Impulsive Gravitational Waves and their Mathematics

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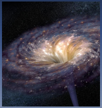

The topic

Study of an interesting class of **exact solutions** of Einstein's equations in General Relativity that are analytically **singular**

- belongs in the recently very active field of

  Nonregular spacetime geometry

  that is, Lorentzian geometry & GR with **metrics of low regularity**.
- joint long-term project with
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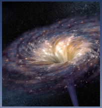

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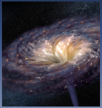
Many of the ideas can be traced back to my Ph.D.,
a time when I received great support by M.O.

This talk is dedicated to you! Happy Birthday!



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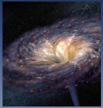

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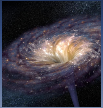

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- infinite curvature along a null hypersurface
- here: **non-expanding** igw's on a constant curvature background
  (i.e., on Minkowski or (anti-)**de Sitter space**)

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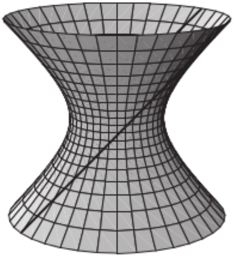
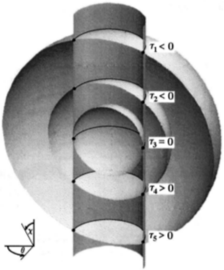
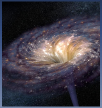


de Sitter universe



propagating wave



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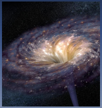

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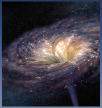

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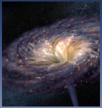
Lipschitz metric in conformally flat coords $(\mathcal{U}, \mathcal{V}, X, Y) \in \mathbb{R}^4$
[Podolský, Griffiths, 99]

$$ds^2 = \frac{\mathbf{g_{ij}}(\mathcal{U}, X^k)\,\mathrm{d}X^i\mathrm{d}X^j - 2\,\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V}}{\left(1 + \frac{\Lambda}{12}(\delta_{ij}X^iX^j - 2\mathcal{U}\mathcal{V} - 2\mathcal{U}_+ G)\right)^2}$$

with

- $g_{ij} = \delta_{ij} + 2\mathcal{U}_+ H_{,ij} + \mathcal{U}_+^{\mathbf{2}}\,\delta^{kl} H_{,ik} H_{,jl}$, $G = H - X^i H_{,i}$
- H smooth fct. of (X, Y), and $\mathcal{U}_+$ the kink-fct.



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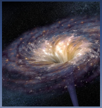

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Distributional metric in a 5D-formalism: de Sitter as a 4D-hyperboloid

$$Z_2^2 + Z_3^2 + Z_4^2 - 2UV = 3/\Lambda,$$

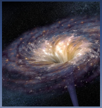
in 5D-Minkowski space with an impulsive *pp*-wave

$$\mathrm{d}s^2 = \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2 - 2\mathrm{d}U\mathrm{d}V + H(Z_2, Z_3, Z_4)\delta(\mathbf{U})\mathrm{d}U^2$$

where $(Z_0, \dots, Z_4) \in \mathbb{R}^5$ are Minkowski coordinates and

$$U = 1/\sqrt{2}\,(Z_0 + Z_1), \quad V = 1/\sqrt{2}\,(Z_0 - Z_1)$$

are null coordinates [Podolský, Ortaggio, 01]



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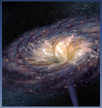

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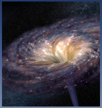
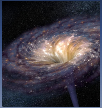
- description by a **Lipschitz continuous** or a **distributional** metric
- Alternative distributional metric in a 4D-formalism
  - even wilder singularities
  - see **Benedict Schinnerl's talk**



Goals & Objectives

Completeness results (all geodesics are globally defined)

- The analytically singular geometries are **geometrically non-singular** in the sense of the [Penrose, 65]-definition.
- Disprove the **Ehlers-Kundt conjecture** in the impulsive case!
  EK: Plane waves (H quadratically) are the only complete *pp*-waves
  Proved only in (very) special cases by [Sánchez, Flores, 17].



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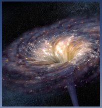
Explicit calculation of particle motion (solve for geodesics)

- Particle scattering at Planck scale & wave memory effect



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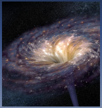
Holy Grail (Make sense of the discontinuous [Penrose, 72]-trsf.)

