The future is not always open

Roland Steinbauer

Department of Mathematics, University of Vienna

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- Outcome (so far): new & sometimes surprising facts

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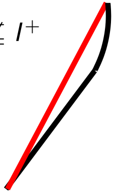
(1) **push up principle fails**

$(J^+ \circ I^+) \not\subseteq I^+$



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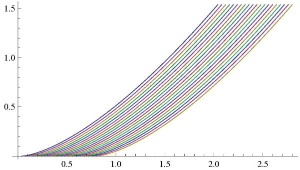

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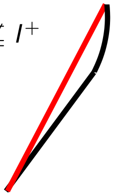

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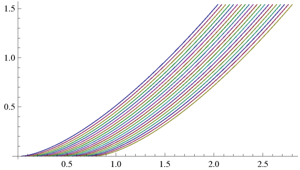

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- many more things to say...

failure of convexity in $C^{1,\alpha}$, cone structures

Techniques and tools

The choice of curves

- classical textbook choice: $\mathcal{C}^1_{\mathrm{pw}}$ (piecew. C^1)
- technically superior **locally Lipschitz** $= \mathcal{L}$
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- **Cylindrical charts** (substitute for normal charts)
  $(\varphi = (t, x^1, \ldots x^n), U)$ with $\varphi(U) = L \times V$ such that

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Lemma. For continuous g we have

$$\check{I}^+(p) = I^+_{\mathcal{C}^1_{\mathrm{pw}}}(p) \text{ which is clearly open.}$$

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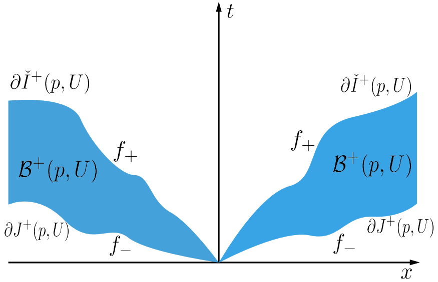

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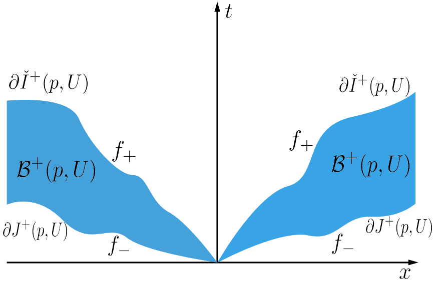

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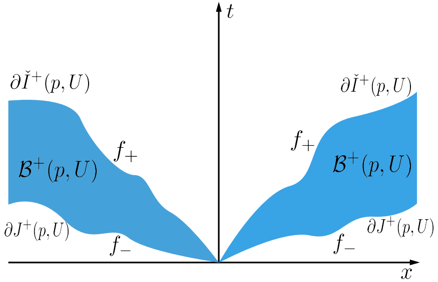

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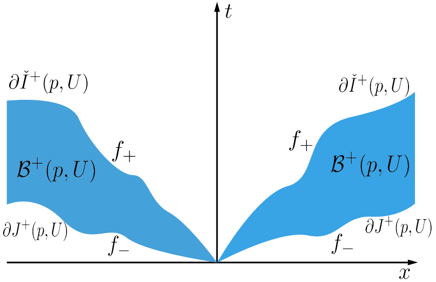

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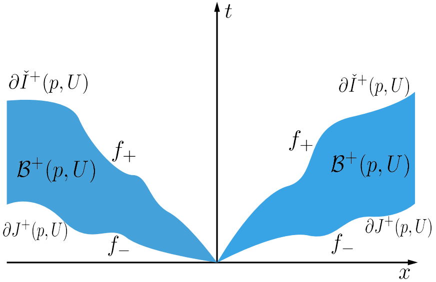

