Schur coefficients of the integral form Macdonald polynomials

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Abstract

In this talk, we consider the combinatorial formula for the Schur coefficients of the integral form of the Macdonald polynomials. As an attempt to prove Haglund's conjecture that $\left\langle \frac{J_\lambda[X;q,q^k]}{(1-q)^n}, s_\mu(X) \right\rangle \in \mathbb{N}[q]$, we have found explicit combinatorial formula for the Schur coefficients in one row case, two column case and certain hook shape cases. A result of Egge-Loehr-Warrington ('2010) gives a combinatorial way of getting Schur expansion of symmetric functions when the expansion of the function in terms of Gessel's fundamental quasi symmetric functions is known. We apply this result to the combinatorial formula for the integral form Macdonald polynomials of Haglund-Haiman-Loehr in quasisymmetric functions to prove the Haglund's conjecture in general cases.

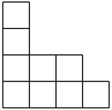
Preliminaries

- A *partition* λ of n is a sequence of weakly decreasing positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $|\lambda| = \sum_{i=1}^{k} \lambda_i = n$. (Notation: $\lambda \vdash n$).

- We identify a partition with the corresponding Young diagram :

(Example $\lambda = (4,3,1,1) \iff$ )

- Dominance order : for $\lambda, \mu \vdash n$, $\lambda \geq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \text{ for all } i.$$

Let λ be a partition.

- A **semi standard Young tableau** (SSYT) T of shape λ is a filling of the diagram of λ with positive integers which is weakly increasing in every row and strictly increasing in every column.
- T has *weight* $\alpha = (\alpha_1, \alpha_2, \dots)$ if T has α_i parts equal to i.
- A **standard Young tableau** (SYT) is a SSYT which contains each number $1, 2, \dots, n$ exactly once, so that its weight is $(1, 1, \dots, 1)$.

The combinatorial definition of Schur functions

A *symmetric function* is a polynomial $f(x_1, x_2, \ldots, x_n)$ which is invariant under the action of the symmetric group, i.e.,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n),$$

or, $\sigma f = f$, for all $\sigma \in S_n$.

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Definition

The *Schur function* $s_\lambda(X)$ of shape λ is the formal power series

$$s_\lambda(X) = \sum_T X^T,$$

summed over all SSYT's T of shape λ.

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summed over all SSYT's T of shape λ.

<u>Note</u>. For any partition λ, the Schur function $s_\lambda(X)$ is a symmetric function.

Let $K_{\lambda\alpha}$ (called **Kostka numbers**) denote the number of SSYT's of shape λ with weight α.

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Note. By the symmetry of the Schur function, we have

$$K_{\lambda\alpha} = K_{\lambda\tilde{\alpha}},$$

where $\tilde{\alpha}$ is any rearrangement of α.

For any partition λ, $|\lambda| = \sum_i \lambda_i = n$,

$$s_\lambda = \sum_{\alpha} K_{\lambda\alpha} X^\alpha = \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu(X),$$

where α ranges over all the weak compositions of n.

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Remark

Note that the Schur functions s_λ are characterized by the following two properties :

(a) *(triangularity)* $s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu$

(b) *(orthogonality)* $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$,
where $\langle\ ,\ \rangle$ *is the Hall scalar product.*

Macdonald Polynomials

Macdonald Polynomials

Theorem ('88, Macdonald)

Given a partition λ, there exists a unique symmetric polynomial $P_\lambda(X; q, t)$ characterized by the following two properties :

(a) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} \xi_{\lambda\mu}(q,t) m_\mu$

(b) $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ *if* $\lambda \neq \mu$, *where*

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda(q,t),$$

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \quad z_\lambda = \prod_{i\geq 1} i^{m_i(\lambda)} \cdot m_i(\lambda)!,$$

where $m_i(\lambda)$ *is the number of parts of* λ *equal to* i.