- Relate distributional to Lipschitz metric in a meaningful way.
- ✓ in Minkowski-background [KS, 99] using nonlinear distributional geometry [GKOS, 01] based on special Colombeau algebras
- ! **Much** more complicated in (anti-)de Sitter space
- Needs Colombeau-solutions of the geodesic eqs. for $\mathcal{D}'$-metric



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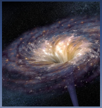
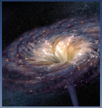


Goals

(1) Geodesic completeness
(2) Explicit calculation of the geodesics



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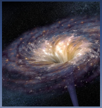
$\mathcal{C}^1$-matching of the geodesics

- Physicists like to derive the geodesics by matching geodesics of background across wave-surface.
- This is **only** possible if the geodesics
  - cross the wave at all
  - are $\mathcal{C}^1$ across the wave-surface
  - are unique



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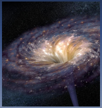
- Obtain (1) & (2) by making the $\mathcal{C}^1$-matching procedure rigorous!

$\lightning$ Lipschitz metric $\rightsquigarrow$ geodesic equations have r.h.s in L^∞, not C^0

! use the [Filippov, 88]-solution concept for ODEs w. discont. r.h.s.



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Filippov solutions: the basic idea

- replace ODE with discont. r.h.s. by a **differential inclusion** relation

$$\dot{x}(t) = F(t, x(t)) \quad \rightsquigarrow \quad \dot{x}(t) \in \mathcal{F}[F](t, x(t))$$

where the **Filippov set-valued map** associated with F is

$$\mathcal{F}[F](t,x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \mathrm{co}\Big(F\big(B_\delta(t,x)\big) \setminus S\Big).$$

(non-empty, closed and convex set)

- A **Filippov solution** of the ODE is an absolutely continuous curve satisfying the inclusion relation almost everywhere.

- Obtain (1) & (2) by making the $\mathcal{C}^1$-matching procedure rigorous!
- ↯ Lipschitz metric $\rightsquigarrow$ geodesic equations have r.h.s in L^∞, not C^0
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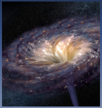
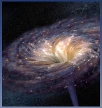


Existence and regularity of geodesics

Every Lipschitz metric has $\mathcal{C}^1$-geodesics **[S, 14]**

Let $(M, \mathbf{g})$ be a $\mathcal{C}^\infty$-manifold with a $\mathcal{C}^{0,1}$-semi Riemannian metric. Then the geodesic equation has Filippov solutions, which are $\mathcal{C}^1$.



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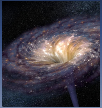
geodesic eq.: $\ddot{x}^i = -\Gamma^i_{jk}\dot{x}^j\dot{x}^k$ (Christoffel symbols $\Gamma \propto \mathbf{g}^{-1}\,\partial\mathbf{g}$)
Rademacher: $\mathbf{g} \in \mathcal{C}^{0,1} \Rightarrow \Gamma \in L^\infty_{\text{loc}}$
Rewrite geodesic equation for in first order form:

$$\dot{z} = F(z(t)) \quad \text{where } z = (x, \dot{x}), \quad F(z) = (\dot{x}^i, -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k)$$

Basic existence theorem provides us with Filippov solutions which are by definition AC-curves.
Hence the geodesics are curves with AC-speeds.



Uniqueness for Filippov solutions

- $g \in \mathcal{C}^{0,1}$ is much below classical threshold for uniqueness ($g \in \mathcal{C}^{1,1}$)
- **But** **g** for igw's is piecewise smooth (C^∞ off hypersrf. $\{U = 0\}$)

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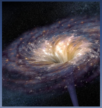
Consider $\dot{x}(t) = f(x(t))$ on $D \subseteq \mathbb{R}^n$ connected

- split into two parts D^+, D^- by a $\mathcal{C}^2$-hypersurface $N = \partial D^+ = \partial D^-$
- $f \in \mathcal{C}^1(D^\pm)$ up to bdr. N, $f^\pm :=$ extensions of $f|_{D^\pm}$ to N
- $f_N^\pm :=$ proj. of $f^\pm$ on the unit normal $\vec{n}$ of N (pt. from D^- to D^+)



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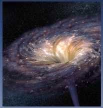
Fillipov's uniqueness results

If $f_N^\pm > 0$ all F-solutions are unique and pass from D^- to D^+.
Analogously for $f_N^\pm < 0$ and passing from D^+ to D^-.
(rules out repulsive trajectories and sliding motion)



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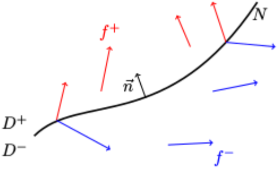

