Some topics in kinetic equations

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Contents

1-INTRO: Kinetic Theory, Vlasov Equation

- The model: ensemble of particles (mean field limit)
  - no internal structure
  - interaction only via collectively created field
- phase space distribution function $f(t,x,v) \geq 0$ ($t \in \mathbb{R}$, $x, v \in \mathbb{R}^3$)

$$\int \int_D f(t,x,v)\, dx\, dv = \# \text{ of particles with } (x,v) \in D \text{ at time } t$$

- no collisions: rate of change along particle parts $\frac{Df}{Dt} = 0$

$$\frac{Df}{Dt} = \partial_t f + \partial_x f \cdot \dot{x} + \partial_v f \cdot \dot{v}$$

- Newton's law: $\rightsquigarrow$ **Vlasov Equation**

$$\partial_t f + v \cdot \partial_x f + F \cdot \partial_v f = 0$$

$F(t,x) \ldots$ force field

1-INTRO: The Vlasov-Poisson system

Vlasov-Poisson: $F = -\nabla u,\ \Delta u = \pm 4\pi\rho = \pm 4\pi \int f\, dv$

$$\begin{aligned} \partial_t f + v \cdot \partial_x f - \partial_x u \cdot \partial_v f &= 0 && \text{(V)} \\ \Delta u &= \pm 4\pi\rho && \text{(P)} \\ \rho &= \int f\, dv && \text{(C)} \end{aligned}$$

$$f(0,x,v) = \overset{\circ}{f}(x,v) \in \mathcal{C}^1_c(\mathbb{R}^6) \qquad \lim_{|x|\to\infty} u(t,x) = 0 \quad \text{(IBC)}$$

- model: galaxy in Newtonian gravity, plasma in electrostatics
- global-in-time classical solutions, i.e., $f \in \mathcal{C}^1([0,\infty) \times \mathbb{R}^6)$ (Pfaffelmoser 1989, Schaeffer 1991, Lions & Perthame 1991)

Relativistic Vlasov-Poisson: replace v by $\hat{v} = v/\sqrt{1+|v|^2}$

- no general existence result; blow up in gravitational case

1-INTRO: Vlasov-type equations—Related System

Vlasov-Maxwell:

$$\begin{array}{rcll} \partial_t f + \hat{v}\cdot\partial_x f + (E + \hat{v}\times B)\cdot\partial_v f & = & 0 & \text{(V)} \\ (\partial_t - \Delta)E & = & -\partial_x\rho - \partial_t j & \\ (\partial_t - \Delta)B & = & \operatorname{rot} j & \text{(M)} \\ \rho = \int f\,dv & & j = \int \hat{v} f\,dv & \text{(C)} \end{array}$$

- model: plasma in relativistic electrodynamics
- global-in-time weak solutions (DiPerna, Lions 1989)
- global-in-time classical solutions **only**
  - with data rectrictions (Glassey, Schaeffer, Strauss, Rein)
  - in lower dimensions (Glassey, Schaeffer)
- no general result !

Vlasov-Einstein: General relativity...

(Andreasson, Rein, Rendall, . . .)

2-RVKG: The relativistic Vlasov-Klein Gordon system

$u(t,x)$ scalar Klein-Gordon field, $f(t,x,v) \geq 0$ distribution function

The RVKG system

$$\begin{array}{rcll} \partial_t f + \hat{v}\cdot\partial_x f - \partial_x u \cdot \partial_v f &=& 0 & \text{(V)} \\ \partial_t^2 u - \Delta u + u &=& -\rho & \text{(KG)} \\ \rho &=& \int f\,dv & \text{(C)} \end{array}$$

Initial data

$$f(0) = \overset{\circ}{f} \in \mathcal{C}^1_c(\mathbb{R}^6) \qquad \begin{array}{rcl} u(0) &=& \overset{\circ}{u}_1 \in \mathcal{C}^3(\mathbb{R}^3) \\ \partial_t u(0) &=& \overset{\circ}{u}_2 \in \mathcal{C}^2(\mathbb{R}^3) \end{array}$$

models an ensemble of collisionless particles

- moving at relativistic speed
- interacting by a quantum mechanical Klein-Gordon field

2-RVKG: Motivation, Aims and Results

Motivation and Aims

- coupling of a single classical particle to a quantum field: dynamics and asymptotics are an active area of research

  (Imaikin, Komech, Markowich, Spohn,. . .)