Properties

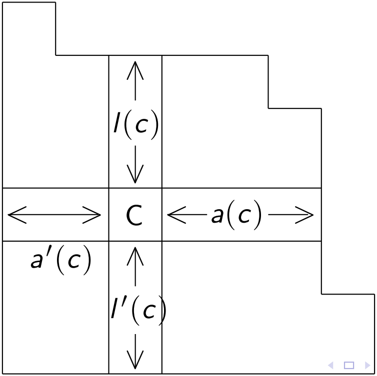
- $q = t$: $P_\lambda(X; t, t) = s_\lambda$; Schur functions
- $t = 1$: $P_\lambda(X; q, 1) = m_\lambda$; monomial symmetric functions
- $q = 1$: $P_\lambda(X; 1, t) = e_{\lambda'}$; elementary symmetric functions
- $\lambda = (1^n)$: $P_{(1^n)}(X; q, t) = e_n(X) = s_{1^n}(X)$
- $q = 0$: $P_\lambda(X; 0, t) = P_\lambda(X; t)$; Hall-Littlewood polynomials

Integral form of Macdonald polynomials

Macdonald also defined the **integral form**

$$J_\mu[X;q,t] = h_\mu(q,t)P_\mu[X;q,t]$$

where $h_\mu(q,t) = \prod_{c\in\mu}(1-q^{a(c)}t^{l(c)+1})$, and for $c \in \mu$, *leg* $l(c)$, *arm* $a(c)$ *(and coleg* $l'(c)$*, coarm* $a'(c)$*)* are defined as follows.



Aside: q, t-Kostka polynomials

In terms of the *modified* Schur functions $s_\lambda[X(1-t)]$,

$$J_\mu[X;q,t] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q,t) s_\lambda[X(1-t)],$$

where $K_{\lambda\mu}(q,t)$ is the q, t-**Kostka polynomials**, and $f[X(1-t)]$ is the image of f under the algebra homomorphism mapping $p_k(X)$ to $(1-t^k)p_k(X)$.

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Macdonald's Positivity Conjecture

$$K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t].$$

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Theorem ('02, Haiman ; '07 Assaf)

$$K_{\lambda\mu}(q, t) \in \mathbb{N}[q, t].$$

Schur Coefficients Conjecture

Conjecture [Haglund]

$$\left\langle \frac{J_\mu[X;q,q^k]}{(1-q)^n}, s_\lambda[x] \right\rangle \in \mathbb{N}[q]$$

Aside

When $t = 0$,

$$J_\mu(X;q,0) = P_\mu(X;q,0) = \sum_{\lambda \vdash \mu} K_{\lambda'\mu'}(q) s_\lambda.$$

Sanderson('98) showed that $K_{\lambda\mu}(q)$ have positive coefficients by realizing $P_\mu(X;q,0)$ as the character of the Demazure module $E_w(\Lambda_0)$.

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Since s_λ is a character of an irreducible $sl(n)$-module, the coefficient of q^j in $K_{\lambda'\mu'}(q)$ is the *multiplicity* of the $sl(n)$-module of highest weight $\mu - j\delta$ in $E_w(\Lambda_0)$.

Partial results from before

Theorem ('12, Y)

When $\mu = (r)$, J_μ has the following Schur expansion

$$\begin{aligned} J_{(r)}[X;q,t] &= \sum_{\lambda \vdash r} \left[\prod_{c \in \lambda} (1 - q^{a'(c) - l'(c)} t) \right] \left(\sum_{T \in SYT(\lambda')} q^{ch(T)} \right) s_\lambda[X] \\ &= \sum_{\lambda \vdash r} \left[\prod_{c \in \lambda} (1 - q^{a'(c) - l'(c)} t) \right] K_{\lambda', 1^r}(q) s_\lambda[X], \end{aligned}$$

where $K_{\lambda\mu}(q)$ is the Kostka-Foulkes polynomial.

Proof.

By using $J_{(r)}[X;q,t] = (t;q)_r P_{(r)}[X;q,t] = (q;q)_r g_r(X;q,t)$. □

$J_{(r,1^s)}$ Formula

Theorem ('12, Y)

For $\mu = (r, 1^s)$, *with* $n = r + s, s \geq r - 3$ *(i.e.,* $r \leq \frac{n+1}{2} + 1$*), we have*

$$J_{(r,1^s)}$$
$$= (t;t)_s \sum_{\substack{\lambda \vdash n \\ \lambda \leqslant \mu}} \left[\prod_{c \in 1^{l(\lambda)}/1^{s+1}} (1 - q^{-l'(c)-1} t) \cdot \prod_{c \in \lambda / 1^{l(\lambda)}} (1 - q^{a'(c)-l'(c)} t) \right]$$
$$\times \quad (1 - q^{n-l(\lambda)} t^{s+1}) \left(\sum_{T \in SSYT(\lambda', \mu')} q^{ch(T)} \right) s_\lambda .$$

Proof.

