Impulsive gravitational waves

From particle motion to geometry

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Long-term project

Non-smooth Spacetime Geometry

Lorentzian Geometry & GR in low regularity

Theoretical branch

- analytic methods:
  causality & singularity thms

  Chruściel, Grant,

  Graf, Grant, Hörmann, Kunzinger, Ohanyan, Vickers,

  Sämann, Schinnerl, Stojković, S.

  Ling, Minguzzi, Sbierski

- synthetic: measured
  Lorentzian length spaces

  Kunzinger, Sämann

  Braun, Cavaletti, Mondino, McCann, Solis,

  Beran, Callist, Ohanyan, Rott, S.

Exact solutions branch

Impulsive
gravitational waves

- particle motion,
  completeness

  Podolský, Sämann, Švarc, S.

- matching spacetimes

  Mars, Manzano, Ohanyan, S.

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- model short but strong pulses of gravitational radiation
- put forward by Roger Penrose in the late 1960's
- shockwave generated by ultrarelativistic particle [AS-1972]

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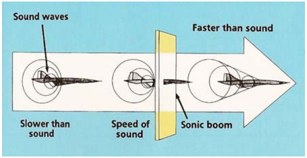

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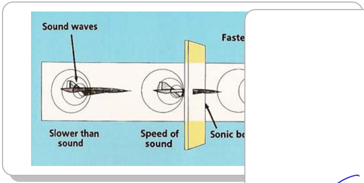

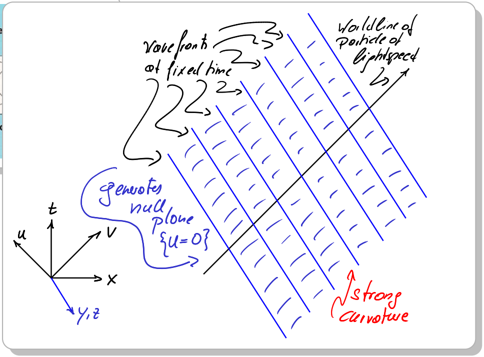

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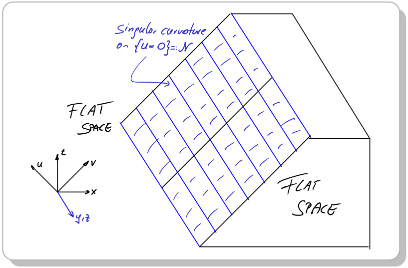

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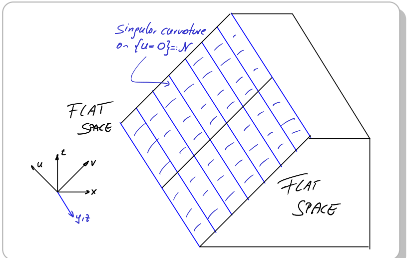


Vivid distributional metric [Pen,72]

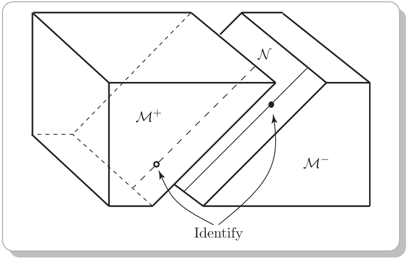
$$\boxed{\mathrm{d}s^2 = -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + \mathrm{d}y^2 + \mathrm{d}z^2 \\ \qquad\qquad +2\,H(y,z)\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2 \qquad (B)}$$

Brinkman-form of impulsive pp-wave

Penrose cut & paste construction

- **cut** Minkowski space $\left(\mathbb{R}^4, ds_0^2 = -2d\mathcal{U}\, d\mathcal{V} + dy^2 + dz^2\right)$ along null plane $\mathcal{N} = \{\mathcal{U} = 0\}$
- **shift** resulting half-spaces M^-, M^+
- **paste** by identifying boundary points in $\mathcal{N}$ according to the **Penrose junction conditions**

$$\mathcal{V} \in \mathcal{M}^- \mapsto \mathcal{V} - H(y,z) \in \mathcal{M}^+$$



Leads to the $C^{0,1}$-**Rosen metric of impuslive pp-waves**

$$\boxed{\mathrm{d}s^2 = 2\left|\mathrm{d}Z + u_+ (H_{,\bar{Z}Z}\,\mathrm{d}Z + H_{,\bar{Z}\bar{Z}}\,\mathrm{d}\bar{Z})\right|^2 - 2\,\mathrm{d}u\,\mathrm{d}v \quad \text{(R)}}$$

$Z \in \mathcal{C}$, complex coordinate to simplify matters ...