Figure 2: repulsive trajectories

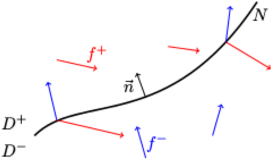

Figure 3: sliding motion

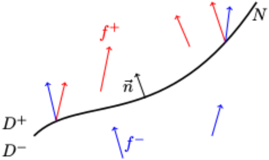

Figure 4: (upward) transversally crossing

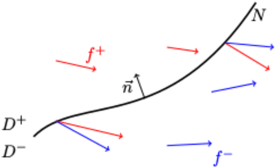

Figure 5: (downward) transversally crossing

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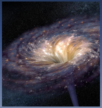

Uniqueness for geodesics

Theorem (g smooth off a totally geodesic hypersrf.)[SS, 18]

Let $(M, \mathbf{g})$ be a $\mathcal{C}^\infty$-manifold with a $\mathcal{C}^{0,1}$-semi Riemannian metric. Assume that

- N is a totally geodesic $\mathcal{C}^2$-hypersurface, and
- $\mathbf{g} \in \mathcal{C}^2(M \setminus N)$.

Then all (Filippov) geodesics (starting not on N) are unique and those who hit N pass through.

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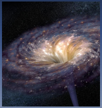
N is called totally geodesic, if
every (F-)geodesic starting tangentially in N stays (initially) in N.



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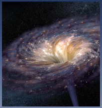

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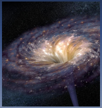
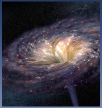
- Locally write $N = \{x^1 = 0\}$, $D^\pm = \{x^1 > \pm 0\}$ then $\vec{n} = e_1$ and for the geodesic $\gamma(t) = (x^1(t), \dots)$
- Rewrite geodesic equation as first order system $\rightsquigarrow$ $f_N^\pm = \dot{x}^1$
  $\rightsquigarrow$ only have to show that $\dot{x}^1 \neq 0$ if $x^1 = 0$
- But this follows for all geodesics starting off N and reaching it since N is totally geodesic.



Completeness & C^1-matching for igw's

- $\mathbf{g} \in \mathcal{C}^{0,1}$ $\Rightarrow$ For any initial condition Filippov solutions to the geodesic equation exist
  $\Rightarrow$ they are curves with AC velocities, in particular $\mathcal{C}^1$
- $\mathbf{g} \in \mathcal{C}^\infty$ off the wave surface $N := \{U = 0\}$
- The wave srfc. N is totally geodesic:
  $\Rightarrow$ All geodesics with data given off N are unique and they cross N
  $\Rightarrow$ The $\mathcal{C}^1$-matching applies
- $\mathbf{g}$ is the background metric off N
  $\Rightarrow$ geodesic completeness



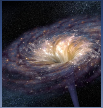
The explicit matching

For the geodesics in non-expanding impulsive gravitational waves on any constant curvature background we obtain

$$
\begin{aligned}
\mathcal{U}_{\mathrm{i}}^{-} &= 0 = \mathcal{U}_{\mathrm{i}}^{+}\,, & \dot{\mathcal{U}}_{\mathrm{i}}^{-} &= \dot{\mathcal{U}}_{\mathrm{i}}^{+}\,,\\
\mathcal{V}_{\mathrm{i}}^{-} &= \mathcal{V}_{\mathrm{i}}^{+} - H_{\mathrm{i}}\,, & \dot{\mathcal{V}}_{\mathrm{i}}^{-} &= \dot{\mathcal{V}}_{\mathrm{i}}^{+} - H_{\mathrm{i},X}\,\dot{x}_{\mathrm{i}}^{+} - H_{\mathrm{i},Y}\,\dot{y}_{\mathrm{i}}^{+}\\
& & &\quad +\tfrac{1}{2}\big((H_{\mathrm{i},X})^2 + (H_{\mathrm{i},Y})^2\big)\,\dot{\mathcal{U}}_{\mathrm{i}}^{+}\,,\\
x_{\mathrm{i}}^{-} &= x_{\mathrm{i}}^{+}\,, & \dot{x}_{\mathrm{i}}^{-} &= \dot{x}_{\mathrm{i}}^{+} - H_{\mathrm{i},X}\,\dot{\mathcal{U}}_{\mathrm{i}}^{+}\,,\\
y_{\mathrm{i}}^{-} &= y_{\mathrm{i}}^{+}\,, & \dot{y}_{\mathrm{i}}^{-} &= \dot{y}_{\mathrm{i}}^{+} - H_{\mathrm{i},Y}\,\dot{\mathcal{U}}_{\mathrm{i}}^{+}\,.
\end{aligned}
$$

w.r.t. the conformally flat coordinates of the background

The wave memory people got this wrong!



