\}$ onto $\{2, \cdots, 2n\}$ such that for each i in $\{1, \cdots, 2n\}$, $p(i) \geq i$. The set of surjective pistols of size $2n$ is denoted by $\mathcal{P}_{2n}$.
- Examples:

$$
\begin{aligned}
\mathcal{P}_2 = &\ \{22\} \\
\mathcal{P}_4 = &\ \{2444,\ 4244,\ 2244\} \\
\mathcal{P}_6 = &\ \{246666,\ 264666,\ 266466,\ 426666,\ 624666,\ 626466, \\
&\ 244666,\ 246466,\ 264466,\ 624466,\ 424666,\ 426466, \\
&\ 224666,\ 226466,\ 244466,\ 424466,\ 224466\}
\end{aligned}
$$

$$
\begin{aligned}
C_1(x) = &\ 1 \\
C_2(x) = &\ 2x + 1 \\
C_3(x) = &\ 6x^2 + 8x + 3 \\
&\ \ldots
\end{aligned}
$$

Non commutative polynomials

Let A be an alphabet.

$$\begin{array}{rcl} \mathbb{R}{<}A{>} & = & \text{Free algebra generated by the letter of } A, \\ \text{Linear basis} & = & \text{Finite words over } A. \end{array}$$

From now on, our alphabet is $A = \{a_i, \text{ for } i \in \mathbb{N}^*\} \bigcup \{a_\omega\}$.

Example:

$$a_2 a_2 + a_2 a_\omega + a_\omega a_2.$$

There is a natural projection Π from $\mathbb{R}{<}A{>}$ onto $\mathbb{R}[x]$:

$$\begin{cases} \Pi(a_\omega) & = & x, \\ \Pi(a_i) & = & 1, \qquad \text{for } i \in \mathbb{N}^*, \\ \Pi(w_1 \cdots w_n) & = & \Pi(w_1) \cdots \Pi(w_n). \end{cases}$$

Non commutative finite difference operator

- The operator $\mathbf{T}_i$:

$$\begin{cases} \mathbf{T}_i(a_\omega) & = & a_\omega + a_i, \\ \mathbf{T}_i(a_j) & = & a_j, \qquad\qquad \text{for j in } \mathbb{N}^*. \end{cases}$$

  For each positive integer i, we have indeed $T \circ \Pi = \Pi \circ \mathbf{T}_i$.

- the operator $\mathbf{\Delta}_i$: $\mathbf{\Delta}_i = \mathbf{T}_i - Id_{\mathbb{R}<A>}$.
- Example: $w = a_2 a_2 a_\omega a_4 a_\omega a_\omega$.

$$\begin{aligned} \mathbf{\Delta}_6(w) &= a_2a_2a_6a_4a_\omega a_\omega + a_2a_2a_\omega a_4a_6a_\omega + a_2a_2a_\omega a_4a_\omega a_6 \\ &\quad + a_2a_2a_6a_4a_6a_\omega + a_2a_2a_\omega a_4a_6a_6 \\ &\quad + a_2a_2a_6a_4a_\omega a_6 + a_2a_2a_6a_4a_6a_6. \end{aligned}$$

Note that for a word w, the terms in $\mathbf{\Delta}_i(w)$ are exactly the words obtained by replacing at least one of the a_ω of w by an a_i.

Non commutative Gandhi polynomials

Let us define the non-commutative Gandhi polynomials as follows:

$$\begin{cases} \mathbf{C}_1 & = & 1, \\ \mathbf{C}_{n+1} & = & \boldsymbol{\Delta}_{2n}(\mathbf{C}_n a_\omega a_\omega), \quad \text{if } n > 1. \end{cases}$$

In particular, we have:

$$\begin{aligned} \mathbf{C}_2 &= a_2a_2 + a_2a_\omega + a_\omega a_2, \\ \mathbf{C}_3 &= a_2a_2a_4a_4 + a_2a_2a_\omega a_4 + a_2a_2a_4a_\omega + a_2a_4a_4a_4 + a_2a_\omega a_4a_4 \\ &+ a_2a_4a_\omega a_4 + a_2a_4a_4a_\omega + a_2a_4a_\omega a_\omega + a_2a_\omega a_4a_\omega \\ &+ a_2a_\omega a_\omega a_4 + a_4a_2a_4a_4 + a_\omega a_2a_4a_4 + a_4a_2a_\omega a_4 \\ &+ a_4a_2a_4a_\omega + a_4a_2a_\omega a_\omega + a_\omega a_2a_4a_\omega + a_\omega a_2a_\omega a_4. \end{aligned}$$

$$\begin{aligned} C_1(x) = &\ 1 \\ C_2(x) = &\ 2x + 1 \\ C_3(x) = &\ 6x^2 + 8x + 3 \\ &\ \dots \end{aligned}$$

