Conceptions of student teachers on the limit concept

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Reliable ideas about fundamental mathematical concepts are an indispensable prerequisite for mathematics teachers to conduct high quality lessons. There are several (normative) basic ideas ("Grundvorstellungen") associated with any mathematical concept, which give meaning to different aspects of the notion in question. In this article, we discuss ideas of learners on the limit of a real sequence. We present an empirical study, which collects student teachers' written statements before and after an intervention by a didactically oriented analysis course. These statements are classified with respect to their coherence with the normative basic ideas. It turns out that explicitly addressing these basic ideas leads to a significant (albeit small) increase in quality of students' statements.

Keywords: Basic ideas, Grundvorstellungen, limit of a sequence, teacher training.

Introduction and research questions

The limit concept is central to the entire field of analysis and its history goes back for thousands of years. Its modern formulation, due to Karl Weierstraß, condenses the long struggle for a proper understanding of the notion into an abstract definition. While indispensable for technically handling the notion, it hides many intuitive and pre-formal ideas. Consequently, also the discussion on the teaching and learning of the limit concept has a long history (e.g., Nardi, 2008), and a number of empirical studies are available for the secondary school sector (e.g., Friedrich, 2001; Marx, 2013; Roh, 2008). Here we extend this line of research into university didactics (cf. also Ostsieker, 2020). More importantly, we complement and extend prior works by presenting an empirical study (see Ableitinger et al., 2022), assessing student teachers' written statements on their ideas of the limit of a real sequence using the framework of basic ideas ("Grundvorstellungen") as a normative measure.

To begin with we review the core literature on the topic and relate it to our work. In a pioneering work, Fischbein (1978) discusses the often contradictory conceptions that learners hold on the notion of infinity. He empirically finds that students' intuitive conceptions focus on the infiniteness of the process rather than on the finiteness of the limit. Monaghan (2001) reports and reflects on studies of concepts of infinity held by high-school students emphasising the significance of the specific formulations in the questionnaire in empirical investigations. As they tend to activate different aspects of the contradictory conceptions held by the test subjects, they might directly influence the result. Monaghan (2001) also states that the inherently inconsistent and unstable conceptions held by university students hardly change during a typical first-year calculus course as is common e.g. in US-colleges. Here we examine whether a didactical intervention can lead to such a change.

> Different investigations which have been carried out show only too clearly that the majority of students do not master the idea of a limit, even at a more advanced stage of their studies. This does not prevent them from working out exercises, solving problems and succeeding in their examinations! (Cornu, 2002, p. 154)

However, many studies find that students when solving problems do not only use the formal definition, but rather rely on intuitive ideas and (everyday) associations. These "spontaneous conceptions" (Cornu, 2002, p. 154) most strongly influence the finding of a solution. Bender (1991) draws similar conclusions reflecting on prior empirical studies mainly among mathematics university students. He identifies dynamic ideas as major source of misconceptions as they lead to an overemphasis on the infinite process at the expense of the finiteness of the limit, cf. Fischbein (1978).

For the effective learning, understanding and teaching of the limit concept it is essential to hold suitable and well-structured ideas of the notion. Propaedeutic ideas or ideas acquired in an everyday context can support the learning and understanding of the formal ε-N-definition, but they can also be hampering if they are not suitable (Roh, 2008). Thus, it seems appropriate to expand the current state of research with regard to students' ideas about the limit concept by investigating the corresponding individual ideas held by student teachers. Moreover, it can be fruitful to assess these ideas with respect to their suitability as a foundation for understanding the limit concept. Consequently, the core issue addressed in this contribution is to survey how teacher students describe the concept of the limit of a (real) sequence in writing after their subject matter and after their didactical training in analysis, respectively. In addition to the existing studies we systematically assess the quality of the collected statements and correlate it with the performance of the students in the associated subject matter course. Finally, we evaluate how the didactical training influences the descriptions given by the students. Explicitly we address the following three research questions:

1. Which ideas on the limit concept (of a real sequence) do teacher students express after their subject matter training and how can these ideas be categorized?
2. What is the correlation between the ideas expressed by the students and their performance in the subject matter course?
3. How do these ideas change after attending a didactically oriented course, in which (normative) basic ideas about the limit of a sequence are explicitly addressed?