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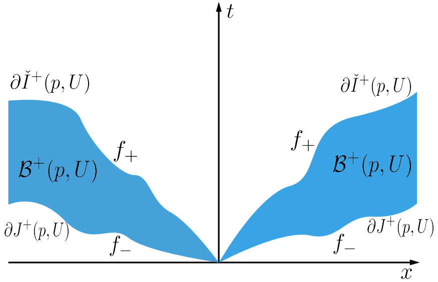


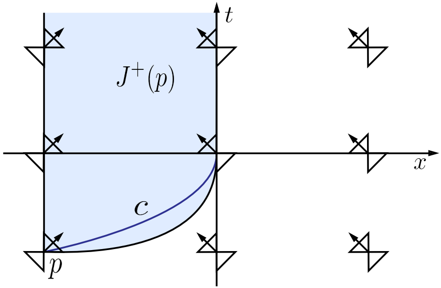
Facts: $\mathcal{B}^+_{\mathrm{int}}(p,U) = \emptyset \;\Leftrightarrow\; \check{I}^+(p,U) = I^+(p,U)$

$\mathcal{B}^+_{\mathrm{ext}}(p,U) = \emptyset \;\Leftrightarrow\;$ push up holds

Counterexample

$$ds^2 = 2\left[-\sin 2\theta(x)\, dt^2 - 2\cos 2\theta(x)\, dx\, dt + \sin 2\theta(x)\, dx^2\right]$$

$\theta(x)$ turns light cones in a Hölder but not Lipschitz way



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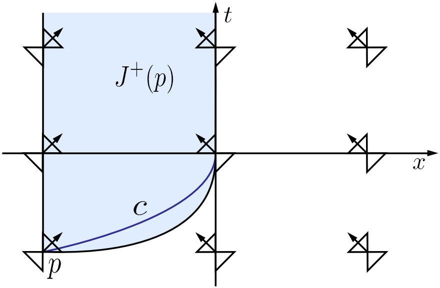

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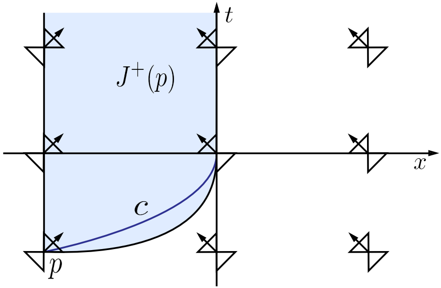

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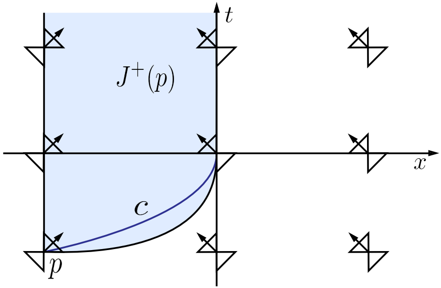

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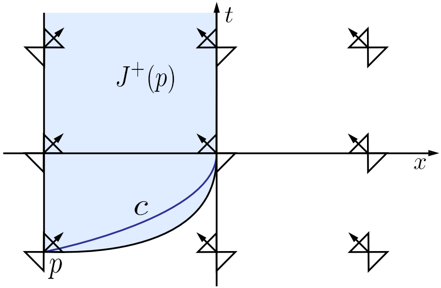

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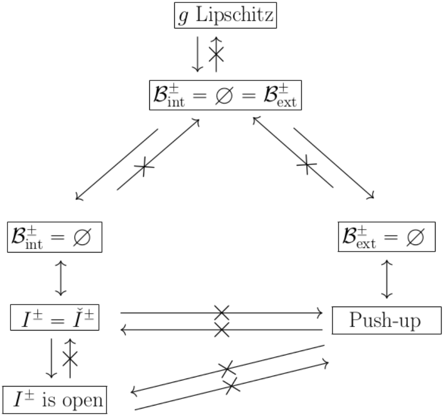
- $\check{I}^+(p)$ does not contain segment $\Rightarrow$ $\check{I}^+(p) \neq I^+(p)$ $\boxed{\text{NO to (Q2)}}$
- $\mathcal{B}^+(p) = \emptyset$ but $\mathcal{B}^-(0) \neq \emptyset$