Bijection between words in $\mathbf{C}_n$ and $\mathcal{P}_{2n}$

A simple bijection

$$\begin{array}{lcl} \mathcal{P}_{2n} & \longleftrightarrow & \mathbf{C}_n \\ 22846688 & \longleftrightarrow & a_2 a_2 a_\omega a_4 a_6 a_6 \end{array}$$

A way to split the set of $\mathcal{P}_{2n+2}$

For a surjective pistol of size $2n$, we set
$\mathcal{D}_p = \{p' \in \mathcal{P}_{2n+2} | p'_i = p_i, \text{ if } p_i < 2n, p'_i \geq 2n, \text{ otherwise}\}$.
For example,
for $p = 226466$, we have:

$$\begin{array}{rcl} \mathcal{D}_p & = & \{22846688,\ 22648688,\ 22646888,\ 22848688, \\ & & 22846888,\ 2264888,\ 22646688\} \end{array}.$$

We have:

$$\mathcal{P}_{2n+2} = \bigcup_{p \in \mathcal{P}_{2n}} \mathcal{D}_p.$$

Proof of $\mathcal{P}_{2n+2} = \bigcup_{p \in \mathcal{P}_{2n}} \mathcal{D}_p$

- If $p \neq p'$, then $\mathcal{D}_p \cap \mathcal{D}_{p'} = \emptyset$:

$$\begin{matrix} p_1 \cdots p_i \cdots p_{2n} \\ p'_1 \cdots p'_i \cdots p'_{2n} \end{matrix} \quad , \; p_i < p'_i \leq 2n.$$

Let q be in $\mathcal{D}_p$, and q' be in $\mathcal{D}_{p'}$. Then:

$$q_i = p_i, \; p'_i \leq q'_i \Longrightarrow q_i < q'_i.$$

- Let p' be in $\mathcal{P}_{2n+2}$:

$$\begin{matrix} p' & = & \cdots & (2n+2) & \cdots p'_i \cdots p'_{2n}(2n+2)(2n+2) \\ \longrightarrow & & \cdots & 2n & \cdots p'_i \cdots p'_{2n} = p. \end{matrix}$$

By construction, $p' \in \mathcal{D}_p$.

Correspondance between words and surjective pistols

Surjective pistols	Words
$p = 226466$ $\mathcal{D}_p = \{22846688,\ 22648688,\ 22646888,\ 22848688,\ 22846888,\ 2264888,\ 22646688\}$	$w = a_2 a_2 a_\omega a_4$ $\mathbf{\Delta}_6(w a_\omega a_\omega) = a_2 a_2 a_\omega a_4 a_6 a_6$ $+ a_2 a_2 a_6 a_4 a_\omega a_6 + a_2 a_2 a_6 a_4 a_6 a_\omega$ $+ a_2 a_2 a_\omega a_4 a_\omega a_6 + a_2 a_2 a_6 a_4 a_\omega a_\omega$ $+ a_2 a_2 a_\omega a_4 a_6 a_\omega + a_2 a_2 a_6 a_4 a_6 a_6$

For surjective pistols we have:

$$\mathcal{P}_{2n+2} = \bigcup_{p \in \mathcal{P}_{2n}} \mathcal{D}_p,$$

and for words, we have:

$$\mathbf{C}_{n+1} = \sum_{w \in C_n} \mathbf{\Delta}_{2n+2}(w a_\omega a_\omega).$$

The Dumont-Foata polynomials

The classical version

The Dumont-Foata polynomials are defined by induction as follows:

$$\begin{cases} DF_1(x,y,z) & = & 1, \\ DF_{n+1}(x,y,z) & = & DF_n(x+1,y,z)(x+z)(x+y) - DF_n(x,y,z)x^2, \end{cases}$$

Since $DF_n(1,1,1) = \#\mathcal{P}_{2n}$, the different variables x, y and z record some statistics on surjective pistols.

