Recent results in Lorentzian geometry and General Relativity in low regularity

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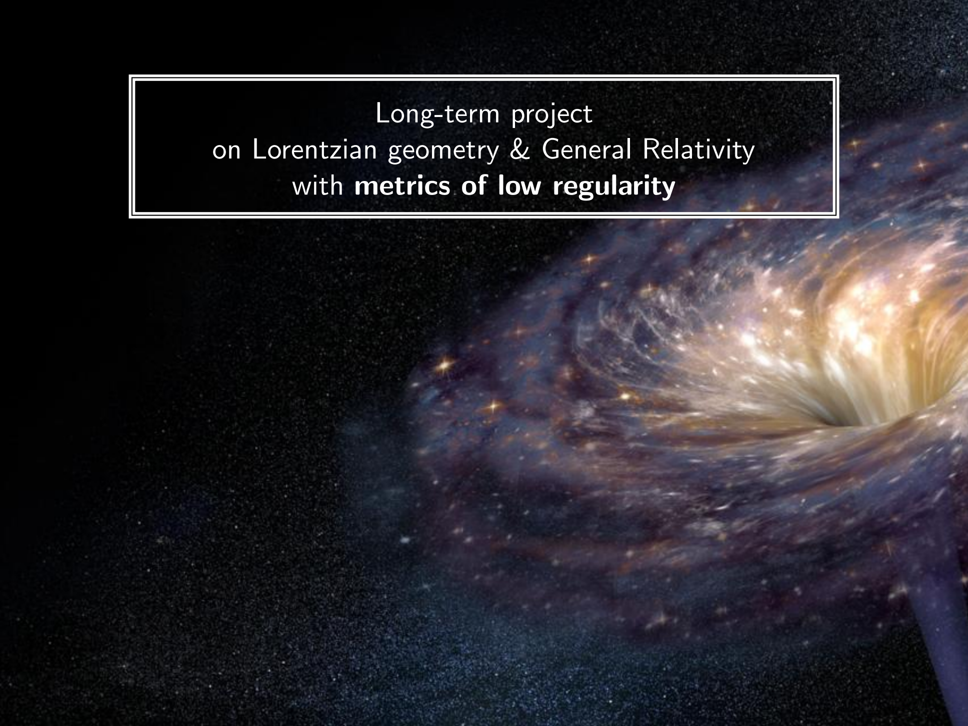
Long-term project
on Lorentzian geometry & General Relativity
with **metrics of low regularity**



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Theoretical branch:
Causality theory and
singularity theorems in $\mathcal{C}^{1,1}$

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M. Graf, G. Hörmann,
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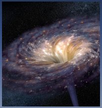
Exact solutions branch:
Impulsive gravitational waves
$\mathbf{g} \in \mathcal{C}^{0,1}$ of even $\mathbf{g} \in \mathcal{D}'$

Vienna & Prague
A. Lecke, C. Sämann,
R. S.

Jiří Podolský, Robert Švarc



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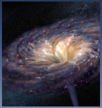

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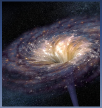
Geos vs. extremal curves I: $g \in C^\infty$

$(\mathbf{M}, \mathbf{g})\ldots$ C^∞-manifold with a R/L metric of certain regularity
geodesics... solutions γ of $\gamma'' = 0$ (straightest possible curves)
extremal curves... γ minimizes the length functional (R)
maximizes length for causally related pts. (L)



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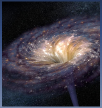
geo eq. loc. uniq. solvable	✓		
geos are locally extremal	✓		
extr. curves are (pre)geos	✓		



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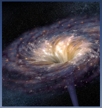
(1) needs exponential map & Gauss lemma
[Minguzzi 2015], [KSS 2014], [KSSV, 2014]



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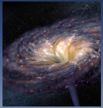
geos are locally extremal	✓	✓(1)	×
extr. curves are (pre)geos	✓	✓	C^1/?(2)

(1) needs exponential map & Gauss lemma
[Minguzzi 2015], [KSS 2014], [KSSV, 2014]
(2R) $g \in C^1 \Rightarrow$ minimizing curves are geodesics $\Rightarrow C^2$
(Du Bois-Reymond trick)
(2L) only for timelike **not causal**

Existence of extremal curves

An issue separate from the geodesic eq. if g below $C^{1,1}$!