The ‘discontinuous coordinate transformation’

- The (B) and (R) forms of the metric are physically equivalent.
- Both possess distributional curvature concentrated on $\mathcal{N}$.
- There is a “discontinuous transformation” between them

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$$\boxed{\begin{aligned} \mathcal{U} &= u \\ \mathcal{V} &= v + \Theta(u)\, H + u_+\, H_{,Z}\, H_{,\bar{Z}} \quad (T) \\ \eta &= Z + u_+\, H_{,\bar{Z}} \end{aligned}}$$

Takes

$$\mathrm{d}s^2 = -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + \mathrm{d}y^2 + \mathrm{d}z^2 + 2\,H(y,z)\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2 \quad (B)$$

to

$$\mathrm{d}s^2 = 2\left|\mathrm{d}Z + u_+(H_{,\bar{Z}Z}\,\mathrm{d}Z + H_{,\bar{Z}\bar{Z}}\,\mathrm{d}\bar{Z})\right|^2 - 2\,\mathrm{d}u\,\mathrm{d}v \quad (R)$$

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- Made rigorous in nonlinear generalized functions [KS,99].
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  (A) Geometric insight:
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  fully nonlinear distributional analysis of the geodesics of (B)

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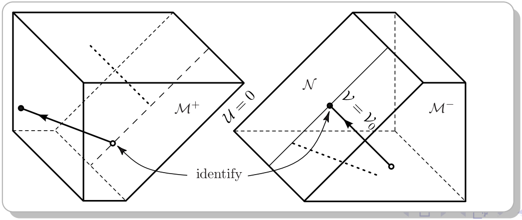

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Goal: Do the same for nonvanishing (positive) cosmological constant Λ.
Analytic core: fixed point argument to solve the
i.v.p. for the geodesic equations of (B)-analog

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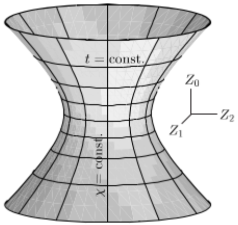
(Non-expanding) Impulsive waves in de Sitter universe

de Sitter space

- vacuum solution of Einstein eqs. with a positive cosmological constant
- simplest cosmological model with observed accelerating expansion
- simplest visualization: **4D hyperboloid in 5D Minkowski space**



Flat $\mathbb{R}^5$ with metric

$$ds_0^2 = -2\mathrm{d}U\mathrm{d}V + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2$$

constraint

$$Z_2^2 + Z_3^2 + Z_4^2 - 2UV = 3/\Lambda,$$

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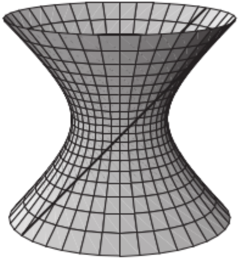

5D pp-wave with metric

$$\mathrm{d}s^2 = -2\mathrm{d}U\mathrm{d}V + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2 + H(Z_2, Z_3, Z_4)\,\delta(U)\,\mathrm{d}U^2$$

(same) constraint

$$Z_2^2 + Z_3^2 + Z_4^2 - 2UV = 3/\Lambda$$

impulse on null hypersurface

$$\{U=0\}: \quad {Z_2}^2 + {Z_3}^2 + {Z_4}^2 = 3/\Lambda$$

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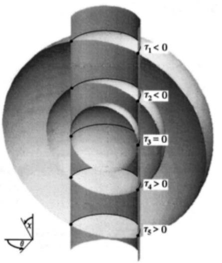

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impulse on null hypersurface

$$\{U = 0\}: \quad {Z_2}^2 + {Z_3}^2 + {Z_4}^2 = 3/\Lambda$$

Regularisation—Generalized spacetime

- **Regularised** 5D impulsive pp-wave $(\bar{M} = \mathbb{R}^5, \bar{g}_\varepsilon)$ with

$$\mathrm{d}\bar{s}_\varepsilon^2 = -2\mathrm{d}U\mathrm{d}V + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \sigma\mathrm{d}Z_4^2 + H(Z_p)\ \delta_\varepsilon(U)\ \mathrm{d}U^2 \qquad (2.1)$$

  with **model delta net**

$$\delta_\varepsilon(x) := \frac{1}{\varepsilon}\,\rho\left(\frac{x}{\varepsilon}\right)$$

- **de Sitter hyperboloid** (M, g_ε) in $(\bar{M}, \bar{g}_\varepsilon)$ as usual:

$$M := \{(U, V, Z_2, Z_3, Z_4) \in \bar{M} : -2UV + Z_2^2 + Z_3^2 + Z_4^2 - a^2 = 0\}$$