  $\longrightarrow$ RVKG generalizes this situation
- close relation to Vlasov-Maxwell and other systems
  $\longrightarrow$ hope to learn more about the general properties of these systems by studying RVKG

Results

1. Existence of local classical solutions plus continuation criterion

   (EJDE, Vol. 2005(1), 1-17, 2005)
2. Existence of global weak solutions for small data

   (Commun. Math. Phys., 238, 367-378, 2003)

2-1 RVKG-class: Solving (V) and (KG)

Solving (V) via method of characteristics

$$f(t,x,v) = \overset{\circ}{f}(X(0,t,x,v), V(0,t,x,v))$$

where $(X(t,s,x,v), V(t,s,x,v))$ is the solution of the characteristic system with initial data $X(s,s,x,v) = x,\ V(s,s,x,v) = v$.

- If the force $F(t,x)$ is nice, so is $f(t)$
- All L^p-norms of f are preserved in time, but **only** the L^1-norm of ρ!

Solution of (KG) $(\xi := \sqrt{(t-s)^2 - |x-y|^2},\ J_1$ Bessel fct.$)$

$$\begin{aligned} u(t,x) &= \text{data} + \frac{1}{4\pi}\int_0^t \int_{|x-y|=t-s} \rho(s,y)\, dS_y \frac{ds}{t-s} \\ &\quad - \frac{1}{4\pi}\int_0^t \int_{|x-y|\le t-s} \rho(s,y)\, \frac{J_1(\xi)}{\xi} dy\, ds \end{aligned}$$

- Problem: No gain in derivatives from ρ to u

2-1 RVKG-class: Towards local classical solutions (1)

A-priori bounds

on f, ρ, $\partial_x u$ and the velocity support $P(t) := \sup\{|v| : (x,v) \in \mathrm{supp} f(t)\}$ by splitting derivatives along and tangential to the light cone

$$S = \partial_t + \hat{v}\partial_x, \qquad T_j = -\frac{y_j - x_j}{|x-y|}\partial_t + \partial_j$$

$\leadsto$ **uniform bounds** applied to a suitable iterative scheeme

Iterative Scheeme

1. $f^{(0)} = \overset{\circ}{f},\ u^{(0)} = \overset{\circ}{u}_1$
2. Define $f^{(n)}$, $u^{(n)}$ recursively via

$$\begin{aligned} (\partial_t + \hat{v}\partial_x) f^{(n)} - \partial_x u^{(n-1)} \cdot \partial_v f^{(n)} &= 0 \\ \partial_t^2 u^{(n)} - \Delta u^{(n)} + u^{(n)} &= -\int f^{(n)}(t,x,v)\,dv \end{aligned}$$

3. $f^{(n)} \in \mathcal{C}^1(\mathbb{R}^+; \mathcal{C}_c^1(\mathbb{R}^6))$, $u^{(n)} \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^3)$ and $\partial_x u^{(n-1)}(t)$ is bounded on cp. sets.

2-1 RVKG-class: Towards local classical solutions (2)

Need also to show that

- $f^{(n)}(t)$, $\partial_x u^{(n)}(t)$ are $||\,||_\infty$-Cauchy (Existence)
- $\partial_{(x,v)} f^{(n)}(t)$, $\partial_x^2 u^{(n)}(t)$ are $||\,||_\infty$-Cauchy (Regularity)

need to control $\partial_x^2 u$ since

$$|\partial_x u^{(n+1)}(x^{(n+1)}) - \partial_x u^{(n)}(x^{(n)})| \leq ||\partial_x^2 u^{(n+1)}||_\infty \, |x^{(n+1)} - x^{(n)}| + ||\partial_x u^{(n+1)} \partial_x u^{(n)}||_\infty$$

Solution

- Use again representation formula.
- Apply derivatives.
- Rewrite ∂_x in terms of S and T.
- Work hard to obtain . . .