$g \in C^0$	local	global
Riem.	length sructure Arzela-Ascoli [Hilbert, 1904]	Hopf-Rinow-Cohn-Vossen in length spaces & [Burtscher 15]
Lor.	existence of globally hyperoblic neighb. [Sämann 16]	continuous Avez-Seifert [Sämann 16]



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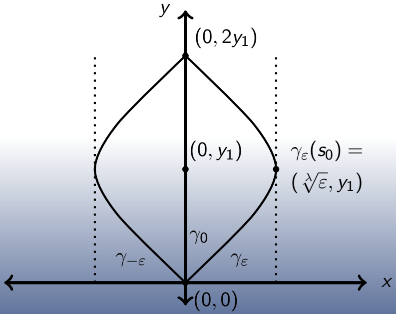
So extremal curves exist for continous metrics
but what is their relation to geodesics if they exist?

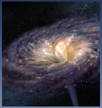
Geos vs. extremal curves II: below $C^{1,1}$

The shocking Hartman&Wintner-example (1951)

$$\mathbf{g}_{ij}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1-|x|^\lambda \end{pmatrix} \qquad \text{on } (-1,1)\times\mathbb{R} \subseteq \mathbb{R}^2$$



- $g \in \mathcal{C}^{1,\lambda-1}$, $\lambda \in (1,2)$ slightly below $\mathcal{C}^{1,1}$
- (nevertheless) geodesic equation uniquely solvable



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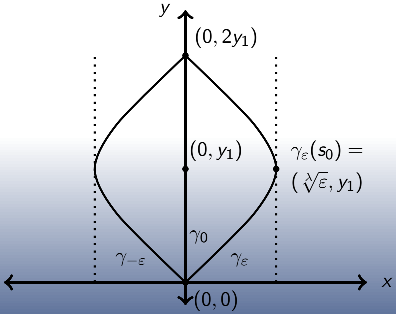

- $g\in\mathcal{C}^{1,\lambda-1}$, $\lambda\in(1,2)$ slightly below $\mathcal{C}^{1,1}$
- (nevertheless) geodesic equation uniquely solvable
- minimizing curves not unique, even locally
- y-axis is geo but non-minimising between any of its points

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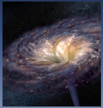
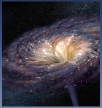


Impulsive gravitational waves

- model short but strong pulses of gravitational radiation propagating in de Sitter universe
- distributional metric on constant curvature background



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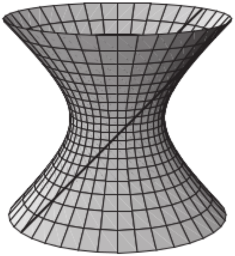
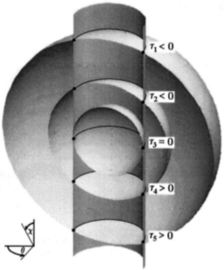
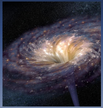


de Sitter universe



propagating wave



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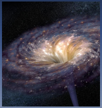

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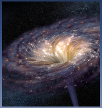
- relevant models of ultrarelativistic particle



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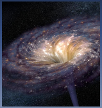
The spacetime: 4D-hyperboloid in 5D-Minkowski with imp. pp-wave

$$\mathrm{d}s^2 = \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2 - 2\mathrm{d}U\mathrm{d}V + H(Z_2, Z_3, Z_4)\delta(U)\mathrm{d}U^2$$

null-coords. $U = \frac{1}{\sqrt{2}}(Z_0 + Z_1)$, $V = \frac{1}{\sqrt{2}}(Z_0 - Z_1)$

$$Z_2^2 + Z_3^2 + Z_4^2 - 2UV = 3/\Lambda,$$

impulse on null hypersurface $\{U = 0\}$: ${Z_2}^2 + {Z_3}^2 + {Z_4}^2 = 3/\Lambda$