By showing the recursion formula. □

Two Column Formula

Theorem ('12, Y)

$$J_{2^b1^{a-b}}[x;q,t]$$

$$= \sum_{k=0}^{b} \left[\frac{(t;t)_{a-b+k}(t;t)_b(q;t)_{a+1}(q-t)(q-t^2)\cdots(q-t^k)}{(t;t)_k(q;t)_{a-b+k+1}} \right] s_{2^{b-k}1^{a-b+2k}}($$

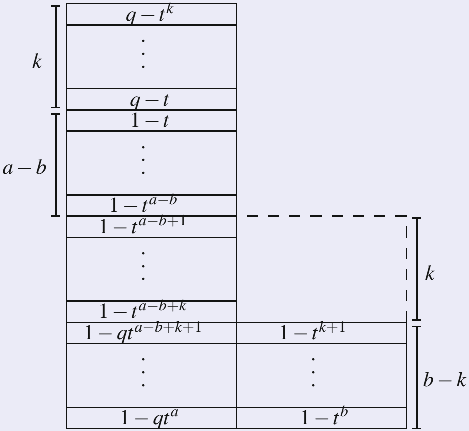
$$= \sum_{k=0}^{b} \left[\frac{(t;t)_{a-b+k}(t;t)_b(q;t)_{a+1}(q^{-1}t;t)_k}{(t;t)_k(q;t)_{a-b+k+1}} \cdot q^k \right] s_{2^{b-k}1^{a-b+2k}}(x).$$

Proof.

By using the Pieri rule of Macdonald polynomials. □

Remark

We can interpret the coefficient of $s_\lambda[X]$ by assigning weights in each cell of λ.



From Quasisymmetric expansion to Schur expansion

(Egge-Loehr-Warrington method)

The bases s_λ and Q_α

- $\alpha \vDash n$: α is a *composition* of n

Definition

$f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is **quasisymmetric** if for any $a_1, \ldots, a_k \in \mathbb{P}$,

$$[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$.

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whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$.

Example

The **monomial quasisymmetric** function M_α is defined by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

- QSym_n is a vector space with a basis $\{F_\alpha : \alpha \vDash n\}$ indexed by *compositions* α of n : F_α is called a **fundamental quasisymmetric function** :

$$F_\alpha = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

  where $S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}$.
- Sym_n is a vector space with a basis $\{s_\lambda : \lambda \vdash n\}$ indexed by partitions λ of n.
- Sym_n is a subspace of QSym_n.

Given $T \in \mathrm{SYT}(\lambda)$, the **reading word** $rw(T)$ is the sequence of entries in T, read in order from left to right in rows, from top to bottom.

- $\mathrm{Des}(T) = \{i \in [n-1] : i+1 \text{ appears to the left of } i \text{ in } rw(T)\}$.
- $\mathrm{Des}'(T) = (i_1, i_2 - i_1, i_3 - i_2, \ldots, n - i_k) \vDash n$, for given $\mathrm{Des}(T) = \{i_1 < i_2 < \cdots < i_k\}$.

Example

For $T \in \mathrm{SYT}(4,3,2)$,

$T =$

8	9		
3	5	7	
1	2	4	6

$rw(T) = 893571246$, $\mathrm{Des}(T) = \{2,4,6,7\}$, $\mathrm{Des}'(T) = (2,2,2,1,2)$

Theorem ('83, Gessel)

$$s_\lambda(X) = \sum_{T \in SYT(\lambda)} F_{Des'(T)}(X).$$

The Kostka matrix

- The *Kostka matrix* K_n is a rectangular matrix of order $p(n) \times c(n)$ with entries

$$K_n(\lambda, \alpha) = K_{\lambda\alpha} = |\text{SSYT}(\lambda, \alpha)|; \text{ Kostka numbers },$$

for $\lambda \vdash n$, $\alpha \vDash n$.

Define

- M_n as a $p(n) \times c(n)$ matrix with

$$M_n(\lambda, \alpha) = |\{T \in \text{ SYT } (\lambda) : \text{Des}'(T) = \alpha\}|$$

- A_n as a $c(n) \times c(n)$ matrix with

$$A_n(\alpha, \beta) = \chi(\beta \text{ is finer than } \alpha).$$

Lemma

For all $n \geq 1$, $M_n A_n = K_n$.

The inverse Kostka matrix

- The *inverse Kostka* matrix K'_n of order $c(n) \times p(n)$ is defined to be a right inverse of K_n with entries $K'_n(\alpha, \lambda)$=the sum of the signs of the special rim-hook tableaux of shape λ that have nonzero rim-hook lengths *in order from bottom to top* given by $\alpha_1, \cdots, \alpha_{l(\alpha)}$, where the sign of a rim-hook spanning r rows is $(-1)^{r-1}$, and the sign of a rim-hook tableau is the product of the signs of the rim-hooks in it.