Theoretical background

Following the studies mentioned above, we base our analysis on Tall and Vinner's (1981) framework of concept image and concept definition: The "concept image" describes "the total cognitive structure that is associated with the concept, which includes all mental pictures and associated properties and processes" (ibid., p. 152), which is expanded or changed through different experiences. On the other hand the "concept definition" is a "form of words used to specify that concept" (ibid., p. 152) in a mathematical way. A refined framework to describe learners' concept image and assess its quality is the one of "basic ideas'' ("Grundvorstellungen") (vom Hofe & Blum, 2016). It provides a theoretical foundation to describe the mental representations of mathematical concepts and to characterize the relationship between mathematical content and individual concept formations. Basic ideas result from a technical and didactical analysis of a mathematical notion and consequently are of a normative nature and represent the part of the concept image that is generally accepted by the scientific community. They stand for the range of ideas to be discussed and, ideally, to be achieved in the classroom. Here, we use the basic ideas of the limit as a reference frame for assessing the quality of students' ideas. Indeed, Greefrath et al. (2016) identify three basic ideas of the limit of a (real) sequence. The first one is the approaching idea, which is based on the concept of the potential infinity and provides the basic intuition for the limit concept, explicitly:

Approaching idea (AI): The image of the elements of the sequence heading towards a fixed value provides the approaching idea as an intuitive idea of the limit.

The image of the dynamic movement of a sequence towards its limit is usually anchored in early experiences with infinite processes, often predating any formal training. Hence, it frequently is an integral part of the concept image of beginner students and continues to have an effect during their formal training. In contrast, the second basic idea of the limit is connected with the static point of view. It focuses on the result of the infinite process rather than on the continuing run through the sequence. It is firmly anchored in the concept of the actual infinity, which provides the idea of the very existence of a result of an infinite process. Moreover, it is very close to the standard definition:

Neighbourhood idea (NI): For every neighbourhood of the limit, no matter how small, there is an index such that from that index on, all sequence elements are in that neighbourhood.

Finally, the third basic idea is the most abstract one. It views the limit as an object that is defined or generated by the sequence and hence again is based on the actual infinity:

Object idea (OI): Limits are viewed as mathematical objects (fixed numbers, for instance), which are constructed or defined by a sequence.

In order to secure that the three basic ideas are a suitable tool to assess the statements of our survey, we have conducted a pilot study. There we have analysed student statements found in empirical works in the field (cf. Introduction) and matched them against the basic ideas, establishing that they indeed provide a suitable framework for the intended data analysis, see Ableitinger et al. (2022).

Setting and data analysis

In our study we have examined student teachers in the teacher training program in mathematics at the secondary level in Austria’s Verbund Nord-Ost, which includes the University of Vienna. This curriculum offers specialized subject matter courses that are separated from undergraduate courses in mathematics. These topical courses cover the central fields of mathematics (geometry and linear algebra, analysis, stochastics), and are followed by so-called school mathematics courses. These didactically oriented courses complement the subject matter education by focusing on all school-relevant aspects of the corresponding topic. To specifically evaluate the corresponding courses in analysis we invited students to fill in a questionnaire in a pre-test-post-test setting, where the first survey took place in the first, and the second in the final unit of the lecture “School Mathematics Analysis”. These points in time were chosen so that the survey was taken (1) after the subject matter training in analysis but before the school mathematics training, and (2) after the school mathematics training, which the majority of students took in their 5^{th} semester. In the questionnaire, we collected ideas of students on mathematical core notions of (one-dimensional) analysis in the form of written statements. We also recorded the grade achieved in the subject matter course and compiled a connected random sample with $n = 59$ subjects.

Here we report on the evaluation of the students’ continuation of the writing stimulus “I imagine *the limit of a sequence* to be ...” (emphasis in the original, in German: „Unter dem *Grenzwert einer Folge* stelle ich mir vor …“). The introductory text and the writing impulse in the questionnaire were designed to especially evoke the ideas held by the subjects rather than asking them for a formal definition. Also, the stimulus contains a highly individual component, which should prevent students from reproducing the basic ideas as formulated in the course. For details on the course see the lecture notes (Steinbauer & Kramer, 2023). In fact, our data confirms that this did not happen at all. At the same time, the writing stimulus is open enough to not specifically activate one of the suspected potentially contradictory conceptions held by the subjects (see the introduction).

The first step in our analysis was to assign the students' statements to one or more of the basic ideas. The first author (a mathematics education researcher) and the third author (a mathematician) first carried out the evaluation separately, and then, in joint discussions, agreed on one mutual coding for each statement. In most cases, it was easy to ascribe the statement to one of the basic ideas. In a second step, the quality of the expression of the respective basic ideas was assessed against a system of six categories of increasing quality, which was generated inductively from the data. The key factor in the creation of the coding guide was in how far the statements actually characterize the limit in the mathematical sense. This approach included the search for (classes of) counterexamples for certain formulations. At the end, the fit of all statements to the complete category system was re-checked. All data was analysed and classified in this way to ensure coding continuity. Table 1 shows our scheme including example formulations for each category. The assigned codes are rank scaled. However, it was not always possible to extract sample formulations from our data. In order to generate a complete coding table, we supplemented missing formulations, which are printed in italics.