Geo equation, regularisation, model

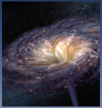
'Distributional equations'

$$\ddot{U} = -\frac{1}{3}\,\Lambda\,U\,e$$

$$\ddot{Z}_p - \frac{1}{2}\,H_{,p}\,\delta\,\dot{U}^2 = -\frac{1}{3}\,\Lambda\,Z_p\,\left(e + \frac{1}{2}\,G\,\delta\,\dot{U}^2\right)$$

$$\ddot{V} - \frac{1}{2}\,H\,\delta'\,\dot{U}^2 - \delta^{pq}\,H_{,p}\,\delta\,\dot{Z}_q\,\dot{U} = -\frac{1}{3}\,\Lambda\,V\,\left(e + \frac{1}{2}\,G\,\delta\,\dot{U}^2\right)$$

where $G := \delta^{pq}\,Z_p\,H_{,q} - H$ and $e = \pm 1, 0$ but $\delta = \delta(U(t))$



Geo equation, regularisation, model

Regularised equations

$$\ddot{U}_\varepsilon = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\dot{\big(H\,\delta_\varepsilon\,U_\varepsilon\big)}\Big)\;\frac{U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

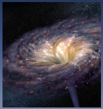
$$\ddot{Z}_{p\varepsilon} - \frac{1}{2}H_{,p}\,\delta_\varepsilon\,\dot{U}_\varepsilon^2 = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\dot{\big(H\,\delta_\varepsilon\,U_\varepsilon\big)}\Big)\;\frac{Z_{p\varepsilon}}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot{V}_\varepsilon - \frac{1}{2}\,H\,\delta_\varepsilon'\,\dot{U}_\varepsilon^2 - \delta^{pq}H_{,p}\,\delta_\varepsilon\,\dot{Z}_q^\varepsilon\,\dot{U}_\varepsilon = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\dot{\big(H\,\delta_\varepsilon\,U_\varepsilon\big)}\Big)\;\frac{V_\varepsilon + H\,\delta_\varepsilon\,U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

where

$$\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t))\,, \quad \delta_\varepsilon' = \delta_\varepsilon'(U_\varepsilon(t))\,,$$

$$\tilde{G}_\varepsilon = \tilde{G}_\varepsilon(U_\varepsilon(t), Z_{p\varepsilon}(t))\,, \quad H = H(Z_{p\varepsilon}(t))\,, \quad \text{and} \quad H_{,p} = H_{,p}(Z_{q\varepsilon}(t))$$



Geo equation, regularisation, model

good news!

$$\ddot{U}_\varepsilon = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\dot{\big)}\Big)\;\frac{U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot{Z}_{p\varepsilon} - \frac{1}{2}H_{,p}\,\delta_\varepsilon\,\dot{U}_\varepsilon^2 = -\Big(e + \frac{1}{2}\,\dot{U}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{U}_\varepsilon\,\big(H\,\delta_\varepsilon\,U_\varepsilon\dot{\big)}\Big)\;\frac{Z_{p\varepsilon}}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}$$

linear & decoupled. . . simply integrate at the end

where

$$\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t))\,, \quad \delta'_\varepsilon = \delta'_\varepsilon(U_\varepsilon(t))\,,$$

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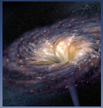

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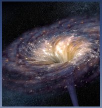


Geo equation, regularisation, model

Model system for $\gamma_\varepsilon \equiv x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathbb{R} \times \mathbb{R}^3$

$$\ddot{u}_\varepsilon = -\left(e + \frac{1}{2}\,\dot{u}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{u}_\varepsilon\,\left(H\,\delta_\varepsilon\,u_\varepsilon\right)^{\boldsymbol{\cdot}}\right)\;\frac{u_\varepsilon}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon}$$

$$\ddot{z}_\varepsilon - \frac{1}{2}DH\,\delta_\varepsilon\dot{u}_\varepsilon^2 = -\left(e + \frac{1}{2}\,\dot{u}_\varepsilon^2\,\tilde{G}_\varepsilon - \dot{u}_\varepsilon\,\left(H\,\delta_\varepsilon\,u_\varepsilon\right)^{\boldsymbol{\cdot}}\right)\;\frac{z_\varepsilon}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon}$$

with

$$H = H(z_\varepsilon) \in C^\infty(\mathbb{R}^3)$$

$$\tilde{G}_\varepsilon(u_\varepsilon, z_\varepsilon) := DH(z_\varepsilon)\,\delta_\varepsilon(u_\varepsilon)\,z_\varepsilon + H(z_\varepsilon)\,\delta'_\varepsilon(u_\varepsilon)\,u_\varepsilon$$