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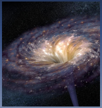
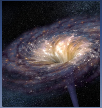


Problems & their solution

- ↯ distributional metric $\rightsquigarrow$ geodesic equations do not make sense
- regularise 5D-ambient space metric



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Distributional metric in a 5D-formalism: de Sitter as a 4D-hyperboloid

$$Z_2^2 + Z_3^2 + Z_4^2 - 2UV = 3/\Lambda,$$

in 5D-Minkowski space with an impulsive *pp*-wave

$$\mathrm{d}s^2 = \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2 - 2\mathrm{d}U\mathrm{d}V + H(Z_2, Z_3, Z_4)\delta(\mathbf{U})\mathrm{d}U^2$$

where $(Z_0, \ldots, Z_4) \in \mathbb{R}^5$ are Minkowski coordinates and

$$U = 1/\sqrt{2}\,(Z_0 + Z_1), \quad V = 1/\sqrt{2}\,(Z_0 - Z_1)$$

are null coordinates [Podolský, Ortaggio, 01]

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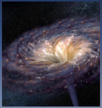

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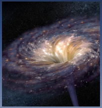

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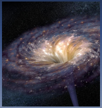

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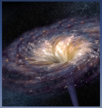

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Regularised equations

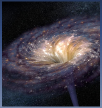
$$\ddot{U}_\varepsilon = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\big)^{\dot{}}\Big)\;\frac{U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot{Z}_{p\varepsilon} - \frac{1}{2}H_{,p}\,\delta_\varepsilon\,\dot{U}_\varepsilon^2 = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\big)^{\dot{}}\Big)\;\frac{Z_{p\varepsilon}}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot{V}_\varepsilon - \frac{1}{2}\,H\,\delta_\varepsilon'\,\dot{U}_\varepsilon^2 - \delta^{pq}H_{,p}\,\delta_\varepsilon\,\dot{Z}_q^\varepsilon\,\dot{U}_\varepsilon = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\big)^{\dot{}}\Big)\;\frac{V_\varepsilon + H\,\delta_\varepsilon U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

where

$$\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t))\,,\quad \delta_\varepsilon' = \delta_\varepsilon'(U_\varepsilon(t))\,,$$
$$\tilde{G}_\varepsilon = \tilde{G}_\varepsilon(U_\varepsilon(t), Z_{p\varepsilon}(t))\,,\quad H = H(Z_{p\varepsilon}(t))\,,\quad \text{and}\quad H_{,p} = H_{,p}(Z_{q\varepsilon}(t))$$



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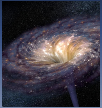

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  but time of existence depends on ε
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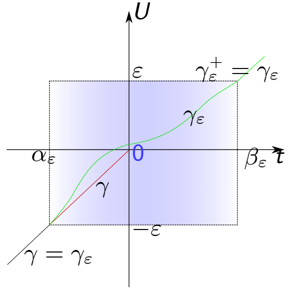

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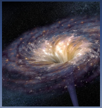

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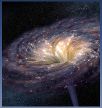
  $\rightsquigarrow$ unclear whether the solution lives long enough to cross the regularised (extended) wave zone
- use a **fixed point argument** and a bag of tricks to obtain a "uniform result".

Details: Benedict Schinnerl's talk



Simple results

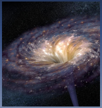
[SSLP, 16]

Theorem (Semi-global existence and uniqueness)

The initial value problem for the geodesic equation has a unique smooth solution

$$\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_\varepsilon) \quad \text{on} \quad [\alpha_\varepsilon, \alpha_\varepsilon + \eta], \quad \text{where} \quad \eta \neq \eta(\varepsilon)$$

for small ε enough. Hence the geodesics extend to the background de Sitter spacetime 'behind' the wave.



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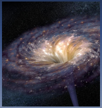
Corollary (Causal completeness)

Every (causal) geodesic is complete, provided the regularisation parameter ε is chosen small enough.



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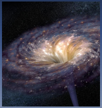
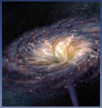
$\lightning$ The smallness condition on ε involves the initial data of the geodesic $\rightsquigarrow$ no **global** completeness result!