- **Generalized** (Colombeau) 5D impulsive pp-wave $(\bar{M} = \mathbb{R}^5, \bar{g})$ with

$$\mathrm{d}\bar{s}^2 = -2dUdV + dZ_2^2 + dZ_3^2 + dZ_4^2 + H(Z_p)\ D(U)\ dU^2 \qquad (2.2)$$

  with $D = [(\delta_\varepsilon)_\varepsilon]$ **generalized δ-function**

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Regularised equations

$$\begin{aligned}
\ddot U_\varepsilon &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\dot{\big)}\Big)\,\frac{U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}\\
\ddot Z_{p\varepsilon} - \frac{1}{2}H_{,p}\,\delta_\varepsilon\,\dot U_\varepsilon^2 &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\dot{\big)}\Big)\,\frac{Z_{p\varepsilon}}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}\\
\ddot V_\varepsilon - \frac{1}{2}\,H\,\delta_\varepsilon^{'}\,\dot U_\varepsilon^2 - \delta^{pq}H_{,p}\,\delta_\varepsilon\,\dot Z_q^\varepsilon\,\dot U_\varepsilon &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\dot{\big)}\Big)\,\frac{V_\varepsilon + H\,\delta_\varepsilon\,U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}
\end{aligned}$$

where

$$\begin{aligned}
&\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t))\,, \quad \delta'_\varepsilon = \delta'_\varepsilon(U_\varepsilon(t))\,, \quad e = 0, \pm 1\,,\\
&\tilde G_\varepsilon = \tilde G_\varepsilon(U_\varepsilon(t), Z_{p\varepsilon}(t))\,, \quad H = H(Z_{p\varepsilon}(t))\,, \quad \text{and} \quad H_{,p} = H_{,p}(Z_{q\varepsilon}(t))
\end{aligned}$$

Geodesic equation

$\gamma : I \to M$ with $\nabla_{\dot\gamma}\dot\gamma = 0$
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good news!

$$\ddot U_\varepsilon = -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\dot{(H\,\delta_\varepsilon\,U_\varepsilon)}\Big)\;\frac{U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot Z_{p\varepsilon} - \frac{1}{2}H_{,p}\,\delta_\varepsilon\,\dot U_\varepsilon^2 = -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\dot{(H\,\delta_\varepsilon\,U_\varepsilon)}\Big)\;\frac{Z_{p\varepsilon}}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

linear & decoupled ... simply integrate at the end

BUT

$$\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t)),\quad \delta'_\varepsilon = \delta'_\varepsilon(U_\varepsilon(t)),\quad e = 0, \pm 1,$$
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Model system for $\gamma_\varepsilon \equiv x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathbb{R} \times \mathbb{R}^3$

$$\ddot u_\varepsilon = -\Big(e + \frac{1}{2}\,\dot u_\varepsilon^2\,\tilde G_\varepsilon - \dot u_\varepsilon\,\big(H\,\delta_\varepsilon\,u_\varepsilon\dot{\big)}\Big)\;\frac{u_\varepsilon}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot z_\varepsilon - \frac{1}{2}DH\,\delta_\varepsilon \dot u_\varepsilon^2 = -\Big(e + \frac{1}{2}\,\dot u_\varepsilon^2\,\tilde G_\varepsilon - \dot u_\varepsilon\,\big(H\,\delta_\varepsilon\,u_\varepsilon\dot{\big)}\Big)\;\frac{z_\varepsilon}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon}$$

with

$$H = H(z_\varepsilon) \in C^\infty(\mathbb{R}^3)$$

$$\tilde G_\varepsilon(u_\varepsilon, z_\varepsilon) := DH(z_\varepsilon)\,\delta_\varepsilon(u_\varepsilon)\,z_\varepsilon + H(z_\varepsilon)\,\delta'_\varepsilon(u_\varepsilon)\,u_\varepsilon$$

Initial data & time of existence

Seed geodesics and initial conditions

The u-component of the seed geodesic γ (black) reaches the regularisation sandwich at

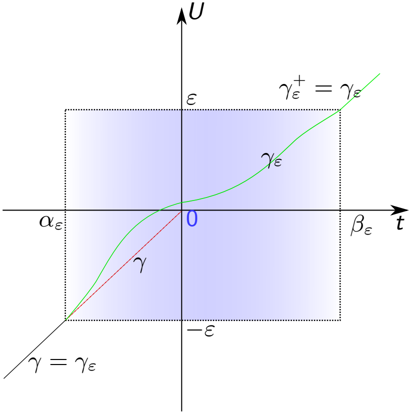
$$t = \alpha_\varepsilon, \text{ i.e., } u(\alpha_\varepsilon) = -\varepsilon.$$

In the background spacetime γ would continue (dotted red) to

$$U = 0 \text{ at } t = 0.$$

In the regularised spacetime γ continues as γ_ε (green) solving the model equations.

Goal: show that γ_ε lives long enough to cross the sandwich for ε small.