Table 1: Coding guidelines for the quality of the expression of the AI, the NI and the OI

Code	Quality of the expression	Example AI	Example NI	Example OI
N	non-specific (no access to the imagination)	value to be aimed for	*Sequence elements are around the limit*	Limit; a number
1 (F)	recognizable misconception	limit for approach (stop sign); never reaching it	most of the sequence elements are in a neighbourhood	the end of a sequence
1	basic idea expressed recognizable	approaches at infinity	all sequence elements must be close to the limit	*the target of a sequence*
2	basic idea weakly developed/formulated imprecisely	comes closer and closer; approaches (with increasing *n*)	from an *n* onwards, all sequence elements remain in one nbd.	*number whose decimal places can be calculated with the sequence*
3	basic idea available but formulated incorrectly	comes infinitely close	from an *n* onwards, all sequence elements remain in each nbh.	*number that can be determined by the sequence to any number of decimal places*
4	basic idea clearly expressed and formulated adequately	finally comes arbitrarily close	almost all the sequence elements are in each small nbh.	*number defined by the sequence*

For example, the formulation "from an n onwards, all sequence members remain in each neighbourhood" for NI, Code 3 does not address the dependency of the index n0 on the size of the ε-neighbourhood, in contrast to the quoted formulation of Code 4. If more than one formulation of the same basic idea was expressed in one statement, we assigned to it the mean value of the codes.

Results

We start with the quantitative results on the frequency and the quality of the expressions. Figure 1 shows the absolute frequencies of statements assigned to the three basic ideas and their respective quality assessments in the two surveys (1) and (2): The AI is the most frequently found in both surveys. This is surprising as the standard definition clearly refers to the NI, which could only be found three times in survey (1), but 14 times in (2). Also, the AI-statements persistently show a lower quality than the NI-formulations, whose quality increases from survey (1) to (2).

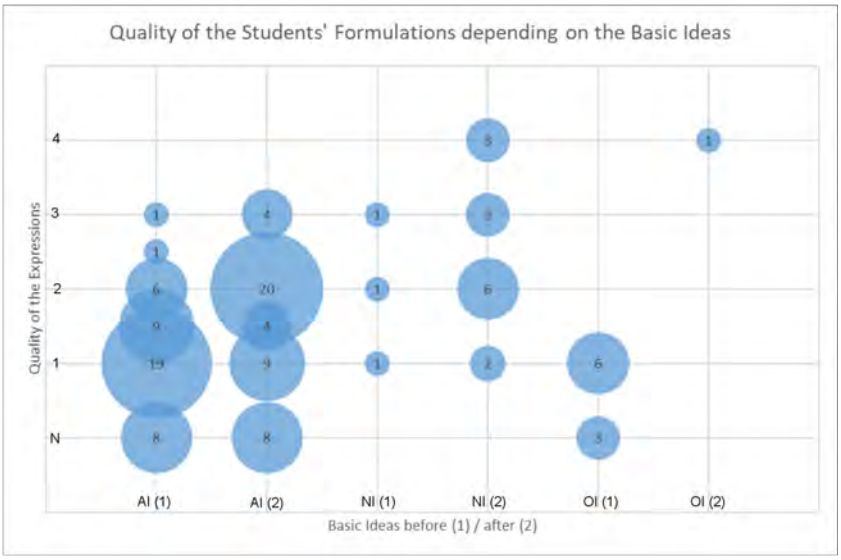

Figure 1: Bubble chart for the quality and frequency of statements in the pre- and post-test

The data of Table 2 supports this impression. The first line gives the absolute frequencies of the assigned basic ideas at times (1) and (2). In the second line, the median of the quality assessment is given, where we have ignored category N (non-specific). In contrast to Figure 1, the median only in the pre-test reflects the higher quality of the NI-expressions as compared with the AI-expressions. It also fits the picture that most misconceptions were found in connection with AI-formulations (Figure 1). Some of the main misconceptions identified by Marx (2013) could also be found in our study, e.g.: limit as a boundary/barrier of approach (twelve counts in (1)/ten in (2)) which cannot be exceeded/reached (13/eight), and the limit as the final element of the sequence (one/two).

