On the Geroch-Traschen class of metrics

Roland Steinbauer
University of Vienna

GF2009, Vienna
September 4, 2009

On the Geroch-Traschen class of metrics

Roland Steinbauer
University of Vienna

GF2009, Vienna
September 4, 2009

joint work with
James Vickers
University of Southampton

Rough metrics for relativity

(1) classically $g \in \mathcal{C}^\infty$ ... but $\mathcal{C}^{1,1}$ is okay

(2) "purely" distributional setting [Marsden, 68], [Parker, 79]
rather resticted

(3) "maximally reasonable" distributional setting
[Geroch&Traschen, 87]

(4) nonlinear distributional geometry
in the framework of Colombeau's special algebra
[Kunzinger, Vickers, Mayerhofer, S., since 02]

??? Compatibility of (3) and (4) ???

Answer: Yes, but ...

Distributional setting(s) for GR

- distributional metric [Marsden, 68], [Parker, 79]

$$g \in \mathcal{D}'^{0}_{2}(M) \cong \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty} \mathcal{T}^0_2(M) \cong L_{\mathcal{C}^\infty}(\mathfrak{X}(M), \mathfrak{X}(M); \mathcal{D}'(M))$$

symmetric and nondegenerate, i.e., $g(X, Y) = 0 \; \forall Y \Rightarrow X = 0$.

$\rightsquigarrow$ no way to define, inverse, curvature, ...

- "maximal reasonable" setting: Geroch-Traschen class

$$g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2 (M)$$

(gt-setting) [Geroch&Traschen, 87], [LeFloch&Mardare, 07]

Pro's: may define curvature $\mathrm{Riem}[\mathrm{g}]$, $\mathrm{Ric}[g]$, $\mathrm{R}[g]$, $W[g]$ in distributions
consistent limits $\rightsquigarrow$ valid modelling

Con's: Bianchi identities fail $\rightsquigarrow$ energy conservation ?
$\dim(\mathrm{supp}(\mathrm{Riem}[\mathrm{g}])) \geq 3 \rightsquigarrow$ thin shells yes, but strings no!

(Special Colombeau) Generalised setting for GR

- **generalised metric:** (technicalities on the index skipped)
  $g \in \mathcal{G}^0_2(M)$ symmetric and $\det(g)$ invertible in $\mathcal{G}$, i.e.,

  $$\forall K \text{ comp. } \exists m: \inf_{p\in K} |\det(g_\varepsilon(p))| \geq \varepsilon^m \qquad (N_\varepsilon)$$

  captures idea of smoothing: locally $\exists$ representative g_ε consisting of smooth metrics and $\det(g)$ invertible in $\mathcal{G}$
- **usual machinery works**, i.e.,
  - pointwise characterization of nondegeneracy
  - raise and lower indices: $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X,\,.\,) \in \mathcal{G}^0_1(M)$
  - $\exists!$ generalised Levi-Civita connection for g
  - generalised curvature $\mathrm{Riem}[g], \mathrm{Ric}[g], \mathrm{R}[g]$ via usual formulae
  - basic $\mathcal{C}^2$-compatibility: $g_\varepsilon \to g$ in $\mathcal{C}^2$, g a vacuum solution of Einstein's equation $\Rightarrow \mathrm{Ric}[g_\varepsilon] \to 0$ in $\mathcal{D}'^1_3$.

The question of compatibility

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ two ways to calculate the curvature
  - (i) gt-setting: coordinate formulae in $\mathcal{D}'$ resp. $W^{m,p}_{\mathrm{loc}}$
    $\rightsquigarrow \mathrm{Riem}[\mathrm{g}] \in \mathcal{D}'^1_3$
  - (ii) $\mathcal{G}$-setting: embed g via convolution with a mollifier
    usual formulae for fixed ε $\rightsquigarrow \mathrm{Riem}[\mathrm{g}_\varepsilon] \in \mathcal{G}^1_3$
- Do we get the same answer?

