Normally hyperbolic operators of low regularity

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1. **Non-smooth Lorentzian geometry**
   - Distributional geometry
   - Nonlinear Distributional Geometry
   - Compatibility

2. **Normally hyperbolic operators & Low Regularity**
   - PDOs with generalised coefficients
   - Generalised wave operators
   - Where to go from here

Distributions on manifolds

[Schwartz, de Rham, Marsden, Parker, ...]

- distributions on manifolds: $\mathcal{D}'(M) = \left[\Omega^n_c(M)\right]'$
- distributional sections of a vector bundle $E \to M$

$$
\begin{aligned}
\mathcal{D}'(M,E) &:= \left[\Gamma(M,E^*) \otimes_{\mathcal{C}^\infty(M)} \Omega^n_c(M)\right]' \\
&\cong L_{\mathcal{C}^\infty(M)}\big(\Gamma(M,E^*), \mathcal{D}'(M)\big) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M,E)
\end{aligned}
$$

- in particular, distributional tensor fields

$$
\begin{aligned}
\mathcal{D}'^r_s(M) &:= \left[\mathcal{T}^s_r(M) \otimes \Omega^n_c(M)\right]' \\
&\cong L_{\mathcal{C}^\infty(M)}\big(\mathcal{T}^s_r, \mathcal{D}'(M)\big) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{T}^r_s(M)
\end{aligned}
$$

- extend operations by continuity: $L_X,\ [\,,\,],\ \wedge,\ \iota_X,\ \{\,,\,\},\ldots$

but with only one $\mathcal{D}'$-factor!

Distributional metrics

Definition (Marsden, 1969)

A distributional metric is a symmetric element

$$g \in \mathcal{D}'^0_2(M) \qquad \text{i.e., } g : \mathfrak{X} \times \mathfrak{X} \to \mathcal{D}'(M),\ \mathcal{C}^\infty(M) - \text{bilinear}$$

which is non-degenerate, i.e., $g(X, Y) = 0\ \forall Y \Rightarrow X = 0$

Problems:

- notion of nondegeneracy is rather weak (non-local)
- can't insert $\mathcal{D}'$-vector fields into $\mathcal{D}'$-metric
- g gives no isomorphism $\mathcal{D}'^1_0 \ni X \mapsto X^\flat := g(X, .) \in \mathcal{D}'^0_1$
- no Levi-Civita connection
- no curvature
- geodesics ?

Geroch-Traschen metrics

Definition (essentially G&T 87)

A distributional metric g is called gt-regular if

$$g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$$

and uniformly nondegenerate, i.e., $|\det g| \geq C$ on cp. sets.

Theorem (G&T 87, LeFloch&Madare 07)

Let g be gt-regular. Then

(i) *There exists a unique Levi-Civita connection in L^2_{loc}.*

(ii) *The Riemann, Ricci and scalar curvature are defined as distributions by the usual formulae.*

(iii) *H^1_{loc}-stability*

(iv) *No-go: codimension of $\mathrm{supp}(Riem) \leq 1$*

Colombeau algebras on manifolds 1

Definition (Scalars, Damsma&deRoever 91)

$$\mathcal{E}_M(M) := \{(u_\varepsilon) \in \mathcal{C}^{\infty(0,1]} : \ \forall K \ \forall P \ \exists I : \ \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{-I})\}$$

$$\mathcal{N}(M) := \{(u_\varepsilon) \in \mathcal{E}_M(M) : \ \forall K \qquad \forall m : \ \sup_{x\in K} | \quad u_\varepsilon(x)| = O(\varepsilon^{m})\}$$

$$\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$$

- Fine sheaf of differential algebras w.r.t. the Lie derivative

$$L_X u := [(L_X u_\varepsilon)_\varepsilon].$$

- Embeddings: $\exists$ injective sheaf morphisms (basically convolution)

$$\iota : \mathcal{C}^\infty(M) \hookrightarrow \mathcal{D}'(M) \hookrightarrow \mathcal{G}(M).$$

Colombeau algebras on manifolds 2

Definition (Sections)

$$\Gamma_{\mathcal{G}}(M,E) := \Gamma_{\mathcal{M}}(M,E)/\Gamma_{\mathcal{N}}(M,E),$$

where moderateness and negligibility are defined analogously.