Table 2: Absolute frequencies and median of the quality of the statements at survey times (1) and (2)

	AI (1)	AI (2)	NI (1)	NI (2)	OI (1)	OI (2)
absolute frequency	44	45	3	14	9	1
median (excluding N)	1	2	2	2	1	4

In total, about 1/3 of the statements contain misconceptions that are similar to those of high school students, suggesting that they were developed in school and persisted the teacher training. Finally, the OI could only rarely be found in the statements, so that an interpretation of their quality does not seem to be appropriate. This extensively answers **research question 1**.

To approach **research question 2**, we have calculated the correlation between the students' grade in the exam on the subject matter course in analysis and the quality of their statements in the pre-test (independent of the assigned basic idea). It results in a Kendall's τ coefficient of $\tau = -0{,}30$ ($p = 0{,}04$), which is a significant, mean negative correlation: Students with good mathematical knowledge also tend to have good ideas about the limit. This connection disappears almost entirely in the post-test where the resulting τ coefficient is $\tau = -0{,}09$, non-significant. On the one hand, our results indicate that the development of mathematical knowledge goes hand in hand with the development of adequate conceptions of the limit. On the other hand, the quality of the ideas formulated by students with initially only weak or medium mathematical knowledge can also be increased later through suitable interventions (e.g. explicit discussion of basic ideas or typical misconceptions).

This leads us to the **research question 3**. In total, a comparison of the pre- and the post-test shows a highly significant improvement in the quality of expression of the student statements, regardless of the basic idea assigned in each case (Wilcoxon signed rank test: $p < 0{,}001$). Therefore, it can be stated that the intervention in the school mathematics course had a clear effect. However, it should be noted that the quality of the statements started from a very low level in the pre-test, and at the end of the school mathematics course only reached an overall mediocre level. If we only consider the subjects who have expressed statements assigned to the AI in both surveys, we also find a highly significant increase in quality (Wilcoxon signed rank test: $p\ =\ 0{,}001$, Figure 1).

Finally, we discuss some of the most frequently observed changes from the pre- to the post-test in the statements of individual subjects. One frequent change was an increase in the quality of AI-formulations. Such a change was observed in a total of 14 subjects (24%), and an increase of at least one full category (see Table 1) in seven subjects (12%). Eleven subjects (19%) increased their formulations from an expression quality of 1 or 1,5 to a value of at least 2. Such an increase was often accompanied by the disappearance of the misconception that the limit should not be reached. As an example, we discuss the statements made by test subject KA0703. In the pre-test, an AI was found and assessed with code 1,5: Figure 2 (a phrase “The value the sequence approaches” encoded with 2 and the misconception “but which it never really reaches” encoded with 1(F)). Moreover, the graphical representation in Figure 2 indicates a strong tie of the student’s idea to the prototype concept of a monotonous and bounded sequence. The word “really” (“wirklich”) in the verbal formulation suggests an additional uncertainty regarding infinite processes.

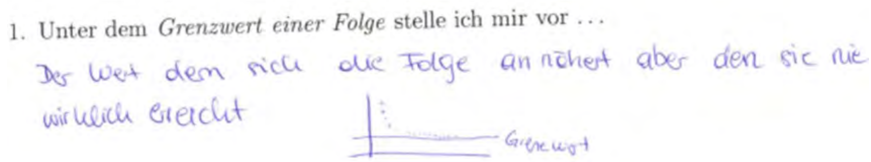

Figure 2: Test subject KA0703, pre-test (Original in German, translation in the text)

In the post-test, subject KA0703 expressed the AI formulation coded with 3 “value to which the sequence elements approach arbitrarily”: Figure 3. In addition to the improved AI, an average NI-formulation (coding 2) “sequence elements are in ε-neighbourhood of this value” was detected.

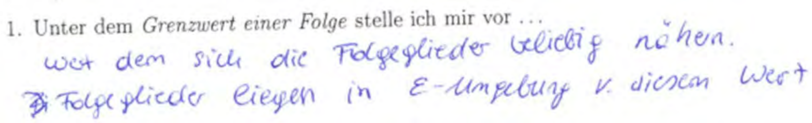

Figure 3: Test subject KA0703, post-test (Original in German, translation in the text)

Another frequently observed change was **from an AI- to an NI-formulation**. In total it was observed in twelve subjects (20%), who in the pre-test gave only one statement which was coded as AI and in the post-test gave at least one which was coded as NI. Four subjects also produced an AI-formulation in both surveys. A change from an AI-formulation coded with N, 1 or 1.5 to an NI-formulation coded with at least 2 could be observed in eight subjects (14%). Conversely, out of six subjects (10%) who expressed an NI-formulation coded 3 or 4 at the post-test, five statements belong to this category. As an example, we single out test person IR0216, who in the first survey expressed the AI-formulation coded with 1 “number, which the sequence elements approach”. In the post-test, an NI-formulation coded with 4 “limit is a number […] in whose arbitrarily small neighbourhood almost all the sequence elements are” was found.