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We look for solutions on

$$J_\varepsilon = [\alpha_\varepsilon, \alpha_\varepsilon + \eta] \quad (\eta > 0)$$

and set data at $t = \alpha_\varepsilon$

$$\gamma_\varepsilon(\alpha_\varepsilon) = (-\varepsilon, z_\varepsilon^0)$$
$$\dot\gamma_\varepsilon(\alpha_\varepsilon) = (\dot u_\varepsilon^0 (> 0), \dot z_\varepsilon^0)$$

where we additionally demand convergence to some seed data

$$(-\varepsilon, z_\varepsilon^0) \to (0, z^0)$$
$$(\dot u_\varepsilon^0 > 0, \dot z_\varepsilon) \to (\dot u^0 > 0, \dot z^0)$$

The full model system

Model system for $\gamma_\varepsilon \equiv x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathbb{R} \times \mathbb{R}^3$

$$\ddot{u}_\varepsilon = -\Big(e + \frac{1}{2}\,\dot{u}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{u}_\varepsilon\,\big(H\,\delta_\varepsilon\,u_\varepsilon\big)^{\boldsymbol{\cdot}}\Big)\,\frac{u_\varepsilon}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot{z}_\varepsilon - \frac{1}{2} DH\,\delta_\varepsilon \dot{u}_\varepsilon^2 = -\Big(e + \frac{1}{2}\,\dot{u}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{u}_\varepsilon\,\big(H\,\delta_\varepsilon\,u_\varepsilon\big)^{\boldsymbol{\cdot}}\Big)\,\frac{z_\varepsilon}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon}$$

with data

$$\begin{aligned} x_\varepsilon(\alpha_\varepsilon) = (u_\varepsilon(\alpha_\varepsilon), z_\varepsilon(\alpha_\varepsilon)) &= (u_\varepsilon^0, z_\varepsilon^0) \\ &= (-\varepsilon, z_\varepsilon^0) \to (0, z^0) \in \mathbb{R} \times \mathbb{R}^3 \end{aligned}$$

$$\dot{x}_\varepsilon(\alpha_\varepsilon) = (\dot{u}_\varepsilon(\alpha_\varepsilon), \dot{z}_\varepsilon(\alpha_\varepsilon)) = (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0) \ \to (\dot{u}^0(> 0), z^0) \in \mathbb{R} \times \mathbb{R}^3$$

Solution space & operator

$$\mathfrak{X}_\varepsilon := \Big\{ x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathcal{C}^1(J_\varepsilon, \mathbb{R}^4) : \ x_\varepsilon(\alpha_\varepsilon) = x_\varepsilon^0, \ \dot{x}_\varepsilon(\alpha_\varepsilon) = \dot{x}_\varepsilon^0$$
$$\|x_\varepsilon - x^0\|_\infty \leq C_1, \ \|\dot{z}_\varepsilon - \dot{z}^0\|_\infty \leq C_2, \ \dot{u}_\varepsilon \in \Big[\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0\Big] \Big\}$$

- prospective solutions assume ε-dependent data
- centred around the 'fixed' data $(0, z^0)$ and $(\dot{u}_0, \dot{z}^0)$
- $\dot{u}_\varepsilon$ forced to stay positive
- $\mathfrak{X}_\varepsilon$ only depends on ε via the domain J_ε and data

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$$A_\varepsilon^1(x_\varepsilon)(t) = -\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{e u_\varepsilon + \frac{1}{2} u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - u_\varepsilon \dot{u}_\varepsilon \left(H\delta_\varepsilon u_\varepsilon\right)^\cdot}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon} \,\mathrm{d}r\,\mathrm{d}s$$

$$+ \dot{u}_\varepsilon^0(t - \alpha_\varepsilon) - \varepsilon$$

$$A_\varepsilon^2(x_\varepsilon)(t) := \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left(\frac{1}{2} DH\delta_\varepsilon \dot{u}_\varepsilon^2 - \frac{e z_\varepsilon + \frac{1}{2} z_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - z_\varepsilon \dot{u}_\varepsilon \left(H\delta_\varepsilon u_\varepsilon\right)^\cdot}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} \right) \mathrm{d}r\,\mathrm{d}s$$

$$+ \dot{z}_\varepsilon^0(t - \alpha_\varepsilon) + z_\varepsilon^0$$

Local existence & uniqueness

A lot of interesting estimates lead to

$$\|(A_\varepsilon)^n(x_\varepsilon) - (A_\varepsilon)^n({x^*}_\varepsilon))\|_{C^1} \leq \frac{1}{\varepsilon}\, \beta_n \, \|x_\varepsilon - x^*_\varepsilon\|_{C^1} \quad \text{with} \sum \beta_n < \infty$$

and so Weissinger's fixed point theorem applies.