Proposition (Characterising sections, Kunzinger&S. 01)

$\Gamma_{\mathcal{G}}(M,E)$ is a fine sheaf of

finitely generated and projective $\mathcal{G}(M)$-modules and

$$\begin{aligned} \Gamma_{\mathcal{G}}(M,E) &\cong L_{\mathcal{C}^\infty(M)}\big(\Gamma(M,E^*),\mathcal{G}(M)\big) \\ &\cong \mathcal{G}(M)\otimes_{\mathcal{C}^\infty}\Gamma(M,E) \\ &\cong L_{\mathcal{G}(M)}\big(\Gamma_{\mathcal{G}}(M,E),\mathcal{G}(M)\big). \end{aligned}$$

Generalised metrics

Definition (Kunzinger&S., 02)

We define a symmetric $g \in \mathcal{G}^0_2(M)$ to be a generalised metric by one of the following equivalent conditions

(i) $\det(g)$ invertible in $\mathcal{G}(M)$

(generalised nondegeneracy)

(ii) for all generalised points $g(\tilde{x})$ is nondegenerate as map $\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$

(pointwise generalised nondegeneracy)

(iii) locally there exists a representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}(M)$

(idea of smoothing)

- technicalities on the index skipped
- g induces an isomorphism $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X, \,.\,) \in \mathcal{G}^0_1(M)$

Levi-Civita connection

Definition

A gen. connection is a map $\nabla : \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}^1_0(M)$ satisfying the ususal conditions.

- extends to entire generalised(!) tensor-algebra
- standard formulas hold, e.g.

$$\nabla_{\partial_i}(Y^j \partial_j) = \big(\partial_i Y^k + \Gamma^k_{ij} Y^j\big)\partial_k \quad (Y^j, \Gamma^k_{ij} \in \mathcal{G}).$$

Theorem (Fundamental Lemma)

For any generalised metric g there exists a unique generalised connection ∇ that is ($X, Y, Z \in \mathcal{G}^1_0(M)$)

(∇_3) *torsion-free i.e.,* $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$

(∇_4) *metric, i.e.,* $\nabla_X g = 0 \Leftrightarrow X\big(g(X, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

It is called Levi-Civita connection and given by Koszul's formula.

Curvature from a generalised connection

Definition (Generalised curvature)

(i) Let g be a generalised metric with generalised Levi-Civita connection ∇. We define the generalised Riemann curvature tensor $R \in \mathcal{G}^1_3(M)$ of g by the usual formula, i.e.,

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

(ii) Ricci and scalar curvature defined as usual.

Observation (Basic compatibility)

Let g be a generalised metric with representative g_ε such that

$$g_\varepsilon \to \tilde{g} \text{ locally in } \mathcal{C}^k.$$

Then any representative R_ε of the Riemann tensor R of g converges to the Riemann tensor $\tilde{R}$ of $\tilde{g}$ locally in $\mathcal{C}^{k-2}$.
The analogue holds true for the Ricci and scalar curvature.

Compatibility with the GT-setting

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ two ways to calculate the curvature
  - (i) gt-setting: coordinate formulae in $\mathcal{D}'$ resp. $W^{m,p}_{\mathrm{loc}}$
    $\rightsquigarrow \mathrm{Riem}[g] \in {\mathcal{D}'}^1_3$
  - (ii) $\mathcal{G}$-setting: embed g via convolution with a mollifier
    usual formulae for fixed ε $\rightsquigarrow \mathrm{Riem}[g_\varepsilon] \in \mathcal{G}^1_3$
- Do we get the same answer?

$$
\begin{array}{ccc}
H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni g & \xrightarrow{\;*\rho_\varepsilon\;} & [g_\varepsilon] \in \mathcal{G} \\
\text{gt-setting} \Big\downarrow & & \Big\downarrow \mathcal{G}\text{-setting} \\
\mathrm{Riem}[g] & \xleftarrow{\;\lim_{\varepsilon\to 0}\;} & \mathrm{Riem}[g_\varepsilon]
\end{array}
$$

Yes, [S.&Vickers, 09]

PDOs with generalised coefficients

- Local approach, i.e. pasting local expressions of the form

$$Q(x,D) = \sum a_\alpha(x) D^\alpha \text{ with } a_\alpha \in \mathcal{G}$$

works immediately.