2-1 RVKG-class: Representation of $\partial_x^2 u$

Lemma. Let u be a $\mathcal{C}^2$ sol. of (KG) and $f \in \mathcal{C}^2$. Then we have

$$\partial_{k\ell} u(t,x) \;=\; F_0^{k\ell} + F_{SS}^{k\ell} + F_{ST}^{k\ell} + F_{TS}^{k\ell} + F_{TT}^{k\ell} + F_{RS}^{k\ell} + F_{RT}^{k\ell} + F_{JR}^{k\ell} + F_{JJ}^{k\ell},$$

for $k, \ell \in \{1,2,3,t\}$ where $(\zeta = t - |x-y|)$

$$\begin{aligned}
F_{SS}^{k\ell} &= \tfrac{-1}{4\pi} \int_{|x-y|\le t} \int c^{k\ell}(\omega,\hat{v})(S^2 f)(\zeta,y,v)dv \tfrac{dy}{|x-y|}, && |c^{k\ell}| \le \tfrac{C}{(1+\omega\cdot\hat{v})^2} \\
F_{ST}^{k\ell} &= \tfrac{1}{4\pi} \int_{|x-y|\le t} \int b_1^{k\ell}(\omega,\hat{v})(Sf)(\zeta,y,v)dv \tfrac{dy}{|x-y|^2}, && |b_1^{k\ell}| \le \tfrac{C}{(1+\omega\cdot\hat{v})^3} \\
F_{TS}^{k\ell} &= \tfrac{1}{4\pi} \int_{|x-y|\le t} \int b_2^{k\ell}(\omega,\hat{v})(Sf)(\zeta,y,v)dv \tfrac{dy}{|x-y|^2}, && |b_2^{k\ell}| \le \tfrac{C}{(1+\omega\cdot\hat{v})^3} \\
F_{TT}^{k\ell} &= \tfrac{1}{4\pi} \int_{|x-y|\le t} \int a^{k\ell}(\omega,\hat{v}) f(\zeta,y,v)dv \tfrac{dy}{|x-y|^3}, && \int a^{k\ell}(\omega,\hat{v})d\omega = 0, \\
F_{RS}^{k\ell} &= \tfrac{1}{8\pi} \int_{|x-y|\le t} \int d^{k\ell}(\omega,\hat{v})(Sf)(\zeta,y,v)dv \tfrac{dy}{|x-y|^2}, && |d^{k\ell}| \le \tfrac{C}{1+\omega\cdot\hat{v}} \\
F_{RT}^{k\ell} &= \tfrac{1}{8\pi} \int_{|x-y|\le t} \int e^{k\ell}(\omega,\hat{v})(Sf)(\zeta,y,v)dv \tfrac{dy}{|x-y|^2}, && |e^{k\ell}| \le \tfrac{C}{(1+\omega\cdot\hat{v})^2} \\
F_{JR}^{k\ell} &= \tfrac{-1}{32\pi} \int_{|x-y|\le t} \rho(\zeta,y)\omega_k\omega_\ell |x-y| dy,
\end{aligned}$$

$$F_{JJ}^{k\ell} = -\frac{1}{4\pi} \int_0^t \int_{|x-y|\le t-s} \rho(s,y) \Big(\frac{J_3(\xi)}{\xi^3}(x_k - y_k)(x_\ell - y_\ell) + \frac{J_2(\xi)}{\xi^2}\delta_{k\ell} \Big) dy\, ds.$$

2-1 RVKG-class: The Result

Theorem. Local existence and uniqueness, continuation criterion

Let $\overset{\circ}{f} \in \mathcal{C}^1_c(\mathbb{R}^6)$, $f \geq 0$, $\overset{\circ}{u}_1 \in \mathcal{C}^3(\mathbb{R}^3)$ and $\overset{\circ}{u}_2 \in \mathcal{C}^2(\mathbb{R}^3)$.

