Some Contributions of Stevan Pilipović to the Structural Theory of Colombeau Algebras

- Happy Birthday Action [Cave, ...]

- Greetings & Thanks. [Stay no. 8; first in 1998, ...]

- Contents. As everybody knows S.P. has a really long list of publications that covers a wide spectrum of topics in analysis. Again as everybody knows his work has deeply influenced the field of algebras of generalized fcts / Colombeau algebras. In particular there is approx. a dozent papers by SP & varying coauthors on the "structural theory" or structure of CA.

Today I will pick up a small portion of these and speak about some work of SP & coauthors that fascinated me by

[bringing fresh methods into the field]

which have not been used before.

Most of the following is contained in "Equalities in algebras of gen. fcts" by S.P., D.S., V.V. which appeared in 2006 in Forum Mathematicum [PSV;06]

- STARTING POINT: a parametrix formula by L. Schwartz

Let $k \in \mathbb{N}_0$, $K \subseteq \mathbb{R}^n$ cp, $0 \in \operatorname{int}(K)$

$\Longrightarrow \exists m \in \mathbb{N}_0$, $\rho \in \mathcal{C}_0^k(\mathbb{R}^n)$, $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ both supported in K:

$$\boxed{\delta = \Delta^m \rho + \theta}$$

Immediate consequence: $f \in \mathcal{D}'(\mathbb{R}^n) \Longrightarrow$

$$\boxed{f = \Delta^m f * \rho + f * \theta}$$

- CONSEQUENCES IN $\mathcal{G}$ ← special Colombeau algebra

(1) (How to detect $\mathcal{G}^\infty$-fcts) ← regular Colombeau fcts $\mathcal{G}^\infty \cap \iota(\mathcal{D}') = \mathcal{C}^\infty$

Let $f \in \mathcal{G}(\Omega)$; suppose $\exists k \in \mathbb{N}_0$: $f * \phi \in \mathcal{G}^\infty$ $\forall \phi \in \mathcal{C}_0^k(\Omega)$

$$\Longrightarrow f \in \mathcal{G}^\infty$$

Immediate from the parametrix formula.

(2) (Equality in the generalized $\mathcal{D}'$-sense)

DEF: $\mathcal{G} \ni f \overset{\mathcal{D}'}{=} 0 :\Longleftrightarrow \int f \cdot \rho = 0 \quad \forall \rho \in \mathcal{C}_0^\infty$

$$\not\Rightarrow f = 0 \quad (\text{in } \mathcal{G})$$

$c(\delta) * c(\delta) - c(\delta)$ [Colombeau '84]

Another result:

– Let $f \in \mathcal{G}(\Omega)$; suppose $\exists k \in \mathbb{N}_0 : \int f \cdot \varphi = 0 \ \forall \varphi \in \mathcal{C}_0^k(\Omega)$

$$\Longrightarrow f = 0$$

In [PSV, 06] one finds an "elementary" proof & an elegant shortcut which uses the parametrix formula

Another one:

$$- f \in \mathcal{G}^\infty, \ f \overset{\infty'}{=} 0 \iff f = 0$$

Let me now come to a characterization of translation invariant generalized fcts.

Thm (Lobster) Let $f \in \mathcal{G}(\mathbb{R}^n)$ s.t.

$$f(\cdot + h) = f(\cdot) \quad \forall h \in \mathbb{R}^n$$

Then f is a (generalized) constant

REMARK. If we had $\forall \tilde{h} \in \tilde{\mathbb{R}}^n \Longrightarrow f = \text{const}$

point value char [Kunzinger, Oberguggenberger 03]

[A gen fct is not characterized by its values on $x \in \mathbb{R}^n$ but $\tilde{x} \in \tilde{\mathbb{R}}^n$]

History.

GF2000@Guadeloupe. M.O. talk on group invariant fcts.
stated the above as a problem that he couldn't solve & offered a lobster to those who could.