- Algebraic approach using the $\mathcal{G}(M)$-module structure of $\Gamma_{\mathcal{G}}(E)$

$$\begin{aligned}\mathrm{PDO}^{(m)}_{\mathcal{G}}(E,F) \quad := \quad \{P : \ & \Gamma_{\mathcal{G}}(E) \to \Gamma_{\mathcal{G}}(F)\ \mathbb{R}\text{-linear and} \\ & \mathrm{ad}(f_0)\cdots\mathrm{ad}(f_m)P = 0 \quad \forall f_i \in \mathcal{G}(M)\}.\end{aligned}$$

- Compatible since

$$\begin{aligned}\mathrm{PDO}^{(1)}_{\mathcal{G}}(M) \quad &\cong \quad \mathrm{Der}(\mathcal{G}(M)) \oplus \mathrm{PDO}^{0}_{\mathcal{G}}(M) \\ &\cong \quad \Gamma_{\mathcal{G}}(TM) \oplus \mathrm{Hom}_{\mathcal{G}}(\mathcal{G}(M))\end{aligned}$$

plus induction.

Lowly regular normally hyperbolic operators

Definition

A PDO of second order with generalised coefficients is called normally hyperbolic if its principal symbol is given by (minus) the (inverse) of a generalised Lorentzian metric.

- Locally $P = g^{ij}\partial_i\partial_j + A^j\partial_j + B$,
  where $g \in \mathcal{G}^0_2(M)$ is a generalised L-metric and A_j, B are generalised coefficients
- gen. metric d'Alembertian/wave operator $\Box_g = g^{ij}\nabla_i\nabla_j$
  where ∇ is the Levi-Civita connection of g
- gen. connection d'Alembertian $\Box^\nabla = -(\mathrm{tr}_g \otimes \mathrm{Id}_E)(\nabla^{T^*M\otimes E} \circ \nabla_g)$
- Weitzenböck formula ???

The wave equation on singular space-times

Results (Existence and uniqueness)

Local existence and uniqueness theorems for the Cauchy problem for the wave operator of "weakly singular" Lorentzian metrics in the Colombeau algebra.

- conical space times
  plus compatibility with distributional results
  [Vickers&Wilson, 00]
- generalisation to essentially locally bounded metrics
  [Grant, Mayerhofer & S., 09]
- generalisation to tensors, refined regularity [Hanel, 10]

Conditions on the metrics

(A) $\forall K \subset\subset U\ \forall k\ \forall \eta_1, \ldots, \eta_k \in \mathfrak{X}(M)$

- $\sup_K ||L\eta_1 \ \ldots \ L\eta_k \ g_\varepsilon|| = O(\varepsilon^{-k})$
- $\sup_K ||L\eta_1 \ldots L\eta_k \ g_\varepsilon^{-1}|| = O(\varepsilon^{-k})$

in particular g_ε, g_ε^{-1} locally uniformly bounded
$\Rightarrow \ 1/M_0 \leq \sqrt{-g_\varepsilon(\xi_\varepsilon, \xi_\varepsilon)} = V_\varepsilon \leq M_0$
existence of unfi. spacelike initial surface with normal ξ

(B) $\forall K \subset\subset U: \ \sup_K ||\nabla_{g^\varepsilon} \xi_\varepsilon|| = O(1)$

$\Rightarrow \ ||L_\xi g_\varepsilon||_{e_\varepsilon} = O(1), \quad ||\nabla_{g^\varepsilon} e_\varepsilon|| = O(1) = ||\nabla_{g^\varepsilon} e_\varepsilon^{-1}||_{e_\varepsilon}$