(i) Then there exists $T > 0$ and a unique solution (f, u) of the relativistic Vlasov-Klein-Gordon system on $[0, T)$

$$(f, u) \in \mathcal{C}^1([0, T); \mathcal{C}^1_c(\mathbb{R}^6)) \times \mathcal{C}^2([0, T) \times \mathbb{R}^3)$$

with the initial data $\overset{\circ}{f}$, $\overset{\circ}{u}_1$ and $\overset{\circ}{u}_2$.

(ii) Choose T maximal. If

$$\sup\{|v| : \ (x, v) \in \mathrm{supp} f(t), \ 0 \leq t < T\} < \infty.$$

Then $T = \infty$.

Moreover we have

- Conservation of mass and energy.
- In one space dimension the criterion (ii) always holds true.

2-2 RVGK-weak: A-Priori Bounds via Energy Conservation

The Energy of RVKG

$$\int \sqrt{1+|v|^2} f \, dxdv + \frac{1}{2} \int \left[|\partial_t u|^2 + |\partial_x u|^2 + |u|^2 \right] dx + \int \rho u dx =: E_K + E_F + E_C$$

- energy is conserved **but** E_C is not positive
- **Idea:** by Hölder, Sobolev and interpolation

$$\left| \int \rho u \, dx \right| \leq \|\rho(t)\|_{6/5} \|u(t)\|_6 \leq C \; E_K(t)^{1/2} \|\partial_x u(t)\|_2$$

  with C depending on $\|\mathring{f}\|_1$ and $\|\mathring{f}\|_\infty$
- **Solution:**
  - construct global classical solutions to regularized system
  - a-priori bounds for small data allow passage to weak limit

2-2 RVGK-weak: Towards Global Weak Solutions (1)

The regularized system

Replace KG by $\quad \partial_t^2 u - \Delta u + u = -\rho * \eta \quad (\eta \in \mathbb{C}_c^\infty(\mathbb{R}^3))$.
For given data $(\mathring{f}, \mathring{u}_1, \mathring{u}_2) \in \mathcal{C}_c^1 \times \mathcal{C}_\mathrm{b}^3 \times \mathcal{C}_\mathrm{b}^2$ there exist global classical solutions

$$(f, u) \in \mathcal{C}^1([0,\infty[\times\mathbb{R}^6) \times \mathcal{C}^2([0,\infty[\times\mathbb{R}^3).$$

Conservation of modified energy

Let $\eta = d * d$ with d even and (f, u) as above with data $(\mathring{f}, \mathring{u}_1 * \eta, \mathring{u}_2 * \eta)$.
Let $\tilde{u}$ be the unique solution to

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} + \tilde{u} = -\rho * d \quad \text{with data } (\mathring{u}_1 * d, \mathring{u}_2 * d).$$

Then

$$\tilde{E} := E_K + \frac{1}{2} \int [|\partial_t \tilde{u}|^2 + |\partial_x \tilde{u}|^2 + |\tilde{u}|^2]\, dx + E_C$$

is conserved.

