The exponential map of a $C^{1,1}$-metric

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Motivation

Semi-Riemannian geometry with metrics of low regularity

GR: field eqs. are hyperbolic PDE$\rightsquigarrow$ regularity issues are vital
Recent interest in low regularity

- *[Le Floch and coworkers, 2007-...]*
- *[Chrusciel, Grant 2012]* **causality theory**

Folklore: "Everything" should work for $\boxed{g \in \mathcal{C}^{1,1}}$
since geodesic equation still uniquely solvable

The exponential map of a smooth Semi-Riemannian metric is a diffeomorphism locally around each point.

- proof rests on the inverse function theorem for *exp*, needs $g \in \mathcal{C}^2$
- however, $exp \in Lip$ only, inverse function theorem? Literatur?

The result

We prove

The exponential map of a $\mathcal{C}^{1,1}$ semi-Riemannian metric retains maximal regularity.
Every point possesses a totally normal neighbourhood.

More precisely

Theorem (KSS 2013)

Let (M, g) be a smooth manifold with a $C^{1,1}$-semi-Riemannian metric. Then locally around each point the exponential map is a bi-Lipschitz homeomorphism.

Strategy of proof

regularization techniques & comparison geometry

Sketch of Proof

step 1 (regularization): Componentwise convolution with a standard mollifier

$$g_\epsilon := g * \rho_\epsilon$$

$(g_\epsilon)_\epsilon$ is a family of smooth Semi-Riemannian metrics
with locally uniformly bounded Riemann curvature tensor.
We have:

- $g_\epsilon \to g$ locally uniformly up to 1st derivative
- $\| Riem\,[g_\epsilon]\|_E \leq K_1$, $\|\Gamma_{g_\epsilon}\|_E \leq K_2$ locally

Now consider the family of exponential maps $\left(exp_p^{g_\epsilon} \right)_\epsilon$

step 2 (common domain): By standard ODE-theory
$exp_p^{g_\varepsilon}$, exp_p^{g} are defined on a **common domain** $B_E(0,\mu) \subseteq T_pM$

The Riemannian case

step 3 (Jacobi field estimates): Using the *Rauch comparison Theorem* we turn bounds on the **sectional curvature** into $(v \in B_E(0,\mu),\ w \in T_pM)$

$$e^{-c}\|w\|_E \leq \|T_v exp_p^{g_\epsilon}(w)\|_E \leq e^c\|w\|_E \qquad (1)$$

This gives
- $exp_p^{g_\epsilon}$ is a **local diffeo** on $B_E(0,\mu)$
- **bi-Lipchitz** estimates (via mean value argument)

step 4 (Injectivity): A Theorem by *Cheeger, Gromov, Taylor* turns the estimates

$$\|Riem[g_\epsilon]\|_{L^\infty(B(p,r))} \leq C_1, \quad Vol_{g_\epsilon}(B(p,r)) \geq C_2$$

into an injectivity radius estimate from below ,i.e.,

$$Inj_{g_\epsilon}(M,p) \geq i(C_1, C_2).$$

The Semi-Riemannian case

step 3' (Jacobi field estimates): Done by hand (sectional curvature unbounded) following ideas by *[Chen, Le Floch 08]*: Using the estimates on Γ_{g_ϵ}, $Riem[g_\epsilon]$ one may bootstrap $\|J_\epsilon(s)\|_E$ to again obtain

$$e^{-c}\|w\|_E \leq \|T_v exp_p^{g_\epsilon}(w)\|_E \leq e^c\|w\|_E \tag{2}$$

which gives
- $exp_p^{g_\epsilon}$ is a **local diffeo**
- **bi-Lipchitz** estimates (via mean value argument)

step 4': (Injectivity): Again done by hand
- using a homoptopy lifting argument on some ball $B_E(0, r_5)$
- needs a tricky nesting of domains

$$exp_p^{g_\epsilon}(\overline{B_E(0, r_5)}) \subseteq B_E(p, r_4) \subseteq exp_p^{g_\epsilon}(\overline{B_E(0, \tilde{r})}) \subseteq exp_p^{g_\epsilon}(B_E(0, r_3))$$

Totally normal neighbourhoods

$U \subseteq M$ is a *normal neighbourhood* around $p \in U$ if there exists $\tilde{U} \subseteq T_pM$ open and starshaped such that

$$exp_p\colon\ \tilde{U} \to U \text{ is a bi-Lipschitz homeomorphism}$$

U is called totally normal, if it is normall for all $p \in U$.

Theorem (KSS 2013)

Let (M, g) be a smooth manifold with a $C^{1,1}$-semi-Riemannian metric. Then each point possesses a basis of totally normal neighbourhoods.

Adaptation of a classical proof by *[Whitehead 1932]*.

Context & Prospects

- From *[Whitehead 1932]* (geometry of paths) it already follows:

  The exponential map of a $\mathcal{C}^{1,1}$-semi-Riemannian metric is a homeomorphism locally around each point.

  different methods, no Lipschitz property

- Very recently *[Minguzzi, arXiv:1308.6675v1 (math.DG)]* proofs the bi-Lipschitz property entirely by ODE-methods.

- Prospect of our work: A $\mathcal{C}^{1,1}$-causality theory with easy to access methods.

References

- B.-L. Chen and P. LeFloch, Injectivity Radius of Lorentzian Manifolds, Commun. Math. Phys. 278 (2008)
- P. Chrusciel and J. Grant, On Lorentzian causality with continuous metrics, Class. Quantum Grav. 29 (2012)
- M. Kunzinger, R. Steinbauer, M.Stojkovic, The exponential map of a $C^{1,1}$ metric, arXiv: 1306.4776v2 [math.DG]
- E. Minguzzi, Convex neighborhoods for Lipschitz connections and sprays, arXiv:1308.6675v1
- J. H. C. Whitehead, Convex regions in the geometry of paths. Q. J. Math. Oxf. Ser. 3 (1932)