Theorem (Existence and uniqueness, (LSSP,16), (SS,17))

The initial value problem for the geodesic equation has a unique smooth solution (for small ε)

$$\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_\varepsilon) \quad \textit{on} \quad [\alpha_\varepsilon, \alpha_\varepsilon + \eta],$$

Moreover γ_ε is uniformly bounded in ε together with $\dot{U}_\varepsilon$ and $\dot{Z}_\varepsilon$.

Local existence & uniqueness

A lo

$$\eta(\cancel{\varepsilon}) := \min\Big\{1,\ \frac{a^2}{24\dot u^0},\ \frac{C_1}{\frac{3}{2}+\dot u^0},\ \frac{2C_1}{54\|\rho\|_1\|DH\|_\infty \dot u^0},\ \frac{a^2\, C_1}{12(|z^0|+C_1)},\ \frac{a^2\, C_2}{8(|z^0|+C_1)},$$
$$\frac{C_1 a^2}{54}\Big(\dot u^0(|z^0|+C_1)(3\|DH\|_\infty\|\rho\|_\infty(|z^0|+C_1)+\|H\|_\infty\|\rho'\|_\infty)\Big)^{-1},\ \frac{C_1}{6(1+|\dot z^0|)},$$
$$\frac{C_1 a^2}{72}\Big((|z^0|+C_1)\Big(3\|DH\|_\infty\|\rho\|_\infty(|\dot z^0|+C_2)+\frac{3}{2}\dot u^0\|H\|_\infty(\|\rho'\|_\infty+\|\rho\|_\infty)\Big)\Big)^{-1}\Big\}$$

and

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A lo

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$$\varepsilon \leq \varepsilon_0 := \min\Big\{ \frac{a^2}{2\|\rho\|_\infty \|H\|_\infty},\ \frac{a^2}{72\dot{u}^0}\Big(3\|DH\|_\infty\|\rho\|_\infty(|z^0| + C_1) + \|H\|_\infty\|\rho'\|_\infty\Big)^{-1},$$
$$\frac{a^2}{96}\Big(3\|DH\|_\infty\|\rho\|_\infty(|\dot{z}^0| + C_2) + \frac{3}{2}\dot{u}^0\|H\|_\infty\big(\|\rho'\|_\infty + \|\rho\|_\infty\big)\Big)^{-1},$$
$$\big(3\|DH\|_\infty\|\rho\|_\infty(|\dot{z}^0| + C_2)\big)^{-1},\ \frac{\eta\dot{u}^0}{6},\ \eta\Big\}.$$

and

Theorem (Existence and uniqueness, (LSSP,16), (SS,17))

The initial value problem for the geodesic equation has a unique smooth solution (for small ε)

$$\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_\varepsilon) \quad \text{on} \quad [\alpha_\varepsilon, \alpha_\varepsilon + \eta],$$

Moreover γ_ε is uniformly bounded in ε together with $\dot{U}_\varepsilon$ and $\dot{Z}_\varepsilon$.

Extension of geodesics

Theorem (Global geodesics (LSSP,16), (SS,17))

The geodesics γ_ε extend to geodesics of the background de Sitter spacetime 'behind' the sandwich wave zone. In particular

- *the solutions γ_ε are global, i.e. defined for all $t \in \mathbb{R}$*
- *the spacetime (M, g_ε) is geodesically complete (i.e. non-singular)*

Proof.

$$U_\varepsilon(\alpha_\varepsilon + \eta) = -\varepsilon + \int_{\alpha_\varepsilon}^{\alpha_\varepsilon + \eta} \dot{U}_\varepsilon(s)\, \mathrm{d}s \geq -\varepsilon + \frac{\eta}{2} \dot{U}^0 \geq -\varepsilon + 3\varepsilon \geq \varepsilon$$

since $\varepsilon \leq \eta\, \dot{U}^0/6$

For such ε, γ_ε leaves the wave zone and extends to a geodesic of the background spacetime since the geodesic equations coincide there. □

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Preliminaries

Recall: γ_ε leaves regularisation strip at some β_ε : $U_\varepsilon(t = \beta_\varepsilon) = \varepsilon$

Lemma (Limits of exit data)