Spacetime completeness result [SS, 17]

Use non-linear distributional geometry (special Colombeau)

- Turn above 'local solution candidate' into a global one



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Globalization Lemma

Let $u : (0,1] \times M \to \mathbb{R}^n$ be a smooth map and let (P) be a property attributable to values $u(\varepsilon, p)$, satisfying:

$$\forall K \subset\subset M \text{ there is } \varepsilon_K > 0 : \text{ (P) holds for all } p \in K \text{ and } \varepsilon < \varepsilon_K.$$

Then there is a smooth map $\tilde{u} : (0,1] \times M \to \mathbb{R}^n$ such that (P) holds globally.
Moreover for each $K \subset\subset M$ there exists some $\varepsilon_K \in (0,1]$ such that $\tilde{u}(\varepsilon, p) = u(\varepsilon, p)$ for all $(\varepsilon, p) \in (0, \varepsilon_K] \times K$.

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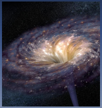

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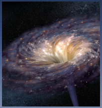

Spacetime completeness result [SS, 17]

Use non-linear distributional geometry (special Colombeau)

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Theorem (Generalized spacetime completeness)

The generalized impulsive wave spacetime (M, g) *given by*

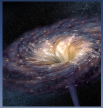
$$-2UV + Z_2^2 + Z_3^2 + Z_4^2 = 3/\Lambda$$

in the 5D-impulsive pp-wave

$$\mathrm{d}s^2 = dZ_2^2 + dZ_3^2 + dZ_4^2 - 2dUdV + H(Z_2, Z_3, Z_4)\mathbf{D}(\mathbf{U})dU^2$$

is geodesically complete.

D, generalized δ-function with model δ-net representative



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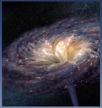


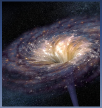
Conclusions & Outlook

Reaching the Holy Grail

- Revealing that the transformation between the 4D-distributional and the Lipschitz metric

$$\mathcal{U} = U\,, \quad \mathcal{V} = V + \Theta\, H + \mathcal{U}_+\, H_{,Z} H_{,\bar{Z}}\,, \quad \eta = Z + \mathcal{U}_+\, H_{,\bar{Z}}\,.$$

  is the limit (shadow) of a generalized diffeomorphism.
- Needs estimates on the dependence of Colombeau-geos on data!
- Until recently available only in the 5-D formalism
  informal calculations (#↯%#!) show that things work out...
- **Now** direct 4-D results available; see **Benedict Schinerl's** talk!



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Gravitational wave memory effect for igw's

- diplomatic mission???
- generalize known results from plane to *pp*-waves & nonvanishing Λ

Some related Literature

[SS,12] C. Sämann, R. Steinbauer, *On the completeness of impulsive gravitational wave spacetimes.* CQG 29 (2012)

[LSŠ,14] A. Lecke, R. Steinbauer, R. Švarc, *The regularity of geodesics in impulsive pp-waves.* GRG 46 (2014)

[PSS,14] J. Podolský, R. Steinbauer, R. Švarc, *Gyratonic pp-waves and their impulsive limit.* PRD 90 (2014)

[S,14] R. Steinbauer, *Every Lipschitz metric has $\mathcal{C}^1$-geodesics.* CQG 31, 057001 (2014)

[PSSŠ,15] J. Podolský, C. Sämann, R. Steinbauer, R. Švarc, *The global existence, uniqueness and $\mathcal{C}^1$-regularity of geodesics in nonexpanding impulsive gravitational waves.* CQG 32 (2015)

[SS,15] C. Sämann, R. Steinbauer, *Geodesic completeness of generalized space-times.* in Pseudo-differential operators and generalized functions. Pilipovic, S., Toft, J. (eds) Birkhäuser/Springer, 2015

[PSSŠ,16] J. Podolský, C. Sämann, R. Steinbauer, R. Švarc, *The global uniqueness and $\mathcal{C}^1$-regularity of geodesics in expanding impulsive gravitational waves.* CQG 33 (2016)

[SSLP,16] C. Sämann, R. Steinbauer, A. Lecke, J. Podolský, *Geodesics in nonexpanding impulsive gravitational waves with Λ, part I*, CQG 33 (2016)

[SSŠ,16] C. Sämann, R. Steinbauer, R. Švarc, *Completeness of general pp-wave spacetimes and their impulsive limit.*, CQG 33 (2016)

[PŠSS,17] J. Podolský, R. Švarc, C. Sämann, R. Steinbauer, *Penrose junction conditions extended: impulsive waves with gyratons*, PRD 96 (2017)

[SS,17] C. Sämann, R. Steinbauer, *Geodesics in nonexpanding impulsive gravitational waves with Λ, part II*, JMP 58 (2017)

[SS,18] C. Sämann, R. Steinbauer, *On geodesics in low regularity*, in IOP: Conference Series 968, (2018)

Hvala na pažnji