$$
\begin{array}{ccc}
H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni g & \xrightarrow{\;*\rho_\varepsilon\;} & [g_\varepsilon] \in \mathcal{G} \\
\text{gt-setting} \Big\downarrow & & \Big\downarrow \mathcal{G}\text{-setting} \\
\mathrm{Riem}[\mathrm{g}] & \xleftarrow{\;\lim_{\varepsilon\to 0}\;} & \mathrm{Riem}[\mathrm{g}_\varepsilon]
\end{array}
$$

On the gt-class of metrics

- $H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is an algebra
- $f \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ invertible $:\Leftrightarrow$ loc. unif. bounded away from 0

$$\forall K \text{ compact } \exists C : \ |f(x)| \geq \ C > 0 \text{ a.e. on } K$$

then f^{-1} is again loc. unif. bded away from 0

Definition (Nondegenerate gt-metrics [LeFM07], [SV09])

A gt-regular metric is a section $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$, which is a Semi-Riemannian metric almost everywhere.
It is called nondegenerate, if

$$\forall K \text{ compact } \exists C : \ |\det g(x)| \geq \ C > 0 \text{ a.e. on } K. \qquad (N)$$

$\Rightarrow \ g^{-1} \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$ and nondegenerate, i.e.,
$\det(g^{-1})$ loc. unif. bded away from 0

Embeddings and association

- scalars on $\Omega \subseteq \mathbb{R}^n$ open: $u \in \mathcal{E}'(\Omega)$

$$\begin{array}{c} u_\varepsilon := u * \rho_\varepsilon \\ \iota(u) := [(u_\varepsilon)_\varepsilon] \end{array} \quad \text{with} \quad \begin{array}{l} \rho \in \mathcal{S}(\mathbb{R}^n),\ \int \rho = 1,\ \rho_\varepsilon := \frac{1}{\varepsilon^n}\rho\left(\frac{\cdot}{\varepsilon}\right) \\ \int x^\alpha \rho(x) dx = 0\ \forall |\alpha| \geq 1 \end{array}$$

- $u \in \mathcal{D}'(\Omega)$: sheaf theoretic construction, or
  set $u_\varepsilon = u * \psi_\varepsilon, \quad \psi_\varepsilon(x) = \chi\left(\frac{x}{\sqrt{\varepsilon}}\right)\rho_\varepsilon(x), \quad \chi$ a cut-off

- ψ_ε is a strict δ-net (moderate, asymptotic vanishing moments)

  (i) $\mathrm{supp}(\psi_\varepsilon) \to \{0\}$ $(\varepsilon \to 0)$ (ii) $\int \psi_\varepsilon \to 1$ $(\varepsilon \to 0)$
  (iii) $\|\psi_\varepsilon\|_{L^1} \leq C$ for all ε (small)

- $g \in \left(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\right)^0_2(M)$: $g^\varepsilon_{ij} := g_{ij} * \psi_\varepsilon$, $\rightsquigarrow$ metric g_ε, $\iota(g) = [(g_\varepsilon)_\varepsilon]$

- association: $\mathcal{G} \ni u \approx v \in \mathcal{D}' :\Leftrightarrow \int u_\varepsilon \omega \to \langle v, \omega \rangle$

Smoothing gt-metrics

Basic properties of smoothing (ψ_ε a strict δ-net)

- $f \in L^1_{\mathrm{loc}} \Rightarrow f_\varepsilon = f * \psi_\varepsilon \in \mathcal{C}^\infty(\Omega_{\psi_\varepsilon})$
- $f \in W^{m,p}_{\mathrm{loc}} \;\Rightarrow\; f_\varepsilon := f * \psi_\varepsilon \to f \quad \text{in} \quad W^{m,p}_{\mathrm{loc}}$ for all m, $1 \le p < \infty$
- $f, h \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \;\Rightarrow\; f_\varepsilon h_\varepsilon \to fh \quad \text{in} \quad H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ for all $p < \infty$

Lemma (Stability of the determinant)

Let g be nondegenerate, gt-regular, then

$$\det(g_\varepsilon) \;\to\; \det g \quad \textit{in} \quad H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}} \textit{ for all } p < \infty.$$

- But (N) for g does not imply (N_ε) for g_ε and $m = 0$, (N^0_ε)!

