HABILITATIONSSCHRIFT

A Nonlinear Distributional Geometry

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Contents

1. *The ultrarelativistic Reissner-Nordstrøm field in the Colombeau algebra*, J. Math. Phys., **38**(3):1614–1622 (1997).

2. *A note on the Penrose junction conditions*, Class. Quant. Grav., **16**(4): 1255–1264 (1999). (with M. Kunzinger)

3. *On the foundations of nonlinear generalized functions I, II*, Mem. Amer. Math. Soc., **153**(729) (2001). (with M. Grosser, E. Farkas and M. Kunzinger)

4. *A global theory of algebras of generalized functions*, Adv. Math., **166**: 50–72 (2002). (with M. Grosser, M. Kunzinger and J. Vickers)

5. *Foundations of a nonlinear distributional geometry*, Acta Appl. Math., **71**:179–206 (2002). (with M. Kunzinger)

6. *Generalized pseudo-Riemannian geometry*, Trans. Amer. Math. Soc. **354**(10):4179–4199 (2002). (with M. Kunzinger)

7. *Intrinsic characterization of manifold–valued generalized functions*, Proc. London Math. Soc., to appear, 2002. (with M. Kunzinger and J. Vickers)

Summary

The work contained in this Habilitationsschrift is centered around the theory of nonlinear generalized functions in a geometric context. It is spanning an arc from applications in mathematical general relativity to the construction of a global theory of (full) diffeomorphism invariant Colombeau algebras on manifolds as well as of spaces of generalized sections (in the special version of the theory) in vector bundles and generalized functions taking values in a manifold.

Introduction

Algebras of generalized functions in the sense of J. F. Colombeau (Colombeau algebras) ([Col84, Col85, Col90, Obe92]) are differential algebras containing the vector space of Schwartz distributions as a subspace while at the same time preserving maximal consistency with respect to classical analysis—as far as possible in the light of L. Schwartz impossibility result ([Sch54]). In particular, derivatives and product extend the classical operations on distributions and smooth functions, respectively. The basic idea of their construction consists in using regularization by nets of smooth functions and of asymptotic estimates in terms of a regularization parameter.
Colombeau algebras were discovered in the early 1980ies in the context of infinite dimensional calculus in locally convex spaces and soon turned out to be an applicable tool in a number of situations involving

(i) differentiation

(ii) nonlinear operations

(iii) singular objects (e.g. non-differentiable functions, distributions).

In particular, first applications came from the field of nonlinear PDEs [Bia92a, Bia92b, Col93, Col94a, Key95] and their numerics [Bia90, Ber95, Ber93]. In the mid 1990ies, however, also applications in mathematical physics and, in particular, in general relativity emerged.

The first of these applications in relativity, considering aspects of ultrarelativistic black hole geometries of the Kerr-Newman family, appeared in paper 1[1]. Such space-times arise by applying a boost with respect to a static observer to the metric and then letting the boost velocity approach the speed of

[1] Numbers refer to the table of contents on page 1.

light (see [Aic71]). In the case of the Reissner-Nordstrøm metric—the unique spherically symmetric solution of the Einstein-Maxwell equations in vacuum, which is the focus of paper 1—a physically motivated scaling of the mass and charge parameters is needed to obtain a distributional metric in the limit (cf. [Lou90]). This scaling in turn forces the electromagnetic field to vanish in the $\mathcal{D}'$-limit while the energy momentum tensor converges to a surface measure supported on a hypersurface orthogonal to the direction of the boost. Since the energy-momentum tensor is roughly given by the square of the electromagnetic field tensor, this situation clearly demonstrates the failure of linear distributional methods and the need for a more refined modeling which in turn is provided by the use of Colombeau generalized functions.
Further applications of algebras of generalized functions in general relativity focused upon the study of conical singularities (e.g. [Cla96, Wil97]) and impulsive pp-waves ([Bal97, Ste98, Kun99]); for an overview see [Vic99a]. In particular, in [Ste98, Kun99] a complete distributional description of impulsive pp-waves—a class of singular space-times where the curvature is concentrated on a null-hypersurface (it contains the ultrarelativistic geometries mentioned above)—has been accomplished. At the heart of these works lies an existence and uniqueness theorem for the geodesic equation in nonlinear generalized functions. In particular, in paper 2 this result was applied to study the relation between two commonly used forms of the line element of pp-waves, namely the (distributional) Brinkmann form and the (continuous) Rosen form. A physically well-motivated but discontinuous (hence ill-defined) transformation between these two forms of the metric introduced by R. Penrose ([Pen72]) indicates the "physical equivalence" of both line elements. Using a global univalence theorem of Gale and Nikaido ([Gal65]) it could be shown that the transformation constitutes a "generalized diffeomorphism" thereby clarifying the exact mathematical meaning of this physical equivalence.

These applications in general relativity were one of the main driving forces leading to an interconnected series of theoretical advances within the theory of nonlinear generalized functions and further applications: Applying algebras of generalized functions in a genuine geometrical context like general relativity adds another requirement to the list introduced above, namely

(iv) diffeomorphism invariance.