The non commutative version

$$\begin{cases} \mathcal{DF}_1 & = 1 \\ \mathcal{DF}_{n+1} & = \mathbf{T}_{2n}(\mathcal{DF}_n)(a_\omega + za_{2n})(a_\omega + ya_{2n}) - \mathcal{DF}_n a_\omega a_\omega, \quad \text{if } n \geq 1. \end{cases}$$

Dumont-Foata polynomials

The first polynomials are:

$$
\begin{aligned}
\mathcal{DF}_2 &= yz \cdot a_2 a_2 + z \cdot a_2 a_\omega + y \cdot a_\omega a_2 \\
\mathcal{DF}_3 &= y^2 z^2 \cdot a_2 a_2 a_4 a_4 + y^2 z \cdot a_2 a_2 a_\omega a_4 + yz^2 \cdot a_2 a_2 a_4 a_\omega \\
&+ yz^2 \cdot a_2 a_4 a_4 a_4 + yz^2 \cdot a_2 a_\omega a_4 a_4 + yz \cdot a_2 a_4 a_\omega a_4 \\
&+ z^2 \cdot a_2 a_4 a_4 a_\omega + z \cdot a_2 a_4 a_\omega a_\omega + z^2 \cdot a_2 a_\omega a_4 a_\omega \\
&+ yz \cdot a_2 a_\omega a_\omega a_4 + y^2 z \cdot a_4 a_2 a_4 a_4 + y^2 z \cdot a_\omega a_2 a_4 a_4 \\
&+ y^2 \cdot a_4 a_2 a_\omega a_4 + yz \cdot a_4 a_2 a_4 a_\omega \; y \cdot a_4 a_2 a_\omega a_\omega \\
&+ yz \cdot a_\omega a_2 a_4 a_\omega + y^2 \cdot a_\omega a_2 a_\omega a_4.
\end{aligned}
$$

Dumont-Foata polynomials

The first polynomials are:

$$
\begin{aligned}
\mathcal{DF}_2 &= yz \cdot a_2 a_2 + z \cdot a_2 a_\omega + y \cdot a_\omega a_2 \\
\mathcal{DF}_3 &= y^2 z^2 \cdot a_2 a_2 a_4 a_4 + y^2 z \cdot a_2 a_2 a_\omega a_4 + yz^2 \cdot a_2 a_2 a_4 a_\omega \\
&+ yz^2 \cdot a_2 a_4 a_4 a_4 + yz^2 \cdot a_2 a_\omega a_4 a_4 + yz \cdot a_2 a_4 a_\omega a_4 \\
&+ z^2 \cdot a_2 a_4 a_4 a_\omega + z \cdot a_2 a_4 a_\omega a_\omega + z^2 \cdot a_2 a_\omega a_4 a_\omega \\
&+ yz \cdot a_2 a_\omega a_\omega a_4 + y^2 z \cdot a_4 a_2 a_4 a_4 + y^2 z \cdot a_\omega a_2 a_4 a_4 \\
&+ y^2 \cdot a_4 a_2 a_\omega a_4 + yz \cdot a_4 a_2 a_4 a_\omega \; y \cdot a_4 a_2 a_\omega a_\omega \\
&+ yz \cdot a_\omega a_2 a_4 a_\omega + y^2 \cdot a_\omega a_2 a_\omega a_4.
\end{aligned}
$$

Dumont-Foata polynomials

The first polynomials are:

$$
\begin{aligned}
\mathcal{DF}_2 &= yz \cdot a_2 a_2 + z \cdot a_2 a_\omega + y \cdot a_\omega a_2 \\
\mathcal{DF}_3 &= y^2 z^2 \cdot a_2 a_2 a_4 a_4 + y^2 z \cdot a_2 a_2 a_\omega a_4 + yz^2 \cdot a_2 a_2 a_4 a_\omega \\
&+ yz^2 \cdot a_2 a_4 a_4 a_4 + yz^2 \cdot a_2 a_\omega a_4 a_4 + yz \cdot a_2 a_4 a_\omega a_4 \\
&+ z^2 \cdot a_2 a_4 a_4 a_\omega + z \cdot a_2 a_4 a_\omega a_\omega + z^2 \cdot a_2 a_\omega a_4 a_\omega \\
&+ yz \cdot a_2 a_\omega a_\omega a_4 + y^2 z \cdot a_4 a_2 a_4 a_4 + y^2 z \cdot a_\omega a_2 a_4 a_4 \\
&+ y^2 \cdot a_4 a_2 a_\omega a_4 + yz \cdot a_4 a_2 a_4 a_\omega \; y \cdot a_4 a_2 a_\omega a_\omega \\
&+ yz \cdot a_\omega a_2 a_4 a_\omega + y^2 \cdot a_\omega a_2 a_\omega a_4 .
\end{aligned}
$$

The Dumont-Foata polynomials

Definitions

Let p be a surjective pistol of size $2n$. A position $i \leq 2n-2$ is:

- a *fixed point* if $p(i) = i$,
- a *surfixed point* if $p(i) = i + 1$,
- a *max point* if $p(i) \geq p(j)\ \forall j \in \{1, \cdots, 2n\}$.