g nondegenerate gt-regular metric $\not\Rightarrow$ g_ε generalised metric

Preserving nondegeneracy (1)

problem (1): preserving positivity for scalars

- want: $0 \le f \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ & loc. unif. bounded away from 0

  $\Rightarrow\ \forall K$ compact $\exists C, \varepsilon_0 :\ f_\varepsilon(x) \ge C > 0 \quad \forall x \in K,\ \varepsilon \le \varepsilon_0 \qquad (N'_\varepsilon)$

  Then $1/f_\varepsilon$ smooth, locally uniformly bounded net, and
  $1/f_\varepsilon \to 1/f$ in $\quad H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}}$ for all $p < \infty$.
- true if $\psi_\varepsilon \ge 0$, but ρ with vanishing moments $\Rightarrow \rho \not\ge 0 \Rightarrow \psi_\varepsilon \not\ge 0$

Lemma (Existence of admissible mollifiers)

There exist moderate strict delta nets ρ_ε with

(i) $\mathrm{supp}(\rho_\varepsilon) \subseteq B_\varepsilon(0)$ *(ii)* $\int \rho_\varepsilon(x)\, dx = 1$

(iii) $\forall j \in \mathbb{N}\ \exists \varepsilon_0 :\ \int x^\alpha \rho_\varepsilon(x)\, dx = 0$ *for all* $1 \le |\alpha| \le j$ *and all* $\varepsilon \le \varepsilon_0$

(iv) $\forall \eta > 0\ \exists \varepsilon_0 :\ \int |\rho_\varepsilon(x)|\, dx \le 1 + \eta$ *for all* $\varepsilon \le \varepsilon_0$.

Convolution with ρ_ε provides an embedding ι_ρ into $\mathcal{G}$ with (N'_ε).

Preserving nondegeneracy (2)

problem (2): preserving nondegeneracy for metrics

- want: $\forall K$ cp. $\exists C, \varepsilon_0 : |\det(g_\varepsilon)| \geq C_K > 0 \, \forall x \in K, \; \varepsilon \leq \varepsilon_0 \quad (N^0_\varepsilon)$

Definition (Stability condition)

Let g be a gt-regular metric and $\lambda_1, \ldots, \lambda_n$ its eigenvalues.

(i) For any compact K we set $\mu_K := \min_{1 \leq i \leq n} \operatorname{essinf}_{x \in K} |\lambda^i(x)|$.

(ii) We call g stable if on K there is A^K continuous, such that

$$\max_{i,j} \operatorname{essup}_{x \in K} |g_{ij}(x) - A^K_{ij}(x)| \leq C < \frac{\mu_K}{2n}.$$

Lemma (Nondegeneracy of smoothed gt-regular metrics)

Let g be a nondegenerate, stable, and gt-regular metric.
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.
Then (N^0_ε) holds, and the embedding $\iota_\rho(g)$ is a gen. metric.

Stability results

Lemma (Stability of the inverse and Christoffel symbols)

Let g be a nondegenerate, stable, and gt-regular metric.
Let g_ε be a smoothing of g with an admissible mollifier $(\rho_\varepsilon)_\varepsilon$.

(i) *The inverse of the smoothing $(g_\varepsilon)^{-1}$ is a smooth and locally uniformly bounded net (on rel. cp. sets for ε small), and*

$$(g_\varepsilon)^{-1} \to g^{-1} \text{ in } H^1_{\mathrm{loc}} \cap L^p_{\mathrm{loc}} \text{ for all } p < \infty.$$

In particular, for any embedding we have that $(\iota_\rho(g))^{-1} \approx g^{-1}$.

(ii) *The Christoffel symbols of the smoothing $\Gamma_{ijk}[g_\varepsilon]$, $\Gamma^i_{jk}[g_\varepsilon]$ are smooth and L^2_{loc}-bounded nets (on rel. cp. sets for ε small), and*

$$\Gamma_{ijk}[g_\varepsilon] \to \Gamma_{ijk} \text{ and } \Gamma^i_{jk}[g_\varepsilon] \to \Gamma^i_{jk} \text{ in } L^2_{\mathrm{loc}}$$

In particular, for any embedding $\Gamma_{ijk}[\iota_\rho(g)] \approx \Gamma_{ijk}[g]$ and $\Gamma^i_{jk}[\iota_\rho(g)] \approx \Gamma^i_{jk}[g]$.