The overall picture & some conclusions



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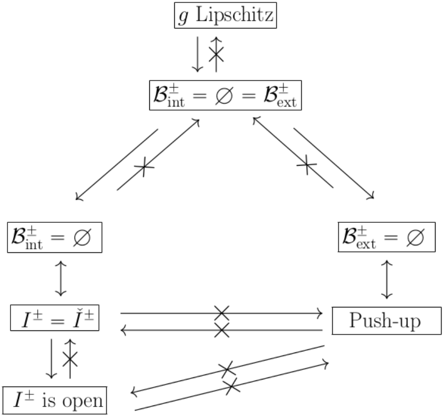

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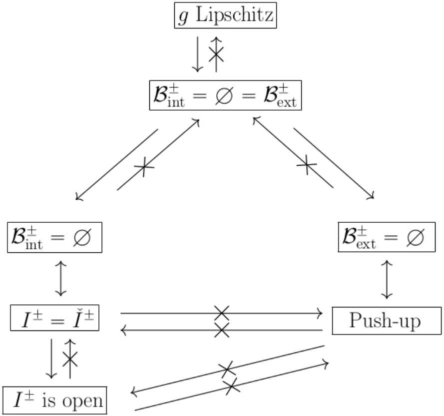

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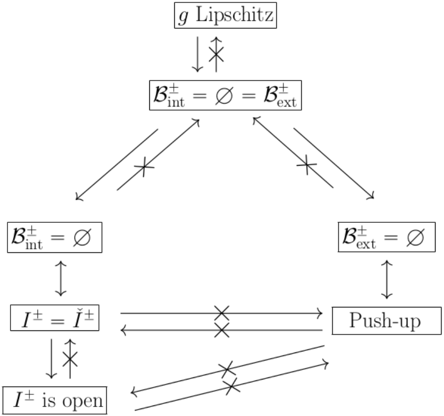

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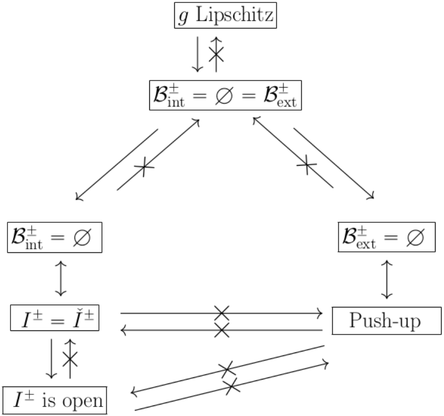

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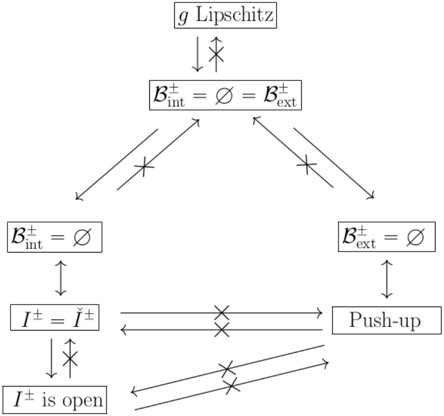
The overall picture & some conclusions



Main message:

- g Lip. $\Rightarrow$ all is fine
- below Lip.
  basics of causality fail

Suggestion:

- metric setting
  $J^{\pm} := J^{\pm}_{\mathcal{L}}$ (lim. crv. thms.)
  $I^{\pm} := I^{\pm}_{\mathcal{C}^1_{\mathrm{pw}}}$ (openness)
- synthetic setting
  C^1-curves not available;
  have to by in the full
  range of these phenomena

El Fin — Muchas Gracias