(C) for each ε : $\Sigma := \Sigma_0$ past cp., spacelike hypersurface s.t.
$\partial J_\varepsilon^+(\Sigma) = \Sigma$
and $\exists A \neq \emptyset$, open $A \subseteq \bigcap_\varepsilon J_\varepsilon^+(\Sigma)$

$\Rightarrow$ existence of classical solutions on common domain

(all norms $\| \ \|$ derived from some classical Riemannian background metric)

The result

Theorem (Grant, Mayerhofer &S., 09)

Let (M, g) be a generalised space-time such that (A)–(C) holds and let $v, w \in \mathcal{G}(\Sigma)$. Then for any $p \in \Sigma$ there exists an open neighbourhood V where

$$\Box_g u = 0, \qquad u|_\Sigma = v, \quad L_{\hat{\xi}}|_\Sigma u = w$$

has a unique solution in $\mathcal{G}(V)$.

- strength: wide class of metric included, e.g. impulsive grav. waves, cosmic strings
- weakness: meaning of condition (C); hard to connect to more classical approaches
- key technology in the proof: higher order energy estimates

General strategy of proof

key ingredient: higher order energy estimates

- use of ε-dependent Sobolev norms
- use of ε-dependent energy-momentum tensors
- switch back to sup-norms

step 1: existence of classical solutions u_ε for fixed ε (by (C))
$\rightsquigarrow$ candidate for $\mathcal{G}$-solution

step 2: existence of $\mathcal{G}$-solutions:
moderate data $\rightsquigarrow$ moderate initial energies
$\rightsquigarrow$ moderate energies for all times $\rightsquigarrow$
$(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(V)$

step 3: uniqueness—part 1:
independence of the choice of the representatives of v, w
negligible data $\rightsquigarrow$ [......] $\rightsquigarrow (u_\varepsilon)_\varepsilon \in \mathcal{N}(V)$

step 2: uniqueness—part 2:
independence of the choice of the representative of g

Key energy estimate

Let u_ε be a solution of $(\Box_\varepsilon)$. Then $\forall k \; \exists C'_k, C''_k, C'''_k$ s.t. $\forall 0 \leq \tau \leq \gamma$

(i) $$E^k_{\tau,\varepsilon}(u_\varepsilon) \leq E^k_{0,\varepsilon}(u_\varepsilon) + C'_k({}^\nabla\|f_\varepsilon\|^{k-1}_{\Omega_\tau,\varepsilon})^2 + C'''_k \int_{\zeta=0}^{\tau} E^k_{\zeta,\varepsilon}(u_\varepsilon)d\zeta + C''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta$$

(ii) $$E^k_{\tau,\varepsilon}(u_\varepsilon) \leq \Big(E^k_{0,\varepsilon}(u_\varepsilon) + C'_k({}^\nabla\|f_\varepsilon\|^{k-1}_{\Omega_\tau,\varepsilon})^2 + C''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} \int_{\zeta=0}^{\tau} E^j_{\zeta,\varepsilon}(u_\varepsilon)d\zeta\Big) e^{C'''_k \tau}$$

(iii) If the initial energy $(E^k_{0,\varepsilon}(u_\varepsilon))_\varepsilon$ is moderate [negligible] then

$$\sup_{0\leq\tau\leq\gamma} (E^k_{\tau,\varepsilon}(u_\varepsilon))_\varepsilon$$

is moderate [negligible].

Outlook & Perspectives

- (semi-)global results by a more clever use of cl. existence theory
  dispense with condition (C)
- general normally hyperbolic operators (Weitzenböck formula ?)
- generalization of Hadamard parametrix construction
  extract regularity
- connect to more classical approaches (GT space-times)
- more general metrics: log-type growth in ε replacing $O(1)$
  Hölder-Zygmund classes
- ...
- generalised singularity theorems
  (extending geodesics in generalised sense)
- go non-linear: Einstein equations
- connections to cosmic censorship hypothesis

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