The full model system

Seed geodesics and initial conditions

The u-component of the seed geodesic γ (black) reaches the regularisation sandwich at

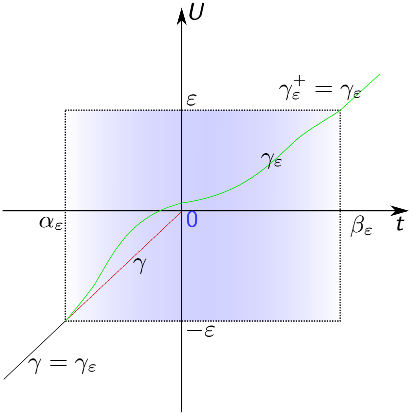
$$t = \alpha_\varepsilon, \text{ i.e., } u(\alpha_\varepsilon) = -\varepsilon.$$

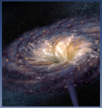
In the background spacetime γ would continue (dotted red) to

$$U = 0 \text{ at } t = 0.$$

In the regularised spacetime γ continues as γ_ε (green) solving the model equations.

Goal: show that γ_ε lives long enough to cross the sandwich for ε small.





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We look for solutions on

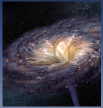
$$J_\varepsilon = [\alpha_\varepsilon, \alpha_\varepsilon + \eta] \quad (\eta > 0)$$

and set data at $t = \alpha_\varepsilon$

$$\gamma_\varepsilon(\alpha_\varepsilon) = (-\varepsilon, z_\varepsilon^0)$$
$$\dot\gamma_\varepsilon(\alpha_\varepsilon) = (\dot u_\varepsilon^0 (> 0), \dot z_\varepsilon^0)$$

where we additionally demand convergence to some seed data

$$(-\varepsilon, z_\varepsilon^0) \to (0, z^0)$$
$$(\dot u_\varepsilon^0 > 0, \dot z_\varepsilon) \to (\dot u^0 > 0, \dot z^0)$$



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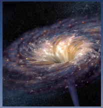

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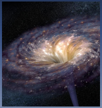
with data

$$\begin{aligned} x_\varepsilon(\alpha_\varepsilon) = (u_\varepsilon(\alpha_\varepsilon), z_\varepsilon(\alpha_\varepsilon)) &= (u_\varepsilon^0, z_\varepsilon^0) \\ &= (-\varepsilon, z_\varepsilon^0) \to (0, z^0) \in \mathbb{R} \times \mathbb{R}^3 \end{aligned}$$

$$\dot{x}_\varepsilon(\alpha_\varepsilon) = (\dot{u}_\varepsilon(\alpha_\varepsilon), \dot{z}_\varepsilon(\alpha_\varepsilon)) = (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0) \;\to (\dot{u}^0 (> 0), z^0) \in \mathbb{R} \times \mathbb{R}^3$$

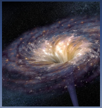


Solution space & operator

$$\mathfrak{X}_\varepsilon := \Big\{ x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathcal{C}^1(J_\varepsilon, \mathbb{R}^4) : \; x_\varepsilon(\alpha_\varepsilon) = x_\varepsilon^0, \; \dot{x}_\varepsilon(\alpha_\varepsilon) = \dot{x}_\varepsilon^0$$

$$\|x_\varepsilon - x^0\|_\infty \leq C_1, \; \|\dot{z}_\varepsilon - \dot{z}^0\|_\infty \leq C_2, \; \dot{u}_\varepsilon \in \Big[\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0\Big] \Big\}$$

- prospective solutions assume ε-dependent data
- centred around the 'fixed' data $(0, z^0)$ and $(\dot{u}_0, \dot{z}^0)$
- $\dot{u}_\varepsilon$ forced to stay positive
- $\mathfrak{X}_\varepsilon$ only depends on ε via the domain J_ε and data



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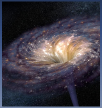
choose constants: $C_1 > 0, \; C_2 = C_2(\|DH\|_{B_{C_1}(z^0)}, \text{fixed data})$