We have that $\beta_\varepsilon \searrow 0$ *and*

$$\begin{aligned} \gamma_\varepsilon(\beta_\varepsilon) &\to \ (0, B + V^0, Z_p^0) \\ \dot\gamma_\varepsilon(\beta_\varepsilon) &\to \ (1, C + \dot V^0, A_p + \dot Z_p^0) \end{aligned}$$

where

$$\begin{aligned} \mathrm{A_p} &= \frac{1}{2}\Big(\, H_{,i}(Z_r^0) + \frac{Z_i^0}{a^2}\big(H(Z_r^0) - \delta^{pq} Z_p^0 H_{,q}(Z_r^0)\big)\Big) \\ \mathrm{B} &= \ \frac{1}{2} H(Z_p^0) \\ \mathrm{C} &= \frac{1}{8}\Big(\sum_{p=1}^{3} H_{,p}(Z_r^0)^2 + \frac{1}{a^2} H(Z_r^0)^2 - \frac{1}{a^2}\left(\delta^{pq} Z_p^0 H_{,q}(Z_r^0)\right)^2\Big) \\ &\qquad - \frac{1}{2a^2}\left(\delta^{pq} Z_p^0 H_{,q}(Z_r^0) - H(Z_r^0)\right) V^0 + \frac{1}{2}\delta^{pq} H_{,p}(Z_r^0) \dot Z_q^0 \end{aligned}$$

Convergence result [LSSP,16]

- in front=seed: $\gamma^-(0) = (0, V^0, Z_p^0)$, $\dot\gamma^-(0) = (1, V^0, Z_p^0)$
- behind: $\gamma^+(0) = (0, V^0 + B, Z_p^0)$, $\dot\gamma^+(0) = (1, V^0 + C, Z_p^0 + A_p)$
- combine:

$$\tilde\gamma(\lambda) = (\tilde U, \tilde V, \tilde Z_p)(\lambda) := \begin{cases} \gamma^-(\lambda), & \lambda \le 0 \\ \gamma^+(\lambda), & \lambda > 0 . \end{cases}$$

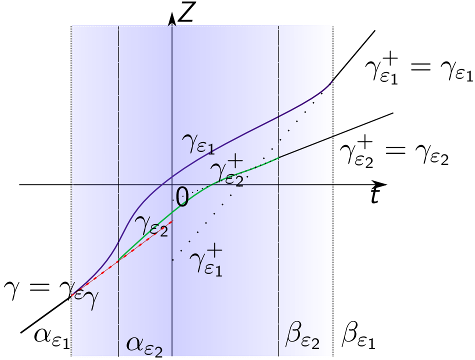
Theorem (Limiting geodesics)

The global solutions converges weakly to the limiting geodesic,

$$\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_{p\varepsilon}) \ \to \ \tilde\gamma = (\tilde U, \tilde V, \tilde Z_p).$$

Moreover, we have $U_\varepsilon \to \tilde U$ in C^1, $Z_{p\varepsilon} \to \tilde Z_p$ in C^0 & this is optimal!

Nice pictures & the null case



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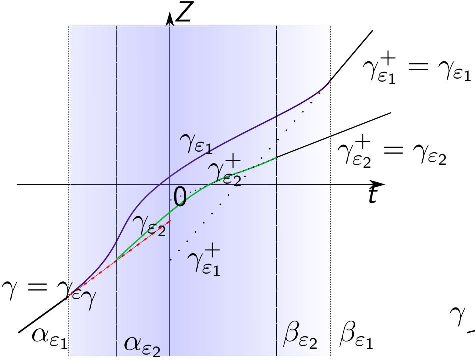

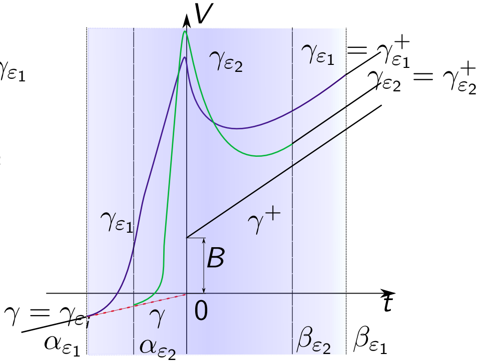

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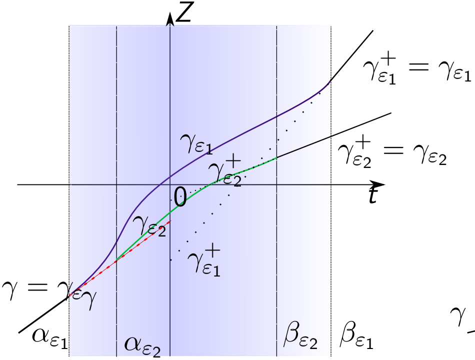