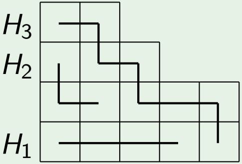
Theorem ('90, Eğecioğlu, Remmel)

For all $n \geq 1$,

$$K_n K'_n = I_{p(n)}$$

Example

For $\lambda = (5, 5, 3, 2)$ and $\alpha = (4, 3, 8)$, we have $K'_{15}(\alpha, \lambda) = +1$.



$\mathrm{sign}(H_1) = (-1)^{1-1} = 1$, $\mathrm{sign}(H_2) = (-1)^{2-1} = -1$,
$\mathrm{sign}(H_3) = (-1)^{4-1} = -1$
$\Rightarrow$ The sign of the rim-hook tableau is $1 \cdot (-1) \cdot (-1) = 1$.

Schur versus quasisymmetric expansions

Define K_n^* to be a $c(n) \times p(n)$ matrix with

$$K_n^*(\alpha, \lambda) = \sum_{\beta \text{ finer than } \alpha} K_n'(\beta, \lambda).$$

Theorem ('10, Egge-Loehr-Warrington)

Suppose $\mathbb{F}$ is a field, and we have a symmetric function

$$f = \sum_{\lambda \vdash n} x_\lambda s_\lambda = \sum_{\alpha \vDash n} y_\alpha F_\alpha$$

with $x_\lambda, y_\alpha \in \mathbb{F}$.
Then the row vectors $\mathbf{x} = (x_\lambda : \lambda \vdash n)$ and $\mathbf{y} = (y_\alpha : \alpha \vDash n)$ satisfy

$$\mathbf{x} M_n = \mathbf{y} \quad \textit{and} \quad \mathbf{x} = \mathbf{y} K_n^*.$$

Thus, $x_\lambda = \sum_{\alpha \vDash n} y_\alpha K_n^*(\alpha, \lambda)$ for all $\lambda \vdash n$.

The combinatorial meaning of K_n^*

We say that a rim-hook tableau S of shape λ and content α is *flat* if each rim-hook of S contains exactly one cell in the first column of the Ferrers diagram of λ.

Theorem ('10, Egge-Loehr-Warrington)

Let $\alpha \vDash n$, $\lambda \vdash n$. If (α, λ) is flat, then

$$K_n^*(\alpha, \lambda) = K_n'(\alpha, \lambda) = \pm 1.$$

Otherwise, $K_n^*(\alpha, \lambda) = 0$. In particular, $K_n^*(\alpha, \lambda) = \chi(\alpha = \lambda)$ when λ is a hook.

Back to Haglund's conjecture

Haglund's combinatorial formula

Theorem (Haglund '08)

$$J_\mu(X;q,t) = \sum_{\substack{\omega \in S_n \\ primary}} F_{Des'(\omega)}(X) \times \prod_{s \in \mu} \left(q^{inv_s(\omega,\mu)} t^{nondes_s(\omega,\mu)} - q^{coinv_s(\omega,\mu)} t^{1+maj_s(\omega,\mu)} \right),$$

where primary means that the entry i occurs in the first i rows of μ,

$$nondes_s(\omega,\mu) = \begin{cases} leg(s)+1, & \text{if } \omega(South(s)) \geq \omega(s) \text{ and } South(s) \in \mu \\ 0, & \text{otherwise} \end{cases},$$

$$maj_s(\omega,\mu) = \begin{cases} leg(s), & \text{if } \omega(North(s)) > \omega(s) \\ 0, & \text{otherwise} \end{cases}$$

Schur coefficients of $J_\mu(X;q,t)$

Let

$$D_{(\omega,\mu)}(q,t)=\prod_{s\in\mu}\left(q^{\mathrm{inv}_s(\omega,\mu)}t^{\mathrm{nondes}_s(\omega,\mu)}-q^{\mathrm{coinv}_s(\omega,\mu)}t^{1+\mathrm{maj}_s(\omega,\mu)}\right).$$

Then,

$$\begin{aligned}J_\mu(X;q,t)&=\sum_{\substack{\omega\in S_n\\ \text{primary}}}D_{(\omega,\mu)}(q,t)F_{\mathrm{Des}'(\omega)}(X)\\&=\sum_{\lambda\vdash n}\left(\sum_{\substack{\omega\in S_n\\ \text{primary}}}D_{(\omega,\mu)}(q,t)K_n'(\mathrm{Des}'(\omega),\lambda)\right)s_\lambda,\end{aligned}$$

where $K_n'(\mathrm{Des}'(\omega),\lambda)$ is ±1 up to the sign of the flat special rim-hook tableau.