Discussion

In this study we have systematized the evaluation of students written statements on the limit of a sequence by using the concept of basic ideas as a normative framework. The basic idea found most frequently is the AI. Since it is not in the focus of the subject matter training, we assumed that it was developed in a pre-formal stage. In addition, the subjects found it difficult to express the AI in a satisfactory quality. Misconceptions also occurred most frequently in the context of the AI. Overall, it can be assumed that a dominant AI, especially if it is of inappropriate quality, impedes a thorough understanding of the limit concept, cf. the findings of Bender (1991).

Our result suggests that the intervention was especially successful in the context of the NI: In the post-test, the NI could be assigned more frequently and was expressed in a consistently higher quality than the AI (Figure 1). This supports former results (Cornu, 2002; Roh, 2008). Overall, we suppose that explicit discussions of the basic ideas of the limit concept, as well as of the dynamic and static aspects of infinity is a fruitful approach to aid students in developing a well-founded concept image. Especially so, is an explicit reference to the advantages of the static aspect with respect to the existence of a finite result of an infinite process.

However, we mention the following limitations of our study. Exclusively analyzing the statements of the students, we have attempted to draw conclusions on the underlying ideas and misconceptions. The extent to which these ideas were actually developed by the students through the intervention and the extent to which they have been ultimately integrated into the students’ repertoire of ideas cannot be answered with our method. Our study design only made external signs visible. In the teaching profession, however, it is essential to verbalize mathematical concepts and, in this respect, the actual statements by the students examined here are highly relevant.

References

Ableitinger, C., Götz, S., & Steinbauer, R. (2022). Vorstellungen von Lehramtsstudierenden zum Grenzwertbegriff [Student teachers' perceptions of the concept of limit values]. *Mathematica Didactica*, *45*. http://doi.org/10.18716/ojs/md/2022.1620

Bender, P. (1991). Fehlvorstellungen und Fehlverständnisse bei Folgen und Grenzwerten [Misconceptions and misunderstandings about consequences and limit values]. *MNU*, *44*(4), 238–243.

Cornu, B. (2002). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Kluwer Academic Publishers. http://doi.org/10.1007/0-306-47203-1_10

Fischbein, E. (1978). Intuition and mathematical education. *Osnabrücker Schriften zur Mathematik 1*, 148–176.

Friedrich, H. (2001). Eine Kategorie zur Beschreibung möglicher Ursachen für Probleme mit dem Grenzwertbegriff [A category to describe possible causes for problems with the limit value concept]. *Journal für Mathematik-Didaktik*, *22*(3-4), 207–230. http://doi.org/10.1007/BF03338936

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., & Weigand, H.-G. (2016). *Didaktik der Analysis: Aspekte und Grundvorstellungen zentraler Begriffe* [Didactics of analysis: Aspects and basic ideas of central concepts]. Springer Spektrum. http://doi.org/10.1007/978-3-662-48877-5

Marx, A. (2013). Schülervorstellungen zu unendlichen Prozessen – Die metaphorische Deutung des Grenzwerts als Ergebnis eines unendlichen Prozesses [Pupils' ideas about infinite processes - The metaphorical interpretation of the limit value as the result of an infinite process]. *Journal für Mathematik-Didaktik, 34*(1), 73–97. http://doi.org/10.1007/s13138-012-0047-5

Monaghan, J. (2001). Young people's ideas of infinity. *Educational Studies in Mathematics, 48*(2-3), 239–257. http://doi.org/10.1023/A:1016090925967

Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level.* Springer. http://doi.org/10.1007/978-0-387-37143-6

Ostsieker, L. (2020). *Lernumgebungen für Studierende zur Nacherfindung des Konvergenzbegriffs. Gestaltung und empirische Untersuchung* [Learning environments for students to reinvent the concept of convergence. Design and empirical investigation]. Springer Spektrum. http://doi.org/10.1007/978-3-658-27194-7

Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217–233. http://doi.org/10.1007/s10649-008-9128-2

Steinbauer, R., & Kramer, S. (2023). *Schulmathematik Analysis* [Lecture notes]. University of Vienna. https://www.mat.univie.ac.at/~stein/teaching/SoSem22/smana-gesamt-2023-01-17.pdf

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. http://doi.org/10.1007/BF00305619

Vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a category of subject-matter didactics. *Journal für Mathematik-Didaktik, 37*(Suppl 1), 225–254. http://doi.org/10.1007/s13138-016-0107-3