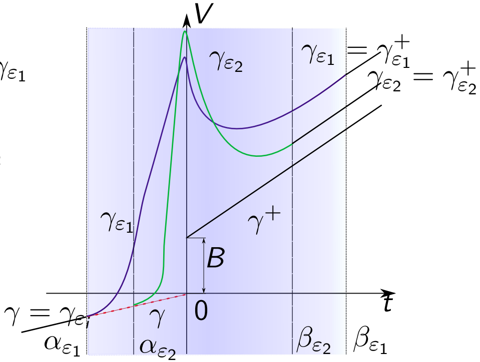

Null case: simple, backgrd. geodesics are straight lines (generators)

$$\tilde{\gamma}(\lambda) = \begin{pmatrix} \lambda \\ V^0 + \dot{V}^0\lambda + \Theta(\lambda)B + C\lambda_+ \\ Z^0_p + \dot{Z}^0_p\lambda + A_p\lambda_+ \end{pmatrix}$$

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(A) The geometric insight [PSSS,19]

'Discontinuous transformation' with Λ

$$\begin{aligned}\mathcal{U} &= u \\ \mathcal{V} &= v + \Theta(u)\,H + u_+\,H_{,Z}\,H_{,\bar Z} \qquad (T) \\ \eta &= Z + u_+\,H_{,\bar Z}\end{aligned}$$

Takes

$$ds^2 = \frac{-2\,d\mathcal{U}\,d\mathcal{V} + 2d\eta d\bar\eta + 2\,H(y,z)\,\delta(\mathcal{U})\,d\mathcal{U}^2}{[\,1 + \frac{1}{6}\Lambda(\eta\bar\eta - \mathcal{U}\mathcal{V})\,]^2} \qquad (B)$$

to

$$ds^2 = \frac{2\left|dZ + u_+(H_{,\bar z z}\,dZ + H_{,\bar z\bar z}\,d\bar Z)\right|^2 - 2\,du\,dv}{[\,1 + \frac{1}{6}\Lambda(Z\bar Z - uv + u_+ G)\,]^2} \qquad (R)$$

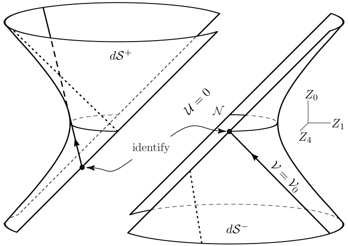
(A) The geometric insight [PSSS,19]

'Discontinuous transformation' with Λ

$$
\begin{array}{rcl}
\mathcal{U} & = & u \\
\mathcal{V} & = & v + \Theta(u)\, H + u_+\, H_{,Z}\, H_{,\bar{Z}} \\
\eta & = & Z + u_+\, H_{,\bar{Z}}
\end{array}
\qquad (T)
$$

limiting geos w. null generators as seed

$$
\tilde{\gamma}(\lambda) = \begin{pmatrix} \lambda \\ V^0 + \dot{V}^0\lambda + \Theta(\lambda)B + C\lambda_+ \\ Z_p^0 + \dot{Z}_p^0\lambda + A_p\lambda_+ \end{pmatrix}
$$



(A) The geometric insight [PSSS,19]

'Discontinuous transformation' with Λ

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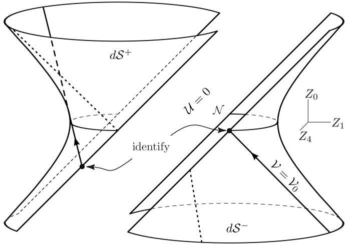
limiting geos w. null generators as seed

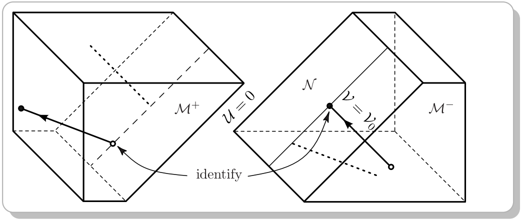
$$\tilde\gamma(\lambda) = \begin{pmatrix} \lambda \\ V^0 + \dot V^0\lambda + \Theta(\lambda)B + C\lambda_+ \\ Z_p^0 + \dot Z_p^0\lambda + A_p\lambda_+ \end{pmatrix}$$



'broken geodesic generators' are
coordinate lines of system in which the metric becomes continuous

(A) The geometric insight & even nicer pictures

The new picture is a direct generalization of Penrose's cut & paste:



(A) The geometric insight & even nicer pictures

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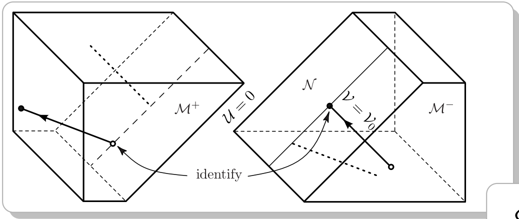

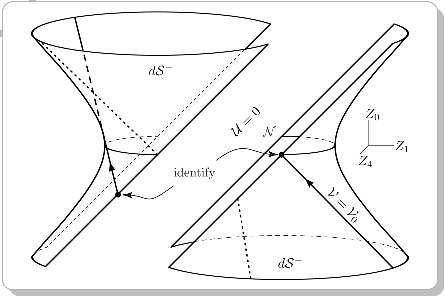

(B) Analytic insight [SSSS,23]