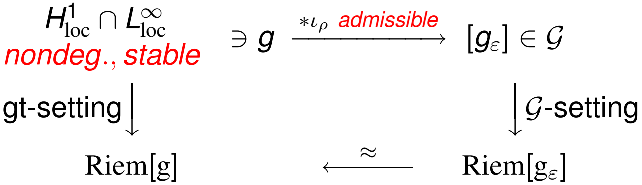
Compatibility results

Theorem (Compatibility of the gt- with the $\mathcal{G}$-setting)

Let g be a nondegenerate, stable, and gt-regular metric, and denote its Riemann tensor by $\mathrm{Riem}[g]$.
Let g_ε *be a smoothing of g with an admissible mollifier* $(\rho_\varepsilon)_\varepsilon$.
Then we have for the Riemann tensor $\mathrm{Riem}[g_\varepsilon]$ *of* g_ε

$$\mathrm{Riem}[g_\varepsilon] \to \mathrm{Riem}[g] \text{ in } \mathcal{D}'^1_3.$$

Hence for any embedding ι_ρ *we have* $\mathrm{Riem}[\iota_\rho(g)] \approx \mathrm{Riem}[g]$.



Discussion

Relation to older stability results: $(g_n)_n$ gt-regular sequence

- [LeFloch&Mardare, 07]
  $g_n \to g$ in H^1_{loc}, $g_n^{-1} \to g^{-1}$ in $L^\infty_{\mathrm{loc}} \Rightarrow \mathrm{Riem[g_n]} \to \mathrm{Riem[g]}$, in $\mathcal{D}'^1_3$.
  for smoothings via convolution $g_n^{-1} \not\to g^{-1}$ in L^∞_{loc}.
- [Geroch&Traschen, 87]
  $g_n \to g$ in H^1_{loc}, $g_n^{-1} \to g^{-1}$ in L^2_{loc}, $g_n,\ g_n^{-1}$ bded in L^∞_{loc} $(*)$
  $$\Rightarrow \mathrm{Riem[g_n]} \to \mathrm{Riem[g]} \text{ in } \mathcal{D}'^1_3.$$

Existence of approximating sequences with $(*)$

- [Geroch&Traschen, 87]
  Only for continuous g, open for general g.
- Positive answer for general g by the above Theorem.

Further prospects

- Jump conditions along singular hypersurfaces in the spirit of [LeFloch&Mardare, 07], [Lichnerowicz, 55-79] in the generalised setting plus compatibility. Applications to gravitational shock waves.

  Diploma thesis of Nastasia Grubic.

- Regularity of generalised solutions to wave equations in singular space-times ([Grant, Mayerhofer, S., 08]).
- Compatibility for connections in fibre bundles ([Kunzinger, Vickers, S., 05]).

Some References

- R. Geroch, J. Traschen, *Strings and Other distributional Sources in General Relativity,* Phys. Rev. D 36, 1987.
- P. LeFloch, C. Mardare, *Definition and Stability of Lorentzian Manifolds With Distributional Curvature,* Port. math. 64, 2007.
- R. Steinbauer, J. Vickers, *On the Geroch-Traschen class of metrics,* Class. Quantum Grav. 26, 2009.
- R. Steinbauer, *A note on distributional semi-Riemannian geometry,* Proceedings of the 12th Serbian Mathematical Congress, Novi Sad, September 2008, arXiv:0812.0173
- R. Steinbauer, J. Vickers, *The Use of Generalized Functions and Distributions in General Relativity,* Class. Q. Grav. 23, 2006.
- M. Kunzinger, R. Steinbauer, J. Vickers, *Generalised Connections and Curvature,* Math. Proc. Cambridge Philos. Soc. 139, 2005.
- M. Kunzinger, R. Steinbauer, *Generalized Pseudo-Riemannian Geometry,* Trans. Amer. Math. Soc. 354, 2002.