2006: Scarpalezos, Pilipović, Valmorin
↳ proof using Baire argument
↳ using parametrix formula & Baire

2008: Vernaeve short & elegant proof (elementary? transl. inv. of the L-measure)
even gives a stronger result:
$\forall h \in \mathbb{R}$ replaced by $h_1, h_2 \in \mathbb{R}$ $h_1/h_2 \in \mathbb{R}^+ \setminus \mathbb{Q}$ algebraic

Proof. (Baire & parametrix) ρ, θ as in the parametrix formula
from the assumption ($\mathbb{R}$-tsl. inv.) f_ε rep of f

$$\varepsilon^{-p} \int \Delta^m f_\varepsilon(y) \left(\check{\rho}(x+y) - \check{\rho}(y)\right) dy = O(1) \quad \forall p \in \mathbb{N}_0 \ \forall x \in \mathbb{R}^n$$

[transform variables...]

$\mathrm{supp}(\rho) \subset B$

We prepare for a first Baire argument

$$A_{p,e} := \left\{ x : \varepsilon^{-p} \left| \int \Delta^m f_\varepsilon(y) \left(\check{\rho}(x+y) - \check{\rho}(y)\right) dy \right| \leq 1 \ \forall \varepsilon < 1/e \right\} \quad \text{[closed]}$$

$$\xrightarrow[\text{inv.}]{\mathbb{R}\text{-tsl.}} \quad \bigcup_e A_{p,e} = \mathbb{R}^n$$

closed ball

$\overset{\text{Baire}}{\Longrightarrow} \exists\, l_0, x_0, r_0 : B(x_0, r_0) \subseteq A_{\rho, l_0}$, i.e.

$$\varepsilon^{-p} \left| \int \Delta^m f_\varepsilon(y) (\check\rho(x+y) - \check\rho(y))\, dy \right| \leq 1 \quad \forall \varepsilon < 1/l_0 \quad \forall x \in B(x_0, r_0)$$

We prepare for a 2^{nd} Baire argument

$$A_{\theta, l} := \left\{ x \in B(x_0, \tfrac{r_0}{2}) : \varepsilon^{-p} \left| \int f_\varepsilon(y) (\check\theta(x+y) - \check\theta(y))\, dy \right| \leq 1 \quad \forall \varepsilon < 1/l \right\}$$

$$\overset{\mathbb{R}\text{-transl}}{\underset{\text{inv.}}{\Longrightarrow}} \bigcup_{l \geq l_0} A_{\theta, l} = B(x_0, r_0/2)$$

$\overset{\text{Baire}}{\Longrightarrow} \exists\, l_1 > l_0,\ x_1 \in B(x_0, r_0/2),\ r_1 < r_0/2 : B(x_1, r_1) \subseteq A_{\theta, l_1}$, i.e.

$$\varepsilon^{-p} \left| \int f_\varepsilon(y) (\check\theta(x+y) - \check\theta(y))\, dy \right| \leq 1 \quad \forall \varepsilon < 1/l_0 \quad \forall x \in B(x_1, r_1)$$

Now we prove the assertion

Want this to be $O(1)$

$$\varepsilon^{-p} \left| f_\varepsilon(-x) - \underbrace{\int (\Delta^m f_\varepsilon(y) \check\rho(y) - f_\varepsilon(y) \check\theta(y))\, dy}_{c_\varepsilon \text{ a generalized constant}} \right|$$

parametrix formula

$$= \Delta^m f_\varepsilon * \rho_\varepsilon(-x) + f_\varepsilon * \theta_\varepsilon(-x)$$

$$\leq \varepsilon^{-p} \left| \int \Delta^m f_\varepsilon(y) \left(\check{\rho}(x+y) - \check{\rho}(y) \right) dy \right|$$

$$+ \varepsilon^{-p} \left| \int f_\varepsilon(y) \left(\check{\sigma}(x+y) - \check{\sigma}(y) \right) dy \right|$$

$$\leq 2 \quad \forall \varepsilon < 1/\ell_1 \text{ on } B(x_1, r_1)$$

$$\xrightarrow[\text{inv.}]{\mathbb{R}\text{-trsl}} \text{ on } B(x, r_1) \quad \forall x \in \mathbb{R}^n$$

$$\Longrightarrow \text{ on all } K \subseteq \mathbb{R}^n \text{ cp.}$$

$\square$

Consequences: This easy characterisation of trsl. inv. perfct translates to an easy characterization of invariance of gen. fcts under reg. Lie group actions

locally diffeomorph group

Has inspired work eg by

[Konjik, Kunzinger], [Vernaeve,]