Modified bounds for E_C

For $p \in]3/2, \infty]$ and $1/p + 1/q = 1$ and (f, u) as above we have

$$\left| \int u(t,x)\,\rho(t,x)\,dx \right| \leq C(\mathring{f})\,\|\partial_x \tilde{u}(t)\|_2\, E_K(t)^{1/2},\ \ t \geq 0,$$

with

$$C(\mathring{f}) := \left(\frac{4q}{\pi}\right)^{1/2} 3^{-7/6} \left(\frac{q+3}{q}\right)^{(q+3)/6} \|\mathring{f}\|_1^{(3-q)/6} \|\mathring{f}\|_p^{q/6}.$$

Velocity-averaging lemma (Golse, Lions, Perthame, Sentis)

to prove that weak limit of approximating sequence solves non-linear(!) V

$$\int f_n(\,.\,,\,.\,,v)\psi(v)\,dv \to \int f(\,.\,,\,.\,,v)\psi(v)\,dv \text{ in } L^2(]0,T[\times B_R(0)).$$

2-2 RVGK-weak: The result

Theorem. Global Weak Solutions

Let

- $\mathring{f} \in L^1_{\mathrm{kin}}(\mathbb{R}^6) \cap L^p(\mathbb{R}^6)$ for some $p \in [2, \infty]$, $\mathring{f} \geq 0$,
- $\mathring{u}_1 \in H^1(\mathbb{R}^3)$, $\mathring{u}_2 \in L^2(\mathbb{R}^3)$,
- $1/p + 1/q = 1$ and

$$\|\mathring{f}\|_1^{(3-q)/3} \|\mathring{f}\|_p^{q/3} < \frac{\pi}{2q} 3^{7/3} \left(\frac{q}{q+3} \right)^{(q+3)/3}$$

Then there exists a unique global weak solution

$$f \in L^\infty([0, \infty[, L^p(\mathbb{R}^6)), \;\; u \in L^\infty([0, \infty[, H^1(\mathbb{R}^3))$$

with

$$\partial_t u \in L^\infty([0, \infty[, L^2(\mathbb{R}^3))$$

of the relativistic Vlasov-Klein-Gordon system with these initial data.

2-2 RVGK-weak: The result—Details

That is, in particular

(a) (f,u) satisfies (V), (KG) in $\mathcal{D}'(]0,\infty[\times\mathbb{R}^6)$.

(b) The mapping

$$[0,\infty[\ni t \mapsto (f(t),u(t),\partial_t u(t)) \in L^2(\mathbb{R}^6)\times L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$$

is weakly continuous with $(f,u,\partial_t u)(0) = (\mathring{f},\mathring{u}_1,\mathring{u}_2)$.

In addition we have

- $f(t)\geq 0$ a.e., $\|f(t)\|_p \leq \|\mathring{f}\|_p,\ t\geq 0$.
- $\partial_t\rho + \mathrm{div} j = 0$ in $\mathcal{D}'(]0,\infty[\times\mathbb{R}^3)$ where $j(t,x) := \int \hat{v} f(t,x,v)\,dv$.
- The weak solution conserves mass: $\|f(t)\|_1 = \|\mathring{f}\|_1$ for a. a. $t\geq 0$.

3-GVP: Recall the Vlasov-Poisson system

Vlasov-Poisson:

$$\begin{aligned} \partial_t f + v\cdot\partial_x f - \partial_x u\cdot\partial_v f &= 0 && \text{(V)}\\ \Delta u &= 4\pi\gamma\rho && \text{(P)}\\ \rho &= \int f\,dv && \text{(C)} \end{aligned}$$

$$f(0,x,v) = \overset{\circ}{f}(x,v) \in \mathcal{C}^1_c(\mathbb{R}^6) \qquad \lim_{|x|\to\infty} u(t,x) = 0 \quad \text{(IBC)}$$

- model: galaxy in Newtonian gravity, plasma in electrostatics
- global-in-time classical solutions, i.e., $f \in \mathcal{C}^1([0,\infty)\times\mathbb{R}^6)$
  (Pfaffelmoser 1989, Schaeffer 1991, Lions & Perthame 1991)

3-GVP: Singular Limits of (VP)

Euler Poisson: $f(t,x,v) = g(t,x)\,\delta(v - w(t,x))$ (velocity field w)

(f,u) solves (VP) $\Leftrightarrow$ (g,w,u) solves Euler-Poisson w. zero pressure

$$\begin{aligned} \partial_t g + \mathrm{div}(gw) &= 0 \\ \partial_t w + (w\partial_x)u &= -\partial_x u \qquad (\mathrm{EP}_0) \\ \Delta u &= 4\pi\gamma g \end{aligned}$$