$\langle J_\mu, s_\lambda \rangle$ when λ is a hook

Egge-Loehr-Warrington's theorem $\Rightarrow K_n^*(\alpha, \lambda) = \chi(\alpha = \lambda)$ if λ is a hook.
Say $\lambda = (k+1, 1^{r-k})$, for $n = r+1$. Then

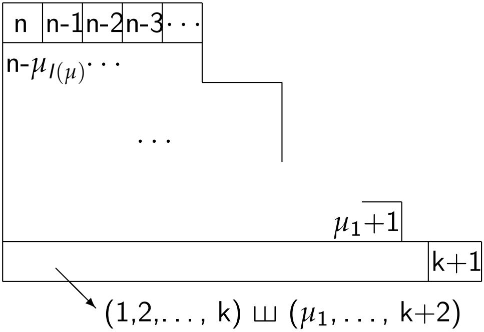
$$\alpha = \lambda = (k+1, 1^{r-k}) \Rightarrow Des(\omega) = \{k+1, k+2, \ldots, n-1\}$$

Thus

$$\omega = [(1, 2, \ldots, k) \sqcup\!\sqcup (n, n-1, \ldots, k+2)] \| k+1,$$

where $\sqcup\!\sqcup$ denotes the shuffle product and $\|$ means the concatenation.

Filling μ with w



$\langle J_\mu, s_{(k+1,1^{n-k-1})} \rangle$

Theorem

$$\langle J_\mu, s_{(k+1,1^{n-k-1})} \rangle = \left[\prod_{i=1}^{\mu_1} q^{(i-1)(\mu_i'-1)} (q^{-(i-1)}t;t)_{\mu_i'-1} \right]$$
$$\times \prod_{\mu_i'>1} (1-q^{k-i+1}t^{\mu_i'}) \cdot (q^{-(\mu_1-k)-1}t;q)_m \cdot q^{\binom{\mu_1-k}{2}} \begin{bmatrix} \mu_1 - 1 \\ k \end{bmatrix}_q ,$$

where $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$ *and*
$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{n-k+1};q)_k}{(q;q)_k}$.

Further results on $J_{(b,a)}(X;q,t)$

Proposition

$$\langle J_{(b,a)}, s_{(b-k,a+k)} \rangle = (t;q)_a (q^{b-a}t^2;q)_a (q^{-1}t;q)_k (t;q)_{b-a-k} q^k \begin{bmatrix} b-a+1 \\ k \end{bmatrix}_q .$$

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Idea of proof.

There are only two special flat rim-hook tableaux of shape $\lambda = (b-k, a+k)$:

Due to the primary condition and the previous lemma, the filling in the first a-columns are fixed, and the difference in the tail part factors nicely.

Proposition

$$\langle J_{(b,a)}, s_{(b,a-k,1^k)} \rangle = (q^{-k}t;q)_a (q^{b-a}t^2;q)_a (t;q)_{b-a} \cdot q^{\binom{k+1}{2}} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q$$

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Idea of proof.

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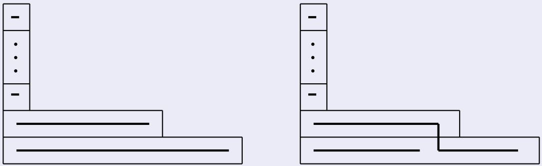

But over the fillings coming from the second tableau, $D_{(\omega,\mu)}(q,t) = 0$ due to the previous lemma. □

Proposition

$$\langle J_{(b,a)}, s_{(b-1,a-k,1^{k+1})}\rangle = (t;q^{-1})_{k+1}(t;q)_{a-k-1}(q^{b-a}t^2;q)_{a-1} \times (q^{-1}t;q)_{b-a}(1-q^{b-k-2}t^2)q^{\binom{k+1}{2}+1}\begin{bmatrix} a \\ k+1 \end{bmatrix}_q [b-a+k]_q$$