Solution space & operator

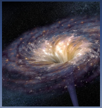
$$\mathfrak{X}_\varepsilon := \Big\{ x_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathcal{C}^1(J_\varepsilon, \mathbb{R}^4) : \; x_\varepsilon(\alpha_\varepsilon) = x_\varepsilon^0, \; \dot{x}_\varepsilon(\alpha_\varepsilon) = \dot{x}_\varepsilon^0$$

$$\|x_\varepsilon - x^0\|_\infty \le C_1, \; \|\dot{z}_\varepsilon - \dot{z}^0\|_\infty \le C_2, \; \dot{u}_\varepsilon \in \Big[\frac{1}{2}\dot{u}^0, \frac{3}{2}\dot{u}^0\Big] \Big\}$$

choose constants: $C_1 > 0$, $C_2 = C_2(\|DH\|_{B_{C_1}(z^0)}, \text{fixed data})$

$$A_\varepsilon^1(x_\varepsilon)(t) = -\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \frac{e u_\varepsilon + \frac{1}{2} u_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - u_\varepsilon \dot{u}_\varepsilon \left(H\delta_\varepsilon u_\varepsilon\right)^{\cdot}}{3/\Lambda - u_\varepsilon^2 H\delta_\varepsilon} \, \mathrm{d}r \, \mathrm{d}s + \dot{u}_\varepsilon^0 (t - \alpha_\varepsilon) - \varepsilon$$

$$A_\varepsilon^2(x_\varepsilon)(t) := \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \left(\frac{1}{2} DH\delta_\varepsilon \dot{u}_\varepsilon^2 - \frac{e z_\varepsilon + \frac{1}{2} z_\varepsilon \dot{u}_\varepsilon^2 \tilde{G}_\varepsilon - z_\varepsilon \dot{u}_\varepsilon \left(H\delta_\varepsilon u_\varepsilon\right)^{\cdot}}{\sigma a^2 - u_\varepsilon^2 H\delta_\varepsilon} \right) \mathrm{d}r \, \mathrm{d}s + \dot{z}_\varepsilon^0 (t - \alpha_\varepsilon) + z_\varepsilon^0$$



Local existence & uniqueness

A lot of interesting estimates lead to

$$\|(A_\varepsilon)^n(x_\varepsilon) - (A_\varepsilon)^n({x^*}_\varepsilon))\|_{C^1} \leq \frac{1}{\varepsilon}\,\beta_n\,\|x_\varepsilon - x_\varepsilon^*\|_{C^1} \quad \text{with} \sum \beta_n < \infty$$

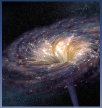
and so Weissinger's fixed point theorem applies.

Theorem (Existence and uniqueness)

The initial value problem for the geodesic equation has a unique smooth solution

$$\gamma_\varepsilon = (U_\varepsilon, V_\varepsilon, Z_\varepsilon) \quad \textit{on} \quad [\alpha_\varepsilon, \alpha_\varepsilon + \eta],$$

provided $\eta \neq \eta(\varepsilon)$ *and* ε *are small enough.*
Moreover γ_ε *is uniformly bounded in* ε *together with* $\dot{U}_\varepsilon$ *and* $\dot{Z}_\varepsilon$*.*



Local existence & uniqueness

$$\eta := \min\Bigg\{1,\ \frac{a^2}{24\dot{u}^0},\ \frac{C_1}{\frac{3}{2}+\dot{u}^0},\ \frac{2C_1}{54\|\rho\|_1\|DH\|_\infty\dot{u}^0},\ \frac{a^2\,C_1}{12(|z^0|+C_1)},\ \frac{a^2\,C_2}{8(|z^0|+C_1)},$$
$$\frac{C_1 a^2}{54}\Big(\dot{u}^0(|z^0|+C_1)(3\|DH\|_\infty\|\rho\|_\infty(|z^0|+C_1)+\|H\|_\infty\|\rho'\|_\infty)\Big)^{-1},\ \frac{C_1}{6(1+|\dot{z}^0|)},$$
$$\frac{C_1 a^2}{72}\Big((|z^0|+C_1)\Big(3\|DH\|_\infty\|\rho\|_\infty(|\dot{z}^0|+C_2)+\frac{3}{2}\dot{u}^0\|H\|_\infty(\|\rho'\|_\infty+\|\rho\|_\infty)\Big)\Big)^{-1}\Bigg\}$$

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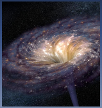