N-body problem: $f(t,x,v) = \sum_{k=1}^{N} \delta\big(x - x_k(t)\big)\,\delta\big(v - v_k(t)\big)$

(f,u) solves (VP) $\Leftrightarrow$ (x_k, v_k) solves N-body problem

$$\begin{aligned} \dot{x}_k &= v_k \\ \dot{x}_k &= \mp \sum_{k \neq j = 1}^{N} \frac{x_k - x_j}{|x_k - x_j|^3} \qquad (\mathrm{N}) \end{aligned}$$

3-GVP: Motivation and Aims

Why Singular Limits of (VP)?

1. Existence-theory of (VP) is much better
2. shell crossing singularities in (EP_0)
3. (N) has only local solutions, time of existence may shrink with growing N

Why Colombeau Generalized Solutions?

1. few rigorous classical results on singular limits of (VP): [Sandor 1996, Dietz/Sandor 1999] on (EP_0) [Neunzert 1984] on (N)
2. "weak" convergence results:
   - not on the whole time of existence of (EP_0)
   - convergence in measure spaces, ...
3. use of ad-hoc weak sol. concepts for nonlinear term $\partial_x u \, \partial_v f$ in (V)

3-GVP: What are Colombeau Algebras?

Algebras of generalized functions in the sense of J.F. Colombeau [Colombeau 1984, 1985] are differential algebras

- that contain the vector space of distributions and
- display maximal consistency with classical analysis (in the sense of L. Schwartz impossibility result).
  In particular
  - the product of $\mathcal{C}^\infty$ function
  - partial derivatives of distributions

  are preserved

Main ideas of the construction are

- regularization of distributions by nets of $\mathcal{C}^\infty$-functions
- asymptotic estimates in terms of a regularization parameter (quotient construction)

3-GVP: On the main result

The algebras used

$$
\begin{aligned}
\mathcal{E}(\mathbb{R}^+ \times \mathbb{R}^n) &:= \mathbb{C}^\infty(\mathbb{R}^+ \times \mathbb{R}^n)^{(0,1]} \\
\mathcal{E}_M^{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^n) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E} : \; \forall K \subset\subset \mathbb{R}^+ \; \forall \alpha \in \mathbb{N}_0^{n+1} \; \exists N \in \mathbb{N} : \\
&\qquad \sup_{(t,z) \in K \times \mathbb{R}^n} |\partial^\alpha u_\varepsilon(t,z)| = O(\varepsilon^{-N})\} \\
\mathcal{N}_{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^n) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E} : \; \forall K \subset\subset \mathbb{R}^+ \; \forall \alpha \in \mathbb{N}_0^{n+1} \; \forall m \in \mathbb{N} : \\
&\qquad \sup_{(t,z) \in K \times \mathbb{R}^n} |\partial^\alpha u_\varepsilon(t,z)| = O(\varepsilon^{m})\} \\
\mathcal{G}_{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^n) &:= \mathcal{G}_M^{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^n) / \mathcal{N}_{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^n).
\end{aligned}
$$

Theorem. Generalized solutions to the spherically symmetric (VP)-System

Let $\mathring{f} \in \mathcal{G}_g(\mathbb{R}^6)$ with a representative $(\mathring{f}_\varepsilon)_\varepsilon$ such that

(i) $(\mathring{f}_\varepsilon)_\varepsilon$ is compactly supported uniformly in ε, non-negative and spherically symmetric,

(ii) $||\mathring{f}_\varepsilon||_1 = M$ (the mass), and

(iii) $||\mathring{f}_\varepsilon||_\infty \leq C\sigma(\varepsilon)^{-1}$, where σ is an appropriate scale.