$$\langle J_{(b,a)}, s_{(b-k-1,a+k,1)}\rangle$$

$$= (1-t)(t;q)_{a-1}(t;q)_{k-1}(q^{b-a}t^2;q)_{a-1}(q^{-1}t;q)_{b-a-k}[a]_q q^{k+1} \times \begin{bmatrix} b-a \\ k \end{bmatrix}_q \frac{[b-a-2k]_q}{[b-a-k+1]_q[k+1]_q} \times \{(1-q^{b-a-k-1}t)(1-q^{b-1}t^2)[k]_q + q^{k-1}(q-t)(1-q^{b-k-2}t^2)[b-a-k+1]_q\}$$
$$+ (t;q)_a(q^{b-a}t^2;q)_a(t;q)_{k-1}(t;q)_{b-a-k-1}(q-t)(q^2-t)q^{k-1} \times \frac{[k]_q[b-a-2k]_q}{[b-a-k+1]_q}\begin{bmatrix} b-a \\ k+1 \end{bmatrix}_q$$

Rectangular Shape Case

Proposition

$$
\begin{aligned}
\langle J_{(b,b)}, s_{(b-r,b-s,1^{r+s})}\rangle = \sum_{k=1}^{r} \Big\{ & (t;q^{-1})_{s+k}(t;q)_{b-s-k} \\
& \times (q^{b-s-r-1}t^2;q)_{s+1}(t^2;q)_{b-s-1} q^{r+\binom{s+k}{2}+\binom{r-k+1}{2}} \\
& \times \frac{[s-r+1]_q}{[r]_q} \begin{bmatrix} b \\ s+k \end{bmatrix}_q \begin{bmatrix} s+k \\ s+1 \end{bmatrix}_q \begin{bmatrix} s \\ r-k \end{bmatrix}_q \Big\} \\
& + \chi(r+s>b)(t;q^{-1})_b(t^2;q))_{b-s-1}(q^{b-r-s-1}t^2;q)_{s+1} \\
& \times q^{r+\binom{b}{2}+\binom{r+s-b+1}{2}} \begin{bmatrix} b \\ r \end{bmatrix}_q \begin{bmatrix} r-1 \\ b-s-1 \end{bmatrix}_q \frac{[s-r+1]_q}{[s+1]_q}
\end{aligned}
$$

Proposition

$$\langle J_{(b^m)}(X;q,t), s_{(b-r,b-s,1^{(r+s+b(m-2))})}\rangle = \left(\prod_{i=1}^{m-2}(q^{-b+1}t^i;q)_b \cdot q^{\binom{b}{2}}\right) \langle J_{(b,b)}, s_{(b-r,b-s,1^{r+s})}\rangle\Big|_{\substack{t\to t^{m-1}\\ t^2\to t^m}},$$

where $\cdot\Big|_{\substack{t\to t^{m-1}\\ t^2\to t^m}}$ *means to replace a single* t *by* t^{m-1} *and* t^2 *by* t^m*.*

Proposition

$$\langle J_{(b^m)}(X;q,t), s_{(b-r,b-s,1^{(r+s+b(m-2))})}\rangle = \left(\prod_{i=1}^{m-2}(q^{-b+1}t^i;q)_b \cdot q^{\binom{b}{2}}\right) \langle J_{(b,b)}, s_{(b-r,b-s,1^{r+s})}\rangle \Big|_{\substack{t\to t^{m-1}\\ t^2\to t^m}},$$

where $\cdot\Big|_{\substack{t\to t^{m-1}\\ t^2\to t^m}}$ *means to replace a single* t *by* t^{m-1} *and* t^2 *by* t^m.

Idea of proof.

The existence of the triple, for $a < b < c$,

makes the factor $\left(q^{\mathrm{inv}_a(w,\mu)}t^{\mathrm{nondes}_a(w,\mu)} - q^{\mathrm{coinv}_a(w,\mu)}t^{1+\mathrm{maj}_a(w,\mu)}\right)$ to be 0. □

Remark

We can apply the same argument to the coefficient of $s_{(b+\alpha-r,b+\beta-s,1^{(r+s+b(m-2))})}$ *in the expansion of* J_μ *when* $\mu = (b+\alpha, b+\beta, b^m)$ *and get*

$$\langle J_{(b+\alpha,b+\beta,b^m)}(X;q,t), s_{(b+\alpha-r,b+\beta-s,1^{(r+s+bm)})}\rangle = \left(\prod_{i=1}^{m}(q^{-b+1}t^i;q)_b \cdot q^{\binom{b}{2}}\right) \langle J_{(b+\alpha,b+\beta)}, s_{(b+\alpha-r,b+\beta-s,1^{r+s})}\rangle\Big|_{\substack{t\to t^{m+1}\\ t^2\to t^{m+2}}}.$$

Thank You