Local existence & uniqueness

$$\varepsilon \leq \varepsilon_0 := \min\Big\{ \frac{a^2}{2\|\rho\|_\infty \|H\|_\infty}, \frac{a^2}{72\dot{u}^0}\Big(3\|DH\|_\infty\|\rho\|_\infty(|z^0| + C_1) + \|H\|_\infty\|\rho'\|_\infty\Big)^{-1},$$
$$\frac{a^2}{96}\Big(3\|DH\|_\infty\|\rho\|_\infty(|\dot{z}^0| + C_2) + \frac{3}{2}\dot{u}^0\|H\|_\infty\big(\|\rho'\|_\infty + \|\rho\|_\infty\big)\Big)^{-1},$$
$$(3\|DH\|_\infty\|\rho\|_\infty(|\dot{z}^0| + C_2))^{-1}, \frac{\eta\dot{u}^0}{6}, \eta\Big\}.$$

Theorem (Existence and uniqueness)

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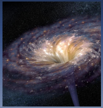


Extension of geodesics

Theorem (Extension of geodesics)

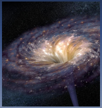
The geodesics γ_ε extend to geodesics of the background de Sitter spacetime 'behind' the sandwich wave zone.

Proof.

$$U_\varepsilon(\alpha_\varepsilon + \eta) = -\varepsilon + \int_{\alpha_\varepsilon}^{\alpha_\varepsilon+\eta} \dot{U}_\varepsilon(s)\,\mathrm{d}s \geq -\varepsilon + \frac{\eta}{2}\dot{U}^0 \geq -\varepsilon + 3\varepsilon \geq \varepsilon$$

since $\varepsilon \leq \eta\, \dot{U}^0/6$

For such ε, γ_ε leaves the wave zone and extends to a geodesic of the background spacetime since the geodesic equations coincide there. □



Basic completeness result

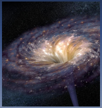
Theorem (Causal completeness)

Every causal geodesic in the entire class of regularised non-expanding impulsive gravitational waves propagating in de Sitter universe (with smooth profile function H) is complete, provided the regularisation parameter ε is chosen small enough.

Drawback: ε depends on the initial data of the very geodesic, so

- no result for periodic spacelike geodesics
- no spacetime completeness result

Extension: Non-smooth H is okay as long as the seed geodesic is not directly aimed at the singularities.



Spacetime completeness result

Use non-linear distributional geometry (simplified Colombeau)

- Turn above 'local solution candidate' into a global one

Spacetime completeness result

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Globalization Lemma
Let $u : (0,1] \times M \to \mathbb{R}^n$ be a smooth map and let (P) be a property attributable to values $u(\varepsilon, p)$, satisfying:

$$\forall K \subset\subset M \text{ there is } \varepsilon_K > 0 : \text{ (P) holds for all } p \in K \text{ and } \varepsilon < \varepsilon_K.$$

Then there is a smooth map $\tilde{u} : (0,1] \times M \to \mathbb{R}^n$ such that (P) holds globally.
Moreover for each $K \subset\subset M$ there exists some $\varepsilon_K \in (0,1]$ such that $\tilde{u}(\varepsilon, p) = u(\varepsilon, p)$ for all $(\varepsilon, p) \in (0, \varepsilon_K] \times K$.

Spacetime completeness result

Use non-linear distributional geometry (simplified Colombeau)

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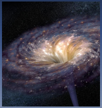
Spacetime completeness result

Use non-linear distributional geometry (simplified Colombeau)

- Turn above 'local solution candidate' into a global one
- prove moderateness and uniqueness



Spacetime completeness result

Use non-linear distributional geometry (simplified Colombeau)

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- prove moderateness and uniqueness

Theorem (Generalized spacetime completeness)

The generalized impulsive wave spacetime (M, g) *given by*

$$-2UV + Z_2^2 + Z_3^2 + Z_4^2 = 3/\Lambda$$

in the 5D-impulsive pp-wave

$$\mathrm{d}s^2 = dZ_2^2 + dZ_3^2 + dZ_4^2 - 2dUdV + H(Z_2, Z_3, Z_4)D(U)dU^2$$

is geodesically complete.

D, generalized δ-function with model δ-net representative

Some related Literature

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Hvala na pažnji