Indeed, for a long time, all known variants of *full*—distinguished by a canonical embedding of the space of distributions—Colombeau algebras lacked the feature of diffeomorphism invariance: some of the basic building blocks of the

construction failed to be invariant under the natural action of a diffeomorphism. As long as there was no diffeomorphism invariant construction of an algebra of generalized functions available there remained the serious objection that there was no way of constructing such algebras on a manifold based on intrinsic terms only. Therefore the applicability of nonlinear generalized functions in any geometric context was highly questionable. In particular, in general relativity the diffeomorphism invariance of the results (i.e., their independence of the coordinate system in which the regularization had been carried out) had to be established "by hand" (cf. [Vic99b]).
The first crucial steps towards a diffeomorphism invariant Colombeau algebra were done by Colombeau and Meril in 1994 ([Col94b]). As an unavoidable tool they had to (re-)introduce infinite dimensional calculus into their construction which they claimed to be diffeomorphism invariant. However, in 1998 J. Jelínek ([Jel99]) pointed out an error in their construction by giving an explicit counter-example and presented another version of the theory. Although avoiding most of the shortcomings of [Col94b] Jelínek's algebra still fell short of establishing diffeomorphism invariance. In particular, he failed in showing that his construction preserves the product of smooth functions while at the same time allowing for a diffeomorphism invariant characterization of the ideal of negligible functions.
This difficulty was finally overcome in paper 3 where the first diffeomorphism invariant Colombeau algebra is presented together with the complete theoretical and technical background. To deal with the unavoidable infinite dimensional calculus on (certain non-Banach) locally convex spaces the construction is built upon calculus in convenient vector spaces ([Kri97]) rather than Silva-differentiability employed so far. In addition to reducing the (still considerable) technical complexity of the construction this decision provides decisive advantages in applications to PDEs (see chapter 11 of paper 3). The construction yields a fine sheaf of differential algebras on open sets of Euclidean space possessing all the properties (i)-(iv) while at the same time providing a canonical embedding of distributions commuting with partial derivatives.

However, applications in relativity add yet one further demand to the above list, namely

(v) intrinsic definition.

More precisely, the construction should be geometric in the sense that all the basic objects should be intrinsically defined on the manifold and the resulting algebra should be a differential algebra with the Lie derivative w. r. t. smooth vector fields commuting with the canonical embedding of distributions.

Such a construction is presented in paper 4. The key ingredients of the local construction of paper 3, i.e., operations like smoothing via convolution or asymptotic vanishing of moments are rephrased to allow for a coordinate invariant description. The notion of smoothing kernels introduced to this end serves to isolate the diffeomorphism-invariant essence of these procedures in the manifold setting. Thus an intrinsic theory of full Colombeau algebras on manifolds enjoying all the distinguishing features (i)–(v), i.e., a global (full) algebra of generalized functions is developed.

Special Colombeau algebras, although lacking a canonical embedding of the space of distributions, allow to model singularities in a nonlinear context in a flexible and efficient way and moreover lend themselves in a very natural way to geometric applications. Hence in any situation where one is willing to do without such a canonical embedding they provide a suitable setting.
Based upon earlier approaches ([De 91]) a systematic study of global analysis in special algebras of generalized functions has been initiated in paper 5. Spaces of generalized sections of vector bundles are introduced and their algebraic structure is studied. In particular, it is shown that the module—over the algebra of generalized functions—of generalized sections is finitely generated and projective. Classical notions of tensor analysis like Lie derivatives (with respect to smooth and generalized vector fields), exterior algebra etc. are extended to the level of generalized functions and a point value description of such functions is established. Finally, the foundations of Hamiltonian mechanics in this setting are laid.
In paper 6, on the other hand, the basic notions of pseudo-Riemannian geometry in generalized functions are introduced: generalized metrics, connections and curvature. In particular, a generalized "Fundamental Lemma of (semi-) Riemannian geometry" is proved and the notion of geodesics of a generalized metric is introduced (understood as generalized curves taking values in a manifold; see below). Finally the paper returns to the field where much of its motivation originated: a guideline to applications in general relativity is given.
Due to the inherent nonlinearities involved, many concepts generalized in papers 5 and 6 lack a distributional counterpart: no comparable theory is (or could be) available. In those cases, however, where distributional analogs—mainly due to G. De Rham ([De 84]) and J. Marsden ([Mar68])—exist, consistency results with respect to the linear setting are given.
One case of particular interest where such analogs do not exist is the notion of generalized functions *taking values* in a differentiable manifold. The need for such functions arises upon considering e.g. flows of generalized vector

fields. Generalized functions taking values in a manifold as well as generalized vector bundle homomorphisms were introduced in [Kun02]. In paper 7 this approach is extended to a functorial theory. Several characterization results are established providing a global approach that in turn is exploited to derive a point value characterization of such functions and to show that composition can be carried out unrestrictedly.

Summarizing, the work presented here provides a nonlinear distributional geometry, i.e., an extension and generalization of many distributional concepts to genuine nonlinear situations in a geometric context. It is well-suited to applications in mathematical physics, in particular, general relativity and non-smooth mechanics. It also opens future lines of research in applications (spherical impulsive gravitational waves, shell crossing singularities in dust solutions of Einstein equations, etc.) as well as in the theory of algebras of generalized functions itself (generalized flows, generalized connections on fiber bundles, etc.).

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