- 'discontinuous trsf.' via limiting geodesics

$$\begin{pmatrix} u \\ v \\ X \\ Y \end{pmatrix} \mapsto \gamma[v, X, Y](u) = \begin{pmatrix} u \\ v + \Theta(u)B + u_+ C \\ X + u_+ H^{\mathrm{i}}{}_{,X} \\ Y + u_+ H^{\mathrm{i}}{}_{,Y} \end{pmatrix} = \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ x \\ y \end{pmatrix}$$

- geometric regularisation of trsf. via regularised geodesics

$$\begin{pmatrix} u \\ v \\ X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} U \\ V \\ Z_p \end{pmatrix} \xrightarrow{(T_\varepsilon)} \gamma^{\varepsilon}_{5D}[V, Z_p](U) = \begin{pmatrix} \bar{U}_\varepsilon \\ \bar{V}_\varepsilon \\ \bar{Z}_{p_\varepsilon} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{U}_\varepsilon \\ \mathcal{V}_\varepsilon \\ x_\varepsilon \\ y_\varepsilon \end{pmatrix}$$

Beware: We have to go via 5D!

The regularized transformation

$$\bar{U}_\varepsilon[V,Z_p](U) = U - \int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \frac{\Delta_\varepsilon(r)}{N_\varepsilon(r)}\,\bar{U}_\varepsilon(r)\,\mathrm{d}r\mathrm{d}s$$

$$\begin{aligned}\bar{V}_\varepsilon[V,Z_p](U) = (1-\beta U)V &+ \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} H(\bar{Z}_{p\varepsilon})(r)\,\delta'_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,\dot{\bar{U}}^2_\varepsilon(r)\,\mathrm{d}r\mathrm{d}s\\ &+ \int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \delta^{pq}\,H_{,p}\big(\bar{Z}_{w\varepsilon}(r)\big)\,\delta_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,\dot{\bar{Z}}_{q\varepsilon}(r)\,\dot{\bar{U}}_\varepsilon(r)\,\mathrm{d}r\mathrm{d}s\\ &- \int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \frac{\Delta_\varepsilon(r)}{N_\varepsilon(r)}\Big(\bar{V}_\varepsilon(r) + H\big(\bar{Z}_{p\varepsilon}(r)\big)\,\delta_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,\bar{U}_\varepsilon(r)\Big)\,\mathrm{d}r\mathrm{d}s\end{aligned}$$

$$\bar{Z}_{p\varepsilon}[V,Z_p](U) = \dots$$

with

$$\Delta_\varepsilon(r) := \frac{1}{2}\,\mathcal{S}\,\dot{\bar{U}}^2_\varepsilon(r)\,\tilde{G}_\varepsilon(r) - \dot{\bar{U}}_\varepsilon(r)\,\frac{d}{dr}\Big(H\big(\bar{Z}_{p\varepsilon}(r)\big)\,\delta_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,\bar{U}_\varepsilon(r)\Big)\ \text{with}$$
$$\tilde{G}_\varepsilon(r) := \delta^{pq}\,H_{,p}\big(\bar{Z}_{w\varepsilon}(r)\big)\,\delta_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,\bar{Z}_{q\varepsilon}(r) + H\big(\bar{Z}_{w\varepsilon}(r)\big)\,\delta'_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,\bar{U}_\varepsilon(r)\,,$$
$$N_\varepsilon(r) := \sigma\,a^2 - \bar{U}^2_\varepsilon(r)\,H\big(\bar{Z}_{p\varepsilon}(r)\big)\,\delta_\varepsilon\big(\bar{U}_\varepsilon(r)\big)\,.$$

The generalized transformation

Theorem (SSSS,23)

The discontinuous coordinate transform $T = [(T_\varepsilon)_\varepsilon]$ is a generalized diffeomorphism on suitable subsets.

That means in particular (on suitable open Ω)

- $T_\varepsilon : \Omega \to T_\varepsilon(\Omega) =: \tilde{\Omega}_\varepsilon$ is a diffeomorphism & there is an open $\tilde{\Omega} \subseteq \bigcap_{\varepsilon \leq \eta} \tilde{\Omega}_\varepsilon$.
- For the inverses T_ε^{-1} there is an open $\Omega_1 \subseteq \bigcap_{\varepsilon \leq \eta} T_\varepsilon^{-1}(\tilde{\Omega})$.

Technically one has to

- estimate the minors of the Jacobian uniformly
- use a global univalence result by Gales & Nikaido
- use results on generalized diffeos by Erlacher & Grosser

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Hvala ljepo