Then there exists a unique solution

$$(f, u) \in \mathcal{G}_{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^6) \times \mathcal{G}_{\tilde{g}}(\mathbb{R}^+ \times \mathbb{R}^3)$$

of (VP) with $f(0, x, v) = \mathring{f}(x, v)$ and u strongly vanishing at infinity. Moreover $f(t)$ is non-negative and spherically symmetric.

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3-GVP: On the Existence of Generalized Solutions

For fixed ε we have global-in-time classical solutions $(f_\varepsilon, u_\varepsilon)$.
To prove existence in $\mathcal{G}_{\tilde{g}}$ we only have to prove moderateness.

Lemma 0. (0^{th} order est.—no problem) $||f_\varepsilon||, ||\rho_\varepsilon||, ||\partial_x u_\varepsilon|| \leq C\sigma^{-N}$

Lemma 1. (1^{st} order estimates—it starts getting bad)

$$||\partial_{(x,v)} f_\varepsilon||_\infty \leq e^{C\sigma^{-2}} \quad ||\partial_x^2 u_\varepsilon||_\infty \leq C\sigma^{-2} \quad ||\partial_x \rho_\varepsilon||_\infty \leq e^{C\sigma^{-2}}$$

Why? Gronwall for the derivatives of the characteristics
$|\partial_x \dot{V}_\varepsilon| \leq ||\partial_x^2 u_\varepsilon|| \, |\partial_x V_\varepsilon| \leq ||\rho_\varepsilon|| \, |\partial_x V_\varepsilon| \leq C\sigma^{-2} \, |\partial_x V_\varepsilon| \rightsquigarrow ||\partial f_\varepsilon|| \leq e^{C\sigma^{-2}}$

Lemma 2. (Higher order estimates—it doesn't get worse)

$$||\partial^\alpha_{(x,v)} f_\varepsilon||_\infty \leq e^{C\sigma^{-2}} \quad ||\partial_x^{\beta+2} u_\varepsilon||_\infty \leq e^{C\sigma^{-2}} \quad ||\partial_x^\beta \rho_\varepsilon||_\infty \leq e^{C\sigma^{-2}}$$

Why? only $||\partial_x^2 u_\varepsilon||$ hence $||\rho||_\infty$ enters Gronwall

3-GVP: On the Uniqueness of Generalized Solutions

How to generalized the boundary condition $\lim_{|x|\to\infty} u(x) = 0 \quad (*)$?

Problem: Every $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}$ (est. on cp. sets!) has a rep. with $(*)$
$\rightsquigarrow$ naive way $\quad \lim_{|x|\to\infty} u_\varepsilon(x) = 0 \ \forall\varepsilon \ (**)$ doesn't work!
non-uniqueness: $\Delta 0 = 0 = \Delta 1$ and both satisfy $(**)$

Solution: Let $\rho \in \mathcal{G}_c(\mathbb{R}^3)$. We call $u \in \mathcal{G}_g(\mathbb{R}^3)$ a solution of (P) *vanishing at infinity* if $\Delta u = 4\pi\rho$ and if there exists a representative $(u_\varepsilon)_\varepsilon$ such that

(i) $\lim_{|x|\to\infty} u_\varepsilon(x) = 0 \quad \forall\varepsilon$, and

(ii) $\operatorname{supp}(\Delta u_\varepsilon)_\varepsilon \subseteq B_{\varepsilon^{-N}}(0)$

Proposition. Let $\rho \in \mathcal{G}_c(\mathbb{R}^3)$. Then there exists one and only one solution of $\Delta u = 0$ in $\mathcal{G}_g(\mathbb{R}^3)$ vanishing at infinity.

3-GVP: Outlook—Future Prospects

- study limits (association) of generalized solutions given by the theorem $\rightsquigarrow$ singular limits of (VP)
- study exact (generalized) solutions of (VP)
- ...
- shell crossing singularities of dust models in general relativity; non-unique continuation of solutions after the singularity regularizing by kinetic models !?! (Brien Nolan, DCU, Dublin)
- ...