For example:

$p = 42468688$, 2 and 6 are fixed points, 3 is a surfixed point, and 5 is a max point. And we have: $fix(p) = 2$, $surfix(p) = 1$, and $max(p) = 1$.

Combinatorial interpretations

$$DF_n(x, y, z) = \sum_{p \in \mathcal{P}_{2n}} x^{max(p)} y^{fix(p)} z^{surfix(p)}.$$

q-analog

Definition

$$D_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0, \\ \Delta_q(D_n(x)x^2) & \text{if } n \geq 1, \end{cases}$$

where $\Delta_q(f)(x) = \frac{f(xq+1)-f(x)}{1+(q-1)x}$.

Examples

$$\begin{aligned} D_2 &= (q+1)x + 1, \\ D_3 &= (q^3 + 2q^2 + 2q + 1)x^2 + (2q^2 + 4q + 2)x + q + 2. \end{aligned}$$

In order to obtain a combinatorial interpretation of this q-analog, we have to find a linear map Π_q such that $(\Pi_q(\mathbf{C}_n))$ satisfies the same recurrence as (D_n).

A linear map for q-analog

Strategy

One way, is to find Π_q satisfying the following conditions for w corresponding to a surjective pistol p of size $2n$:

- $\Pi_q \circ \mathbf{\Delta}_{2n}(w a_\omega a_\omega) = \Delta_q(\Pi_q(w) x^2)$,
- $\Pi_q(w) = q^{s(p)} x^{max(p)}$,
- $\Pi_q(w a_\omega a_\omega) = \Pi_q(w) x^2$.

Examples on the first cases

for $n = 2$, $D_2 = (q+1)x + 1$:

	x	1
1	2444	2244
q	4244	

or

	x	1
1	4244	2244
q	2444	

.

A linear map for q-analog

For $n = 3$, $D_3 = (q^3 + 2q^2 + 2q + 1)x^2 + (2q^2 + 4q + 2)x + q + 2$:

	x^2	x	1	
$\mathcal{D}_{2244}$		224666	224466	1
		226466		q
$\mathcal{D}_{2444}$	246666	244666	244466	1
	264666	246466		q
		264466		q
	266466			q^2
$\mathcal{D}_{4244}$	426666	424666	424466	q
	624666	426466		q^2
		624466		q^2
	626466			q^3

A linear map for q-analog

For $n = 3$, $D_3 = (q^3 + 2q^2 + 2q + 1)x^2 + (2q^2 + 4q + 2)x + q + 2$:

	x^2	x	1	
$\mathcal{D}_{2244}$		224666	224466	1
		226466		q
$\mathcal{D}_{2444}$	246666	244666	244466	1
	264666	246466		q
		264466		q
	266466			q^2
$\mathcal{D}_{4244}$	426666	424666	424466	q
	624666	426466		q^2
		624466		q^2
	626466			q^3

A linear map for q-analog

For $n = 3$, $D_3 = (q^3 + 2q^2 + 2q + 1)x^2 + (2q^2 + 4q + 2)x + q + 2$:

	x^2	x	1	
$\mathcal{D}_{2244}$		224666	224466	1
		226466		q
$\mathcal{D}_{2444}$	246666	244666	244466	1
	264666	246466		q
		264466		q
	266466			q^2
$\mathcal{D}_{4244}$	426666	424666	424466	q
	624666	426466		q^2
		624466		q^2
	626466			q^3

A linear map for q-analog

Special inversions

A special inversion is a pair (i, j) such that $i > j$, $p_i < p_j$, and i is the rightmost position corresponding to the value p_i. We denote by $sinv(p)$ the number of special inversions of a surjective pistol p.

A definition of Π_q

Let w be in $\mathbf{C}_n$, and p be the corresponding surjective pistol.
We define:

$$\Pi_q(w) = q^{sinv(p)} \Pi(w).$$

Combinatorial interpretation of D_n

$$D_n(x) = \sum_{p \in \mathcal{P}_{2n}} q^{sinv(p)} x^{max(p)}.$$